A characterization of the Anderson metal-insulator transport transition

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Dedicated to Jean-Michel Combes on the occasion of his Sixtieth Birthday

Abstract. We investigate the Anderson metal-insulator transition for random Schrödinger operators. We define the *strong insulator region* to be the part of the spectrum where the random operator exhibits strong dynamical localization in the Hilbert-Schmidt norm. We introduce a local transport exponent $\beta(E)$, and set the *metallic transport region* to be the part of the spectrum with nontrivial transport (i.e., $\beta(E) > 0$). We prove that these insulator and metallic regions are complementary sets in the spectrum of the random operator, and that the local transport exponent $\beta(E)$ provides a characterization of the *metal-insulator transport transition.* Moreover, we show that if there is such a transition, then $\beta(E)$ has to be discontinuous at a *transport mobility edge.* More precisely, we show that if the transport is nontrivial then $\beta(E) \geq \frac{1}{2d}$, where d is the space dimension. These results follow from a proof that slow time evolution of quantum waves irandom media implies the starting hypothesis for the authors' bootstrap multiscale analysis. We also conclude that the strong insulator region coincides with the part of the spectrum where we can perform a bootstrap multiscale analysis, proving that the multiscale analysis is valid all the way up to a transport mobility edge.

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1. Introduction

In his seminal 1958 article [An], Anderson argued that a Schrödinger operator in a highly disordered medium would exhibit exponentially localized eigenstates, in contrast to the extended eigenstates of a Schrödinger operator in a periodic medium. For a Schrödinger operator with a random potential and spectrum of the form $[E_0, \infty)$, the following picture (e.g., [LGP, Section 4.2]) is widely accepted: The region of the spectrum near the bottom of the spectrum E_0 results from large fluctuations of the potential, with the corresponding states localized primarily in the regions of such fluctuations. But in three or more dimensions, at very large energies the kinetic term should dominate the fluctuations of the potential to produce extended states. Thus a transition must occur from an *insulator regime*, characterized by localized states, to a very different *metallic regime* characterized by extended states. The energy E_{me} at which this *metal-insulator transition* occurs is called the *mobility edge*. The medium should have zero conductivity in the *insulator region* $[E_0, E_{\mathrm{me}}]$ and nonzero conductivity in the *metallic region* $[E_{\mathrm{me}}, \infty)$. The standard mathematical interpretation of this picture is that the random Schrödinger operator should have pure point spectrum with exponentially decaying eigenstates in the interval $[E_0, E_{\mathrm{me}}]$ and absolutely continuous spectrum on the interval $[E_{\mathrm{me}}, \infty)$.

Fortysome years have passed since Anderson's article, but our mathematical understanding of this picture is still unsatisfactory and one-sided: we know that there exists an energy $E_1 > E_0$ such that the random Schrödinger operator exhibits exponential localization (i.e., pure point spectrum with exponentially decaying eigenstates) in the interval $[E_0, E_1]$ (e.g., [GMP, KS, FS, HM, FMSS, CKM, vDK, AM, Ai, CH1, Klo2, KSS, Wa2, GK1, Klo3, Klo4, GK3]). But up to now there are no mathematical results on the existence of continuous spectrum and a metal-insulator transition. (Except for the special case of the Anderson model on the Bethe lattice, where one of us has proved that for small disorder the random operator has purely absolutely continuous spectrum in a nontrivial interval [Kle1] and exhibits ballistic behavior [Kle2].) The existence of a mobility edge separating pure point spectrum from pure absolutely continuous spectrum remains a conjecture. Moreover, the issue of the nature of the metal-insulator transition, if it exists, is widely open. The possibility of an interval of singular continuous spectrum interpolating between the pure point spectrum and the expected absolutely continuous spectrum cannot be ruled out, nor can the possible coexistence of spectra of different type (but see [JaL]).

The intuitive physical notion of localization has also a dynamical interpretation: an initially localized wave packet should remain localized under time evolution. In a periodic medium there is ballistic motion: the n-th moment of an initially localized wave packet grows with time as t^n [AK,KL]. In a random medium the insulator regime should exhibit *dynamical localization*: all moments of an initially localized wave packet are uniformly bounded in time.

Exponential and dynamical localization are not equivalent notions. Although dynamical localization implies pure point spectrum by the RAGE Theorem (e.g., the argument in [CFKS, Theorem 9.21]), the converse is not true. Dynamical localization is actually a strictly stronger notion than pure point spectrum: exponential localization can take place whereas a quasi-ballistic motion is observed [DR+1, DR+2]. Dynamical localization always excludes transport, but exponential localization may allow transport. (See also [DR+2, Tc] for an analysis of the difference between the two notions of localization.) These considerations raise the question of what is the appropriate characterization of the insulator region.

But in spite of the differences between exponential and dynamical localization, it turns out that for the Anderson model, the most commonly studied random Schrödinger operator, wherever exponential localization has been proved, so far, so has dynamical localization, even strong (i.e., in expectation) dynamical localization, both on the lattice [Ai, ASFH] and on the continuum [GDB, DSS, GK1]. (For similar results in related contexts see [Ge, DBG, JiL, GJ, DSS].) In fact, one can always proves more: strong dynamical localization in the Hilbert-Schmidt norm [GK1].

There are similar questions about the metallic region. Absolutely continuous spectrum (and more generally uniformly α-Hölder continuous spectrum with $\alpha \in (0,1]$)) is known to force nontrivial transport [Gu, Co, La]. But the situation is not clear as far as point or singular continuum spectrum is concerned, since either kind of singular spectrum may or may not give rise to nontrivial transport. It is possible to go through different types of spectra while the transport properties remain essentially the same. (E.g., [GKT], where an example is given of a random decaying potential which exhibits a transition from pure point to singular continuous spectrum, with the Hausdorff dimension going from 0 to 1, but for which the lower asymptotic transport exponent $\beta(E)$ (see (2.20)) is equal to 1 everywhere on the spectrum.) Thus a *spectral* transition is far from being sufficient to determine a *transport* transition.

In this article we present a new approach to the metal-insulator transition based on transport instead of spectral properties. This new point of view, in addition to being closer to the physical meaning of a "metal-insulator" transition, allows for a better understanding of the transition as shown in this paper. We define the *strong insulator region* to be the part of the spectrum where the random operator exhibits strong dynamical localization in the Hilbert-Schmidt norm, and hence no transport. We introduce a local transport exponent $\beta(E)$, and set the *metallic transport region* to be the part of the spectrum with nontrivial transport (i.e., $\beta(E) > 0$). We prove that these insulator and metallic regions are complementary sets in the spectrum of the random operator. (This rules out the possibility of trivial transport, i.e., transport with $\beta(E) = 0$.) Since the strong insulator region is defined as a relatively open subset of the spectrum, there is a natural definition of a *transport mobility edge*. We thus show that the local transport exponent $\beta(E)$ provides a characterization of the *metal-insulator transport transition*. Moreover, we show that if there is such a transition, then $\beta(E)$ has to be discontinuous at a transport mobility edge. More precisely, we show that if the transport is nontrivial then $\beta(E) \geq \frac{1}{2d}$, where d is the space dimension.

These results follow from a proof that slow time evolution of quantum waves in random media implies the starting hypothesis for the authors' bootstrap multiscale analysis [GK1]. We also conclude that the strong insulator region coincides with the part of the spectrum where we can perform a bootstrap multiscale analysis, proving that the multiscale analysis is valid all the way to a transport mobility edge.

It turns out that the strong insulator region may be defined by a large number of very natural properties, all equivalent. There is an appealing analogy with classical statistical mechanics: the energy is the parameter that corresponds to the temperature, the region of exponential localization is the analogous concept to the single phase region with exponentially decaying correlation functions, and the strong insulator region corresponds to the region of complete analyticity [DS1, DS1], possessing every possible virtue we can imagine!

In this article our results are stated for random Schrödinger operators in the continuum, but the present analysis remains valid in the more general setting when there is a Wegner estimate and the bootstrap multiscale analysis can be performed. In particular, it applies to the Anderson model in the lattice, to classical waves in random media as in [FK1,FK2,KK1,KK2], and to Landau Hamiltonians with random potentials as in [CH2,Wa1,GK3].

This paper is organized as follows: In Section 2 we first introduce the random Schrödinger operators we consider in this article; their relevant properties are proven in Appendix A. We then define the strong insulator and the metallic transport regions, and state our main results: Theorems 2.8, 2.10, and 2.11. The first two theorems are consequences of the third, which is proven in Section 6. In Section 3 we study properties of transport exponents. Section 4 is devoted to the study of the strong insulator region; we show that it may be defined by a large number of natural properties, all equivalent (Theorem 4.2). In Section 5 we give a characterization of the metallic transport region, and a criterion for an energy to be in this region (Theorem 5.1).

2. Statement of main results

In this article a *random Schrödinger operator* will be a random operator of the form

$$H_\omega = -\Delta + V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x), \tag{2.1}$$

where Δ is the d-dimensional Laplacian operator and V_ω is a random potential, i.e., $\{V_\omega(x);\ x \in \mathbb{R}^d\}$ is a real valued measurable process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that:

(R) Regularity: $V_\omega = V_\omega^{(1)} + V_\omega^{(2)}$, where $\{V_\omega^{(i)}(x);\ x \in \mathbb{R}^d\}$, $i = 1, 2$, are real valued measurable processes on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for $\mathbb{P}$-a.e. ω we have:
  - (R_1) $0 \le V_\omega^{(1)} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x)$.
  - (R_2) $V_\omega^{(2)}$ is relatively form-bounded with respect to $-\Delta$ with relative bound < 1.

(E) $\mathbb{Z}^d$-ergodicity: There is an ergodic family $\{\tau_y;\ y \in \mathbb{Z}^d\}$ of measure preserving transformations on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $V^{(i)}_{\tau_y \omega}(x) = V_\omega^{(i)}(x - y)$ for $i = 1, 2$ and all $y \in \mathbb{Z}^d$.

(IAD) Independence at a distance: There exists $\varrho > 0$ such that for any bounded subsets B_1, B_2 of $\mathbb{R}^d$ with $\operatorname{dist}(B_1, B_2) > \varrho$ the processes $\{V_\omega(x);\ x \in B_1\}$ and $\{V_\omega(x);\ x \in B_2\}$ are independent.

It follows from (R) that H_ω is defined as a semi-bounded self-adjoint operator for $\mathbb{P}$-a.e. ω. Note that using (R_2) and the ergodicity given in (E), we conclude that there are nonnegative constants $\Theta_1 < 1$ and Θ_2 such that for all $\psi \in \mathcal{D}(\nabla)$ we have

$$\left| \left\langle \psi, V_\omega^{(2)} \psi \right\rangle \right| \le \Theta_1 \|\nabla \psi\|^2 + \Theta_2 \|\psi\|^2 \quad \text{for } \mathbb{P}\text{-a.e. } \omega\,. \tag{2.2}$$

Thus $H_\omega \ge -\Theta_2$ for $\mathbb{P}$-a.e. ω. Moreover, H_ω is a random operator, i.e., the mappings $\omega \to f(H_\omega)$ are strongly measurable for all bounded measurable functions on $\mathbb{R}$. (That $H_\omega^{(2)} = -\Delta + V_\omega^{(2)}$ is a random operator follows from [KM, Proposition 6]. Using $H_\omega^{(2)} \ge -\Theta_2$ and the Trotter product formula for $\mathrm{e}^{-t\left(H_\omega^{(2)} + V_\omega^{(1)}\right)}$, we conclude that H_ω is a random operator as in [KM, Proof of Proposition 4].) In view of (E), it now follows from [KM, Theorem 1] that there exists a nonrandom set Σ such that $\sigma(H_\omega) = \Sigma$ with probability one, and that the decomposition of $\sigma(H_\omega)$ into pure point spectrum, absolutely continuous spectrum, and singular continuous spectrum is also independent of the choice of ω with probability one.

A typical example is given by an Anderson-type Hamiltonian, a random Schrödinger operator with a random potential of the form

$$V_\omega = V_{\mathrm{per}} + W_\omega \ , \tag{2.3}$$

where V_{per} is a periodic potential (by rescaling we take the period to be one), and

$$W_\omega(x) = \sum_{i \in \mathbb{Z}^d} \lambda_i(\omega) u(x-i), \tag{2.4}$$

where u is a real valued measurable function with compact support, and the $\{\lambda_i(\omega);\ i \in \mathbb{Z}^d\}$ are independent identically distributed random variables; see [CH1,Klo2,KSS]. We require $V_{\mathrm{per}} = V_{\mathrm{per}}^{(1)} + V_{\mathrm{per}}^{(2)}$, with $V_{\mathrm{per}}^{(i)}$, $i = 1, 2$, periodic with period one, $0 \le V_{\mathrm{per}}^{(1)} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x)$, $V_{\mathrm{per}}^{(2)}$ relatively form-bounded with respect to $-\Delta$ with relative bound < 1 (e.g., $V_{\mathrm{per}}^{(2)} \in L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x)$ with $p > \frac{d}{2}$ if $d \ge 2$ and $p = 2$ if $d = 1$), $u \in \mathrm{L}^q(\mathbb{R}^d, \mathrm{d}x)$ with $q > \frac{d}{2}$ if $d \ge 2$ and $q = 2$ if $d = 1$, and take the random variables $\lambda_i(\omega)$ to be bounded. It follows that W_ω is a potential in Kato class for $\mathbb{P}$-a.e. ω (see [Si]), and $H_\omega = V_{\mathrm{per}} + W_\omega$ is a random Schrödinger operator satisfying conditions (R), (E), and (IAD). Other examples of random Schrödinger operators are studied in [Klo1,CH1,CHM,CHN,HK,CHKN].

Remark 2.1. Although in this paper we only treat explicitly random Schrödinger operators on the continuum, our results also apply to random Schrödinger operators on the lattice. These are of the form $H_\omega = -\Delta + V_\omega$ on $\ell^2(\mathbb{Z}^d)$, where Δ is the discrete Laplacian and $\{V_\omega(x);\ x \in \mathbb{Z}^d\}$ is a real valued stochastic process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We still require conditions (E) and (IAD), but (R) is not needed since the discrete Laplacian is a bounded operator. (It is not hard to see that the results of [GK2] hold for any Schrödinger operator on the lattice, with the constants in the estimates independent of the potential.) Such operators include the usual Anderson model (e.g., [FS,FMSS,MS,vDK,AM,Ai,ASFH, Wa2,Klo3]).

A random Schrödinger operator satisfies all the requirements for the bootstrap multiscale analysis [GK1] with the possible exception of a Wegner estimate (Theorem A.1). It also satisfies an interior estimate (Lemma A.2) and the kernel polynomial decay estimate of [GK2, Theorem 2] (see Theorem A.5).

In this article a Wegner estimate in an open interval (Assumption W in [GK1]) will be an explicit hypothesis in our theorems. To state it we need to consider the restriction of a random Schrödinger operator H_ω to a finite box. By $\Lambda_L(x)$ we denote the open box (or cube) of side $L > 0$:

$$\Lambda_L(x) = \{y \in \mathbb{R}^d;\ \ \|y - x\| < L/2\}, \tag{2.5}$$

and by $\overline{\Lambda}_L(x)$ the closed box, where Throughout this paper we use the sup norm in $\mathbb{R}^d$:

$$\|x\| = \max\{|x_i|,\ i = 1, \ldots, d\}\ . \tag{2.6}$$

(We will use $|x|$ to denote the usual Euclidean norm.) *In this article we will always take boxes with side* $L \in 2\mathbb{N}$. The operator $H_{\omega,x,L}$ is defined as the restriction of H_ω, either to the open box $\Lambda_L(x)$ with Dirichlet boundary condition, or to the closed box $\overline{\Lambda}_L(x)$ with periodic boundary condition. (We consistently work with either Dirichlet or periodic boundary condition, and denote by $\|\ \|_{x,L}$ the norm or the operator norm on $\mathrm{L}^2(\Lambda_L(x), \mathrm{d}y)$.) To see that $H_{\omega,x,L}$ is well defined as a semi-bounded self-adjoint operator on $\mathrm{L}^2(\Lambda_L(x), \mathrm{d}y)$, note that if $\nabla_{x,L}$ is the gradient operator restricted to either to the open box $\Lambda_L(x)$ with Dirichlet boundary condition, or to the closed box $\overline{\Lambda}_L(x)$ with periodic boundary condition, then it follows from (2.2) that for all $\psi \in \mathcal{D}(\nabla_{x,L})$ we have

$$\left| \left\langle \psi, V_\omega^{(2)} \psi \right\rangle_{x,L} \right| \leq \Theta_1 \|\nabla_{x,L}\psi\|_{x,L}^2 + \Theta_2 \|\psi\|_{x,L}^2 \quad \text{for } \mathbb{P}\text{-a.e. } \omega\,. \tag{2.7}$$

(For Dirichlet boundary condition (2.7) follows immediately from (2.2) for all boxes with the same Θ_1 and Θ_2 as in (2.2). For periodic boundary condition (2.7) follows from (2.2) by using a smooth partition of the identity on the torus, with the same Θ_1 but with Θ_2 enlarged by a finite constant depending only on the dimension d, so we can modify Θ_2 in (2.2) so (2.2) and (2.7) hold with the same Θ_1 and Θ_2 for all boxes $\Lambda_L(x)$.) We write $R_{\omega,x,L}(z) = (H_{\omega,x,L} - z)^{-1}$ for the resolvent.

We say that the random Schrödinger operator H_ω satisfies a Wegner estimate in an open interval $\mathcal{I}$ if for every $E \in \mathcal{I}$ there exists a constant Q_E, bounded on compact subintervals of $\mathcal{I}$, such that

$$\mathbb{P}\{\operatorname{dist}(\sigma(H_{\omega,x,L}), E) \leq \eta\} \leq Q_E \eta L^d\ , \tag{2.8}$$

for all $\eta > 0$, $x \in \mathbb{Z}^d$, and $L \in 2\mathbb{N}$.

Remark 2.2. Wegner estimates have been proven for a large variety of random operators [Weg, HM, CKM, CL, PF, CH1, Klo2, CH2, CHM, Ki, FK1, FK2, Wa1, KSS, St, CHN, HK, CHKN, KK2]. In some of these estimates one gets L^{bd} instead of L^d in the right-hand-side of (2.8), with $b > 1$. Recently the expected volume dependency (i.e., L^d) has been obtained for certain random operators, at the price of loosing a bit in the η dependency [CHN, HK, CHKN]. In this paper, we shall use (2.8) as stated, the modifications in our methods required for these other forms of (2.8) being obvious (see Remark 2.13). Our methods may also accomodate (2.8) being valid only for large L, and/or only for $\eta < \eta_L$ with $\eta_L = L^{-r}$, $r > 0$.

Remark 2.3. For Bernoulli and other singular potentials, there are Wegner-type estimates, but they are not of the same form as (2.8); they only estimate the probabilities of sub-exponentially small distances to the spectrum, i.e., $\eta = \mathrm{e}^{-L^{\beta}}$ with $0 < \beta < 1$ [CKM, LKS, DSS]. While the bootstrap multiscale analysis may still be performed with these Wegner-type estimates (see [GK1, Remark 3.13, Theorems 5.6 and 5.7], and hence applied to the random Schrödinger operators in [CKM, LKS, DBG, DSS], the results of this paper are not applicable to such operators with our proof of Theorem 2.11. This is due to the fact that whereas we only have polynomial decay for the operator kernels of smooth functions of these operators (see Theorem A.5), the bootstrap multiscalse analysis for such operators requires sub-exponentially small probabilities for bad events.

If $x \in \mathbb{R}^d$ we write $\langle x \rangle = \sqrt{1+|x|^2}$. We use $\langle X \rangle$ to denote the operator given by multiplication by the function $\langle x \rangle$. By χ_x we denote the characteristic function of the the cube of side 1 centered at $x \in \mathbb{R}^d$. Given an open interval $I \subset \mathbb{R}$, we denote by $C_c^\infty(I)$ the class of real valued infinitely differentiable functions on $\mathbb{R}$ with compact support contained in I, with $C_{c,+}^\infty(I)$ being the subclass of nonnegative functions. The Hilbert-Schmidt norm of an operator A is written as $\|A\|_2$, i.e., $\|A\|_2^2 = \operatorname{tr} A^*A$. $C_{a,b,\ldots}$ will always denote some finite constant depending only on $a, b, \ldots$.

We start by defining the (random) moment of order $n \geq 0$ at time t for the time evolution in the Hilbert-Schmidt norm, initially spatially localized in the cube of side one around the origin, and "localized" in energy by the function $\mathcal{X} \in C_{c,+}^\infty(\mathbb{R})$, by

$$M_\omega(n, \mathcal{X}, t) = \left\| \langle X \rangle^{\frac{n}{2}} \mathrm{e}^{-itH_\omega} \mathcal{X}(H_\omega) \chi_0 \right\|_2^2 , \tag{2.9}$$

its expectation by

$$\mathbf{M}(n, \mathcal{X}, t) = \mathbb{E}\{ M_\omega(n, \mathcal{X}, t) \} , \tag{2.10}$$

and its time averaged expectation by

$$\mathcal{M}(n, \mathcal{X}, T) = \frac{2}{T} \int_0^\infty \mathrm{e}^{-\frac{2t}{T}} \mathbf{M}(n, \mathcal{X}, t)\, \mathrm{d}t. \tag{2.11}$$

These quantities are always finite for $\mathcal{X} \in C_{c,+}^\infty(\mathbb{R})$ (see Proposition 3.1).

Definition 2.4. *The random Schrödinger operator H_ω exhibits strong HS-dynamical localization in the open interval I if for all $\mathcal{X} \in C^\infty_{c,+}(I)$ we have*

$$\mathbb{E}\left\{\sup_{t\in\mathbb{R}} M_\omega(n, \mathcal{X}, t)\right\} < \infty \quad \textit{for all } n \geq 0\,. \tag{2.12}$$

The random operator H_ω exhibits strong HS-dynamical localization at the energy $E \in \mathbb{R}$ if there exists an open interval I, with $E \in I$, such that there is strong HS-dynamical localization in the open interval I.

The intuitive idea behind the last definition is that the moments of an initially localized wave packet remain uniformly bounded under time evolution "localized" in an open interval around the energy E. By taking the Hilbert-Schmidt norm we take into account all possible wave packets localized in a given bounded region.

Note that H_ω exhibits strong HS-dynamical localization in an open interval I if and only if H_ω exhibits strong HS-dynamical localization at every energy $E \in I$, as it should. The "if" part can be shown by using the compacteness of the support of functions $\mathcal{X} \in C^\infty_{c,+}(I)$ and a smooth partition of unity.

Definition 2.5. *The strong insulator region Σ_{SI} for H_ω is defined as*

$$\Sigma_{\mathrm{SI}} = \{E \in \Sigma\,;\ H_\omega \textit{ exhibits strong HS-dynamical localization at } E\}\,.$$

Note that Σ_{SI} is a relatively open subset of the spectrum Σ.

The existence of a nontrivial strong insulator region is now proven for the usual random Schrödinger operators. It is a consequence of well established results on Anderson localization and of the bootstrap multiscale analysis [GK1, Theorem 3.4] that yields strong HS-dynamical localization [GK1, Corollary 3.10]. The relevant results, adapted for this article, are stated in Theorem 4.1.

Definition 2.6. *The multiscale analysis region Σ_{MSA} is defined as the set of energies where we can perform the bootstrap multiscale analysis:*

$$\Sigma_{\mathrm{MSA}} = \{E \in \Sigma\,;\ \textit{ the hypotheses of Theorem 4.1 hold at } E\}\,.$$

Note that the conclusion of Theorem 4.1 is that $\Sigma_{\mathrm{MSA}} \subset \Sigma_{\mathrm{SI}}$.

On the lattice strong HS-dynamical localization turns out to be the same as strong dynamical localization (of wave packets) and was originally proven by the Aizenman-Molchanov method [Ai, ASFH]. Note that if Σ_{AM} denotes the set of energies in the spectrum satisfying the the starting hypothesis of the Aizenman-Molchanov method [AM, Ai, ASFH], we have $\Sigma_{\mathrm{AM}} = \Sigma_{\mathrm{MSA}}$.

On the continuum strong dynamical localization of operators (not just of wave packets) appears to be the appropriate notion. The most natural definition from the point of view of applicability was given in Definition 2.4 and uses the Hilbert-Schmidt norm. But it is also natural to use operator norms, we may say that H_ω exhibits strong operator-norm-dynamical localization in an open interval or at an energy if we replace condition (2.12) in Definition 2.4 by

$$\mathbb{E}\left\{\sup_{t\in\mathbb{R}}\left\|\langle X\rangle^{\frac{n}{2}}\mathrm{e}^{-itH_\omega}\mathcal{X}(H_\omega)\chi_0\right\|\right\}<\infty \quad \text{for all } n\geq 0\,. \tag{2.13}$$

It turns out that for random Schrödinger operators the two notions are equivalent (see Theorem 4.2), as pointed out to the authors by B. Simon with a different proof.

In Section 4 we show that the strong insulator region is defined by a large number of very natural properties, all equivalent. In the analogy with classical statistical mechanics: the strong insulator region corresponds to the region of complete analyticity [DS1,DS1].

We now turn to transport properties. If $\mathcal{X}\in C^\infty_{c,+}(\mathbb{R})$, we have that $\mathcal{X}(H_\omega)$ is either $=0$ or $\neq 0$ with probability one. To measure the rate of growth of moments of initially spatially localized wave packets under the time evolution, "localized" in energy by $\mathcal{X}\in C^\infty_{c,+}(\mathbb{R})$ with $\mathcal{X}(H_\omega)\neq 0$, we compute the upper and lower transport exponents

$$\beta^+(n,\mathcal{X})=\limsup_{T\to\infty}\frac{\log\mathcal{M}(n,\mathcal{X},T)}{n\log T}\,, \tag{2.14}$$

$$\beta^-(n,\mathcal{X})=\liminf_{T\to\infty}\frac{\log\mathcal{M}(n,\mathcal{X},T)}{n\log T}\,. \tag{2.15}$$

(Note that we normalize by n.) If $\mathcal{X}(H_\omega)=0$ we set $\beta^\pm(n,\mathcal{X})=0$. We define the *$n$-th upper and lower transport exponents* in an open interval I by

$$\beta^\pm(n,I)=\sup_{\mathcal{X}\in C^\infty_{c,+}(I)}\beta^\pm(n,\mathcal{X})\,, \tag{2.16}$$

and the *n-th local upper and lower transport exponents* at the energy E by

$$\beta^\pm(n,E)=\inf_{I\ni E}\beta^\pm(n,I)\,. \tag{2.17}$$

Roughly speaking, the exponents $\beta^\pm(n,E)$ provide a measure of the rate of transport for which E is responsible. (We discuss an inversion formula for (2.17) in Remark 3.3.)

In Proposition 3.2 we show that each exponent is increasing in n and prove the ballistic bound

$$0\leq\beta^\pm(n,\mathcal{X}),\ \beta^\pm(n,I),\ \beta^\pm(n,E)\leq 1\,. \tag{2.18}$$

Note that $\beta^\pm(n,E)=0$ if $E\notin\Sigma$.

The *asymptotic upper and lower transport exponents* may thus be defined by

$$\beta^\pm(I)=\lim_{n\to\infty}\beta^\pm(n,I)=\sup_n\beta^\pm(n,I)\,, \tag{2.19}$$

and the *local asymptotic upper and lower transport exponents* by

$$\beta^{\pm}(E) = \lim_{n\to\infty} \beta^{\pm}(n,E) = \sup_n \beta^{\pm}(n,E)\,, \tag{2.20}$$

and we have $0 \le \beta^{\pm}(I), \beta^{\pm}(E) \le 1$, with $\beta^{\pm}(E) = 0$ if $E \notin \Sigma$. Note that $\beta^{\pm}(E) > 0$ if and only if $\beta^{\pm}(n,E) > 0$ for some $n > 0$.

In this paper we will mostly work with the lower transport exponents; for convenience we will drop the superscript and use simply β for β^-, i.e., we will write $\beta(E)$ for $\beta^-(E)$, etc.

Definition 2.7. *The metallic transport region Σ_{MT} for H_ω is defined as the set of energies with nontrivial transport:*

$$\Sigma_{\mathrm{MT}} = \{E \in \mathbb{R},\ \beta(E) > 0\} = \{E \in \Sigma,\ \beta(E) > 0\}\,. \tag{2.21}$$

Its complementary set in the spectrum will be called the trivial transport region Σ_{TT} (note that logarithmic transport is not excluded a priori):

$$\Sigma_{\mathrm{TT}} = \Sigma \backslash \Sigma_{\mathrm{MT}} = \{E \in \Sigma,\ \beta(E) = 0\}\,. \tag{2.22}$$

It follows from the definitions and [GK1, Corollary 3.10] that

$$\Sigma_{\mathrm{MSA}} \subset \Sigma_{\mathrm{SI}} \subset \Sigma_{\mathrm{TT}}\,. \tag{2.23}$$

Our first theorem states that if we have a Wegner estimate in an open interval $\mathcal{I}$, then we have equality in (2.23) inside $\mathcal{I}$. We use the notation $B^{\mathcal{I}} = B \cap \mathcal{I}$ for a subset B of $\mathbb{R}$.

Theorem 2.8. *Let H_ω be a random Schrödinger operator satisfying a Wegner estimate in an open interval $\mathcal{I}$. Then $\Sigma_{\mathrm{TT}}^{\mathcal{I}} \subset \Sigma_{\mathrm{MSA}}^{\mathcal{I}}$ and hence*

$$\Sigma_{\mathrm{MSA}}^{\mathcal{I}} = \Sigma_{\mathrm{SI}}^{\mathcal{I}} = \Sigma_{\mathrm{TT}}^{\mathcal{I}}\,. \tag{2.24}$$

In particular, the strong insulator region and the metallic transport region are complementary sets in the spectrum $\Sigma^{\mathcal{I}}$ of H_ω in $\mathcal{I}$, i.e.,

$$\Sigma_{\mathrm{SI}}^{\mathcal{I}} \cap \Sigma_{\mathrm{MT}}^{\mathcal{I}} = \emptyset \quad \textit{and} \quad \Sigma_{\mathrm{SI}}^{\mathcal{I}} \cup \Sigma_{\mathrm{MT}}^{\mathcal{I}} = \Sigma^{\mathcal{I}}\,. \tag{2.25}$$

The equality (2.24) shows that the strong insulator region is canonical in the sense that it may be defined by three equivalent conditions or properties, all very natural. In fact we will see in Theorem 4.2 that the number of such conditions/properties is actually much larger.

Remark 2.9. Theorem 2.8 asserts that the range of applicability of the multiscale analysis is optimal in the sense that it includes the whole strong insulator region. From this point of view, Theorem 2.8 may be regarded as the converse to the multiscale analysis introduced in [FS], of which the bootstrap version of [GK1] is the most powerful version. By showing that the input and the conclusion of the bootstrap multiscale analysis are equivalent, Theorem 2.8 shows that the multiscale analysis functions everywhere in the strong insulator region, and thereby, all the way to a metal-insulator transport transition (if any). This leads to a characterization of the metallic transport region (Theorem 5.1).

Since the strong insulator region is a relatively open subset of the spectrum, we have

$$\Sigma_{\mathrm{SI}} = \left\{ \bigcup_{j=1}^{N} I_j \right\} \cap \Sigma \,, \tag{2.26}$$

where the I_j's are disjoint open intervals; N may be either finite or infinite. An energy $\tilde{E} \in \Sigma$ that is an edge of one of the intervals I_j will be called a *transport mobility edge*. If we have a Wegner estimate on an open interval $\mathcal{I}$, then if $\tilde{E} \in \mathcal{I}$ is a transport mobility edge it follows from Theorem 2.8 that we must have $\tilde{E} \in \Sigma_{\mathrm{MT}}$.

Our second theorem shows that the transport exponents have a discontinuity at a transport mobility edge, by providing an estimate on the minimal rate of transport that can exist inside the metallic transport region.

Theorem 2.10. *Let H_ω be a random Schrödinger operator satisfying a Wegner estimate in an open interval $\mathcal{I}$. If $\beta(E) > 0$ for some $E \in \mathcal{I}$ then $\beta(E) \geq \frac{1}{2d}$, i.e. the metallic transport region in $\mathcal{I}$ is given by*

$$\Sigma_{\mathrm{MT}}^{\mathcal{I}} = \left\{ E \in \mathcal{I},\ \beta(E) \geq \frac{1}{2d} \right\} . \tag{2.27}$$

In fact, if $\beta(E) > 0$ for some $E \in \mathcal{I}$, then $\beta(n, E) \geq \frac{1}{2d} - \frac{11}{2n}$ for all $n \geq 0$.

Theorem 2.8 shows that the local transport exponent $\beta(E)$ provides a characterization of the metal-insulator transport transition. Theorem 2.10 says that if this transition occurs, $\beta(E)$ has to be discontinuous at a transport mobility edge.

To put this result in perspective, note that existence of absolutely continuous spectrum would imply $\beta(E) \geq \frac{1}{d}$ [Gu, Co]. In fact, the existence of uniformly α-Hölder continuous spectrum ($\alpha \in (0, 1]$) implies $\beta(E) \geq \frac{\alpha}{d}$ [La]. (While the Guarnieri-Combes-Last bound is stated for a fixed self-adjoint operator, the same bound follows for random operators using Fatou's Lemma and Jensen's inequality.) But the converse is not true, a lower bound on the local transport exponent does not specify the spectrum (e.g., [DR+2, La, DBF, BGT, CM, GKT]).

We stress that the lower bound on the local transport exponent supplied by Theorem 2.10 is obtained without any knowledge of the type of spectrum that may exist in Σ_{MT}.

Theorems 2.8 and 2.10 are consequences of our main technical result, namely Theorem 2.11 below, which ensures that slow transport cannot take place for random Schrödinger operators satisfying our assumptions. Note that a weaker form of this result has been discussed by Martinelli and Scoppola [MS, Section 8] for the discrete Anderson model.

Theorem 2.11. *Let H_ω be a random Schrödinger operator satisfying a Wegner estimate in an open interval $\mathcal{I}$. Let $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$, with $\mathcal{X} \equiv 1$ on some open interval $J \subset \mathcal{I}$, $\alpha \geq 0$, and $n > 2d\alpha + 11d$. If*

$$\liminf_{T \to \infty} \frac{1}{T^\alpha} \mathcal{M}(n, \mathcal{X}, T) < \infty \,, \tag{2.28}$$

then $J \cap \Sigma \subset \Sigma_{\mathrm{MSA}}$, and hence $J \cap \Sigma \subset \Sigma_{\mathrm{SI}}$.

Remark 2.12. If we only have (2.28) for some $n > 2d\alpha + 8d$ (rather than $n > 2d\alpha + 11d$), the proof of Theorem 2.11 shows that $\Sigma_{\mathrm{MSA}} \backslash J$ has Lebesgue measure zero, and hence under the hypotheses of [CHM, Corollary 1.3] we have Anderson localization in $J \cap \Sigma$ by spectral averaging.

Remark 2.13. If we have a Wegner estimate with L^{bd} instead of L^d in the right-hand-side of (2.8), as in [HM,MS], where $b = \frac{d}{2} + 2$, and [Ki, FK1,FK2,KSS,St,KK2], where $b = 2$, the only changes in our results would be that in Theorem 2.11 we would need $n > 2bd\alpha + (9b + 2)d$, and hence we would have $\beta(E) \geq \frac{1}{2bd}$ in (2.27). If we have $Q_E \eta^s L^d$ with $0 < s < 1$ in the right-hand-side of (2.8), as in [CHN,CHKN, HK], there are no changes in our results.

Theorem 2.11 has the following immediate corollary, which can be read as follows: if the transport at an energy E is too slow (i.e., $\beta(n, E) < \frac{1}{2d} - \frac{11}{2n}$ for some $n > 11d$), then strong HS-dynamical localization has to hold at E.

Corollary 2.14. *Let H_ω be a random Schrödinger operator satisfying a Wegner estimate in an open interval $\mathcal{I}$. If $E \in \mathcal{I} \cap \Sigma$ and $\beta(n, E) < \frac{1}{2d} - \frac{11}{2n}$ for some $n > 11d$, then $E \in \Sigma_{\mathrm{MSA}} \subset \Sigma_{\mathrm{SI}}$.*

Proof. If $\beta(n_0, E) < \frac{1}{2d} - \frac{11}{2n_0}$ for a given $n_0 > 11d$, we pick $\alpha > 0$ such that $\beta(n_0, E) < \frac{\alpha}{n_0} < \frac{1}{2d} - \frac{11}{2n_0}$. It follows from (2.17) and (2.16) that there is an open interval $I \ni E$, such that $\beta(n_0, \mathcal{X}) < \frac{\alpha}{n_0}$ for all $\mathcal{X} \in C^\infty_{c,+}(I)$, and hence we have (2.28) with $n_0 > 2d\alpha + 11d$ for all $\mathcal{X} \in C^\infty_{c,+}(I)$. Since we can pick $\mathcal{X} \in C^\infty_{c,+}(I)$ such that $\mathcal{X} \equiv 1$ on some open interval $J \subset \mathcal{I}$, with $E \in J$, we can apply Theorem 2.11 to conclude that $E \in \Sigma_{\mathrm{MSA}} \subset \Sigma_{\mathrm{SI}}$. $\square$

Theorem 2.8 follows immediately from Corollary 2.14, since $\beta(E) = 0 \Rightarrow \beta(n, E) = 0$ for all $n \geq 0$. The same is true for Theorem 2.10, since if $\beta(n, E) < \frac{1}{2d} - \frac{11}{2n}$ for some $n > 11d$, it follows from Corollary 2.14 that $E \in \Sigma_{\mathrm{SI}}$ and hence $\beta(E) = 0$.

Theorem 2.11 is proven in Section 6. We show that (2.28) implies the starting hypothesis for the bootstrap multiscale analysis of [GK1] (recalled below as Theorem 4.1), which only requires polynomial decay of the finite volume resolvent at some large scale with probability close to one (how close being independent of the scale). The kernel polynomial decay estimate, which follows from [GK2], plays an important role in the proof. The Wegner estimate also plays a major role, in particular, it is used to rule out a possible set of energies of zero Lebesgue measure of singular energies where the starting hypothesis for the bootstrap multiscale analysis may not hold.

3. Transport exponents

In this section we study properties of the moments (2.9)-(2.11) and of the transport exponents (2.14)-(2.17) of a random Schrödinger operator H_ω.

Proposition 3.1. *Let $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$ such that $\mathcal{X}(H_\omega) \neq 0$ with probability one. Then*

$$0 \leq M_\omega(0, \mathcal{X}, 0) \leq M_\omega(n, \mathcal{X}, t) \leq C_{d,\Theta_1,\Theta_2,\mathcal{X},n} \langle t\rangle^{[n+\frac{3d}{2}]+3} \quad (3.1)$$

for $\mathbb{P}$-a.e. ω ,

$$0 < \mathbf{M}(0, \mathcal{X}, 0) \leq \mathbf{M}(n, \mathcal{X}, t) \leq C_{d,\Theta_1,\Theta_2,\mathcal{X},n} \langle t\rangle^{[n+\frac{3d}{2}]+3}, \quad (3.2)$$

$$0 < \mathbf{M}(0, \mathcal{X}, 0) \leq \mathcal{M}(n, \mathcal{X}, T) \leq C'_{d,\Theta_1,\Theta_2,\mathcal{X},n} \langle T\rangle^{[n+\frac{3d}{2}]+3}, \quad (3.3)$$

where $[u]$ denotes the largest integer $\leq u$.

Proof. It is easy to see that

$$0 \leq M_\omega(0, \mathcal{X}, 0) = M_\omega(0, \mathcal{X}, t) \leq M_\omega(n, \mathcal{X}, t)\,. \quad (3.4)$$

Since $\mathbf{M}(0, \mathcal{X}, 0) = \mathbb{E}\left(\|\mathcal{X}(H_\omega)\chi_x\|_2^2\right)$ for all $x \in \mathbb{Z}^d$ as H_ω is $\mathbb{Z}^d$-ergodic, it follows that $\mathbf{M}(0, \mathcal{X}, 0) > 0$. Thus we also have the first two inequalities in (3.2).

To prove the last inequality in (3.1), note that if $\mathcal{Y} \in C^\infty_c(\mathbb{R}; \mathbb{C})$, we have

$$\begin{aligned}\|\chi_x \mathcal{Y}(H_\omega)\chi_0\|_2^2 &= \operatorname{tr}\left(\chi_x \mathcal{Y}(H_\omega)\chi_0 \mathcal{Y}(H_\omega)\chi_x\right) \\ &\leq \|\chi_x \mathcal{Y}(H_\omega)\chi_0\| \|\chi_0 \mathcal{Y}(H_\omega)\chi_x\|_1\,,\end{aligned} \quad (3.5)$$

where $\|B\|_1$ denotes the trace norm of the operator B. We now pick $\nu > \frac{d}{4}$ and use

$$\|\chi_{x_0} \langle X\rangle^n \| \le 2^{\frac{n}{2}} \langle x_0\rangle^n \tag{3.6}$$

to get

$$\begin{aligned}\|\chi_x \mathcal{Y}(H_\omega)\chi_0\|_1 &= \|\chi_x \langle X\rangle^{2\nu} \langle X\rangle^{-2\nu} \mathcal{Y}(H_\omega) \langle X\rangle^{-2\nu} \langle X\rangle^{2\nu} \chi_0\|_1 \\ &\le 2^{2\nu} \langle x\rangle^{2\nu} \| \langle X\rangle^{-2\nu} \mathcal{Y}(H_\omega) \langle X\rangle^{-2\nu} \|_1 \\ &\le 2^{2\nu} \langle x\rangle^{2\nu}\, 4 \| \langle X\rangle^{-2\nu} |\mathcal{Y}|(H_\omega) \langle X\rangle^{-2\nu} \|_1 \\ &\le 2^{2\nu+2} \mathcal{T}_{\nu,d,\Theta_1,\Theta_2} \|\mathcal{Y}\Phi_{d,\Theta_1,\Theta_2}\|_\infty \langle x\rangle^{2\nu} ,\end{aligned} \tag{3.7}$$

where we used (A.12).

It now follows from (3.5), (3.7) and the kernel decay estimate (A.15) that

$$\|\chi_x \mathcal{Y}(H_\omega)\chi_0\|_2^2 \le C_{\nu,d,\Theta_1,\Theta_2,k} \|\mathcal{Y}\Phi_{d,\Theta_1,\Theta_2}\|_\infty |||\mathcal{Y}|||_{k+2} \langle x\rangle^{-k+2\nu} \tag{3.8}$$

for $\mathbb{P}$-a.e. ω and all $k = 1, 2, \ldots$.

We conclude from (3.6) and (3.8) that for $\mathbb{P}$-a.e. ω

$$\begin{aligned} M_\omega(n, \mathcal{X}, t) &\le 2^{\frac{n}{2}} \sum_{x\in\mathbb{Z}^d} \langle x\rangle^n \|\chi_x \mathrm{e}^{-itH_\omega} \mathcal{X}(H_\omega)\chi_0\|_2^2 \\ &\le 2^{\frac{n}{2}} C_{\nu,d,\Theta_1,\Theta_2,k} \|\mathcal{X}\Phi_{d,\Theta_1,\Theta_2}\|_\infty ||| \mathrm{e}^{-itu}\mathcal{X}|||_{k+2} \sum_{x\in\mathbb{Z}^d} \langle x\rangle^{-k+n+2\nu} \end{aligned} \tag{3.9}$$

for all $k = 1, 2, \ldots$ and $n \ge 0$. Recalling (A.16), we see that

$$||| \mathrm{e}^{-itu}\mathcal{X}|||_{k+2} \le C_{\mathcal{X},k} \langle t\rangle^{k+2} . \tag{3.10}$$

Given $n \ge 0$, we pick $\nu = \frac{d}{4} + \frac{1}{4}\left(1 + \left[n + \frac{3d}{2}\right] - \left(n + \frac{3d}{2}\right)\right)$, and choose $k = \left[n + \frac{3d}{2}\right] + 1$; note $k - n - 2\nu > d$. It follows from (3.9) and (3.10) that for $\mathbb{P}$-a.e. ω

$$M_\omega(n, \mathcal{X}, t) \le C_{d,\Theta_1,\Theta_2,\mathcal{X},n} \langle t\rangle^{[n+\frac{3d}{2}]+3} , \tag{3.11}$$

which is the last inequality in (3.1).

The inequalities in (3.2) follow immediately from (3.1). The inequalities in (3.3) follow from (3.2) by averaging in time and using $\langle tT\rangle \le \langle t\rangle \langle T\rangle$. □

Proposition 3.2. *Let $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$, I an open interval, and $E \in \mathbb{R}$. Then*

(i) $\beta^\pm(n, \mathcal{X})$, $\beta^\pm(n, I)$ *and* $\beta^\pm(n, E)$ *are monotone increasing in* $n \ge 0$.

(ii) $0 \le \beta^\pm(n, \mathcal{X}), \beta^\pm(n, I), \beta^\pm(n, E) \le 1$.

Proof. It suffices to prove the proposition for $\beta^{\pm}(n,\mathcal{X})$. We may assume $\mathcal{X}(H_\omega) \neq 0$ without loss of generality. We first show that (ii) follows from (i) and (3.3). To see that, note that (3.3) yields

$$\beta^{\pm}(n,\mathcal{X}) \leq 1 + \frac{3d+6}{2n}\,. \tag{3.12}$$

Thus, since $\beta^{\pm}(n,\mathcal{X})$ is increasing in n by (i),

$$\beta^{\pm}(n,\mathcal{X}) \leq \lim_{m\to\infty} \beta^{\pm}(m,\mathcal{X}) \leq 1\,. \tag{3.13}$$

We now turn to (i). If for $F \in C_c(\mathbb{R}^d)$ we also use F to denote the operator given by multiplication by the function $F(x)$, then

$$L_{\mathcal{X},T}(F) = \tag{3.14}$$
$$\frac{2}{T}\int_0^\infty \mathrm{e}^{-\frac{2t}{T}} \mathbb{E}\left(\operatorname{tr}\left(\chi_0 \mathcal{X}(H_\omega) e^{iH_\omega t} F \mathrm{e}^{-itH_\omega} \mathcal{X}(H_\omega)\chi_0\right)\right)\,\mathrm{d}t$$

gives a positive linear functional on $C_c(\mathbb{R}^d)$ by Proposition 3.1, and hence there exists a Borel measure $\mu_{\mathcal{X},T}$ on $\mathbb{R}^d$ such that

$$L_\omega(\mathcal{X},t)(F) = \int_{\mathbb{R}^d} F(x)\,\mathrm{d}\mu_{\mathcal{X},T}(x) \quad \text{for any } F \in C_c(\mathbb{R}^d). \tag{3.15}$$

We thus have by monotone convergence that

$$\mathcal{M}(n,\mathcal{X},T) = \int_{\mathbb{R}^d} \langle X\rangle^n \mathrm{d}\mu_{\mathcal{X},T}(x)\,, \tag{3.16}$$

and we can see that $\mu_{\mathcal{X},T}$ is a finite measure, as

$$\int_{\mathbb{R}^d} \mathrm{d}\mu_{\mathcal{X},T}(x) = \mathbf{M}(0,\mathcal{X},0) < \infty\,. \tag{3.17}$$

Let $m \geq n \geq 0$. We may use Jensen's inequality with respect to the finite measure $\mu_{\mathcal{X},T}$ to conclude that

$$\mathcal{M}(n,\mathcal{X},T)^{\frac{m}{n}} \leq \mathbf{M}(0,\mathcal{X},0)^{\frac{m}{n}-1}\mathcal{M}(m,\mathcal{X},T)\,, \tag{3.18}$$

and hence that

$$\mathcal{M}(n,\mathcal{X},T)^{\frac{1}{n}} \leq \mathbf{M}(0,\mathcal{X},0)^{\frac{1}{n}-\frac{1}{m}}\mathcal{M}(m,\mathcal{X},T)^{\frac{1}{m}}\,. \tag{3.19}$$

It follows that $\beta^{\pm}(n,\mathcal{X})$ is monotone increasing in n. □

We may thus define

$$\beta^{\pm}(I) = \lim_{n\to\infty} \beta^{\pm}(n,I) = \sup_n \beta^{\pm}(n,I)\,, \tag{3.20}$$

$$\beta^{\pm}(E) = \lim_{n\to\infty} \beta^{\pm}(n,E) = \sup_n \beta^{\pm}(n,E)\,, \tag{3.21}$$

and we have $0 \leq \beta^{\pm}(I), \beta^{\pm}(E) \leq 1$, with $\beta^{\pm}(E) = 0$ if $E \notin \Sigma$.

Remark 3.3. It is natural to wonder whether one can recover $\beta^{\pm}(n,I)$ from the *local* transport exponents $\beta^{\pm}(n,E)$ for $E \in I$. One can easily check that this is true *locally*, i.e. that for each E one can find a small intervall J containing E such that $\beta^{\pm}(n,J) \approx \sup_{E' \in J} \beta^{\pm}(n,E')$. More precisely, one shows that for any E, $n \in \mathbb{N}$ and $\nu > 0$, there exists $J_{E,n,\nu} \ni E$ such that: $\sup_{E' \in J_{E,n,\nu}} \beta^{\pm}(n,E') \le \beta^{\pm}(n, J_{E,n,\nu}) \le \sup_{E' \in J_{E,n,\nu}} \beta^{\pm}(n,E') + \nu$, or, using the monotonicity of the function $\beta^{\pm}(n,I)$ in I:

$$\beta^{\pm}(n,E) = \inf_{I \ni E} \sup_{E' \in I} \beta^{\pm}(n,E'). \tag{3.22}$$

(This should compared to (2.17).) This local inversion formula turns into a global one for upper exponents:

$$\beta^{+}(n,I) = \sup_{E \in I} \beta^{+}(n,E). \tag{3.23}$$

This can be seen by combining the local inversion formula and a compactness argument, plus the fact that

$$\beta^{+}(n, I_1 \cup I_2) = \max(\beta^{+}(n,I_1), \beta^{+}(n,I_2)) \,. \tag{3.24}$$

Note also that (3.23) trivially extends to the upper asymptotic transport exponents: $\beta^{+}(I) = \sup_{E \in I} \beta^{+}(E)$, using Proposition 3.2 (i).

4. The strong insulator region

In this section we show that the strong insulator region may be defined by a large number of very natural properties, all equivalent, courtesy of the bootstrap multiscale analysis and Theorem 2.11.

The characteristic function of a set $\Lambda \subset \mathbb{R}^d$ is denoted by χ_Λ. If $x \in \mathbb{R}^d$, $L \in 2\mathbb{N}$, and $\Lambda_L(x)$ is a finite box as in (2.5), we set

$$\chi_{x,L} = \chi_{\Lambda_L(x)} \quad \left(\chi_x = \chi_{x,1} = \chi_{\Lambda_1(x)}\right), \tag{4.1}$$

$$\Upsilon_L(x) = \left\{ y \in \mathbb{Z}^d;\ \|y - x\| = \tfrac{L}{2} - 1 \right\}, \tag{4.2}$$

and define its boundary belt by

$$\tilde{\Upsilon}_L(x) = \overline{\Lambda}_{L-1}(x) \backslash \Lambda_{L-3}(x) = \bigcup_{y \in \Upsilon_L(x)} \overline{\Lambda}_1(y) \ ; \tag{4.3}$$

it has the characteristic function

$$\Gamma_{x,L} = \chi_{\tilde{\Upsilon}_L(x)} = \sum_{y \in \Upsilon_L(x)} \chi_y \ \ a.e. \tag{4.4}$$

We will also need an inner boundary belt and and its characteristic function, given by

$$\widehat{\Upsilon}_L(x) = \overline{\Lambda}_{L-\frac{3}{2}}(x) \backslash \Lambda_{L-\frac{5}{2}}(x) = \bigcup_{y \in \Upsilon_L(x)} \overline{\Lambda}_{\frac{1}{2}}(y)\,, \tag{4.5}$$

$$\widehat{\Gamma}_{x,L} = \chi_{\widehat{\Upsilon}_L(x)}\,. \tag{4.6}$$

Given $\theta > 0$, $E \in \mathbb{R}$, $x \in \mathbb{Z}^d$, and $L \in 6\mathbb{N}$, we say that the box $\Lambda_L(x)$ is (θ, E)-*suitable* for H_ω if $E \notin \sigma(H_{\omega,x,L})$ and

$$\| \Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,L/3} \|_{x,L} \leq \frac{1}{L^\theta}\,. \tag{4.7}$$

The bootstrap multiscale analysis [GK1, Theorem 3.4] yields strong HS-dynamical localization [GK1, Corollary 3.10]. In Theorem A.1 we show that random Schrödinger operators as defined in Section 2 satisfies Assumptions SLI, EDI, IAD, NE, and SGEE of [GK1], so the relevant results of that article may be restated as follows:

Theorem 4.1 ([GK1]). *Let H_ω be a random Schrödinger operator satisfying a Wegner estimate in an open interval $\mathcal{I}$. Given $\theta > d$, for each $E \in \mathcal{I}$ there exists a finite scale $\overline{\mathcal{L}}_\theta(E)$ (depending only on θ, E, Q_E, d, Θ_1, Θ_2), bounded in compact subintervals of $\mathcal{I}$, such that, if for some $E \in \Sigma^{\mathcal{I}}$ we can verify at some finite scale $\mathcal{L} > \overline{\mathcal{L}}_\theta(E)$ that*

$$\mathbb{P}\{\Lambda_{\mathcal{L}}(0) \ \textit{ is } (\theta, E)\textit{-suitable}\} > 1 - \frac{1}{841^d}\,, \tag{4.8}$$

then $E \in \Sigma_{\mathrm{SI}}$.

In the next theorem we give a long list of properties of an energy in the strong insulator region; they are all equivalent and any of them may be used to define the strong insulator region (hence the analogy with the the region of complete analyticity in classical statistical mechanics [DS1,DS1]). We use $\mathcal{B}_1(\mathbb{R})$ to denote the bounded real-valued Borel functions f of a real variable with $\sup_{t\in\mathbb{R}} |f(t)| \leq 1$. We also write $C^\infty_{c,+,1}(I)$ for the functions in $C^\infty_{c,+}(I)$ which are bounded by 1.

Theorem 4.2. *Let H_ω be a random Schrödinger operator satisfying a Wegner estimate in an open interval $\mathcal{I}$. If $E \in \Sigma^{\mathcal{I}}$, the following conditions are equivalent:*

(i) For all $\theta > d$ we have

$$\limsup_{L\to\infty} \mathbb{P}\{\Lambda_L(0) \ \textit{ is } (\theta, E)\textit{-suitable}\} = 1\,. \tag{4.9}$$

(ii) For some $\theta > d$ we have

$$\limsup_{L\to\infty} \mathbb{P}\{\Lambda_L(0) \ \textit{ is } (\theta, E)\textit{-suitable}\} = 1\,. \tag{4.10}$$

(iii) For some $\theta > d$ we have

$$\limsup_{L\to\infty} \mathbb{P}\{\Lambda_L(0) \ \textit{ is } (\theta,E)\textit{-suitable}\} > 1 - \frac{1}{841^d}\,. \tag{4.11}$$

(iv) $E \in \Sigma_{\mathrm{MSA}}$, i.e., for some $\theta > d$ we can verify (4.8) at some finite scale $\mathcal{L} > \overline{\mathcal{L}}_\theta(E)$, where $\overline{\mathcal{L}}_\theta(E)$ is given in Theorem 4.1.
(v) There exists $\delta > 0$ such that for each $0 < \zeta < 1$ we have

$$\mathbb{E}\left(\sup_{f\in\mathcal{B}_1(\mathbb{R})} \|\chi_x f(H_\omega)E_{H_\omega}([E-\delta,E+\delta])\chi_y\|_2^2\right) \le C_\zeta\, \mathrm{e}^{-|x-y|^\zeta} \tag{4.12}$$

for all $x,y \in \mathbb{Z}^d$, with $C_\zeta < \infty$.
(vi) For some $0 < \zeta < 1$ there exists $\delta > 0$ such that

$$\mathbb{E}\left(\sup_{f\in\mathcal{B}_1(\mathbb{R})} \|\chi_x f(H_\omega)E_{H_\omega}([E-\delta,E+\delta])\chi_y\|_2^2\right) \le C\, \mathrm{e}^{-|x-y|^\zeta} \tag{4.13}$$

for all $x,y \in \mathbb{Z}^d$, with $C < \infty$.
(vii) There exists $\delta > 0$ such that for each $p = 1,2,\ldots$ we have

$$\mathbb{E}\left(\sup_{f\in\mathcal{B}_1(\mathbb{R})} \|\chi_x f(H_\omega)E_{H_\omega}([E-\delta,E+\delta])\chi_y\|_2^2\right) \le \frac{C_p}{\langle x-y\rangle^p} \tag{4.14}$$

for all $x,y \in \mathbb{Z}^d$, with $C_p < \infty$.
(viii) There exists $\delta > 0$ such that we have

$$\mathbb{E}\left(\sup_{f\in\mathcal{B}_1(\mathbb{R})} \left\|\langle X\rangle^{\frac{n}{2}} f(H_\omega)E_{H_\omega}([E-\delta,E+\delta])\chi_0\right\|_2^2\right) < \infty \tag{4.15}$$

for all $n \ge 0$.
(ix) There exists $\delta > 0$ such that we have

$$\mathbb{E}\left(\sup_{\mathcal{X}\in C^\infty_{c,+,1}((E-\delta,E+\delta))} \left\|\langle X\rangle^{\frac{n}{2}} \mathcal{X}(H_\omega)\chi_0\right\|_2^2\right) < \infty \tag{4.16}$$

for all $n \ge 0$.
(x) $E \in \Sigma_{\mathrm{SI}}$, i.e., H_ω exhibits strong HS-dynamical localization at E.
(xi) H_ω exhibits strong operator-norm-dynamical localization at E (see (2.13)).
(xii) $\beta^+(E) = 0$.
(xiii) $E \in \Sigma_{\mathrm{TT}}$, i.e., $\beta(E) = 0$.
(xiv) For some $n > 11d$ we have $\beta(n,E) = 0$.
(xv) For some $n > 11d$ we have $\beta(n,E) < \frac{1}{2d} - \frac{11}{2n}$.

(xvi) There exist $\mathcal{X} \in C^{\infty}_{c,+}(\mathbb{R})$, with $\mathcal{X} \equiv 1$ on some open interval containing E, $\alpha \geq 0$, and $n > 2d\alpha + 11d$, such that we have (2.28).

Proof. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), that (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii), that (viii) $\Rightarrow$ (ix) $\Rightarrow$ (x) $\Rightarrow$ (xii) $\Rightarrow$ (xiii) $\Rightarrow$ (xiv) $\Rightarrow$ (xv) $\Rightarrow$ (xvi), and that (x) $\Rightarrow$ (xi). The proof of [GK1, Corollary 3.10] shows that (vii) $\Rightarrow$ (viii).

To see that (xi) $\Rightarrow$ (x), note that, proceeding as in the proof of Proposition 3.1 and using (A.12) as in (3.7),

$$\begin{aligned} M_\omega(n,\mathcal{X},t) &= \operatorname{tr}\left(\chi_0 \mathcal{X}(H_\omega)\mathrm{e}^{itH_\omega}\langle X\rangle^n \mathrm{e}^{-itH_\omega}\mathcal{X}(H_\omega)\chi_0\right) \\ &\leq 2^{2\nu}\|\langle X\rangle^{n+2\nu}\mathrm{e}^{-itH_\omega}\mathcal{X}(H_\omega)\chi_0\|\|\langle X\rangle^{-2\nu}\mathcal{X}(H_\omega)\mathrm{e}^{itH_\omega}\langle X\rangle^{-2\nu}\|_1 \\ &\leq 2^{2\nu+2}\mathcal{T}_{\nu,d,\Theta_1,\Theta_2}\left\|\mathcal{X}\Phi_{d,\Theta_1,\Theta_2}\right\|_\infty \|\langle X\rangle^{n+2\nu}\mathrm{e}^{-itH_\omega}\mathcal{X}(H_\omega)\chi_0\|\,. \end{aligned} \tag{4.17}$$

The nontrivial content of the theorem is that (iv) $\Rightarrow$ (v), and (xvi) $\Rightarrow$ (i). The first implication is the content of [GK1, Theorem 3.8]. To prove the second implification, we note first that (xvi) $\Rightarrow$ (iv) by Theorem 2.11. To finish the proof, we must show that (iv) $\Rightarrow$ (i). This can be proved by, either adapting the proof of [GK1, Theorems 5.1 and 5.2], or by using the already established fact that (iv) $\Rightarrow$ (xiii), that it follows from (xiii) that there exists $\mathcal{X} \in C^{\infty}_{c,+}(\mathbb{R})$, with $\mathcal{X} \equiv 1$ on some open interval containing E, such that for all $\alpha \geq 0$ and $n > 2d\alpha + 11d$ we have (2.28), and hence that (i) follows from the proof of Theorem 2.11 since given arbitrary $\theta > d$ we can pick $\alpha \geq 0$ and $n > 2d\alpha + 11d$ such that the proof of Theorem 2.11 gives (4.9) with this θ (see (6.47) and (6.54)). □

Remark 4.3. In [GK1] we used (viii) as the definition of strong HS-dynamical localization in the interval $[E-\delta, E+\delta]$.

5. The metallic transport region

In this section we give a characterization of the metallic transport region, and a criterion for an energy to be in it. Roughly speaking, the criterion says that if the finite volume resolvent does not decay faster than the inverse of the volume of the box, then the energy E must be in the metallic transport region. More precisely, it says that if (4.10) is violated with $\theta = d$, then $E \in \Sigma_{\mathrm{MT}}$.

Given $\theta > 0$, $E \in \mathbb{R}$, $x \in \mathbb{Z}^d$, and $L \in 6\mathbb{N}$, we say that the box $\Lambda_L(x)$ is (θ,E)-*unsuitable* for H_ω if it is not (θ,E)-*suitable* for H_ω (see (4.7)), i.e., if either $E \in \sigma(H_{\omega,x,L})$ or

$$\|\Gamma_{x,L}R_{\omega,x,L}(E)\chi_{x,L/3}\|_{x,L} > \frac{1}{L^\theta}\,. \tag{5.1}$$

Theorem 5.1. *Let H_ω be a random Schrödinger operator satisfying a Wegner estimate in an open interval $\mathcal{I}$. Then the metallic transport region in $\mathcal{I}$ is given by*

$$\Sigma_{\mathrm{MT}}^{\mathcal{I}} = \tag{5.2}$$
$$\left\{ E \in \mathcal{I};\ \liminf_{L \to \infty} \mathbb{P}\{\Lambda_L(0) \ \ \textit{is } (\theta, E)\textit{-unsuitable}\} > 0 \textit{ for some } \theta > d \right\}.$$

Thus, if $E \in \mathcal{I}$ is such that

$$\liminf_{L \to \infty} \mathbb{P}\{\Lambda_L(0) \ \ (d, E)\textit{-unsuitable}\} > 0 \,, \tag{5.3}$$

then $E \in \Sigma_{\mathrm{MT}}$, and hence $\beta(E) \geq \frac{1}{2d}$.

Proof. (5.2) follows from Theorems 2.8 and 4.2(i). Now, if for $\mu > 0$ we set $g(\mu) = \mathbb{P}\{\Lambda_L(0) \ \text{ is } (\mu, E)\text{-unsuitable}\}$, the function $g(\mu)$ is clearly nondecreasing. Thus condition (5.3) implies $E \in \Sigma_{\mathrm{MT}}$ according to (5.2). The last statement of Theorem 5.1 now follows from Theorem 2.10. □

6. The main proof

In this section we prove our main technical result, Theorem 2.11. Its main hypothesis, condition (2.28), is formulated in terms of the dynamics, but the starting hypothesis of the bootstrap multiscale analysis, condition (4.8), is stated in terms of resolvents. We start by reformulating condition (2.28) in terms of resolvents.

Proposition 6.1. *Let H_ω be a random Schrödinger operator, $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$, and set $R_\omega(z) = (H_\omega - z)^{-1}$. For any $n > 0$ and $T > 0$ we have*

$$\mathcal{M}(n, \mathcal{X}, T) = \frac{1}{\pi T} \int_{\mathbb{R}} \mathbb{E} \left\| \langle X \rangle^{\frac{n}{2}} R_\omega(E + i\tfrac{1}{T}) \mathcal{X}(H_\omega) \chi_0 \right\|_2^2 \mathrm{d}E \,. \tag{6.1}$$

In particular, if we set

$$\Omega_\varepsilon(n, \mathcal{X}, E) = \mathbb{E} \left(\left\| \langle X \rangle^{\frac{n}{2}} R_\omega(E + i\varepsilon) \mathcal{X}(H_\omega) \chi_0 \right\|_2^2 \right) , \tag{6.2}$$

condition (2.28) in Theorem 2.11 is the same as

$$\Omega \equiv \liminf_{\varepsilon \to 0^+} \varepsilon^{1+\alpha} \int_{\mathbb{R}} \Omega_\varepsilon(n, \mathcal{X}, E) \, \mathrm{d}E < \infty \,. \tag{6.3}$$

Proof. The equality (6.1) follows from Lemma 6.3 below, applied to the family of operators $\langle X \rangle_N^{\frac{n}{2}}$, a cut-off of $\langle X \rangle^{\frac{n}{2}}$ inside the ball of radius N centered at the origin, and the Monotone Convergence Theorem. □

Remark 6.2. Proposition 6.1 is the main reason for the use of Hilbert-Schmidt norms in our definitions. Their use is justified by [GK1, Corollary 3.10] and Theorem 4.2.

Lemma 6.3. *Let A, B be bounded operators, H a self-adjoint operator, and $R(z) = (H - z)^{-1}$. Then for all $T > 0$ we have*

$$\int_0^\infty \mathrm{e}^{-\frac{2t}{T}} \left\| A e^{-iHt} B \right\|_2^2 \mathrm{d}t = 2\pi \int_{\mathbb{R}} \left\| AR(E + i\tfrac{1}{T}) B \right\|_2^2 \mathrm{d}E\,. \qquad (6.4)$$

Proof. By the spectral theorem,

$$\left(H - \left(E + i\tfrac{1}{T}\right)\right)^{-1} = i \int_0^\infty \mathrm{e}^{itE} \mathrm{e}^{-it\left(H - i\frac{1}{T}\right)}\, \mathrm{d}t\,. \qquad (6.5)$$

Multiplying on the left by the operator A and on the right by the operator B, taking matrix elements of both sides, and applying Plancherel's Theorem we get that for any vector $\psi \in \mathcal{H}$ we have

$$\int_0^\infty \mathrm{e}^{-\frac{2t}{T}} \left\| A e^{-iHt} B\psi \right\|^2 \mathrm{d}t = 2\pi \int_{\mathbb{R}} \left\| AR(E + i\tfrac{1}{T}) B\psi \right\|^2 \mathrm{d}E\,. \qquad (6.6)$$

The lemma then follows from the definition of the Hilbert-Schmidt norm: $\|T\|_2^2 = \sum_n \|Te_n\|^2$, where $(e_n)_{n\in\mathbb{N}}$ is any orthonormal basis for the Hilbert space. $\square$

The proof of Theorem 2.11 requires that we obtain the finite volume condition (4.8) out of the infinite volume condition (6.3). The following lemma will play an important role in estimating finite volume probabilities out of infinite volume expectations.

We recall our notation for finite volume introduced in (2.5) and in Section 4 (see (4.1) - (4.6)). Following [FK1,FK2,KK1], we equip each cube $\Lambda_L(x)$ with functions $\phi_{x,L}$ and $\rho_{x,L} \in C_c^1(\mathbb{R}^d)$, such that $\operatorname{supp}\phi_{x,L}$, $\operatorname{supp}\rho_{x,L} \subset \Lambda_L(x)$, $0 \le \phi_{x,L}$, $\rho_{x,L} \le 1$, and

$$\chi_{x,\frac{L}{2}-\frac{5}{4}}\phi_{x,L} = \chi_{x,\frac{L}{2}-\frac{5}{4}}\,, \quad \chi_{x,\frac{L}{2}-\frac{3}{4}}\phi_{x,L} = \phi_{x,L}\,, \qquad (6.7)$$

$$\widehat{\Gamma}_{x,L}\left(\nabla\phi_{x,L}\right) = \nabla\phi_{x,L}\,, \quad |\nabla\phi_{x,L}| \le 3\sqrt{d}\,, \qquad (6.8)$$

$$\widehat{\Gamma}_{x,L}\,\rho_{x,L} = \widehat{\Gamma}_{x,L}\,, \quad \Gamma_{x,L}\,\rho_{x,L} = \rho_{x,L}\,, \qquad (6.9)$$

$$|\nabla\rho_{x,L}| \le 5\sqrt{d}\,. \qquad (6.10)$$

In what follows we work with boxes centered at 0 and omit the center from the notation.

Lemma 6.4. *Let H_ω be a random Schrödinger operator satisfying a Wegner estimate in an open interval $\mathcal{I}$. Let $p_0 > 0$ and $\gamma > d$. For each $E \in \mathcal{I}$ there exists $\mathcal{L}_1 = \mathcal{L}_1(d, \Theta_1, \Theta_2, E, Q_E, \gamma, p_0)$, bounded on compact subsets of $\mathcal{I}$, such that given $L \in 2\mathbb{N}$ with $L \ge \mathcal{L}_1$, and* *subsets B_1 and B_2 of Λ_L with $B_1 \subset \Lambda_{L/2-5/4}$ and $\tilde{\Upsilon}_L \subset B_2$, then for each $a > 0$ and $0 < \varepsilon \leq 1$ we have*

$$\mathbb{P}\left(\|\chi_{B_2} R_{\omega,L}(E+i\varepsilon)\chi_{B_1}\|_L > \frac{a}{4}\right) \leq \frac{L^\gamma}{a}\,\mathbb{E}(\|\chi_{B_2} R_\omega(E+i\varepsilon)\chi_{B_1}\|) + \frac{p_0}{10}, \tag{6.11}$$

and

$$\mathbb{P}\left(\|\chi_{B_2} R_{\omega,L}(E)\chi_{B_1}\|_L > \frac{a}{2}\right) \leq \frac{L^\gamma}{a}\,\mathbb{E}(\|\chi_{B_2} R_\omega(E+i\varepsilon)\chi_{B_1}\|) + \frac{p_0}{10} + 2Q_E\left(\frac{\varepsilon}{a}\right)^{\frac{1}{2}} L^d. \tag{6.12}$$

We shall use Lemma 6.4 with B_2 equal to either $\tilde{\Upsilon}_L$ (recall $\Gamma_L = \chi_{\tilde{\Upsilon}_L}$) or $\Lambda_L \backslash \Lambda_{\frac{2L}{3}}$, and B_1 equal to either $\Lambda_{\frac{2L}{3}}$ or $\Lambda_{\frac{L}{3}}$.

Proof of Lemma 6.4. We write χ_1 and χ_2 for χ_{B_1} and χ_{B_2}, note that $\phi_L\chi_1 = \chi_1$ and $\Gamma_L \leq \chi_2$. We start by estimating the quantity $\|\chi_2 R_{\omega,L}(E+i\varepsilon)\chi_1\|_L$ in terms of $\|\chi_2 R_\omega(E+i\varepsilon)\chi_1\|$. (The estimate is given in (6.18) below.) To do so, we proceed as in [KK1, Lemma 3.7], obtaining

$$\begin{aligned} R_\omega(E-i\varepsilon)\phi_L J_L = {} & J_L\phi_L R_{\omega,L}(E-i\varepsilon) \\ & + R_\omega(E-i\varepsilon)(\nabla\phi_L)^* J_L \nabla_L R_{\omega,L}(E-i\varepsilon) \\ & - R_\omega(E-i\varepsilon)\nabla^* J_L(\nabla\phi_L) R_{\omega,L}(E-i\varepsilon), \end{aligned} \tag{6.13}$$

as bounded operators from $\mathrm{L}^2(\Lambda_L, \mathrm{d}x)$ to $\mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x)$, where $J_L : \mathrm{L}^2(\Lambda_L, \mathrm{d}x) \to \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x)$ is the canonical injection. Taking adjoints, we get

$$\begin{aligned} & \chi_2 R_{\omega,L}(E+i\varepsilon)\chi_1 J_L^* = \chi_2 R_{\omega,L}(E+i\varepsilon)\phi_L J_L^*\chi_1 = \\ & \chi_2 J_L^*\phi_L R_\omega(E+i\varepsilon)\chi_1 - \chi_2 R_{\omega,L}(E+i\varepsilon)\nabla_L^* J_L^*(\nabla\phi_L) R_\omega(E+i\varepsilon)\chi_1 \\ & \quad + \chi_2 R_{\omega,L}(E+i\varepsilon)(\nabla\phi_L)^* J_L^* \nabla R_\omega(E+i\varepsilon)\chi_1. \end{aligned} \tag{6.14}$$

Thus, proceeding as in the proof of [KK1, Lemma 3.8], and recalling (6.7)-(6.10),

$$\begin{aligned} & \|\chi_2 R_{\omega,L}(E+i\varepsilon)\chi_1\|_L \leq \|\chi_2 R_\omega(E+i\varepsilon)\chi_1\| + \\ & 3\sqrt{d}\,\{\|\chi_2 R_{\omega,L}(E+i\varepsilon)\nabla_L^*\rho_L\|_L \|\Gamma_L R_\omega(E+i\varepsilon)\chi_1\| \\ & + \|\chi_2 R_{\omega,L}(E+i\varepsilon)\Gamma_L\|_L \|\rho_L \nabla R_\omega(E+i\varepsilon)\chi_1\|\}. \end{aligned} \tag{6.15}$$

We now use Lemma A.2 (choosing always an appropriate $a > 0$ in (A.2)) to obtain

$$\|\rho_L \nabla R_\omega(E+i\varepsilon)\chi_1\| \leq C^{(1)}_{d,\Theta_1,\Theta_2}(1+|E+i\varepsilon|)\|\Gamma_L R_\omega(E+i\varepsilon)\chi_1\|, \tag{6.16}$$

where we used $\rho_L \chi_1 = 0$, and

$$\begin{aligned} \|\chi_2 R_{\omega,L}(E+i\varepsilon)\nabla_L^* \rho_L\|_L &= \|\rho_L \nabla_L R_{\omega,L}(E-i\varepsilon)\chi_2\|_L \qquad (6.17)\\ &\le \|\Gamma_L \chi_2\|_L + C^{(2)}_{d,\Theta_1,\Theta_2}(1+|E+i\varepsilon|)\|\Gamma_L R_{\omega,L}(E+i\varepsilon)\chi_1\| \\ &\le 1 + C^{(2)}_{d,\Theta_1,\Theta_2}(1+|E+i\varepsilon|)\|\Gamma_L R_{\omega,L}(E+i\varepsilon)\chi_1\| \,. \end{aligned}$$

It follows, using $\Gamma_L \le \chi_2$, that

$$\begin{aligned} &\|\chi_2 R_{\omega,L}(E+i\varepsilon)\chi_1\|_L \le \qquad (6.18)\\ &\quad C_{d,\Theta_1,\Theta_2}\|\chi_2 R_\omega(E+i\varepsilon)\chi_1\| \left(1+(1+|E+i\varepsilon|)\|R_{\omega,L}(E+i\varepsilon)\|_L\right) \,. \end{aligned}$$

(In our notation $C^{(1)}_{d,\Theta_1,\Theta_2}$, $C^{(2)}_{d,\Theta_1,\Theta_2}$, C_{d,Θ_1,Θ_2} are constants depending only on d, Θ_1, Θ_2.)

We now fix $\gamma > d$ and $a > 0$. Using Chebychev's inequality and the Wegner estimate (2.8), we obtain

$$\begin{aligned} &\mathbb{P}\left(\|\chi_2 R_{\omega,L}(E+i\varepsilon)\chi_1\|_L > \frac{a}{4}\right) \le \mathbb{P}\left(\|\chi_2 R_\omega(E+i\varepsilon)\chi_1\| > \frac{a}{L^\gamma}\right) \\ &\quad + \mathbb{P}\left(C_{d,\Theta_1,\Theta_2}\left(1+(1+|E+i\varepsilon|)\|R_{\omega,L}(E+i\varepsilon)\|_L\right) > \frac{1}{4}L^\gamma\right) \qquad (6.19)\\ &\quad \le \frac{L^\gamma}{a}\,\mathbb{E}\left(\|\chi_2 R_\omega(E+i\varepsilon)\chi_1\|\right) + 8C_{d,\Theta_1,\Theta_2}(1+|E+i\varepsilon|)Q_E L^{-(\gamma-d)} \end{aligned}$$

for $L^\gamma > 8C_{d,\Theta_1,\Theta_2}$. The estimate (6.11) follows if if L is large enough, depending on $d, \Theta_1, \Theta_2, E, Q_E, p_0, \gamma$.

We turn to (6.12). If $E \notin \sigma(H_L)$, we have

$$R_{\omega,L}(E) = R_{\omega,L}(E+i\varepsilon) - i\varepsilon R_{\omega,L}(E)R_{\omega,L}(E+i\varepsilon)\,, \qquad (6.20)$$

hence

$$\|\chi_2 R_{\omega,L}(E)\chi_1\|_L \le \|\chi_2 R_{\omega,L}(E+i\varepsilon)\chi_1\|_L + \varepsilon\|R_{\omega,L}(E)\|_L^2 \,. \qquad (6.21)$$

Thus

$$\begin{aligned} &\mathbb{P}\left(\|\chi_2 R_{\omega,L}(E)\chi_1\|_L > \frac{a}{2}\right) \qquad (6.22)\\ &\le \mathbb{P}\left(\|\chi_2 R_{\omega,L}(E+i\varepsilon)\chi_1\|_L > \frac{a}{4}\right) + \mathbb{P}\left(\|R_{\omega,L}(E)\| > \frac{1}{2}\left(\frac{a}{\varepsilon}\right)^{\frac{1}{2}}\right) \\ &\le \mathbb{P}\left(\|\chi_2 R_{\omega,L}(E+i\varepsilon)\chi_1\|_L > \frac{a}{4}\right) + 2Q_E\left(\frac{\varepsilon}{a}\right)^{\frac{1}{2}} L^d \,, \end{aligned}$$

where we used the Wegner estimate (2.8). The estimate (6.12) now follows from (6.11). □

We are now ready to prove Theorem 2.11.

Proof of Theorem 2.11. Suppose condition (2.28) holds for given $\mathcal{X} \in C^{\infty}_{c,+}(\mathbb{R})$, with $\mathcal{X} \equiv 1$ in an open interval $J \subset \mathcal{I}$, $\alpha \geq 0$, and $n > 2d\alpha + 11d$, so we have (6.3) by Proposition 6.1. To prove Theorem 2.11, it suffices to show that for each $E \in J$ there is some $\theta > d$ such that condition (4.10) is satisfied, i.e.,

$$\limsup_{L\to\infty} \mathbb{P}\left(\|\Gamma_L R_{\omega,L}(E)\chi_{L/3}\|_L \leq \frac{1}{L^\theta} \right) = 1 \,, \tag{6.23}$$

so the starting condition (4.8) of the bootstrap multiscale analysis holds at some finite scale $L > \overline{\mathcal{L}}_\theta(E)$.

So let $E \in J$, $\theta > d$, and $L \in 6\mathbb{N}$. We start by estimating

$$P_{E,L} = \mathbb{P}\left(\|\Gamma_L R_{\omega,L}(E)\chi_{L/3}\|_L > \frac{1}{2L^\theta} \right). \tag{6.24}$$

Using (6.12) in Lemma 6.4 with $a = 2L^{-\theta}$ would provide an estimate for $P_{E,L}$. But later on we would need $n > 3d\alpha + 11d$ to conclude the proof. To work with $n > 2d\alpha + 11d$, we squeeze a bit more from Lemma 6.4. We use the resolvent identity (6.20), plus

$$\chi_L = \chi_{\frac{2L}{3}} + \chi_{L\setminus\frac{2L}{3}} \,, \qquad \text{where} \quad \chi_{L\setminus\frac{2L}{3}} \equiv \chi_{\Lambda_L\setminus\Lambda_{\frac{2L}{3}}} \,, \tag{6.25}$$

to obtain

$$\|\Gamma_L R_{\omega,L}(E)\chi_{L/3}\|_L \leq \|\Gamma_L R_{\omega,L}(E+i\varepsilon)\chi_{L/3}\|_L \tag{6.26}$$
$$+\, \varepsilon \, \|\Gamma_L R_{\omega,L}(E)\chi_{2L/3}\|_L \, \|R_{\omega,L}(E+i\varepsilon)\|_L \tag{6.27}$$
$$+\, \varepsilon \, \|R_{\omega,L}(E)\|_L \, \|\chi_{L\setminus 2L/3} R_{\omega,L}(E+i\varepsilon)\chi_{L/3}\|_L \,. \tag{6.28}$$

We now estimate $\|\Gamma_L R_{\omega,L}(E+i\varepsilon)\chi_{L/3}\|_L$ in (6.26) using (6.11) with $a = L^{-\theta}$, $\|\Gamma_L R_{\omega,L}(E)\chi_{2L/3}\|_L$ in (6.27) by (6.12) with $a = 1$, and $\|\chi_{L\setminus 2L/3} R_{\omega,L}(E+i\varepsilon)\chi_{L/3}\|_L$ in (6.28) using (6.11) with $a = 1$. The probability that $\varepsilon\|R_{\omega,L}(E)\|_L$ is greater than $\frac{1}{4}L^{-\theta}$ is estimated by (2.8). We obtain

$$\begin{aligned} P_{E,L} &\leq L^{\theta+\gamma}\, \mathbb{E}\left(\|\Gamma_L R_\omega(E+i\varepsilon)\chi_{L/3}\|\right) + \\ &L^\gamma \mathbb{E}\left(\|\Gamma_L R_\omega(E+i\varepsilon)\chi_{2L/3}\|\right) + L^\gamma \mathbb{E}\left(\|\chi_{L\setminus 2L/3} R_\omega(E+i\varepsilon)\chi_{L/3}\|\right) \\ &+\; 4Q_E\varepsilon L^{\theta+d} + 2Q_I\varepsilon^{\frac{1}{2}}L^d + \frac{3\,p_0}{10} \,, \end{aligned} \tag{6.29}$$

where $\gamma > d$, $0 < \varepsilon \leq 1$, and $0 < p_0 < 1$. The estimate is valid for $L > \mathcal{L}_1$, where $\mathcal{L}_1 = \mathcal{L}_1(d, \Theta_1, \Theta_2, E, Q_E, \gamma, p_0)$ is as in Lemma 6.4.

Compared to the direct use of (6.12) with $a = L^{-\theta}$, the gain lies in the fact that now L will be chosen such that $\varepsilon \approx L^{-\theta-d}$ (recall $\theta > d$) instead of $\varepsilon \approx L^{-\theta-2d}$. This will allow us to work with $n > 2d\alpha + 11d$ instead of $n > 3d\alpha + 11d$.

Let I be a compact subinterval of J. We will estimate the right-hand-side of (6.29) for $E \in I$. To do so, let

$$Q_I = \sup_{E \in I} Q_E < \infty \,, \tag{6.30}$$

where Q_E is given in (2.8). We need to estimate the expression $L^{(\theta+\gamma)} \mathbb{E}\left(\|\Gamma_L R_\omega(E+i\varepsilon)\chi_{L/3}\|\right)$, plus two similar terms. To do so, we use

$$\begin{aligned} \mathbb{E}\left(\|\Gamma_L R_\omega(E+i\varepsilon)\chi_{L/3}\|\right) &\le \mathbb{E}\left(\|\Gamma_L R_\omega(E+i\varepsilon)\mathcal{X}(H_\omega)\chi_{L/3}\|\right) \\ &\quad + \mathbb{E}\left(\|\Gamma_L R_\omega(E+i\varepsilon)(1-\mathcal{X}(H_\omega))\chi_{L/3}\|\right) \,. \end{aligned} \tag{6.31}$$

To estimate the last term, note that since $\mathcal{X}(u) = 1$ for all $u \in J$, the function

$$f_{E,\varepsilon}(u) = (u - (E+i\varepsilon))^{-1}(1-\mathcal{X}(u)) \tag{6.32}$$

is a bounded, infinitely differentiable function on the real line for $E \in J$ and $\varepsilon \in \mathbb{R}$. Moreover, it is easy to see that

$$\sup_{E\in I} \sup_{|\varepsilon|\le 1} \| f_{E,\varepsilon} \|_k < \infty \quad \text{for all} \quad k = 1, 2, \ldots . \tag{6.33}$$

(The norms are defined in (A.16).) It follows from (A.15) and (6.33) that for all $E \in I$ and $|\varepsilon| \le 1$ we h ve

$$\sup_{|\varepsilon|\le 1} L^{\theta+\gamma} \mathbb{E}\left(\|\Gamma_L R_\omega(E+i\varepsilon)(1-\mathcal{X}(H_\omega))\chi_{L/3}\|\right) \le \frac{p_0}{10} \tag{6.34}$$

if $L \ge \mathcal{L}_2(I)$, with $\mathcal{L}_2(I) = \mathcal{L}_2(d, \Theta_1, \Theta_2, \mathcal{X}, I, \theta, \gamma, p_0) < \infty$.

On the other hand,

$$\begin{aligned} &\mathbb{E}\left(\|\Gamma_L R_\omega(E+i\varepsilon)\mathcal{X}(H_\omega)\chi_{L/3}\|\right) \\ &\le \sum_{y \in \mathbb{Z}^d \cap \Lambda_{\frac{L}{3}}(0)} \mathbb{E}(\|\Gamma_L R_\omega(E+i\varepsilon)\mathcal{X}(H_\omega)\chi_y\|) \\ &= \sum_{y \in \mathbb{Z}^d \cap \Lambda_{\frac{L}{3}}(0)} \mathbb{E}\left(\|\chi_{\tilde{\Upsilon}_L - y} R_\omega(E+i\varepsilon)\mathcal{X}(H_\omega)\chi_0\|\right) \\ &\le \left(\frac{L}{3} - \frac{3}{2}\right)^{-\frac{n}{2}} \sum_{y \in \mathbb{Z}^d \cap \Lambda_{\frac{L}{3}}(0)} \mathbb{E}\left(\||\langle X\rangle^{\frac{n}{2}} \chi_{\tilde{\Upsilon}_L - y} R_\omega(E+i\varepsilon)\mathcal{X}(H_\omega)\chi_0\|\right) \\ &\le \left(\frac{L}{3} - \frac{3}{2}\right)^{-\frac{n}{2}} \left(\frac{L}{3}\right)^d \mathbb{E}\left(\||\langle X\rangle^{\frac{n}{2}} R_\omega(E+i\varepsilon)\mathcal{X}(H_\omega)\chi_0\|\right) \\ &\le \frac{12^n}{3^d} L^{-\frac{n}{2}+d} \mathbb{E}\left(\|\langle X\rangle^{\frac{n}{2}} R_\omega(E+i\varepsilon)\mathcal{X}(H_\omega)\chi_0\|_2\right) \,, \end{aligned} \tag{6.35}$$

where we used $L \ge 6$ and $\|A\| \le \|A\|_2$.

Combining (6.31), (6.34), (6.35), and (6.2), we conclude that for $E \in I$ and $0 < \varepsilon \leq 1$ we have

$$\begin{aligned} & L^{\theta+\gamma}\,\mathbb{E}\left(\|\Gamma_L R_\omega(E+i\varepsilon)\chi_{L/3}\|\right) && (6.36)\\ & \leq \frac{12^n}{3^d} L^{-\frac{n}{2}+d+\theta+\gamma}\,\mathbb{E}\left(\|\langle X\rangle^{\frac{n}{2}} R_\omega(E+i\varepsilon)\mathcal{X}(H_\omega)\chi_0\|_2\right) + \frac{p_0}{10} \\ & \leq \frac{12^n}{3^d} L^{-\frac{n}{2}+d+\theta+\gamma}\,\left(\mathbb{E}\left(\|\langle X\rangle^{\frac{n}{2}} R_\omega(E+i\varepsilon)\mathcal{X}(H_\omega)\chi_0\|_2^2\right)\right)^{1/2} + \frac{p_0}{10} \\ & \leq \frac{12^n}{3^d}\Omega_\varepsilon(n,\mathcal{X},E)^{\frac{1}{2}} L^{-\frac{n}{2}+d+\theta+\gamma} + \frac{p_0}{10}\,, \end{aligned}$$

for $L \geq \mathcal{L}_2(I)$.

The two other similar terms in (6.29), namely the terms given by $L^\gamma\,\mathbb{E}(\|\Gamma_L R_{\omega,L}(E+i\varepsilon)\chi_{\frac{2L}{3}}\|)$ and $L^\gamma\,\mathbb{E}(\|\chi_{L\setminus 2L/3} R_{\omega,L}(E+i\varepsilon)\chi_{\frac{L}{3}}\|)$, are estimated in the same way, using the fact that $\operatorname{dist}(\tilde{\Upsilon}_L, \Lambda_{\frac{2L}{3}}) \geq \frac{L}{6} - \frac{3}{2}$ and $\operatorname{dist}(\Lambda_{L\setminus\frac{2L}{3}}, \Lambda_{\frac{L}{3}}) \geq \frac{L}{6}$. We conclude that there is $\mathcal{L}_3(I) = \mathcal{L}_3(d, \Theta_1, \Theta_2, \mathcal{X}, I, Q_I, \theta, \gamma, p_0)$, such that if $L \geq \mathcal{L}_3(I)$ we have

$$\begin{aligned} & P_{E,L} \leq && (6.37)\\ & C_{n,d}\,\Omega_\varepsilon(n,\mathcal{X},E)^{\frac{1}{2}} L^{-\frac{1}{2}n+d+\theta+\gamma} + 4Q_I\,\varepsilon L^{\theta+d} + 2Q_I\varepsilon^{\frac{1}{2}} L^d + \frac{3p_0}{5} \end{aligned}$$

for all $E \in I$ and $0 < \varepsilon \leq 1$.

For a given $\varepsilon > 0$ we set (with $[K]_{6\mathbb{N}} = \max\{L \in 6\mathbb{N};\ L \leq K\}$)

$$L(I,\varepsilon) = \left[\left(\frac{p_0}{40 Q_I\,\varepsilon}\right)^{\frac{1}{\theta+d}}\right]_{6\mathbb{N}}. \tag{6.38}$$

Thus $4Q_I\varepsilon L(I,\varepsilon)^{\theta+d} \leq p_0/10$. In addition, since $\theta > d$, $2Q_I\varepsilon^{\frac{1}{2}} L(I,\varepsilon)^d \leq \frac{p_0}{10}$ for ε small enough, depending on Q_I, θ, d, p_0. From (6.38) we also have

$$\frac{p_0}{160\,Q_I} L(I,\varepsilon)^{-(\theta+d)} \leq \varepsilon \leq \frac{p_0}{80\,Q_I} L(I,\varepsilon)^{-(\theta+d)} \tag{6.39}$$

for ε small enough, depending on Q_I, p_0, d. It follows from (6.37) and (6.39) that there exists $\tilde{\varepsilon}(I) = \tilde{\varepsilon}((d, \Theta_1, \Theta_2, \mathcal{X}, I, Q_I, \theta, \gamma, p_0) > 0$, such that for all $0 < \varepsilon \leq \tilde{\varepsilon}(I)$ and $E \in I$ we have

$$P_{E,L(I,\varepsilon)} \leq C_{n,d}\,\Omega_\varepsilon(n,\mathcal{X},E)^{\frac{1}{2}} L(I,\varepsilon)^{-\frac{n}{2}+d+\theta+\gamma} + \frac{4}{5}p_0\,. \tag{6.40}$$

At this point we might be tempted to conclude the proof by noting that applying Fatou's Lemma to condition (6.3) yields

$$\int_{\mathbb{R}} \liminf_{\varepsilon\to 0+} \varepsilon^{1+\alpha}\Omega_\varepsilon(n,\mathcal{X},E)\,\mathrm{d}E < +\infty, \tag{6.41}$$

hence $\liminf_{\varepsilon\to 0+} \varepsilon^{1+\alpha}\Omega_\varepsilon(n,\mathcal{X},E) < \infty$ for for a.e. E. If $n > 2d\alpha+8d$, it follows from (6.40), (6.39), and (6.41) that we can choose $\theta > d$ and $\gamma > d$ such that

$$\liminf_{L\to\infty} P_{E,L} \le \liminf_{\varepsilon\to 0+} P_{E,L(E,\varepsilon)} \le p_0 \tag{6.42}$$

for a.e. $E \in I$, and hence, since I is an arbitrary compact subinterval of J, for a.e. $E \in J$. Since p_0 is also arbitrary, the starting condition (6.23) for the bootstrap multiscale analysis is satisfied at a.e. $E \in J$. It follows from [GK1, Corollary 3.10 and Theorem 3.11] that there is a open subset G of J, such that $J\backslash G$ has zero Lebesgue measure and the random operator H_ω exhibits pure point spectrum and strong HS-dynamical localization in G. (If we assumed the hypotheses of [CHM, Corollary 1.3], we could even conclude that H_ω has pure point spectrum in the whole interval J.) But we cannot conclude that there is strong HS-dynamical localization in J; we cannot rule out the possibility of energies where the multiscale analysis cannot be performed (i.e., energies outside Σ_{MSA}), although the set of such singular energies must have zero measure.

To overcome this difficulty we use two facts: a) the Wegner estimate (2.8) holds everywhere in the interval J, and, b) the information that we have at our disposal is stronger than (6.41), we are given the finiteness of the right-hand-side of the inequality in Fatou's Lemma, namely condition (6.3).

We shall prove that there are no singular energies, i.e., that (6.23) holds for all $E \in J$. In view of (6.3) we can pick a sequence $\varepsilon_k \to 0+$ such that

$$\varepsilon_k^{1+\alpha}\int_{\mathbb{R}} \Omega_{\varepsilon_k}(n,\mathcal{X},E)\,\mathrm{d}E \le 2\Omega \quad \text{for} \quad k = 1,2,\ldots. \tag{6.43}$$

Given a compact subinterval of I of J and $M > 0$, we set

$$A_{k,I,M} = \{E \in I;\ \ \varepsilon_k^{1+\alpha}\Omega_{\varepsilon_k}(n,\mathcal{X},E) \le M\}\,. \tag{6.44}$$

In view of (6.43), we have

$$|I\backslash A_{k,I,M}| \le \frac{2\,\Omega}{M} \quad \text{for all } k \text{ and } M\,, \tag{6.45}$$

where $|A|$ denotes the Lebesgue measure of the set A. It follows from (6.40) and (6.39) that for each $E \in A_{k,I,M}$ we have

$$P_{E,L_k(I)} \le C_{n,d} M^{1/2} L_k(I)^{-m} + \frac{4}{5}p_0\,, \tag{6.46}$$

where $L_k(I) = L(I,\varepsilon_k)$, and

$$m = \frac{n}{2} - d - \theta - \gamma - \frac{1}{2}(1+\alpha)(\theta+d), \tag{6.47}$$

with $m > 0$ for suitable n's. We set

$$M_{k,I} = 2\Omega L_k(I)^{\theta+2\gamma}\,, \tag{6.48}$$
$$A_k(I) = A_{k,I,M_{k,I}}\,. \tag{6.49}$$

From (6.46) we see that

$$P_{E,L_k(I)} \le C_{n,d}(2\Omega)^{1/2} L_k(I)^{-m+\frac{1}{2}\theta+\gamma} + \frac{4}{5}p_0 \quad \text{if} \quad E \in A_k(I)\,, \tag{6.50}$$

hence there exists $k(I) = k(d, I, n, \alpha, \Omega, \theta, \gamma, p_0) < \infty$, such that if $k \ge k(I)$ we have $P_{E,L_k(I)} \le p_0$ for all $E \in A_k(I)$.

Let $E' \in I$. It follows from (6.45) and (6.48) that we can find $E \in A_k(I)$ such that

$$|E - E'| \le \frac{2\Omega}{M_k} = L_k(I)^{-\theta-2\gamma}\,. \tag{6.51}$$

The resolvent identity gives

$$\begin{aligned} &\|\Gamma_L R_{\omega,L}(E')\chi_{\frac{L}{3}}\|_L \\ &\le \|\Gamma_L R_{\omega,L}(E)\chi_{\frac{L}{3}}\|_L + |E-E'|\|R_{\omega,L}(E')\|_L\|R_{\omega,L}(E)\|_L \\ &\le \|\Gamma_L R_{\omega,L}(E)\chi_{\frac{L}{3}}\|_L + L_k(I)^{-\theta-2\gamma}\|R_{\omega,L}(E')\|_L\|R_{\omega,L}(E)\|_L\,, \end{aligned} \tag{6.52}$$

for any L. If $\operatorname{dist}(E, \sigma(H_{L_k(I)}) > 2L_k(I)^{-\gamma}$, it follows from (6.51) that $\operatorname{dist}(E', \sigma(H_{L_k(I)})) > L_k(I)^{-\gamma}$ for k large enough, depending only on θ and γ. Using the Wegner estimate (2.8) with the estimates (6.52) and (6.50), we see that we have

$$\begin{aligned} &\mathbb{P}\left(\left\|\Gamma_{L_k(I)} R_{\omega,L_k(I)}(E')\chi_{\frac{L_k(I)}{3}}\right\|_{L_k(I)} > \frac{1}{L_k(I)^\theta}\right) \\ &\quad\le P_{E,L_k(I)} + 2Q_I L_k(E)^{-\gamma+d} \\ &\quad\le C_{n,d}(2\Omega)^{1/2} L_k(I)^{-m+\frac{1}{2}\theta+\gamma} + \frac{4}{5}p_0 + 2Q_I L_k(I)^{-\gamma+d} \\ &\quad\le C_{n,d}(2\Omega)^{1/2} L_k(I)^{-m+\frac{1}{2}\theta+\gamma} + \frac{9}{10}p_0\,, \end{aligned} \tag{6.53}$$

for all k large enough, depending on Q_I, θ, γ, p_0, but independent of the energy $E' \in I$.

Given $n > n(\alpha) = 2d\alpha + 11d$, we now choose $\theta > d$ and $\gamma > d$ such that

$$m > \frac{1}{2}\theta + \gamma\,. \tag{6.54}$$

It follows from (6.53) that for all $E' \in I$ we have

$$\limsup_{k\to\infty} \mathbb{P}\left(\left\| \Gamma_{L_k(I)} R_{\omega,L_k(I)}(E') \chi_{\frac{L_k(I)}{3}} \right\|_{L_k(I)} > \frac{1}{L_k(I)^\theta} \right) \le p_0 \,. \tag{6.55}$$

Since $0 < p_0 < 1$ is arbitrary, we conclude that (6.23) holds for each $E' \in I$.

Theorem 2.11 is proven. □

A. Properties of random Schrödinger operators

In this appendix we verify properties of random Schrödinger operators that are needed for the bootstrap multiscale analysis [GK1] (justifying Theorem 4.1) and for the proof of Theorem 2.11.

Theorem A.1. *Let H_ω be a random Schrödinger operator (as defined in Section 2, satifying conditions (R), (E), and (IAD)). Then H_ω is a $\mathbb{Z}^d$-ergodic random self-adjoint operator satisfying Assumptions SLI, EDI, IAD, NE, and SGEE of [GK1]. The constants γ_{I_0} in Assumption SLI and $\tilde{\gamma}_{I_0}$ in Assumption EDI are given by $\gamma_{I_0} = \tilde{\gamma}_{I_0} = \sup_{E\in I_0} \gamma_E$, with*

$$\gamma_E = 6\sqrt{\frac{2d}{1-\Theta_1}}\sqrt{|E| + \Theta_2 + \frac{200d}{1-\Theta_1}} \,. \tag{A.1}$$

Proof. We saw in Section 2 that H_ω is a random self-adjoint operator; $\mathbb{Z}^d$-ergodicity is the content of condition (E), and Assumption IAD of [GK1] was built into condition (IAD). (Condition (IAD) is not needed to prove any of the other properties.)

We consider a (nonrandom) Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^d, \mathrm{d}x)$, where the potential $V = V^{(1)} + V^{(2)}$ satisfies the regularity condition (R) of Section 2, with constants $0 \le \Theta_1 < 1$ and $0 \le \Theta_2 < \infty$ as in (2.7). We will prove properties for H that will then be valid for random Schrödinger operators with probability one, proving Assumptions SLI, EDI, NE, and SGEE.

We start with an interior estimate, which is also used in the proof of Lemma 6.4.

Lemma A.2. *Let $x \in \mathbb{R}^d$, $L \in 2\mathbb{N}\cup\{\infty\}$, and η a real valued, continuously differentiable function on $\mathbb{R}^d$ with compact support $K \subset \Lambda_{x,L}$ and $\|\eta\|_\infty \le 1$. Then for any $a > 0$ we have*

$$\|\eta\nabla_{x,L}\psi\|_{x,L}^2 \le \tag{A.2}$$
$$a\,\|\chi_K H_{x,L}\psi\|_{x,L}^2 + \tfrac{2}{(1-\Theta_1)^2}\left(\tfrac{1}{2a} + \Theta_2(1-\Theta_1) + 8\|\nabla\eta\|_\infty^2\right)\|\chi_K\psi\|_{x,L}^2$$

for all $\psi \in \mathcal{D}(H_{x,L})$.

Proof. We adapt the proof of [Wei, Auxiliary Theorem 10.26]. In the following the constants $r, s, t > 0$ will be chosen later on. We have (we omit x, L from the norms)

$$\begin{aligned} r\left\|\eta H_{x,L}\psi\right\|^2 + \tfrac{1}{r}\left\|\eta\psi\right\|^2 &\geq 2\mathrm{Re}\left\langle H_{x,L}\psi, \eta^2\psi\right\rangle \\ &= 2\mathrm{Re}\left\{\left\langle \nabla_{x,L}\psi, \nabla_{x,L}\eta^2\psi\right\rangle + \left\langle \eta\psi, V\eta\psi\right\rangle\right\} \\ &\geq 2\mathrm{Re}\left\{\left\langle \nabla_{x,L}\psi, \eta^2\nabla_{x,L}\psi\right\rangle + 2\left\langle \eta\nabla_{x,L}\psi, (\nabla\eta)\,\psi\right\rangle + \left\langle \eta\psi, V^{(2)}\eta\psi\right\rangle\right\} \\ &\geq 2\left\{(1-s)\left\|\eta\nabla_{x,L}\psi\right\|^2 - \tfrac{1}{s}\left\|(\nabla\eta)\,\psi\right\|^2 - \Theta_1\left\|\nabla_{x,L}\eta\psi\right\|^2 - \Theta_2\left\|\eta\psi\right\|^2\right\} \\ &\geq 2\Big\{(1-s-(1+t)\Theta_1)\left\|\eta\nabla_{x,L}\psi\right\|^2 - \left(\tfrac{1}{s} + \left(1+\tfrac{1}{t}\right)\Theta_1\right)\left\|(\nabla\eta)\,\psi\right\|^2 \\ &\qquad - \Theta_2\left\|\eta\psi\right\|^2\Big\}\,, \end{aligned} \tag{A.3}$$

where we used (2.7) and the fact that

$$\left\|\nabla_{x,L}\eta\psi\right\|^2 \leq (1+t)\left\|\eta\nabla_{x,L}\psi\right\|^2 + \left(1+\tfrac{1}{t}\right)\left\|(\nabla\eta)\,\psi\right\|^2\,. \tag{A.4}$$

The desired estimate (A.2) now follows from (A.3) if $\|\eta\|_\infty \leq 1$, by choosing $s = \frac{1-\Theta_1}{4}$, $t = \frac{s}{\Theta_1}$, and $r = 2a(1-s-(1+t)\Theta_1) = a(1-\Theta_1)$. □

Assumption SLI can now be proven for H exactly as in [KK1, Lemma 3.8], using Lemma A.2 instead of [KK1, Lemma 3.4]. The constant γ_E corresponding to [KK1, eq. (3.80)], i.e., γ_E such that the constant γ_{I_0} in [GK1, eq. (2.9)] is given by $\gamma_{I_0} = \sup_{E\in I_0}\gamma_E$, is given by (A.1)

Similarly, Assumption EDI is proven as in [KK1, Lemma 3.9], with the constant $\tilde{\gamma}_{I_0}$ appearing in [GK1, eq. (2.15)] being the same as γ_{I_0} in Assumption SLI.

We now turn to Assumption NE, and prove a deterministic estimate. Since it follows from (2.7) that for $x \in \mathbb{R}^d$, $L \in 2\mathbb{N}\cup\{\infty\}$ we have

$$H_{x,L} \geq -(1-\Theta_1)\Delta_{x,L} - \Theta_2, \tag{A.5}$$

it follows by standard arguments (e.g., [KK1, Lemma 3.3]) that

$$\begin{aligned} \operatorname{tr}\left\{\chi_{(-\infty,E)}(H_{x,L})\right\} &\leq \operatorname{tr}\left\{\chi_{\left(-\infty,\frac{E+\Theta_2}{1-\Theta_1}\right)}(-\Delta_{x,L})\right\} \\ &\leq C_d\left(\frac{(E+\Theta_2)\vee 0}{1-\Theta_1}\right)^{\frac{d}{2}} L^d\,, \end{aligned} \tag{A.6}$$

for some constant C_d depending on d only. Assumption NE follows.

It remains to prove Assumption SGEE. This will be done in two lemmas.

Lemma A.3. *The set* $\mathcal{D}_0(H) = \{\phi \in \mathcal{D}(H);\ \phi$ *has compact support*$\}$ *is an operator core for* H.

Proof. Given a semi-bounded self-adjoint operator W, the corresponding quadratic form will be denoted by $W(\varphi,\psi)$, with $\varphi,\psi \in \mathcal{Q}(W)$, the quadratic form domain of W.

Since we have $\mathcal{Q}(H) = \mathcal{D}(\nabla) \cap \mathcal{Q}(V^{(1)})$, it follows that $\eta\,\mathcal{Q}(H) \subset \mathcal{Q}(H)$ if η is a real valued, twice continuously differentiable function on $\mathbb{R}^d$ which is bounded with bounded first and second derivatives. Thus, if $\psi \in \mathcal{D}(H)$ and $\varphi \in \mathcal{Q}(H)$, we have

$$\begin{aligned} H(\varphi,\eta\psi) &= \langle \nabla\varphi, \nabla\eta\psi\rangle + \langle \varphi, V\eta\psi\rangle \\ &= \langle \nabla\eta\varphi, \nabla\psi\rangle + \langle \eta\varphi, V\psi\rangle - \langle \varphi, (\nabla\eta)\cdot\nabla\psi\rangle + \langle \nabla\varphi, (\nabla\eta)\psi\rangle \\ &= \langle \eta\varphi, H\psi\rangle - 2\langle \varphi, (\nabla\eta)\cdot\nabla\psi\rangle - \langle \varphi, (\Delta\eta)\psi\rangle\,. \end{aligned} \tag{A.7}$$

Thus, using also (A.5), we get

$$\begin{aligned} &|H(\varphi,\eta\psi)| \le \\ &\left\{ \left(\|\eta\|_\infty + \tfrac{2\|\nabla\eta\|_\infty}{\sqrt{1-\Theta_1}} \right) \|H\psi\| + \left(\|\Delta\eta\|_\infty + \tfrac{2\sqrt{\Theta_2}\|\nabla\eta\|_\infty}{\sqrt{1-\Theta_1}} \right) \|\psi\| \right\} \|\varphi\|\,. \end{aligned} \tag{A.8}$$

thus $\eta\psi \in \mathcal{D}(H)$ and

$$\begin{aligned} &\|H\eta\psi\| \le \\ &\left\{ \left(\|\eta\|_\infty + \tfrac{2\|\nabla\eta\|_\infty}{\sqrt{1-\Theta_1}} \right) \|H\psi\| + \left(\|\Delta\eta\|_\infty + \tfrac{2\sqrt{\Theta_2}\|\nabla\eta\|_\infty}{\sqrt{1-\Theta_1}} \right) \|\psi\| \right\}. \end{aligned} \tag{A.9}$$

We now pick a real valued, twice continuously differentiable function ρ on $\mathbb{R}^d$ with compact support, such that $0 \le \rho \le 1$ and $\rho(0) = 1$. For each $n = 1, 2, \ldots$ we set $\rho_n(x) = \rho(\frac{1}{n}x)$ and $\eta_n = 1 - \rho_n$. Given $\psi \in \mathcal{D}(H)$. we let $\psi_n = \rho_n\psi$. Then $\psi_n \in \mathcal{D}_0(H)$, $\|\psi - \psi_n\| \to 0$, and $\|H(\psi - \psi_n)\| = \|H(\eta_n\psi)\| \to 0$ by (A.9). Thus $\mathcal{D}_0(H)$ is a core for H. $\square$

Let $\nu > d/4$, $\mathcal{H}_+ = \mathrm{L}^2(\mathbb{R}^d, \langle x\rangle^{4\nu}\mathrm{d}x)$, and $\mathcal{D}_+(H) = \{\phi \in \mathcal{D}(H) \cap \mathcal{H}_+;\ H\phi \in \mathcal{H}_+\}$. Since if $\psi \in \mathcal{D}_0(H)$ we can see that $H\psi$ has compact support by looking at the quadratic form, we have $\mathcal{D}_0(H) \subset \mathcal{D}_+(H)$, and hence $\mathcal{D}_+(H)$ is a core for H. It is also easy to see that $\mathcal{D}_0(H)$, and hence also $\mathcal{D}_+(H)$, is dense in $\mathcal{H}_+$. This proves the first part of Assumption GEE in [GK1, page 425], which is also the first part of SGEE. The second part of SGEE, [GK1, eq. (2.32)], will follow from the following lemma. Note that the lemma proves the stronger [GK1, eq. (2.36)].

Lemma A.4. *Let* $\nu > \frac{d}{4}$. *There is a finite constant* $\mathcal{T}_{\nu,d,\Theta_1,\Theta_2}$, *depending only on the indicated constants, such that*

$$\operatorname{tr}\left(\langle X\rangle^{-2\nu} \left(H + \Theta_2 + (1-\Theta_1)\right)^{-2[[\frac{d}{4}]]} \langle X\rangle^{-2\nu} \right) \le \mathcal{T}_{\nu,d,\Theta_1,\Theta_2}\,, \tag{A.10}$$

where $[[\frac{d}{4}]]$ *is the smallest integer* $> \frac{d}{4}$. *Thus, letting*

$$\Phi_{d,\Theta_1,\Theta_2}(E) = (E + \Theta_2 + (1 - \Theta_1))^{2[[\frac{d}{4}]]}\ , \tag{A.11}$$

we have

$$\operatorname{tr}\left(\langle X\rangle^{-2\nu} f(H)\,\langle X\rangle^{-2\nu}\right) \le \mathcal{T}_{\nu,d,\Theta_1,\Theta_2} \| f\Phi_{d,\Theta_1,\Theta_2}\|_\infty < \infty \tag{A.12}$$

for every bounded measurable function $f \ge 0$ *on the real line with compact support.*

Proof. It follows form (A.5) that for any $L \in 2\mathbb{N}$ we have

$$H_{0,L} + \Theta_2 + (1 - \Theta_1) \ge (1 - \Theta_1)(-\Delta_{0,L} + 1)\,. \tag{A.13}$$

Recalling that for self-adjoint operators T and S, if $0 \le S \le T$ then $\operatorname{tr} f(S) \le \operatorname{tr} f(T)$ for any positive decreasing function on $[0,\infty)$, we get

$$\begin{aligned} &\operatorname{tr}\left(H_{0,L} + \Theta_2 + (1 - \Theta_1)\right)^{-2[[\frac{d}{4}]]} \le \\ &\qquad (1 - \Theta_1)^{-2[[\frac{d}{4}]]} \operatorname{tr}\left(-\Delta_{0,L} + 1\right)^{-2[[\frac{d}{4}]]} < \infty. \end{aligned} \tag{A.14}$$

(The finiteness is easy to see for periodic boundary condition.)

We may now proceed as in the proof of [KKS, Theorem 1.1] and obtain (A.10). The bound (A.12) is an immediate consequence. □

Theorem A.1 is proven. □

The following kernel polynomial decay estimate follows immediately from [GK2, Theorem 2]:

Theorem A.5. *Let* H_ω *be a random Schrödinger operator (as defined in Section 2) satifying conditions (R) and (E). There is a finite constant* $\widetilde{C}_d$*, depending only on the dimension* d *(thus independent of* H_ω*), such that for all infinitely differentiable functions* f *on the real line we have*

$$\|\chi_x f(H_\omega)\chi_y\| \le \widetilde{C}_d \, |||f|||_{k+2} \left(\frac{3k\,(\Theta_2 + 8)}{\sqrt{1 - \Theta_1}}\right)^k \langle x - y\rangle^{-k} \tag{A.15}$$

for $\mathbb{P}$*-a.e.* ω*, all* $k = 1, 2, \ldots$*, and all* $x, y \in \mathbb{R}^d$*, where* Θ_1 *and* Θ_2 *are given in (2.2), and*

$$|||f|||_n = \sum_{r=0}^{n} \int_{\mathbb{R}} |f^{(r)}(u)| \langle u\rangle^{r-1} \mathrm{d}u \ , \quad n = 1, 2, \ldots. \tag{A.16}$$

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References

[Ai] Aizenman, M.: Localization at weak disorder: some elementary bounds. Rev. Math. Phys. **6**, 1163-1182 (1994)

[AM] Aizenman, M., Molchanov, S.: Localization at large disorder and extreme energies: an elementary derivation. Commun. Math. Phys. **157**, 245-278 (1993)

[ASFH] Aizenman, M., Schenker, J., Friedrich, R., Hundertmark, D.: Finite volume fractional-moment criteria for Anderson localization. Commun. Math. Phys. **224**, 219-253 (2001)

[An] Anderson, P.: Absence of diffusion in certain random lattices. Phys. Rev. **109**, 1492-1505 (1958)

[AK] Asch, J., Knauf, A.: Motion in periodic potentials. Nonlinearity **11**, 175-200 (1998)

[BGT] Barbaroux, J-M., Germinet, F., Tcheremchantsev, S. : Fractal dimensions and the phenomenon of intermittency in quantum dynamics. Duke Math. J. **110**, 161-193 (2001)

[CKM] Carmona, R., Klein, A., Martinelli, F.: Anderson localization for Bernoulli and other singular potentials. Commun. Math. Phys. **108**, 41-66 (1987)

[CL] Carmona, R, Lacroix, J.: *Spectral theory of random Schrödinger operators.* Boston: Birkhaüser, 1990

[Co] Combes, J.M.: Connection between quantum dynamics and spectral properties of time evolution operators in *Differential Equations and Applications in Mathematical Physics*, Eds. W.F. Ames, E.M. Harrel, J.V. Herod, Academic Press, 59-69 (1993)

[CH1] Combes, J.M., Hislop, P.D.: Localization for some continuous, random Hamiltonian in d-dimension. J. Funct. Anal. **124**, 149-180 (1994)

[CH2] Combes, J.M., Hislop, P.D.: Landau Hamiltonians with random potentials: localization and the density of states. Commun. Math. Phys. **177**, 603-629 (1996)

[CHKN] Combes, J.M., Hislop, P.D., Klopp, F. Nakamura, S.: The Wegner estimate and the integrated density of states for some random operators. Preprint

[CHM] Combes, J.M., Hislop, P.D., Mourre, E.: Spectral averaging, perturbation of singular spectra, and localization. Trans. Amer. Math. Soc. **348**, 4883-4894 (1996)

[CHN] Combes, J.M., Hislop, P.D., Nakamura, S.: The L^p-theory of the spectral shift function, the Wegner estimate and the integrated density of states for some random operators. Commun. Math. Phys. **218**, 113-130 (2001)

[CM] Combes, J.M., Mantica, G.: Fractal dimensions and quantum evolution associated with sparse potential Jacobi matrices. Long time behaviour of classical and quantum systems (Bologna, 1999), 107–123, Ser. Concr. Appl. Math., 1, World Sci. Publishing, River Edge, NJ, 2001

[CFKS] Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B.: *Schrödinger operators.* Heidelberg: Springer-Verlag, 1987

[DSS] Damanik, D., Sims, R., Stolz, G.: Localization for one dimensional, continuum, Bernoulli-Anderson models. Duke Math. J. To appear

[DSS] Damanik, D., Stollmann, P.: Multi-scale analysis implies strong dynamical localization. Geom. Funct. Anal. **11**, 11-29 (2001)

[DBF] De Bièvre, S., Forni, G.: Transport properties of kicked and quasiperiodic Hamiltonians. J. Statist. Phys. **90**, 1201-1223 (1998)

[DBG] De Bièvre, S., Germinet, F.: Dynamical localization for random dimer Schrödinger operator. J. Stat. Phys. **98**, 1135-1147 (2000)

[DR+1] Del Rio, R., Jitomirskaya, S., Last, Y., Simon, B.: What is Localization? Phys. Rev. Lett. 75, 117-119 (1995)

[DR+2] Del Rio, R., Jitomirskaya, S., Last, Y., Simon, B.: Operators with singular continuous spectrum IV: Hausdorff dimensions, rank one pertubations and localization. J. d'Analyse Math. **69**, 153-200 (1996)

[DS1] Dobrushin, R., Shloshman, S.: Completely analytical Gibbs fields. Prog in Phys. **10**, 347-370 (1985)

[DS1] Dobrushin, R., Shloshman, S.: Completely analytical interactions. J. Stat. Phys. **46**, 983-1014 (1987)

[vDK] von Dreifus, H., Klein, A.: A new proof of localization in the Anderson tight binding model. Commun. Math. Phys. **124**, 285-299 (1989)

[FK1] Figotin, A., Klein, A.: Localization of classical waves I: Acoustic waves. Commun. Math. Phys. **180**, 439-482 (1996)

[FK2] Figotin, A., Klein, A.: Localization of classical waves II: Electromagnetic waves. Commun. Math. Phys. **184**, 411-441 (1997)

[FMSS] Fröhlich, J., Martinelli, F., Scoppola, E., Spencer, T.: Constructive proof of localization in the Anderson tight binding model. Commun. Math. Phys. **101**, 21-46 (1985)

[FS] Fröhlich, J., Spencer, T.: Absence of diffusion with Anderson tight binding model for large disorder or low energy. Commun. Math. Phys. **88**, 151-184 (1983)

[Ge] Germinet, F.: Dynamical localization II with an application to the almost Mathieu operator. J. Stat Phys. **95**, 273-286 (1999)

[GDB] Germinet, F., De Bièvre, S.: Dynamical localization for discrete and continuous random Schrödinger operators. Commun. Math. Phys. **194**, 323-341 (1998)

[GJ] Germinet, F., Jitomirskaya, S.: Strong Dynamical Localization for the almost Mathieu model. Rev. Math. Phys. **13**, 755-765 (2001)

[GKT] Germinet, F., Kiselev, A., Tcheremchantsev, S.: In preparation

[GK1] Germinet, F., Klein, A.: Bootstrap Multiscale Analysis and Localization in random media. Commun. Math. Phys. **222**, 415-448 (2001)

[GK2] Germinet, F, Klein, A.: Decay of operator-valued kernels of functions of generalized Schrödinger and other operators. Proc. Amer. Math. Soc. To appear

[GK3] Germinet, F., Klein, A.: Finite volume criteria for localization in random media. In preparation.

[GMP] Gol'dsheid, Ya., Molchanov, S., Pastur, L.: Pure point spectrum of stochastic one dimensional Schrödinger operators. Funct. Anal. Appl. **11**, 1-10 (1977)

[Gu] Guarneri, I., Spectral properties of quantum diffusion on discrete lattices, Europhys. Lett. 10 (1989), 95-100; On an estimate concerning quantum diffusion in the presence of a fractal spectrum, Europhys. Lett. **21**, 729-733 (1993).

[HK] Hislop, P.D., Klopp, F.: The integrated density of states for some random operators with nonsign definite potentials. Preprint

[HM] Holden, H., Martinelli, F.: On absence of diffusion near the bottom of the spectrum for a random Schrödinger operator. Commun. Math. Phys. **93**, 197-217 (1984)

[JaL] Jaksic, V., Last, Y.: Spectral structure of Anderson type Hamiltonians. Invent. Math. **141** 561–577 (2000).

[JiL] Jitomirskaya, S., Last, Y.: Anderson localization for the almost Mathieu equation, III. Semi-uniform localization, continuity of gaps, and measure of the spectrum. Commun. Math. Phys. **195**, 1-14 (1998)

[Ki] Kirsch, W.: Wegner estimates and Anderson localization for alloy-type potentials. Math. Z. **221**, 507–512 (1996)

[KM] Kirsch, W., Martinelli, F. : On the ergodic properties of the spectrum of general random operators. J. Reine Angew. Math. **334**, 141-156 (1982)

[KSS] Kirsch, W., Stolz, G., Stollmann, P.: Localization for random perturbations of periodic Schrödinger operators. Random Oper. Stochastic Equations **6**, 241-268 (1998)

[KL] Kiselev, A., Last, Y.: Solutions, spectrum, and dynamics for Schrödinger operators on infinite domains. Duke Math. J. **102**, 125-150 (2000)

[Kle1] Klein, A.: Extended states in the Anderson model on the Bethe lattice. Adv. Math. **133**, 163–184 (1998)

[Kle2] Klein, A.: Spreading of wave packets in the Anderson model on the Bethe lattice. Commun. Math. Phys. **177**, 755–773 (1996)

[KK1] Klein, A., Koines, A.: A general framework for localization of classical waves: I. Inhomogeneous media and defect eigenmodes. Math. Phys. Anal. Geom. **4**, 97-130 (2001)

[KK2] Klein, A., Koines, A.: A general framework for localization of classical waves: II. Random media. In preparation.

[KKS] Klein, A., Koines, A., Seifert, M.: Generalized eigenfunctions for waves in inhomogeneous media. J. Funct. Anal. To appear

[Klo1] Klopp, F.: Localization for semiclassical continuous random Schrödinger operators. II. The random displacement model. Helv. Phys. Acta **66**, 810–841 (1993)

[Klo2] Klopp, F.: Localization for continuous random Schrödinger operators. Commun. Math. Phys. **167**, 553-569 (1995)

[Klo3] Klopp, F.:Weak disorder localization and Lifshitz tails. Preprint

[Klo4] Klopp, F.: Weak disorder localization and Lifshitz tails: continuous Hamiltonians. Preprint

[KS] Kunz, H., Souillard, B.: Sur le spectre des operateurs aux differences finies aleatoires. Commun. Math. Phys. **78**, 201-246 (1980)

[LKS] Lacroix, J., Klein, A., Speis, A.: Localization for the Anderson model on a strip with singular potentials. J. Funct. Anal. **94**, 135-155 (1990)

[La] Last, Y: Quantum dynamics and decompositions of singular continuous spectra. J. Funct. Anal. **142**, 406-445 (1996)

[LGP] Lifshits, I.M., Gredeskul, A.G., Pastur, L.A.: *Introduction to the Theory of Disordered Systems*, Wiley-Interscience, New York, 1988.

[MS] Martinelli, F., Scoppola, E.: Introduction to the mathematical theory of Anderson localization. Riv. Nuovo Cimento **10**, 1-90 (1987)

[Pa] Pastur, L.: Spectral properties of disordered systems in one-body approximation. Commun. Math. Phys. **75**, 179-196 (1980)

[PF] Pastur, L., Figotin, A.: *Spectra of Random and Almost-Periodic Operators.* Heidelberg: Springer-Verlag, 1992

[Si] Simon, B.: Schrödinger semi-groups. Bull. Amer. Math. Soc. **Vol.7**, 447-526 (1982)

[St] Stollmann. P.: Wegner estimates and localization for continuum Anderson models with some singular distributions. Arch. Math. (Basel) **75**, 307-311 (2000)

[Tc] Tcheremchantsev, S.: How to prove dynamical localization. Commun. Math. Phys. **221**, 27-56 (2001)

[Wa1] Wang, W.-M.: Microlocalization, percolation, and Anderson localization for the magnetic Schrödinger operator with a random potential. J. Funct. Anal. **146**, 1-26 (1997)

[Wa2] Wang, W.-M.: Localization and universality of Poisson statistics near Anderson transition. Invent. Math. **146**, 365-398 (2001)

[Weg] Wegner, F.: Bounds on the density of states in disordered systems. Z. Phys. B **44**, 9-15 (1981)

[Wei] Weidmann, J.: *Linear Operators in Hilbert Spaces.* Berlin: Springer-Verlag, 1980