SPHERICALLY SYMMETRIC SOLUTIONS OF A BOUNDARY VALUE PROBLEM FOR MONOPOLES

ANTONELLA MARINI AND LORENZO SADUN

Abstract. In this paper we study spherically symmetric monopoles, which are critical points for the Yang-Mills-Higgs functional over a disk in 3 dimensions, with prescribed degree and covariant constant at the boundary. This is a 3-dimensional gauge-theory generalization of the Ginzburg-Landau model in 2 dimensions.

1. Introduction

In this paper we treat a 3-dimensional analogue of the vortex equations in 2 dimensions and look for solutions with spherical symmetry as described in §2. The domain considered is a 3-dimensional disk and we prescribe the degree of the monopole at the boundary. Unlike in 2 dimensions, where Ginzburg-Landau-type functionals appear either with or without gauge potentials, the problem in 3 dimensions is well-posed only in gauge theory. Without a curvature term in the action, a minimizing sequence for the action would yield a trivial limit [2]. In the presence of a gauge potential, this problem is well-posed, natural and has physical meaning.

The most general Yang-Mills-Higgs functional takes the form

$$\mathcal{YMH}(A,\phi) = \frac{\epsilon}{2}\|F\|^2_{L^2} + \frac{\rho}{2}\|D_A\phi\|^2_{L^2} + \frac{\lambda}{8}\|\,|\phi|^2 - a^2\|^2_{L^2}\ , \tag{1.1}$$

for appropriate constants ϵ, ρ, λ and a. Working on $\mathbb{R}^3$, one usually applies a rescaling of ϕ, a rescaling of space, and a rescaling of the action to set $\epsilon = \rho = a = 1$, so the action functional depends on a single parameter, λ. On the unit ball, however, we cannot rescale space, so we can only eliminate two of the four parameters. We set $\rho = a = 1$, and obtain a 2-paramater family of functionals

$$\mathcal{YMH}_{\epsilon,\lambda}(A,\,\phi) = \frac{\epsilon}{2}\|F\|^2_{L^2(B^3)} + \frac{1}{2}\|D_A\phi\|^2_{L^2(B^3)} + \frac{\lambda}{8}\|\,|\phi|^2 - 1\|^2_{L^2(B^3)}\ . \tag{1.2}$$

(Alternatively, we could work on a sphere of radius R. One can then rescale to set $\epsilon = 1$, at the cost of varying R. We then obtain a 2-parameter family of functionals indexed by λ and R.) We know from the general theory for monopoles (cf. [2] for $\epsilon = 1$, $\lambda \geq 0$) that there exists a minimum for this functional which satisfies the Euler Lagrange equations

$$\begin{aligned} \epsilon * D_A * F &= [D_A\phi,\,\phi] \\ *D_A * D_A\phi &= \frac{\lambda}{2}(|\phi|^2 - 1)\phi \end{aligned} \tag{1.3}$$

and suitable boundary conditions on $\partial B^3 \equiv S^2$ (cf. Section 4 and [2, 3]), and is smooth. In this paper we prove the existence, and describe the form, of spherically symmetric solutions to these equations.

We note that, even for $\lambda = 0$, these are not solutions to the Bogomolnyi equations found in [1]. The Bogomolnyi solutions are obtained only in the limit $\lambda \to 0$, $R \to \infty$, or equivalently $\lambda \to 0$, $\epsilon \to 0$.

2. Spherically symmetric connections, monopoles, and gauge transformations

We work on the trivial principal $SU(2)$-bundle $P = B^3 \times SU(2)$ and its associated vector bundles. A is a connection on P, which can be viewed as a 1-form on B^3 with values in $su(2)$. The Higgs field ϕ is a section of the adjoint bundle, i.e., a map $\phi : B^3 \to su(2)$. Here $su(2) \equiv \{X \in \mathcal{M}_{2\times 2} \, : \, tr\, X = 0\, ; \, X + \bar{X}^T = 0\}$ is the Lie algebra of $SU(2)$. We identify $su(2)$ with the imaginary quaternions $Im\,\mathbb{H} \equiv \{x_1 i + x_2 j + x_3 k : \, (x_1, x_2, x_3) \in \mathbb{R}^3\}$ by identifying the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, with i, j, k, respectively, and extending this mapping to a Lie-algebra isomorphism. It is also convenient to define the 3-vector of Lie-algebra elements $\vec{\sigma} = (i, j, k)$.

The symmetry group $SO(3)$ acts on pairs $(A,\, \phi)$, simultaneously rotating the 3-dimensional base space and the 3-dimensional Lie algebra. That is, the triple (i, j, k) transform in the same way as the triple (x_1, x_2, x_3), so quantities such as $\vec{\sigma} \cdot \vec{x}$ are invariant.

We are interested in finding Yang-Mills-Higgs fields (A, ϕ) which are invariant under this group action. To this purpose, one needs to specify the value of the connection one-form $A \, : \, B^3 \to \Lambda^1(B^3) \otimes Im\,\mathbb{H}$ and of the Higgs field $\phi \, : \, B^3 \to Im\,\mathbb{H}$ at one point of each group orbit (on the base) and impose invariance under the isotropy group of that point. We find it convenient to fix the values of (A, ϕ) on the slice $L \equiv \{\vec{x} \in B^3 \, : \, x_2 = x_3 = 0, x_1 \in (0, 1]\}$. The isotropy group at $(x, 0, 0) \in L$ is $SO(2)$, i.e. rotations about the x_1-axis (and about the i axis in the Lie-algebra). For the Higgs field ϕ one has in general

$$\phi(x, 0, 0) = \varphi_1(x) i + \varphi_2(x) j + \varphi_3(x) k \in Im\,\mathbb{H} \simeq su(2)\,,$$

where $\varphi_l(x)\,,\; l = 1, 2, 3$ are real-valued functions. Imposing invariance under $SO(2)$ forces $\varphi_2(x) = \varphi_3(x) = 0$ for all $x \in (0, 1]$. Applying the action of $SO(3)$, one obtains the symmetric form of the Higgs field ϕ

$$\phi = \frac{\varphi(r)}{r} \vec{\sigma} \cdot \vec{x} \equiv \frac{\varphi(r)}{r} \left(x_1 i + x_2 j + x_3 k\right), \tag{2.1}$$

with $r \equiv |\vec{x}|$, $\vec{x} \equiv (x_1, x_2, x_3) \in \mathbb{R}^3$, $\vec{\sigma} \equiv (i, j, k) \in Im\mathbb{H}$.

An $su(2)$-valued connection on the slice L is given in general by

$$\begin{aligned} A(x,0,0) &= a_{11}(x)i\,dx_1 + a_{12}(x)i\,dx_2 + a_{13}(x)i\,dx_3 + a_{21}(x)j\,dx_1 + \\ &\quad + a_{22}(x)k\,dx_2 + a_{23}(x)j\,dx_3 + a_{31}(x)k\,dx_1 + a_{32}(x)k\,dx_2 + +a_{33}(x)k\,dx_3 \end{aligned} \tag{2.2}$$

Imposing SO(2)-invariance yields $a_{12} = a_{13} = a_{21} = a_{31} = 0$, $a_{22} = a_{33}$, and $a_{23} = -a_{32}$. Thus the final form of the $SO(2)-$invariant connection evaluated at the points $(x,0,0) \in L$ is

$$A(x,0,0) = a(x)i\,dx_1 + b(x)(j\,dx_2 + k\,dx_3) + c(x)(k\,dx_2 - j\,dx_3)\,.$$

Transporting this slice via the $SO(3)$ group action on B^3 one obtains the invariant version

$$A(x_1,x_2,x_3) = \frac{\alpha(r)}{r}\vec{\sigma}\cdot\vec{dx} + \frac{\beta(r)}{r^3}(\vec{x}\times\vec{dx})\cdot(\vec{x}\times\vec{\sigma}) + \frac{\gamma(r)}{r^2}\vec{\sigma}\cdot(\vec{x}\times\vec{dx})\,, \tag{2.3}$$

where"$\times$" denotes the cross product of vectors and $\vec{dx} \equiv (dx_1, dx_2, dx_3)$.

At this point there is still some gauge freedom available to further specify the connection A, namely that provided by "symmetric" gauge transformations. Such a transformation g is determined by its values on the slice L, which must be $SO(2)$-invariant. This yields $i\,g(x,0,0) = g(x,0,0)\,i$, thus $g(x,0,0) = exp\,(i\,h(x))$. Therefore, symmetric gauge transformations are of the type

$$g(x_1,x_2,x_3) = exp\,(h(r)\,\frac{\vec{\sigma}\cdot\vec{x}}{|\vec{x}|})\;. \tag{2.4}$$

where $h(r)$ is an arbitrary function of the radius r. Performing such a gauge transformation does not change the form (2.1) of the Higgs field ϕ. However, A transforms nontrivially. In particular, setting $h(r) \equiv \int \frac{\alpha(r)}{r}\,dr$, exactly cancels the α-piece in $g^{-1}dg + g^{-1}A\,g$. Thus, one can impose $\alpha(r) = 0$. The final version of A is then

$$A(x_1,x_2,x_3) = \frac{\beta(r)}{r^3}(\vec{x}\times\vec{dx})\cdot(\vec{x}\times\vec{\sigma}) + \frac{\gamma(r)}{r^2}\vec{\sigma}\cdot(\vec{x}\times\vec{dx})\,. \tag{2.5}$$

The only gauge freedom remaining is from the constant of integration in the indefinite integral $\int \frac{\alpha(r)}{r}\,dr$ (cf. Section 4).

Note that the β and γ terms have opposite parities. The isometry $\vec{x} \to -\vec{x}$ sends β to $-\beta$ but sends γ to $+\gamma$.

3. The Yang-Mills-Higgs functional on spherically symmetric configurations

In this section we explicity compute the Yang-Mills-Higgs functional, and the resulting equations of motion, for symmetric pairs (A,ϕ). Our connection is a sum of two terms,

$$A = B + C\,,$$

where

$$B=\frac{\beta(r)}{r^3}(\vec{x}\times\vec{dx})\cdot(\vec{x}\times\vec{\sigma})=\frac{\beta(r)}{r^3}[(r^2-x_1^2)i-x_1x_3k-x_1x_2j]d\,x_1+ \tag{3.1}$$
$$+\frac{\beta(r)}{r^3}[(r^2-x_2^2)j-x_2x_3k-x_1x_2i]d\,x_2+\frac{\beta(r)}{r^3}[(r^2-x_3^2)k-x_1x_3i-x_2x_3j]d\,x_3\,,$$

and

$$C=\frac{\gamma(r)}{r^2}\vec{\sigma}\cdot(\vec{x}\times\vec{dx})=\frac{\gamma(r)}{r^2}(x_3j-x_2k)d\,x_1+ \tag{3.2}$$
$$+\frac{\gamma(r)}{r^2}(x_1k-x_3i)d\,x_2+\frac{\gamma(r)}{r^2}(x_2i-x_1j)d\,x_3$$

Preliminary computations

The curvature of A is given by

$$F=dA+A\wedge A=dB+dC+B\wedge B+C\wedge C\,,$$

since the cross term $B\wedge C+C\wedge B$ is identically zero.

Computing F_{12}, the various nonzero terms are

$$(d\,B)_{12}=\frac{\beta'}{r^2}(-x_2i+x_1j) \tag{3.3}$$

$$(d\,C)_{12}=\big(\frac{-\gamma}{r^2}\big)'\big(\frac{x_1x_3}{r}\,i+\frac{x_2x_3}{r}\,j\big)+\big(\frac{2\gamma}{r^2}+\big(\frac{\gamma}{r^2}\big)'\big(\frac{x_2^2}{r}+\frac{x_1^2}{r}\big)\big)\,k \tag{3.4}$$

$$(B\wedge B)_{12}=\frac{2\beta^2}{r^4}\left[x_1x_3i+x_2x_3j+x_3^2k\right] \tag{3.5}$$

$$(C\wedge C)_{12}=\frac{2\gamma^2}{r^4}\left[x_1x_3i+x_2x_3j+x_3^2k\right]. \tag{3.6}$$

By rotational symmetry, the contributions to F_{13} and F_{23} are similar.

The covariant derivative of the monopole $\phi=\frac{\varphi(r)}{r}\vec{\sigma}\cdot\vec{x}$ is given by

$$D_A\phi\equiv d\phi+[B,\phi]+[C,\phi]\,, \tag{3.7}$$

where

$$d\phi=\frac{\vec{\sigma}\cdot\vec{x}}{r}\,d\varphi+\varphi\,d\big(\frac{\vec{\sigma}\cdot\vec{x}}{r}\big)\,, \tag{3.8}$$

$$[B,\phi]\equiv B\phi-\phi B=\frac{2\beta\varphi}{r^2}(-x_3j+x_2k)\,d\,x_1+\text{cyclic permutations,} \tag{3.9}$$

$$[C,\phi]\equiv C\phi-\phi C=\frac{2\gamma\varphi}{r^3}[(r^2-x_1^2)i-x_1x_3k-x_1x_2j]d\,x_1+\text{cyclic permutations.} \tag{3.10}$$

At this point we are ready to compute the three terms involved in the Yang-Mills-Higgs functional (1.2). They are

1) $|F|^2\equiv F\wedge *F$
2) $|D\phi|^2\equiv D\phi\wedge *D\phi$

3) $(|\phi|^2-1)^2$.

Computation of 1): One shows easily that

$$dB\wedge *dC = dC\wedge *dB = 0\,,$$

$$dB\wedge *(B\wedge B) = (B\wedge B)\wedge *dB = 0\,,$$

$$dB\wedge *(C\wedge C) = (C\wedge C)\wedge *dB = 0\,,$$

thus

$$\begin{aligned}|F|^2 =& |dB|^2+|dC|^2+dC\wedge *(B\wedge B)+dC\wedge *(C\wedge C)+(B\wedge B)\wedge *(B\wedge B)\\ &+(B\wedge B)\wedge *(C\wedge C)+(C\wedge C)\wedge *(C\wedge C)\\ =&2\frac{\beta'^2}{r^2}+2\frac{\gamma'^2}{r^2}+4\frac{\gamma^2}{r^4}+4\frac{\beta^2\gamma}{r^4}+4\frac{\gamma^3}{r^4}+4\frac{\beta^4}{r^4}+8\frac{\beta^2\gamma^2}{r^4}+4\frac{\gamma^4}{r^4}\\ =&\frac{2\beta'^2}{r^2}+\frac{2\gamma'^2}{r^2}+\frac{4(\beta^2+\gamma^2+\gamma)^2}{r^4}.\end{aligned} \tag{3.11}$$

Computation of 2): One shows that

$$d\phi\wedge *[B,\phi] = [B,\phi]\wedge *d\phi = 0\,,$$

$$[B,\phi]\wedge *[C,\phi] = [C,\phi]\wedge *[B,\phi] = 0\,,$$

thus

$$|D\phi|^2 = |d\phi|^2+|[B,\phi]|^2+|[C,\phi]|^2+2\,d\phi\wedge *[C,\phi] \tag{3.12}$$

$$= \left(\varphi'^2+2\frac{\varphi^2}{r^2}\right)+8\frac{\beta^2\varphi^2}{r^2}+8\frac{\gamma^2\varphi^2}{r^2}+8\frac{\gamma\varphi^2}{r^2}. \tag{3.13}$$

Computation of 3): One easily obtains

$$(|\phi|^2-1)^2 = \big(\frac{\varphi^2}{r^2}-1\big)^2.$$

Collecting terms, the Yang-Mills-Higgs functional calculated on spherically symmetric configurations is given by

$$\begin{aligned}\mathcal{SYMH}(\gamma,\varphi) = 4\pi\int_0^1 [\ 2\epsilon\left(\beta'^2+\gamma'^2+\frac{2}{r^2}(\beta^2+\gamma^2+\gamma)^2\right)+r^2\varphi'^2+\\ 2\varphi^2[1+4(\beta^2+\gamma^2+\gamma)]+\lambda r^2(\varphi^2-1)^2\ ]dr\ .\end{aligned} \tag{3.14}$$

4. Further gauge transformations and the Euler-Lagrange equations

We want to search for absolute minima of the functional (3.14) among all finite-action spherically symmetric configurations (β,γ,φ). The following theorem restricts the possibilities:

Theorem 4.1. *If the functional* (3.14) *has a minimum, then this minimum is achieved by functions* (β,γ,φ) *with* β *identically zero and* $\gamma(0)=0$.

Proof. First we find a gauge transformations that yields $\beta(0) = \gamma(0) = 0$. In that gauge, we then show that minimization requires $\frac{\beta}{\gamma+\frac{1}{2}}$ to be constant, hence for β to be identically zero.

If $\mathcal{SYMH}(\beta, \gamma, \varphi) < \infty$, then β and γ much approach well-defined limits as $r \to 0$, and $\beta^2(0) + \gamma^2(0) + \gamma(0) = 0$. If $\beta(0)$ and $\gamma(0)$ are not already zero, we let θ be the argument of the complex number $\beta(0) + i\gamma(0)$, and define

$$g = exp(\theta \frac{\vec{\sigma} \cdot \vec{x}}{|\vec{x}|}) \equiv exp(\theta \frac{x_1 i + x_2 j + x_3 k}{|\vec{x}|}) \equiv \cos(\theta) + \sin(\theta) \frac{\vec{\sigma} \cdot \vec{x}}{|\vec{x}|} \ . \tag{4.1}$$

Then

$$d\, g(x_1, x_2, x_3) = \sin(\theta) \left[\frac{\vec{\sigma} \cdot \vec{dx}}{|\vec{x}|} + d(\frac{1}{|\vec{x}|}) \vec{\sigma} \cdot \vec{x} \right] \ , \tag{4.2}$$

and evaluating on our slice gives

$$d\, g(r, 0, 0) = \frac{\sin(\theta)}{r} (j\, d\, x_2 + k\, d\, x_3). \tag{4.3}$$

Our transformed connection on the slice is then

$$\begin{aligned} [g^{-1} d\, g + g^{-1} A\, g](r, 0, 0) = & \left[\frac{\cos(2\theta)\, \beta(r)}{r} + \frac{\sin(2\theta)\, \gamma(r)}{r} + \frac{\cos(\theta) \sin(\theta)}{r} \right] (j\, d\, x_2 + k\, d\, x_3) \\ & + \left[-\frac{\sin(2\theta)\, \beta(r)}{r} + \frac{\cos(2\theta)\, \gamma(r)}{r} - \frac{\sin^2(\theta)}{r} \right] (k\, d\, x_2 - j\, d\, x_3) \\ \equiv & \frac{\hat{\beta}(r)}{r} (j\, d\, x_2 + k\, d\, x_3) + \frac{\hat{\gamma}(r)}{r} (k\, d\, x_2 - j\, d\, x_3). \end{aligned} \tag{4.4}$$

Plugging in the values of $\sin(\theta)$, $\cos(\theta)$, etc., gives

$$\hat{\beta}(r) \ = \ \frac{(\beta^2(0) - \gamma^2(0))\beta(r) + 2\beta(0)\gamma(0)\gamma(r) + \beta(0)\gamma(0)}{\beta^2(0) + \gamma^2(0)}, \tag{4.5}$$

$$\hat{\gamma}(r) \ = \ \frac{(-2\beta(0)\gamma(0))\beta(r) + (\beta^2(0) - \gamma^2(0))\gamma(r) - \gamma^2(0)}{\beta^2(0) + \gamma^2(0)}. \tag{4.6}$$

As $r \to 0$, both these terms go to zero, since $\beta^2(0) + \gamma^2(0) + \gamma(0) = 0$.

Having set $\beta(0) = \gamma(0) = 0$, we now show that β must be identically zero. We choose polar coordinates in the $(\beta, \gamma + \frac{1}{2})-$plane:

$$\begin{cases} \beta = \nu \cos t \\ \gamma + \dfrac{1}{2} = \nu \sin t \end{cases} \tag{4.7}$$

In these coordinates, the functional (3.14) becomes

$$\begin{aligned} \mathcal{SYMH}(\nu, t) = 4\pi \int_0^1 [\ & 2\epsilon \left(\nu'^2 + t'^2 \nu^2 + \frac{2}{r^2} (\nu^2 - \frac{1}{4})^2 \right) + r^2 \varphi'^2 + \\ & 8\varphi^2 \nu^2 + \lambda r^2 (\varphi^2 - 1)^2 \ ] dr \ , \end{aligned} \tag{4.8}$$

with $\nu^2(0) = \frac{1}{4}$. The only dependence on t is in the $t'^2\nu^2$ term, which is minimized by setting $t =$ constant. But then $\cot(t) = \frac{\beta}{\gamma+\frac{1}{2}}$ must be constant, and equal to $\frac{\beta(0)}{\gamma(0)+\frac{1}{2}} = 0$, so β is identically zero. $\square$

We may therefore restrict our attention to the functional

$$\mathcal{S}(\gamma,\varphi) = 4\pi \int_0^1 [\ 2\epsilon\left(\gamma'^2 + \frac{2}{r^2}(\gamma^2+\gamma)^2\right) + r^2\varphi'^2 + 2\varphi^2(1+2\gamma)^2 + \lambda r^2(\varphi^2-1)^2\ ]dr\ . \tag{4.9}$$

The Euler-Lagrange equations for this functional together with the appropriate boundary conditions are then

$$\begin{cases} \gamma'' - \dfrac{2}{\epsilon}\varphi^2(1+2\gamma) - \dfrac{2}{r^2}(\gamma^2+\gamma)(1+2\gamma) = 0 & \text{on } (0,1) \\ \varphi'' + \dfrac{2\varphi'}{r} - \dfrac{2\varphi}{r^2}(1+2\gamma)^2 - 2\lambda\varphi(\varphi^2-1) = 0 & \text{on } (0,1) \\ \gamma(1) = -\dfrac{1}{2} \\ \gamma(0) = 0 \\ \varphi(1) = +1 \quad (\text{ or } \varphi(1) = -1) \end{cases} \tag{4.10}$$

The boundary conditions above come directly from the variational principle. In fact, to cancel the boundary terms one needs to either restrict the space of connections to those with prescribed boundary data, or to impose Neumann-type boundary conditions. The boundary condition $\varphi(1) = \pm 1$ comes from $|\phi| = 1$ on ∂B^3 and $\gamma(1) = -\frac{1}{2}$ comes from

$$(D\phi)_\tau = (d\phi)_\tau + [A_\tau, \phi] = (1+2\gamma)\phi\ d\left(\frac{\vec{\sigma}\cdot\vec{x}}{r}\right) = 0 \tag{4.11}$$

on ∂B^3, where the subscript τ denotes tangential components (cf. [2, 3]).

Observation: After some computation, the system (4.10) could also be obtained by imposing spherical symmetry in (1.3), thus critical symmetric points for the action (4.9) are symmetric critical points (not necessarily minima) for (1.2). This is also known apriori from the "principle of symmetric criticality" [4].

5. Existence of spherically symmetric monopoles

Our basic existence result is

Theorem 5.1. *For all values of $\lambda \geq 0$, $\epsilon > 0$, there exists a symmetric solution of*

$$\begin{cases} \epsilon * D_A * F = [D_A\phi,\ \phi] & \text{on } B^3 \\ * D_A * D_A\phi = \dfrac{\lambda}{2}(|\phi|^2 - 1)\phi & \text{on } B^3 \\ (D\phi)_\tau = 0 & \text{on } \partial B^3 \\ |\varphi| = 1 & \text{on } \partial B^3 \end{cases} \tag{5.1}$$

Observation: These equations do not reduce to the Bogomolnyi equations, even if $\lambda = 0$. The Bogolmolnyi argument involves an integration by parts; on a finite domain, this results in a boundary contribution [1].

Proof. Because of the derivative terms in the action, the natural space for γ (denoted $\mathcal{H}_\gamma$) is $H^1(0,1)$, while the natural space $\mathcal{H}_\varphi$ for φ is the weighted Sobolev space $H^1((0,1), r^2dr)$. By the Sobolev embedding theorem, functions in $\mathcal{H}_\gamma$ are continuous on $[0,1]$. Functions in $\mathcal{H}_\varphi$ are continuous on $(0,1]$, but may not have a limit at $r = 0$. We may therefore apply boundary conditions to γ at $r = 0$ and at $r = 1$, and to φ at $r = 1$.

Let

$$\mathcal{F} = \{(\gamma, \varphi) \in \mathcal{H}_\gamma \times \mathcal{H}_\varphi \ : \ \gamma(1) = -\frac{1}{2}, \, \gamma(0) = 0, \, \varphi(1) = 1.\} \tag{5.2}$$

The action functional (4.9) is well-defined on $\mathcal{F}$, and is finite whenever φ is bounded. In particular, $\mu \equiv Inf_{\mathcal{F}} \mathcal{S}$ is finite. We follow the direct method in the calculus of variations. That is, take a minimizing sequence for $\mathcal{S}$, show that it converges weakly in $\mathcal{F}$, and then show that the weak limit minimizes the action and so solves the Euler-Lagrange equations.

Let (γ_n, φ_n) be a minimizing sequence for $\mathcal{S}$. Since $\lambda \geq 0$, the action is not increased if we make the replacement

$$\varphi(r) \to \begin{cases} -1, & \text{if } \varphi(r) < -1; \\ \varphi(r), & \text{if } -1 \leq \varphi(r) \leq 1; \\ 1, & \text{if } \varphi(r) > 1. \end{cases} \tag{5.3}$$

As a result, we can assume that each $\varphi_n(r)$ is bounded in magnitude by 1. Under these circumstances, the sequence (γ_n, φ_n) is bounded in $\mathcal{F} \subset \mathcal{H}_\gamma \times \mathcal{H}_\varphi$. However, balls in $\mathcal{H}_\gamma$ are weakly compact, as are balls in $\mathcal{H}_\varphi$, so the pair (γ_n, φ_n) converges weakly in $\mathcal{H}_\gamma \times \mathcal{H}_\varphi$ to a limit $(\gamma_\infty, \varphi_\infty)$.

By Sobolev, $\gamma_n(r)$ and $\varphi_n(r)$ converge pointwise to $\gamma_\infty(r)$ and $\varphi_\infty(r)$, so the limiting values $\gamma(0)$, $\gamma(1)$, and $\varphi(1)$ are preserved, and $(\gamma_\infty, \varphi_\infty) \in \mathcal{F}$. Moreover, terms in $\mathcal{S}(\gamma_n, \varphi_n)$ that don't involve derivatives converge to the corresponding terms in $\mathcal{S}(\gamma_\infty, \varphi_\infty)$. The derivative terms are quadratic, hence weakly semicontinuous. As a result, $\mathcal{S}(\gamma_\infty, \varphi_\infty)$ is bounded above by μ, and therefore must equal μ.

Showing that γ_∞ and φ_∞ satisfy the Euler-Lagrange equations (4.10) is then a standard exercise in the calculus of variations. Smoothness of $(\gamma_\infty, \varphi_\infty)$ away from $r = 0$ follows by elliptic regularity of the equations (4.10). Smoothness at $r = 0$ follows from regular singular-point analysis, combined with the fact that both functions are bounded (see §6 for details). This in turn implies that the connection and Higgs field (A, ϕ) constructed from $(\gamma_\infty, \varphi_\infty)$ comprise a smooth, symmetric classical solution to the PDE system (5.1). (Alternatively, one can establish regularity of (A, ϕ) from the ellipticity of the PDE system (5.1), since we are working in a gauge with $d^*A = 0$.)

□

6. Regular Singular Point Analysis

In §5 we demonstrated the existence of symmetric monopoles for arbitrary $\lambda \geq 0$ and $\epsilon > 0$. In this section we explore their form near the regular singular point of the equations (4.10), namely $r = 0$.

Theorem 6.1. *Let* (γ, φ) *be a bounded finite-action solution to the ODE system* (4.10) *for some fixed* $\epsilon > 0$ *and* $\lambda \geq 0$*. Then there exist constants* a_1 *and* b_2 *such that*

$$\begin{aligned} \varphi(r) =& a_1 r + O(r^3); \\ \gamma(r) =& b_2 r^2 + O(r^4); \\ \gamma'(r) =& 2b_2 r + O(r^3); \end{aligned} \tag{6.4}$$

near $r = 0$*. In particular,* $\varphi(0) = \gamma'(0) = 0$*.*

Proof. We begin with the equation for φ, which we write as

$$\varphi'' + \frac{2\varphi'}{r} - \frac{2\varphi}{r^2} = \frac{8\varphi}{r^2}(\gamma + \gamma^2) + 2\lambda\varphi(\varphi^2 - 1). \tag{6.5}$$

Since φ is bounded and $\gamma(0) = 0$, the terms on the right hand side are less singular than those on the left hand side, and to leading order φ resembles the solution to the homogeneous equation

$$\varphi'' + \frac{2\varphi'}{r} - \frac{2\varphi}{r^2} = 0. \tag{6.6}$$

The general solution to this equation is $\varphi = a_1 r + a_{-2} r^{-2}$. However, φ is bounded by assumption, so $a_{-2} = 0$. Thus the solution to (6.5) is, to leading order, $a_1 r$.

Next we turn to the equation for γ, namely

$$\gamma'' - \frac{2\gamma}{r^2} = \frac{2}{\epsilon}\varphi^2(1 + 2\gamma) + \frac{2\gamma}{r^2}(2\gamma^2 + 3\gamma). \tag{6.7}$$

Again, since φ is bounded and $\gamma(0) = 0$, this may be viewed as a perturbation of the homogeneous linear equation

$$\gamma'' - \frac{2\gamma}{r^2} = 0, \tag{6.8}$$

whose solution is $\gamma = b_{-1} r^{-1} + b_2 r^2$. Since γ is bounded, $b_{-1} = 0$. Thus our solution to (6.7) is, to leading order, $b_2 r^2$.

With these basic results, we can estimate the right hand sides of (6.5) and (6.7). The right hand side of (6.5) is $O(r)$, which gives an $O(r)$ correction to φ'', hence an $O(r^3)$ correction to φ. The right hand side of (6.7) is $O(r^2)$, thus giving an $O(r^3)$ correction to γ' and an $O(r^4)$ correction to γ. $\square$

One can do an expansion for φ and γ in powers of r. Indeed, only odd powers contribute to φ and only even powers contribute to γ. This is seen by induction. By Theorem 6.1, φ is odd and γ is even through order r^2. However, if φ is odd and γ is even through order r^k, then the right hand sides of (6.5) and (6.7) are odd and even, respectively, to order r^k, and so φ and γ are odd and even, respectively, to order r^{k+2}. Thus φ and γ are odd and even to all orders in r.

We can therefore write an asymptotic expansion

$$\begin{aligned} \varphi(r) &\sim \sum_{n \text{ odd}} a_n r^n, \\ \gamma(r) &\sim \sum_{n \text{ even}} b_n r^n. \end{aligned} \tag{6.9}$$

Plugging this expansion into equations (6.5) and (6.7) and equating coefficients of r^{n-2} gives recursion relations of the form

$$\begin{aligned} (2k)(2k+3)a_{2k+1} &= \text{algebraic expression involving } a_1, b_2, \dots, b_{2k}, \\ (2k+1)(2k-2)b_{2k} &= \text{algebraic expression involving } a_1, b_2, \dots, b_{2k-2}. \end{aligned} \tag{6.10}$$

These relations do not constrain a_1 or b_2. However, once we have a_1 and b_2, the remaining coefficients are determined. The first few are

$$\begin{aligned} a_3 =& (4a_1b_2 - \lambda a_1)/5; \\ b_4 =& \left(3b_2^2 + \epsilon^{-1}a_1^2\right)/5; \\ a_5 =& (4a_1b_4 + 4a_3b_2 + 4a_1b_2^2 + \lambda(a_1^3 - a_3))/14; \\ b_6 =& \left(b_2^3 + 3b_2b_4 + \epsilon^{-1}(a_1a_3 + a_1^2b_2)\right)/7; \\ a_7 =& [4(a_1b_6 + a_3b_4 + a_5b_2 + a_3b_2^2 + 2a_1b_2b_4) + \lambda(3a_1^2a_3 - a_5)]/27; \\ b_8 =& \left[3(2b_2^2b_4 + b_4^2 + 2b_2b_6) + \epsilon^{-1}(a_3^2 + 2a_1a_5 + 2a_1^2b_4 + 4a_1a_3b_2)\right]/27; \\ a_9 =& \big[4(a_1b_8 + a_3b_6 + a_5b_4 + a_7b_2 + a_5b_2^2 + 2a_3b_2b_4 + a_1b_4^2 + 2a_1b_2b_6) \\ &+ \lambda(3a_1^2a_5 + 3a_1a_3^2 - a_7)\big]/44; \\ b_{10} =& \big[3(b_2^2b_6 + b_2b_4^2 + b_2b_8 + b_4b_6) \\ &+ \epsilon^{-1}(a_1a_7 + a_3a_5 + a_1^2b_6 + a_3^2b_2 + 2a_1a_3b_4 + 2a_1a_5b_2)\big]/22. \end{aligned} \tag{6.11}$$

7. Symmetries and Stability

The action functional and the resulting Euler-Lagrange equations are invariant under two natural symmetries.

$$\varphi(r) \to -\varphi(r) \qquad \gamma(r) \to +\gamma(r); \tag{7.12}$$

$$\varphi(r) \to \varphi(r) \qquad \gamma(r) \to -1-\gamma(r). \tag{7.13}$$

The first symmetry (7.12) comes from the isometry $\vec{x} \to -\vec{x}$ of B^3, which of course respects rotational symmetry. Since $\vec{\sigma}\cdot\vec{x}$ is odd and $\vec{\sigma}\cdot(\vec{x}\times\vec{dx})$ is even, pulling the pair (A,ϕ) back by this isometry flips the sign of φ while preserving γ. Using this symmetry, we can fix the sign of $\varphi(1)$, which we henceforth take to be positive.

The second symmetry (7.13) is a gauge transformation by $(\vec{\sigma}\cdot\vec{x})/r$. This is of the form of (4.2), with $\theta=\pi/2$. From (4.4) it is clear that this transformation sends γ to $-1-\gamma$ without generating a β term or changing φ. Applying this to a connection with $\gamma(0)=0$ yields a new connection with $\gamma(0)=-1$. This connection has finite action but is singular at the origin, reflecting the singular gauge transformation that generated it.

We now consider stability properties of the ODE system (4.10). These ODEs have several fixed points, namely

$$(\gamma,\varphi) = \ (-\tfrac{1}{2},1),\ (-\tfrac{1}{2},-1),\ (-\tfrac{1}{2},0),\ (0,0) \text{ or } (-1,0). \tag{7.14}$$

The point $\left(-\frac{1}{2},-1\right)$ is related to $\left(-\frac{1}{2},1\right)$ by the symmetry (7.12), while $(-1,0)$ is related to $(0,0)$ by (7.13), so we do not need to study these. What remains is $\left(-\frac{1}{2},1\right)$, $\left(-\frac{1}{2},0\right)$, and (0,0).

For the $\gamma=-1/2$ fixed points, we define $\delta=\gamma+1/2$, and the equation for γ becomes

$$\delta'' = 4\delta\left(\frac{\varphi^2}{\epsilon} - \frac{1}{4r^2} + \frac{\delta^2}{r^2}\right). \tag{7.15}$$

The fixed point $\gamma=-1/2$ is stable for $\varphi=1$ when $r^2<\epsilon/4$, but is unstable if $r^2>\epsilon/4$. This defines a natural length scale to the problem, namely $\sqrt{\epsilon}/2$. We should expect our solutions to behave qualitatively differently for r less than or greater than this length scale. Of course, if $\epsilon>4$, then all radii r are less than this length scale. In the case of $\varphi=0$, the value $\gamma=1/2$ is always stable, regardless of ϵ or r.

Near $\gamma=-1/2$, the equation for φ becomes

$$\varphi'' + 2\frac{\varphi'}{r} = 2\varphi\left(\frac{4\delta^2}{r^2} + \lambda(\varphi^2-1)\right). \tag{7.16}$$

The behavior of this fixed point depends on the value of φ. Near $\varphi=0$ we have

$$(r\varphi)'' = -2\lambda r\varphi + O(\delta^2) + O(\varphi^2),$$

which is stable for all positive values of r. Near $\varphi = 1$, however, we write $\varphi = 1 + \zeta$ and have

$$(r\zeta)'' = 4\lambda(r\zeta) + O(\zeta^2) + O(\delta^2).$$

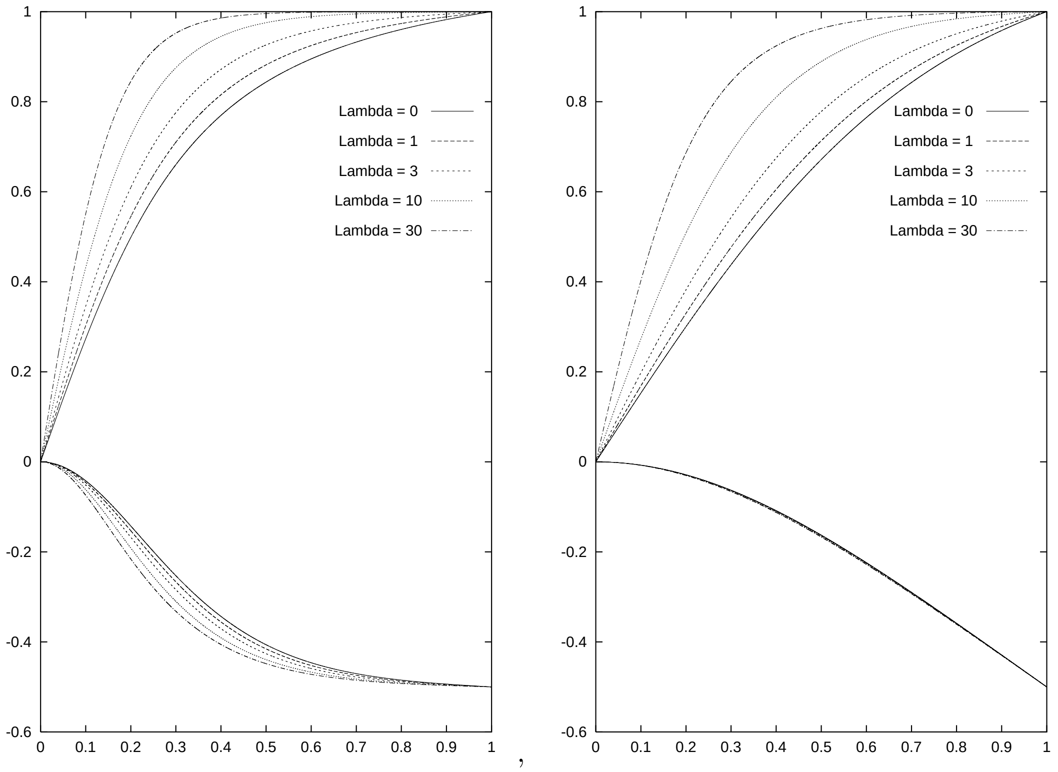
This is unstable as long as $\lambda > 0$, and has natural length scale $1/\sqrt{4\lambda}$.

To summarize, the fixed point $(-1/2, 0)$ is stable, while the fixed point $(-1/2, 1)$ is unstable. If $r^2 < \epsilon/4$, then there is one unstable mode, corresponding to growth of $\varphi - 1$. If $r^2 > \epsilon/4$, then there are two unstable modes, one for φ and one for γ.

Finally, there is the fixed point (0,0). Near (0,0) we have

$$\begin{aligned} \gamma'' &= 2\gamma/r^2 + \text{ higher order}, \\ \varphi'' + 2\varphi'/r &= 2\varphi(r^{-2} - \lambda) + \text{ higher order}. \end{aligned} \tag{7.17}$$

This fixed point is always unstable, with γ growing rapidly. φ will grow exponentially if $r < 1/\sqrt{\lambda}$, and will oscillate if $r > 1/\sqrt{\lambda}$.



$\epsilon = 0.1$ $\epsilon = 10$

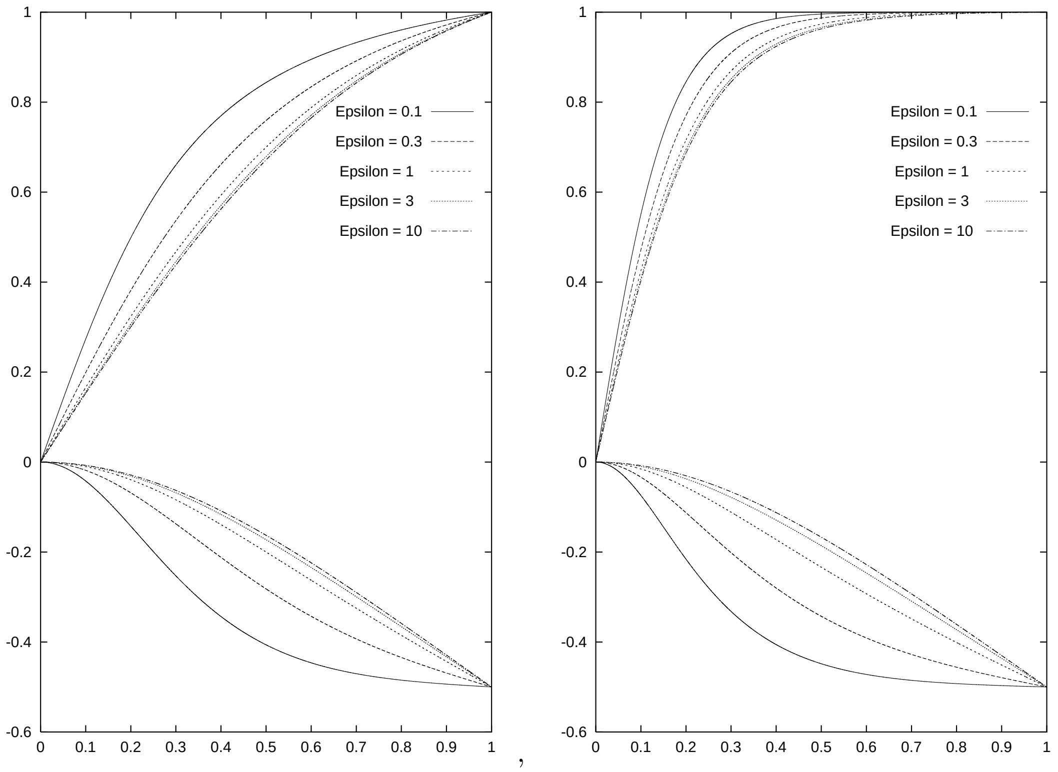
Figure 1. Trajectories with fixed ϵ.

8. Numerical Results and Qualitative Analysis

For any fixed ϵ and λ, and given a_1 and b_2, one can in principle integrate the differential equations out to $r = 1$. In practice, numerical errors due to the discretization of the interval $[0, 1]$ can be very bad near the origin, due to the singular nature of the ODE system there. A better method is to use the power series (6.9) in a neighborhood of the origin and to numerically integrate from there. In a discretization of $10,000$ points, we use the power series out to $r = 0.01$, or 100 lattice spacings from the origin.

In this way we get a pair $(\gamma(1), \varphi(1))$ for each (a_1, b_2). Using Newton's method, we then find values of (a_1, b_2) such that $(\gamma(1), \varphi(1)) = (-1/2, 1)$. Table 1 lists the correct values of a_1 and b_2 for several values of ϵ and λ.

The resulting functions $\varphi(r)$ and $\gamma(r)$ are sketched in Figures 1 and 2. Figure 1 shows the functions for different values of λ and ϵ fixed at 0.1 or at 10. Figure 2 is similar, only with λ fixed and ϵ variable. In each case the positive function is φ and the negative function is γ.



$\lambda = 0$ $\qquad$ $\lambda = 30.$

Figure 2: Trajectories with λ fixed.

From these figures several qualitative features are clear. Although φ depends significantly on both ϵ and λ, γ is practically independent of λ, especially when ϵ is large. The length scale

ϵ	λ	a_1	b_2
0.1	0	2.82909077	-4.47460232
0.1	1	3.14773551	-4.92072556
0.1	3	3.62692766	-5.57110938
0.1	10	4.62892407	-6.81947999
0.1	30	6.19274693	-8.46474894
0.3	0	2.01904955	-1.88549902
0.3	1	2.26118176	-2.04994984
0.3	3	2.66517994	-2.31673622
0.3	10	3.59550462	-2.86817001
0.3	30	5.12510342	-3.58374045
1	0	1.67098122	-1.02894746
1	1	1.85973704	-1.07504639
1	3	2.19572981	-1.15577783
1	10	3.04898441	-1.34041824
1	30	4.55384341	-1.58910470
3	0	1.57081044	-0.80615986
3	1	1.74184236	-0.82078859
3	3	2.05156143	-0.84695210
3	10	2.86750186	-0.90924487
3	30	4.35776101	-0.99551235
10	0	1.53622287	-0.73146686
10	1	1.70099654	-0.73576432
10	3	2.00102288	-0.74350147
10	10	2.80198139	-0.76219107
10	30	4.28571713	-0.78838676

TABLE 1. Taylor coefficients (a_1, b_2) for various values of (ϵ, λ).

on which γ changes from 0 to $-1/2$ is the smaller of $\sqrt{\epsilon}$ and 1. The length scale on which φ changes from 0 to 1 is the smallest of $\sqrt{\epsilon}$, $1/\sqrt{\lambda}$, and 1. Thus changing λ has the greatest effect when λ is greater than 1, while changing ϵ has the greatest effect when $\epsilon < 1$.

The source code for these numerical results can be obtained from the authors.

Acknowledgements

The authors thank Prof. Karen K. Uhlenbeck for useful conversations on this subject.

References

[1] A. Jaffe, C. Taubes, *Vortices and Monopoles,* Progress in Physics; **2**, Birkhäuser, Boston, 1980.
[2] A. Marini, *A boundary value problem for monopoles over a 3-dimensional disk*, Preprint.
[3] A. Marini, *Regularity theory for the generalized Neumann problem for Yang-Mills Connections- Non-trivial examples in dimensions 3 and 4*, Mathemathische Annalen, **317**, 173-193 (2000);. 6.
[4] R. S. Palais, *The principle of symmetric criticality,* Comm. Math. Phys. **69** (1979) 19–30.

Dipartimento Di Matematica, Universita' di L'Aquila, 67100 L'Aquila, Italy
E-mail address: marini@univaq.it

Department of Mathematics, University of Texas, Austin, TX
E-mail address: sadun@math.utexas.edu