The geometric structure of the Landau bands

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We have proposed a semiclassical explanation of the geometric structure of the spectrum for the two-dimensional Landau Hamiltonian with a two-periodic electric field without any additional assumptions on the potential. Applying an iterative averaging procedure we approximately, with any degree of accuracy, separate variables and describe a given Landau band as the spectrum of a Harper-like operator. The quantized Reeb graph for such an operator is used to obtain the following structure of the Landau band: localized states on the band wings and extended states near the middle of the band. Our approach also shows that different Landau bands have different geometric structure.

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The standard theories of the integer quantum Hall effect [1] are modelled by the Hamiltonian for a charged particle in a heterostructure with a uniform magnetic field; they are based on the following assumption concerning the structure of the spectrum: the spectrum consists of broadened Landau levels (Landau bands) generated by extended states near the middle of each band and by localized states in the band wings; then pinning the Fermi level by the localized states causes the occurrence of plateaus in the conductance curve (see [2] for details). This specific spectral structure is usually related to the presence of impurities or other defects in the quantum Hall system [3]. Indeed, for the Gauss or Lloyd ensembles of impurities the density of states has the behavior required for the explanation of the quantum Hall effect [4]. Moreover, using ideas from [5] it can be rigorously shown that the desired spectral structure is displayed by the two-dimensional Landau Hamiltonian with substitutional point impurities arranged at the nodes of a lattice, assuming a quasi-periodic distribution of the coupling constants [6]. The same holds for point impurities with random positions on the system plane and with random distribution of the coupling constants [7].

The aim of the present article is to show that this special spectral structure can also be verified by semiclassical approximation for a purely periodic Hamiltonian with a uniform magnetic field without any additional assumptions on the potential. Note that a semiclassical approach to the quantum Hall effect has been considered in [8] for a Hamiltonian with a disordered potential, within the framework of percolation theory on the Chalker–Coddington network. The results of [8] confirm the intuitive picture which relates the extended states in the band center with the central drift of an open cyclotron orbit, and the localized states in the band wings with closed orbits. This intuitive picture was rigorously justified in [10] using Maslov's canonical operator [11]. Below we offer some consequences of the results from [10] which describe in detail the semiclassical spectrum of the periodic Landau Hamiltonian and show, in particular, the desired spectral structure.

We use the Landau gauge for the Hamiltonian of a magneto-Bloch electron,

$$\hat{H}=\frac{\hbar^2}{2m}\big((-i\partial_1+(eB/c\hbar)x_2)^2-\partial_2^2\big)+V(x_1,x_2),$$

where the potential V is periodic with respect to the lattice Λ generated by the vectors $\mathbf{a}_1=(L,0)$ and $\mathbf{a}_2$; we assume that V is a real-analytic function. Introducing the cyclotron frequency $\omega_c=|eB|/cm$, the magnetic length $l_M=(\hbar/m\omega_c)^{1/2}$, and passing to dimensionless potential energy $v=V/\max V$ and to coordinates $\mathbf{X}=\mathbf{x}/L$, we rewrite $\hat{H}$ in the form $\hat{H}=mL^2\omega_c^2\hat{H}^0$ where

$$\hat{H}^0=\frac{1}{2}\left[\Big(\hat{P}_1+X_2\Big)^2+\hat{P}_2^2\right]+\varepsilon_V v(X_1,X_2)\,.$$

Here $\hat{P}_j=-i\varepsilon_B\partial/\partial X_j$ and the dimensionless quantities $\varepsilon_B=(l/L)^2$ and $\varepsilon_V=\varepsilon_B\max|V|/\hbar\omega_c$ will be considered as small parameters in our spectral problem. Let us estimate these parameters in a typical situation: for a field strength $B\simeq 10$T the magnetic length is of order 10nm; the characteristic length L for periodic arrays of quantum dots or antidots in GaAs is of order 200–500nm, such that $\varepsilon_B\sim 10^{-3}$; and using for GaAs the electron effective mass $m=0.067m_e$ we estimate $\hbar\omega_c$ as 15meV, implying that for V with values in the range 1–25 meV [13] we have $\varepsilon_V\lesssim\varepsilon_B$.

The main difficulty of the problem under study consists in the non-integrability of the corresponding classical system. At first sight, the presence of the small parameter ε_V suggests the use of perturbation theory; but it is readily seen that this will only be possible if we impose additional relations between ε_B and ε_V, because ε_B is also small. Moreover, using perturbation theory we can obtain only rather rough spectral information.

The smallness of ε_B, however, will allow us to describe the fine structure of each Landau band by using a semi-classical approach.

The trajectories of the classical system with the unperturbed Hamiltonian $\frac{1}{2}\big((P_1+X_2)^2+P_2^2\big)$ on the plane (X_1, X_2) are the cyclotron orbits with radius $\sqrt{2I}$ and angle φ centered at a point (y_1, y_2). Therefore, it is reasonable to introduce new canonical variables: the generalized momenta I, y_1 (or p, y_1) and the positions φ, y_2 (or q, y_2) according to the formulas

$$X_1 = q + y_1, \quad P_1 = -y_2, \quad X_2 = p + y_2, \quad P_2 = -q$$
$$p = \sqrt{2I}\cos\varphi, \quad q = \sqrt{2I}\sin\varphi\,.$$

In these variables the classical Hamiltonian H^0 takes the form $H^0 = I + \varepsilon_V v(\sqrt{2I_1}\sin\varphi + y_1, \sqrt{2I}\cos\varphi + y_2)$.

This representation suggests that we average H_0 over the angle φ, and describe the central drift of the cyclotron orbits by means of the averaged Hamiltonian

$$\begin{aligned}\mathcal{H}^{\rm av}(I, y_1, y_2; \varepsilon_V) &= \frac{1}{2\pi}\int\limits_0^{2\pi} H^0\, d\varphi \\ &= I + \varepsilon_V J_0(\sqrt{-2I\Delta_{\mathbf{y}}})v(y_1, y_2),\end{aligned} \tag{1}$$

where J_0 denotes the Bessel function of order zero; here the second term on the right-hand side results from the action of the pseudodifferential operator $J_0(\sqrt{-2I\Delta_{\mathbf{y}}})$ on v. Next, it can be shown that there is a canonical change of variables $(p', q', \mathbf{y}') = (p, q, \mathbf{y}) + O(\varepsilon_V)$ such that

$$H^0(p, q, \mathbf{y}; \varepsilon_V) = \mathcal{H}^{\rm av}\big(\frac{1}{2}(p'^2 + q'^2), \mathbf{y}'; \varepsilon_V\big) + O(\varepsilon_V^2),$$

but the error estimate $O(\varepsilon_V^2)$ is not good enough to describe the desired fine structure of the spectrum of H^0. The crucial point, then, is the following assertion which is proved in [10] by suitably iterating the averaging procedure:

There is a canonical change of variables $(P, Q, \mathbf{Y}) = (p, q, \mathbf{y}) + O(\varepsilon_V)$ *such that in the new coordinates* $(P, Q, \mathbf{Y})$ *the classical Hamiltonian system,* H^0, *becomes integrable modulo* $O(\varepsilon_V^\infty)$. *More precisely,* $H^0(p, q, \mathbf{y}; \varepsilon_V) = \mathcal{H}^0(\frac{1}{2}(P^2+Q^2), Y_1, Y_2; \varepsilon_V) + O(\varepsilon_V^\infty)$, *where the first term is real-analytic and all the terms are two-periodic in* $\mathbf{Y}$.

It is worthwhile to note that the first step of the mentioned iteration procedure gives just the function $\mathcal{H}^{\rm av}$.

Now we quantize $\mathcal{H}^0$ considering the pairs $\{P, Q\}$ and $\{Y_1, Y_2\}$ as pairs of canonically conjugated variables, i.e. we set $\hat{P} = -i\varepsilon_B\partial/\partial Q$, $\hat{Y}_1 = -i\varepsilon_B\partial/\partial Y_2$. By the Correspondence Principle, a canonical transformation in classical mechanics should correspond to a unitary transformation of the quantum mechanical state space. In our case this expectation is expressed in the fact that the spectra of the operators $\hat{H}^0$ and $\hat{\mathcal{H}}^0$ coincide semiclassically, up to $O(\varepsilon_B^\infty + \varepsilon_V^\infty)$. Now, since $\hat{P}$ and $\hat{Q}$ commute with $\hat{Y}_j$ $(j = 1, 2)$, the quantum Hamiltonian $\hat{\mathcal{H}}^0$ commutes with the Hamiltonian of the harmonic oscillator $\frac{1}{2}(\hat{P}^2 + \hat{Q}^2)$. Hence the eigenfunctions Ψ of $\hat{\mathcal{H}}^0$ may be chosen in the form $\Psi(Q, Y_2) = \psi_n(Q)\varphi_n(Y_2)$, where ψ_n are the eigenfunctions of the harmonic oscillator corresponding to the eigenvalues $E_n = (n+1/2)\varepsilon_B$, $n = 0, 1, \ldots$, and the functions φ_n satisfy the equation

$$\hat{\mathcal{H}}_n\varphi_n = E\varphi_n. \tag{2}$$

Here $\hat{\mathcal{H}}_n$ is a pseudodifferential operator obtained as the quantization of the classical Hamiltonian $\mathcal{H}_n(Y_1, Y_2) = \mathcal{H}^0(E_n, Y_1, Y_2, \varepsilon_V)$. As $mL^2\omega_c^2\varepsilon_B = \hbar\omega_c$, the number E_n is nothing but the n-th Landau level. Thus the spectrum of each operator $\hat{\mathcal{H}}_n$ arises from broadening the n-th Landau level into a band under the influence of the periodic potential. More precisely, in our approach each Landau band is described by the one-dimensional equation (2), which depends on the band, and the original spectral problem is reduced to a family of one-dimensional spectral problems. This reduction makes it now possible to describe the fine structure of each Landau band using established methods of semiclassical approximation, since we only have to deal with an integrable Hamiltonian.

In view of the established periodicity, $\mathcal{H}_n$ can be treated as a Hamiltonian on the two-torus; therefore, only the following three types of trajectories occur for the corresponding Hamiltonian system: (1) contractible closed curves and extremum points, (2) non-contractible closed curves, (3) separatrices and saddle points. The concept of the *Reeb graph* [14] has proved a very useful tool in describing the topological structure of the space of trajectories. Namely, points of the Reeb graph are in a one-to-one correspondence with connected components of the level sets of the Hamiltonian, in such a way that separatrices correspond to the branching points of the graph. If one represents $\mathcal{H}_n$ as the height function of a suitably deformed torus, then construction of the Reeb graph becomes especially evident, which is illustrated in Fig. 1A and 1B. .

Obviously, contractible trajectories on the torus are covered by families of *closed* trajectories on the plane (Y_1, Y_2), and non-contractible ones are covered by families of *non-closed* (but periodic) trajectories on this plane. Hence, to obtain the semiclassical approximation to the spectrum of $\hat{\mathcal{H}}_n$, i. e., to obtain the semiclassical structure of the n-th Landau band, one has to quantize only the closed trajectories, using the Bohr-Sommerfeld rule; thus we obtain the so-called quantized Reeb graph

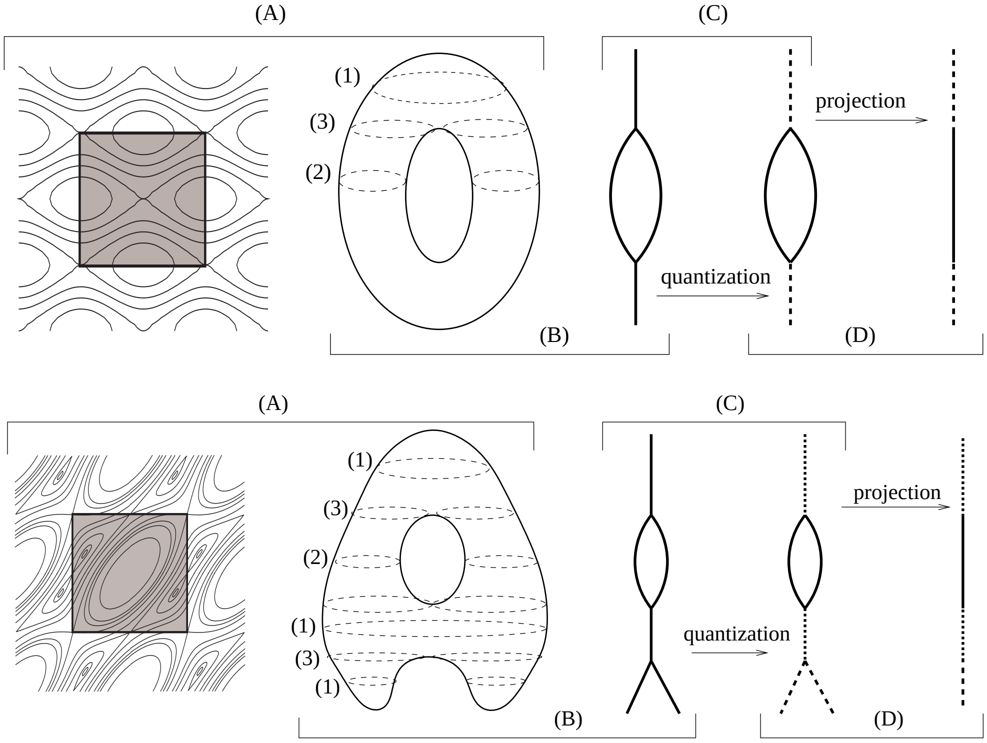

FIG. 1: Two examples of the geometric structure of the Landau band: (A) Level curves of a Hamiltonian on the torus and their realization through the height function; (B) Trajectories on the torus: (1) contractible closed ones, (2) non-contractible closed ones, (3) separatrices, and the Reeb graph of the Hamiltonian; (C) Applying the Bohr-Sommerfeld quantization; (D) Relationship between the Reeb graph and the Landau band

(Fig. 1C). Returning to the original coordinates $(\mathbf{P}, \mathbf{X})$ we see that the closed trajectories of $\mathcal{H}_n$ generate two-dimensional tori. It is well known that the semiclassical eigenfunctions corresponding to these tori decrease exponentially outside the classically allowed regions (i. e. the projections of the tori onto the $\mathbf{X}$-plane), hence correspond to localized states. On the other hand, open trajectories in the $\mathbf{Y}$-plane correspond to two-dimensional cylinders in the phase space $(\mathbf{P}, \mathbf{X})$. Since those are non-compact the corresponding states must be extended; they can be constructed using Maslov's canonical operator [11].

Our considerations show that the nth Landau band in the spectrum of $\hat{H}$ is structurally completely determined by the Reeb graph of $\mathcal{H}_n$ (see Fig. 1D). It is evident from this correspondence that, for a *generic* two-periodic potential V, the "highest" and the "lowest" points of the Reeb graph correspond to extremal points of $\mathcal{H}_n$. This means that "'boundary vertices" typically correspond to closed trajectories such that, consequently, the wings of the Landau band are filled by semi-classically localized states. On the other hand, the "middle region" of the Reeb graph contains loops or nodes which correspond to the open trajectories or separatrices such that the central part of the Landau band consists of extended states.

Using the previous considerations one can describe the spectrum globally. To do this, one has to construct the Reeb graph of $\mathcal{H}^0(I, Y_1, Y_2, \varepsilon_V)$ for each I; as I runs through $[0, +\infty)$, the Reeb graph is transformed and traces out a certain surface (the *Reeb surface* of the potential v). We illustrate this by considering the potential $v(X_1, X_2) = \cos X_1 + \cos X_2 + 2\cos(X_1 + X_2)$. Using Eq. (1), we get

$$\mathcal{H}^{\mathrm{av}}(I, Y_1, Y_2) = I + \varepsilon_V \big(J_0(\sqrt{2I})(\cos Y_1 + \cos Y_2) + 2J_0(\sqrt{4I}) \cos(Y_1 + Y_2)\big).$$

Use first $\mathcal{H}^{\mathrm{av}}$ instead of $\mathcal{H}^0$, then the Reeb surface has the form shown in Fig. 2. Quantizing I by the Bohr–Sommerfeld rule $I = E_n$ we select a discrete subset of the Reeb graphs, which can be obtained by cutting the Reeb surface by the plane $I = E_n$. For different indices n we obtain different types of Reeb graphs, some of which are shown in Fig. 3.

It is apparent that in all these cases the Landau band has the asserted structure. For some n, the domain of infinite motion may be very small and tend to a point (these points are emphasized in Fig. 2), but it always exists. The inclusion of the correction $\mathcal{H}^0 - \mathcal{H}^{\mathrm{av}}$ may lead to the appearance of new vertices and new edges in the Reeb graph near the branching points; but these edges

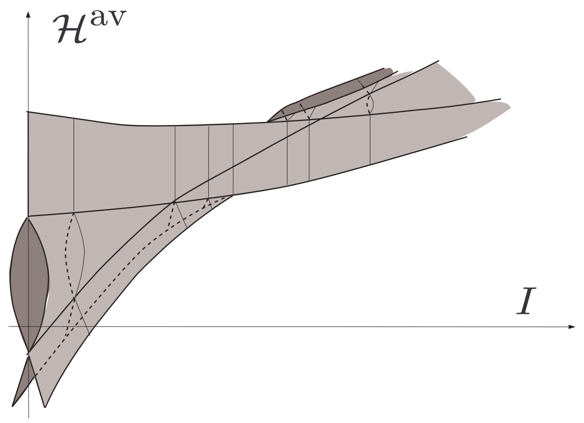

FIG. 2: The Reeb surface for the potential $v(X_1, X_2) = \cos(X_1) + \cos(X_2) + 2\cos(X_1 + X_2)$

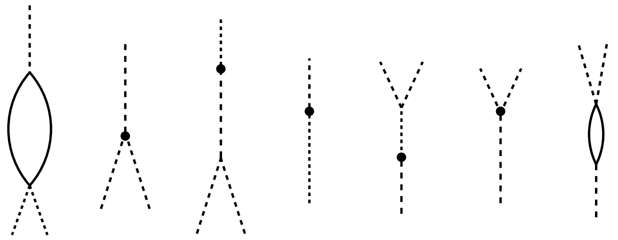

FIG. 3: The structure of different Landau bands for the potential $v(X_1, X_2) = \cos(X_1)+\cos(X_2)+2\cos(X_1+X_2)$

have length $O(\varepsilon_V^2)$ and hence do not change the asserted spectral structure. Clearly, spectral regions, corresponding to different Landau bands may intersect.

To summarize, we have proposed a semiclassical explanation of the geometric structure of the spectrum for the two-dimensional Landau Hamiltonian with a two-periodic electric field. Applying an iterative averaging procedure we approximately, with any degree of accuracy, separate variables and reduce the original spectral problem to an infinite family of one-dimensional eigenvalues problems for Harper-like operators; each of these operators is the quantization of a classical Hamiltonian having a torus as the phase space. Such a reduction provides a convenient tool for describing the geometric structure of each Landau band. Namely, the space of trajectories of each Hamiltonian on the torus is represented by the so-called Reeb graph. The projection of the Bohr-Sommerfeld quantization of the graph onto the energy axis gives the corresponding Landau band. Clearly, open edges of the graph represent contractible trajectories; quantization of these trajectories leads to localized states in the Landau band. Therefore, the band wings are filled by the localized states. On the contrary, open trajectories are always represented by closed edges, which lie in the middle of the graph; they generate the extended states near the middle of the Landau band.

It should be particularly emphasized that different Landau bands are described, generally speaking, by topologically different graphs and, therefore, can have different structure. Such a distinction of the geometric structure of the different Landau band was established numerically in [15] for various kinds of potentials.

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