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Perturbation Theory for the Quantum Time-Evolution in Rotating Potentials

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ABSTRACT. The quantum mechanical time-evolution is studied for a particle under the influence of an explicitly time-dependent rotating potential. We discuss the existence of the propagator and we show that in the limit of rapid rotation it converges strongly to the solution operator of the Schrödinger equation with the averaged rotational invariant potential.

1. The model, rotating frames

We consider the dynamics of a quantum mechanical particle of mass m moving in $\mathbb{R}^\nu,\ \nu \geq 2$, with kinetic energy $H_0 = H_0(\mathbf{p}) = h(|\mathbf{p}|)$ under the influence of a "rotating" potential $V_{\omega t}(\mathbf{x}) = V_0(\mathcal{R}(\omega t)^{-1}\,\mathbf{x})$. One may think of an atom or molecule interacting, e.g., with the blades of a rotating fan or with another rotating (heavy) object which is not significantly influenced by the (light) quantum particle. The Schrödinger operator $H(\omega t) = H_0 + V_{\omega t}$ is explicitly time-dependent.

In this paper we continue the investigation of [**1**] and address mainly two questions: (i) existence of a unitary propagator $U(t;t_0)$ which describes the time evolution of the system, (ii) the limit of rapid rotation where we show that the time evolution is well approximated by the evolution with the rotational invariant average potential. Applications to scattering theory will be treated in a subsequent paper.

We will first introduce the model in more detail before we state the main results in Theorems 5.2 and 6.2.

The coordinates are chosen in such a way that the rotation with constant angular velocity ω takes place in the x_1, x_2-plane, i.e.,

$$\mathcal{R}(\omega t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & \mathbb{1}_{\nu-2} \end{pmatrix}.$$

We denote by $\psi(\mathbf{x})$ the square integrable configuration space wave function of the (abstract) state in Hilbert space $\Psi \in \mathcal{H} \cong L^2(\mathbb{R}^\nu)$ and by $\hat{\psi}(\mathbf{p})$ its isometric Fourier transform, i.e., the momentum space wave function. The standard representation of this group

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of rotations as a strongly continuous one-parameter group of unitary operators $R(\omega t)$ on $\mathcal{H}$ is

$$R(\omega t)\,\Psi = e^{-i\omega t\, J}\ \Psi, \qquad (R(\omega t)\,\psi)(\mathbf{x}) = \psi\left(\mathcal{R}(\omega t)^{-1}\,\mathbf{x}\right). \tag{1.1}$$

The self-adjoint generator J with domain $\mathcal{D}(J)$ is essentially self-adjoint on the following sets which are dense in $L^2(\mathbb{R}^\nu)$ and invariant under rotation:

$$\mathcal{D} := \left\{\Psi \in \mathcal{H} \mid \hat{\psi} \in C_0^\infty(\mathbb{R}^\nu)\right\} \subset \mathcal{D}(H_0) \cap \mathcal{D}(J), \tag{1.2}$$

see, e.g., [**5**, Theorem VIII.11]. On suitable states $J\,\Psi = [x_1 p_2 - x_2 p_1]\,\Psi$. When using Cartesian coordinates in the plane of rotation

$$(J\psi)(\mathbf{x}) = [x_1(-i\partial/\partial x_2) - x_2(-i\partial/\partial x_1)]\ \psi(\mathbf{x}), \tag{1.3}$$

$$(J\hat{\psi})(\mathbf{p}) = [\,p_2(i\partial/\partial p_1) - p_1(i\partial/\partial p_2)]\ \hat{\psi}(\mathbf{p}), \tag{1.4}$$

and in polar coordinates $(\sqrt{x_1^2 + x_2^2},\, \phi_x)$ or $(\sqrt{p_1^2 + p_2^2},\, \phi_p)$, respectively,

$$J = -i\,\partial/\partial\phi_x \quad \text{or} \quad J = -i\,\partial/\partial\phi_p.$$

The free Hamiltonian H_0 is assumed to be a rotational symmetric continuously differentiable function of the momentum operator, $H_0 = H_0(\mathbf{p}) = h(|\mathbf{p}|)$ which has an unbounded velocity operator, i.e., h' is unbounded. Standard examples are

$$H_0^{\mathrm{NR}} = \frac{|\mathbf{p}|^2}{2m} \qquad \text{or} \qquad H_0(\mathbf{p}) = \frac{1}{\beta}|\mathbf{p}|^\beta, \ \ \beta > 1 \tag{1.5}$$

for nonrelativistic or more general kinematics with velocity operator $\nabla H_0(\mathbf{p}) = \mathbf{p}/m$ or $\nabla H_0(\mathbf{p}) = |\mathbf{p}|^{(\beta-2)}\,\mathbf{p}$, respectively (in units with $\hbar = 1$). The relativistic free Hamiltonian $H_0^{\mathrm{Rel}} = \sqrt{|\mathbf{p}|^2 c^2 + m^2 c^4}$ should be considered only for potentials of compact support inside a ball of radius R and for bounded angular velocities such that $R\,\omega/2\pi$ does not exceed the speed of light c. We will not treat the latter case here.

The dynamics are governed by the rotating potential, the explicitly time-dependent multiplication operator in configuration space

$$V_{\omega t}(\mathbf{x}) := V_0\left(\mathcal{R}(\omega t)^{-1}\,\mathbf{x}\right) = R(\omega t)\ V_0(\mathbf{x})\ R(\omega t)^* \tag{1.6}$$

with domain $R(\omega t)\,\mathcal{D}(V_0)$. The assumptions about V_0 will be stated later.

In the *inertial frame*–for an observer at rest–the free time evolution is $\exp(-itH_0)$. We are looking for a unitary *propagator* or solution operator $U(t;t_0)$, that is, it has to satisfy

$$U(t_0;t_0) = \mathbb{1}, \quad U(t;t_0) = U(t;t_1)\,U(t_1;t_0), \quad \forall\ t, t_0, t_1 \in \mathbb{R}, \tag{1.7}$$

which solves in some sense the Schrödinger equation for Hamiltonians $H(\omega t)$

$$i\partial_t\ U(t;t_0) = H(\omega t)\,U(t;t_0), \quad H(\omega t) = H_0 + V_{\omega t}. \tag{1.8}$$

Unless $V_{\omega t}$ and $H(\omega t)$ have some smoothness in their dependence on t the question of existence of such a propagator U for general or even periodic Hamiltonians is a hard question. See, e.g., [**6**], [**9**] and references therein where a wide class of potentials is covered.

For the special case of rotating potentials one may use alternatively a *rotating frame* where the observer rotates with the same angular velocity around the origin as the potential does. This is a common approach both in classical and quantum mechanics, see, e.g., [**2, 8**] for related investigations. Then the potential becomes time-independent according to (1.6) but the unperturbed evolution is more complicated instead: If the observer rotates like $\mathcal{R}(\omega t)\,\mathbf{x}$ in configuration space then a fixed state Ψ looks for him like turning in the opposite direction: $\psi\left(\mathcal{R}(\omega t)^{+1}\,\mathbf{x}\right) = (R(\omega t)^*\,\psi)(\mathbf{x}) = (R(\omega t)^{-1}\,\psi)(\mathbf{x})$.

The free time-evolution for a state with initial condition Ψ at time zero is described for the observer at rest by

$$e^{-it\,H_0}\;\Psi \qquad\qquad \text{(inertial frame)}$$

and for the rotating observer by

$$R(\omega t)^*\;e^{-it\,H_0}\;\Psi \qquad\qquad \text{(rotating frame).}$$

Since we have assumed that the free Hamiltonian H_0 is invariant under rotations the change of the evolution comes merely from the fact that $R(\omega t)^*\,e^{-it\,H_0}$ describes the combined change in time due to the free evolution and to the changing orientation of the observer. To avoid confusion with the free motion in any frame we will call $R(\omega t)^*\,e^{-it\,H_0}\;\Psi$ the *unperturbed motion* in the rotating frame.

Since all operators in the groups $\{R(\omega t)^* \mid t \in \mathbb{R}\}$ and $\{e^{-it\,H_0} \mid t \in \mathbb{R}\}$ commute their product $\{R(\omega t)^*\;e^{-it\,H_0} \mid t \in \mathbb{R}\}$ is a unitary strongly continuous one-parameter group as well. By Stone's Theorem it has a self-adjoint generator which we denote by H_ω with domain $\mathcal{D}(H_\omega)$:

$$R(\omega t)^*\;e^{-it\,H_0} =: e^{-it\,H_\omega}, \qquad t \in \mathbb{R}. \tag{1.9}$$

The sets given in equation (1.2) are dense and invariant under this group. Consequently, H_ω is essentially self-adjoint on both of them. Differentiation yields the operator sum

$$H_\omega = H_0 - \omega J \qquad \text{on} \quad \mathcal{D}(H_0) \cap \mathcal{D}(J) \subsetneq \mathcal{D}(H_\omega) \tag{1.10}$$

and similarly the form sum on $\mathcal{Q}(H_0) \cap \mathcal{Q}(J) \subsetneq \mathcal{Q}(H_\omega)$. Due to cancellations the domains $\mathcal{D}(H_\omega)$ and $\mathcal{Q}(H_\omega)$ are strictly larger than $\mathcal{D}(H_0) \cap \mathcal{D}(J)$ and $\mathcal{Q}(H_0) \cap \mathcal{Q}(J)$, respectively, for any $\omega \neq 0$, see, e.g., the explicit construction in [**1**, Section 3]. In particular, H_ω is not bounded below, its essential spectrum is $\sigma^{\mathrm{ess}}(H_\omega) = \mathbb{R}$ for $\omega \neq 0$.

2. The concept of solution

A formal calculation yields that the family of operators

$$\begin{aligned} U(t;t_0) :&= R(\omega t)\;e^{-i(t-t_0)(H_\omega+V_0)}\;R(\omega t_0)^* \\ &= R(\omega(t-t_0))\;\;e^{-i(t-t_0)(H_\omega+V_{\omega t_0})} \\ &= e^{-i(t-t_0)(H_\omega+V_{\omega t})}\;R(\omega(t-t_0)) \end{aligned} \tag{2.1}$$

actually is a propagator in the sense of equation (1.7) and it satisfies the Schrödinger equation (1.8),

$$\begin{aligned} i\partial_t\;U(t;t_0)\;\Psi &= R(\omega t)\;\{\omega J + H_\omega + V_0\}\;e^{-i(t-t_0)(H_\omega+V_0)}\;R(\omega t_0)^*\;\Psi \\ &= \{H_0 + V_{\omega t}\}\;U(t;t_0)\;\Psi. \end{aligned} \tag{2.2}$$

All this is justified if, e.g., the sum $H_\omega + V_0$ is defined as a self-adjoint operator, $R(\omega t_0)^*\;\Psi$ is contained in $\mathcal{D}(H_\omega + V_0)$, and if $e^{-i(t-t_0)(H_\omega+V_0)}\;R(\omega t_0)^*\;\Psi$ lies in $\mathcal{D}(J) \cap \mathcal{D}(H_0) \cap \mathcal{D}(V_0)$ such that $\omega J + H_\omega + V_0 = H_0 + V_0 = R(\omega t)\,(H_0 + V_{\omega t})\,R(\omega t)^*$ makes sense there, see equations (1.10) and (1.6). It will be difficult to verify these or other sufficient domain properties for a suitable dense set of vectors Ψ unless the potentials are not too singular.

The terms on the right hand side of (2.1) are all equal by (1.6) as soon as the expression $H_\omega + V_{\omega t} = R(\omega t)\;(H_\omega + V_0)\;R(\omega t)^*$ is defined as a self-adjoint operator for one (and then all) ωt.

We will not study how one might extend “differentiability” when domain problems are present but we propose here to consider equation (2.1) as a *definition* of a propagator which “solves” the Schrödinger equation (1.8). This point of view takes advantage of the special form of the time-dependence and–as equation (2.2) shows–it is consistent with the usual concept of solution for sufficiently regular potentials. Alternatively, one may consider instead of the differential equations the corresponding more regular integral equations. The explicitly time-dependent Schrödinger equation (1.8) corresponds to the Duhamel formula for U considered as a perturbation of the free evolution

$$U(t;t_0) = e^{-i(t-t_0)H_0} - i\int_{t_0}^{t} d\tau\; e^{-i(t-\tau)H_0}\; R(\omega\tau)\, V_0\, R(\omega\tau)^*\; U(\tau,t_0). \tag{2.3}$$

Multiplication from the left by $R(\omega t)^*$ and from the right by $R(\omega t_0)$ yields for $\widetilde{U}(t,t_0) := R(\omega t)^*\, U(t;t_0)\, R(\omega t_0)$ the integral equation

(2.4)

$$\widetilde{U}(t,t_0) = R(\omega(t-t_0))^*\; e^{-i(t-t_0)H_0} - i\int_{t_0}^{t} d\tau\; R(\omega(t-\tau))^*\; e^{-i(t-\tau)H_0}\; V_0\; \widetilde{U}(\tau,t_0).$$

Using (1.9) this turns out to be the Duhamel formula for $\widetilde{U}$ viewed as a perturbation of $\exp\{-i(t-t_0)H_\omega\}$ which corresponds to the following time-independent differential equation

$$i\partial_t\, \widetilde{U}(t,t_0) = (H_\omega + V_0)\, \widetilde{U}(t,t_0), \quad \widetilde{U}(t,t_0) = e^{-i(t-t_0)(H_\omega+V_0)}. \tag{2.5}$$

The different ways in (2.1) of writing the propagator give rise to different integral equations. Their solutions are equal as long as the property $\exp\{-i(t-t_0)(H_\omega+V_0)\}\,\Psi \in \mathcal{D}(V_0)$ holds for a dense set of vectors Ψ or similarly for quadratic forms.

It remains to study the question for which potentials V_0 the sum $H_\omega + V_0$ can be defined as a self-adjoint operator. We will treat an easier special case in Sections 3–5 where uniformity in ω is needed and provide preliminary results for more general singular potentials in Section 6.

3. Rapid rotation, averaged potential

In this section we will introduce the averaged potential as a preparation for the next two sections where the limiting behavior of the system as $\omega \to \infty$ will be studied.

The leading part of the potential can be obtained by averaging over one period

$$\begin{aligned} \overline{V}(\mathbf{x}) :&= \frac{\omega}{2\pi}\int_{t_0}^{t_0+2\pi/\omega} ds\; V_{\omega s}(\mathbf{x}) = R(\omega t_0)\frac{\omega}{2\pi}\int_0^{2\pi/\omega} ds\; V_{\omega s}(\mathbf{x})\;\; R(\omega t_0)^* \\ &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi\; V_0(\mathcal{R}(\varphi)^{-1}\,\mathbf{x}). \end{aligned} \tag{3.1}$$

Due to the periodicity in time this multiplication operator is independent of ω and t_0 and it is invariant under rotation. With $W_0 := V_0 - \overline{V}$ we have

$$V_{\omega t} = \overline{V} + W_{\omega t}, \quad H(\omega t) = H_0 + V_{\omega t} = (H_0 + \overline{V}) + W_{\omega t}. \tag{3.2}$$

Thus, only the remainder term W is responsible for the explicit time-dependence of the Hamiltonian.

Here we are interested in statements which hold uniformly in ω. For simplicity of presentation we assume throughout this and the following two sections that the time-independent potential $\overline{V}$ is operator bounded relative to the free Hamiltonian H_0 with relative bound less than one and that the remainder W is a bounded operator. Any free

Hamiltonian as specified above (see, e.g., (1.5)) is admissible here. Its properties enter only indirectly through the Kato-boundedness of $\overline{V}$ relative to H_0. By the Kato-Rellich Theorem both domains in (1.2) are cores for each of the operators H_0, $H_\omega = H_0 - \omega J$, $H_0 + \overline{V}$, $H(\omega t)$, and $\omega J + W_0$. The operator sums act pointwise on these domains.

Analogously to (1.9) and (1.10) the invariance under rotations of $H_0 + \overline{V}$ implies that

$$R(\omega t)\, e^{-it\,(H_0+\overline{V})} =: e^{-it\,(H_\omega+\overline{V})} \tag{3.3}$$

is a unitary one-parameter group which leaves the domain $\mathcal{D}(H_0)\cap\mathcal{D}(J)$ invariant. Consequently, its self-adjoint generator "$H_\omega + \overline{V}$" is essentially self-adjoint there:

$$H_\omega + \overline{V} = H_0 - \omega J + \overline{V} \qquad \text{on its core} \quad \mathcal{D}(H_0)\cap\mathcal{D}(J). \tag{3.4}$$

The same applies to $H_\omega+\overline{V}+W_0$ as a bounded perturbation thereof. The Duhamel integral equation for the propagator U as a perturbation of $\exp\{-i(t-t_0)(H_0+\overline{V})\}$ is evidently well defined:

$$U(t;t_0) = e^{-i(t-t_0)(H_0+\overline{V})} - i\int_{t_0}^{t} d\tau\; e^{-i(t-\tau)(H_0+\overline{V})}\; W_{\omega\tau}\; U(\tau,t_0) \tag{3.5}$$

and similarly for $\widetilde{U}$, compare (2.3) and (2.4).

Next we show that the splitting $V = \overline{V}+W$ corresponds to a splitting into the diagonal and off-diagonal parts w.r.t. the eigenspaces of J. We define the orthogonal projections P_j by

$$P_j\,\mathcal{H} := \{\Psi\in\mathcal{D}(J)\mid J\,\Psi = j\,\Psi\} \quad j\in\sigma(J)=\mathbb{Z}\,, \quad \sum_{j\in\mathbb{Z}} P_j = \mathbb{1}. \tag{3.6}$$

When using polar coordinates in the x_1, x_2-plane of $\mathbb{R}^\nu$ the eigenfunctions of J are of the form

$$\psi(r\cos\varphi, r\sin\varphi, x_3,\ldots x_\nu) = e^{i\varphi\, j}\;\tilde{\psi}(r, x_3,\ldots, x_\nu).$$

LEMMA 3.1. *With $V_0 = \overline{V} + W_0$ and P_j as defined in* (3.1), (3.6)

$$\overline{V} = \sum_{j\in\mathbb{Z}} P_j\; V_0\; P_j\,, \tag{3.7}$$

$$W_0 = \sum_{j\in\mathbb{Z}} (\mathbb{1}-P_j)\; V_0\; P_j = \sum_{j\in\mathbb{Z}} P_j\; V_0\; (\mathbb{1}-P_j). \tag{3.8}$$

PROOF. Due to rotational invariance of $\overline{V}$ we have

$$\overline{V} = \overline{V}\;\sum_{j\in\mathbb{Z}} P_j = \sum_{j\in\mathbb{Z}} P_j\;\overline{V}\;P_j\,.$$

The rotation simplifies to a phase factor $\exp(it\omega\, j)$ on the range of P_j ,

$$\begin{aligned} P_j\;\overline{V}\;P_j &= P_j\;\frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\; R(\omega t)\; V_0\; R(\omega t)^*\; P_j \\ &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\; P_j\; V_0\; P_j = P_j\; V_0\; P_j. \end{aligned}$$

This shows (3.7) and as a simple consequence (3.8). □

For rotational invariant operators we obtain the following limiting behavior.

LEMMA 3.2. *For H_0, H_ω and $\overline{V}$ as introduced above and for any $\ell \in \mathbb{Z}$, $\zeta \in \mathbb{R} \setminus \{0\}$*

$$\operatorname*{s-lim}_{\omega\to\infty} (H_\omega + \overline{V} + \omega\ell - i\zeta)^{-1} = (H_0 + \overline{V} - i\zeta)^{-1}\, P_\ell = P_\ell\, (H_0 + \overline{V} - i\zeta)^{-1}\, P_\ell. \tag{3.9}$$

Note that the right hand side of (3.9) is not a resolvent. The lemma does not state strong resolvent convergence unless we restrict the operators to mappings on the invariant subspaces $P_\ell\,\mathcal{H}$.

PROOF. Denote by $E(\mu)$ the resolution of the identity for the operator $H_0 + \overline{V}$, i.e., $H_0 + \overline{V} = \int \mu\ dE(\mu)$. To show strong convergence it is sufficient to consider a total set of states. We use $\Phi = P_j\ \Phi = \int_{|\mu|<M} dE(\mu)\ \Phi$ for some $j \in \mathbb{Z},\ M < \infty$. Then $\Phi \in \mathcal{D}(H_0 + \overline{V}) \cap \mathcal{D}(J) \subset \mathcal{D}(H_\omega + \overline{V})$ and $(H_\omega + \overline{V})\ \Phi = (H_0 + \overline{V} - \omega j)\ \Phi$. This equality holds as well for Φ replaced by $(H_0 + \overline{V} + \omega(\ell - j) - i\zeta)^{-1}\ \Phi$ because the latter has the same qualitative properties as assumed above for Φ. The resolvent identity then yields the first of the following equations:

$$\begin{aligned}(H_\omega + \overline{V} + \omega\ell - i\zeta)^{-1}\ \Phi &= (H_0 + \overline{V} + \omega(\ell - j) - i\zeta)^{-1}\ P_j\ \Phi\\ &= \begin{cases}(H_0 + \overline{V} - i\zeta)^{-1}\ P_\ell\ \Phi & \text{for } j = \ell,\\ \longrightarrow 0 \quad \text{as } \omega \to 0 & \text{for } j \neq \ell.\end{cases}\end{aligned}$$

The last limit follows from the fact that for $\ell - j \neq 0$

$$\lim_{\omega\to\infty}\ \sup_{|\mu|<M} |(\mu + \omega(\ell - j) - i\zeta)^{-1}| = 0.$$

$\square$

4. Product formulas

The Trotter product formula for operator sums of self-adjoint operators A, B with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$ states that

$$\operatorname*{s-lim}_{n\to\infty} \left\{e^{-iTA/n}\ e^{-iTB/n}\right\}^n = e^{-iT(A+B)} \tag{4.1}$$

uniformly in T from compact intervals provided that $A + B$ is essentially self-adjoint on $\mathcal{D}(A) \cap \mathcal{D}(B)$, see, e.g., [**5**, Theorem VIII.31]. This theorem can be used directly as stated for the form (2.1) of the propagator U as follows. Let

$$H_\omega + V_{\omega t_0} = (H_0 + \overline{V}) + (-\omega J + W_{\omega t_0}) =: A + B \tag{4.2}$$

where $A = H_0 + \overline{V}$ is self-adjoint on $\mathcal{D}(H_0)$ and $B = -\omega J + W_0$ is self-adjoint on $\mathcal{D}(J)$ and both operators are essentially self-adjoint on $\mathcal{D}(H_0) \cap \mathcal{D}(J)$ (and on $\mathcal{D}$ as given in (1.2)) by the Kato-Rellich theorem. Moreover, this set is left invariant under the unitary one-parameter group (a product of two commuting groups)

$$e^{-it(H_0+\overline{V})}\ e^{+it\omega J} =: e^{-it(H_0+\overline{V}-\omega J)}.$$

Thus, its generator "$H_0 + \overline{V} - \omega J$" is essentially self-adjoint on $\mathcal{D}(H_0) \cap \mathcal{D}(J)$ and it coincides with the operator sum there. The same applies to the bounded perturbation thereof: $A + B = H_0 + \overline{V} - \omega J + W_{\omega t_0}$. Thus, all assumptions for (4.1) are satisfied.

Application to U as given in (2.1) yields

$$
\begin{aligned}
U(t_0+T,t_0) &= R(\omega T)\, e^{-iT(H_\omega+V_{\omega t_0})} \\
&= \operatorname*{s-lim}_{n\to\infty} R(\omega T)\left\{e^{-iT(H_0+\overline{V})/n}\, e^{-iT(-\omega J+W_{\omega t_0})/n}\right\}^n \\
&= \operatorname*{s-lim}_{n\to\infty} \prod_{k=0}^{n-1}\left[e^{-iT(H_0+\overline{V})/n}\, R(\omega T/n)\, e^{-iT(-\omega J+W_{\omega t_0+k\omega T/n})/n}\right]. \qquad (4.3)
\end{aligned}
$$

The product in the last line is to be understood as ordered with increasing k from right to left. The last equality holds because $R(k\omega T/n)\, W_{\omega t_0}\, R(k\omega T/n)^* = W_{\omega t_0+k\omega T/n}$.

Consider now the case where one of the operators, say $B(t)$, is explicitly time-dependent and belongs to a family of pairwise commuting bounded operators $\{B(t)\}_{t\in\mathbb{R}}$, then the exponential function of the integral satisfies the differential equation

$$
i\frac{d}{dt}\,\exp\left\{-i\int_{t_1}^{t} ds\, B(s)\right\} = B(t)\,\exp\left\{-i\int_{t_1}^{t} ds\, B(s)\right\}.
$$

The idea behind the Trotter product formula (4.1) is the following approximation argument. To find a solution of the initial value problem $i(d/dt)\,\mathcal{U}(t) = (A+B)\,\mathcal{U}(t)$ for a finite time interval of length T one may split the interval into subintervals and first solve $i(d/dt)\,\mathcal{U}(t) = B\,\mathcal{U}(t)$ for the short time T/n, then solve $i(d/dt)\,\mathcal{U}(t) = A\,\mathcal{U}(t)$ and continue alternating between the two differential equations n times. In the strong limit as $n\to\infty$ one obtains the desired result. Translating this to the "non-autonomous" situation the product in (4.1) should be replaced for the interval $[t_0, t_0+T]$ by

$$
\begin{aligned}
&e^{-iTA/n}\,\exp\left\{-i\int_{t_0+(n-1)T/n}^{t_0+T} ds\, B(s)\right\}\cdot\ldots\cdot e^{-iTA/n}\,\exp\left\{-i\int_{t_0}^{t_0+T/n} ds\, B(s)\right\} \\
&\qquad = \prod_{k=0}^{n-1} e^{-iTA/n}\,\exp\left\{-i\int_{t_0+kT/n}^{t_0+(k+1)T/n} ds\, B(s)\right\}. \qquad (4.4)
\end{aligned}
$$

The factors in the product are again ordered with k increasing from right to left.

If, e.g., A is self-adjoint and $\{B(t)\}$ is a family of bounded pairwise commuting self-adjoint operators then the modified Trotter product formula reads

$$
\operatorname*{s-lim}_{n\to\infty}\prod_{k=0}^{n-1} e^{-iTA/n}\,\exp\left\{-i\int_{t_0+kT/n}^{t_0+(k+1)T/n} ds\, B(s)\right\} = \mathcal{U}(t_0+T;t_0) \qquad (4.5)
$$

where $i(d/dt)\,\mathcal{U}(t_0+t;t_0) = (A+B(t))\,\mathcal{U}(t_0+t;t_0)$ in the sense of (3.5). This should be part of the folklore but we are not aware of a reference to such a result. One can adjust Nelson's proof ([**4**] or [**5**, Theorem VIII.30]) to show (4.5). However, in our application where $A = H_0+\overline{V}$ and $B(t) = W_{\omega t}$ it is simpler to observe that the products in (4.3) and (4.5) actually are the same. We will show that

$$
R(\omega t)\, e^{-it(-\omega J+W_{\omega t_1})} = \exp\left\{-i\int_{t_1}^{t_1+t} ds\, W_{\omega s}\right\}. \qquad (4.6)
$$

To show equality of the two families of operators we observe that they both equal the identity operator for $t=0$ and that they satisfy the same differential equation when applied to an arbitrary vector $\Psi\in\mathcal{H}$. For Φ in the dense set $\mathcal{D}(J)$ the time derivative of the term on the left hand side is

$$
\begin{aligned}
&i\frac{d}{dt}\ (\Phi,\ R(\omega t)\ \exp\{-it(-\omega J+W_{\omega t_1})\}\ \Psi) \\
&\qquad = (\Phi,\ R(\omega t)\ \{\omega J+(-\omega J+W_{\omega t_1})\}\ \exp\{-it(-\omega J+W_{\omega t_1})\}\ \Psi) \\
&\qquad = \bigl(\Phi,\ W_{\omega(t_1+t)}\ R(\omega t)\ \exp\{-it(-\omega J+W_{\omega t_1})\}\ \Psi\bigr)\ .
\end{aligned}
$$

Thus, the vector valued function is strongly differentiable with uniformly bounded derivative:

$$
\begin{aligned}
&i\frac{d}{dt}\ R(\omega t)\ \exp\{-it(-\omega J+W_{\omega t_1})\}\ \Psi \\
&\qquad = W_{\omega(t_1+t)}\ R(\omega t)\ \exp\{-it(-\omega J+W_{\omega t_1})\}\ \Psi.
\end{aligned} \tag{4.7}
$$

For the right hand side we get the same result:

$$
i\frac{d}{dt}\ \exp\left\{-i\int_{t_1}^{t_1+t} ds\ W_{\omega s}\right\}\Psi = W_{\omega(t_1+t)}\ \exp\left\{-i\int_{t_1}^{t_1+t} ds\ W_{\omega s}\right\}\Psi\,.
$$

Thus, equation (4.6) holds for all $t,\,t_1\,\in\,\mathbb{R}$. Setting $t\,=\,T/n$ and $t_1\,=\,t_0\,+\,kT/n$ verifies that the factors in the products in equations (4.3) and (4.5) are the same as was to be expected.

Summing up we have shown the following product formula. Recall that the precise assumptions for these sections were stated in the first two paragraphs of Section 3.

PROPOSITION 4.1. *For H_0 and V_0 as specified in Section 3 the propagator U satisfies*

$$
U(t_0+T,t_0) = \operatorname*{s-lim}_{n\to\infty}\prod_{k=0}^{n-1}\left[e^{-iT(H_0+\overline{V})/n}\ \exp\left\{-i\int_{t_0+kT/n}^{t_0+(k+1)T/n} ds\ W_{\omega s}\right\}\right]. \tag{4.8}
$$

The factors in the product are ordered with k increasing from right to left.

Observe that for $T/n\,=\,\ell\,2\pi/\omega,\ \ell\,\in\,\mathbb{Z}$, the integrals vanish because the average of $W_{\omega s}$ over a period is zero. In this case the product simplifies to $e^{-iT(H_0+\overline{V})}$. The same holds for the norm-limit as $\omega\to\infty$ for each of the factors. To show $\operatorname{s-lim}_{\omega\to\infty} U(t_0+T,t_0)=\exp\{-iT(H_0+\overline{V})\}$ as we will do in the next section we need the limits in the other order. In that case there is another product formula which is better suited and has the advantage that the convergence is in norm for bounded perturbations W. We define

$$
\widetilde{u}(t_2,t_1) := \exp\Bigl\{-i\int_0^{t_2-t_1} ds\ e^{is(H_0+\overline{V})}\ W_{\omega(t_1+s)}\ e^{-is(H_0+\overline{V})}\Bigr\} \tag{4.9}
$$

and its first order approximation

$$
\widetilde{\widetilde{u}}(t_2,t_1) := \mathbb{1} - i\int_0^{t_2-t_1} ds\ e^{is(H_0+\overline{V})}\ W_{\omega(t_1+s)}\ e^{-is(H_0+\overline{V})}. \tag{4.10}
$$

The exponential $\widetilde{u}$ has the advantage of being unitary even for unbounded W, but for the present case of bounded $\|W_0\|$ the linear approximation $\widetilde{\widetilde{u}}$ with $\left\|\widetilde{\widetilde{u}}(t_2,t_1)\right\|\le 1+|t_2-t_1|\,\|W_0\|$ is easier to handle.

PROPOSITION 4.2. *For H_0 and V_0 as specified in Section 3 the propagator U satisfies*

$$
\left\|U(t_0+T,t_0)-\prod_{k=0}^{n-1}\left[e^{-iT(H_0+\overline{V})/n}\ \widetilde{u}\Bigl(\frac{(k+1)T}{n},\frac{kT}{n}\Bigr)\right]\right\| \le \frac{(T\,\|W_0\|)^2}{n}\,, \tag{4.11}
$$

(4.12)
$$\left\| U(t_0+T,t_0) - \prod_{k=0}^{n-1}\left[e^{-iT(H_0+\overline{V})/n}\ \widetilde{\widetilde{u}}\Big(\frac{(k+1)T}{n},\frac{kT}{n}\Big)\right]\right\| \le \frac{(T\,\|W_0\|)^2}{2n}\ e^{(T\,\|W_0\|)}\,.$$

The factors in the product are ordered with k increasing from right to left.

PROOF. From the Duhamel formula (3.5) one immediately reads off that
$$\left\| U(t_2;t_1) - e^{-i(t_2-t_1)(H_0+\overline{V})}\right\| \le (t_2-t_1)\ \|W_0\|.$$
We write down the same Duhamel formula again and use the above estimate to derive a good approximation.
$$\begin{aligned} U(t_2;t_1) =& e^{-i(t_2-t_1)(H_0+\overline{V})}\left[\mathbb{1} - i\int_0^{t_2-t_1} ds\ e^{is(H_0+\overline{V})}\ W_{\omega(t_1+s)}\ e^{-is(H_0+\overline{V})}\right] \\ &- i\int_0^{t_2-t_1} ds\ e^{-i(t_2-t_1-s)(H_0+\overline{V})}\ W_{\omega(t_1+s)}\left\{U(t_1+s;t_1) - e^{-is(H_0+\overline{V})}\right\} \end{aligned}$$
In the last line we use the estimate above which gives with the shorthand (4.10)
$$\begin{aligned} &\|U(t_2;t_1) - e^{-i(t_2-t_1)(H_0+\overline{V})}\ \widetilde{\widetilde{u}}(t_2,t_1)\| \\ &\le \int_0^{t_2-t_1} ds\ \|W_{\omega(t_1+s)}\|\ s\ \|W_0\| = [(t_2-t_1)\ \|W_0\|\,]^2/2. \end{aligned} \tag{4.13}$$
With $|e^{-i\alpha} - (1-i\alpha)| \le \alpha^2/2$ for $\alpha \in \mathbb{R}$ we get
$$\begin{aligned} \|\widetilde{u}(t_2,t_1) - \widetilde{\widetilde{u}}(t_2,t_1)\| &\le \left\| \int_0^{t_2-t_1} ds\ e^{is(H_0+\overline{V})}\ W_{\omega(t_1+s)}\ e^{-is(H_0+\overline{V})}\right\|^2 /2 \\ &\le [(t_2-t_1)\ \|W_0\|\,]^2/2. \end{aligned} \tag{4.14}$$
Combining (4.13) with (4.14) yields
$$\|U(t_2;t_1) - e^{-i(t_2-t_1)(H_0+\overline{V})}\ \widetilde{u}(t_2,t_1)\| \le [(t_2-t_1)\ \|W_0\|\,]^2\,. \tag{4.15}$$
Now we split the time interval into n equal parts. The order in the products is always with k increasing from right to left.
$$\begin{aligned} &\prod_{k=0}^{n-1} U\Big(t_0+\frac{(k+1)T}{n}; t_0+\frac{kT}{n}\Big) - \prod_{k=0}^{n-1}\left[e^{-iT(H_0+\overline{V})/n}\ \widetilde{u}\Big(t_0+\frac{(k+1)T}{n}; t_0+\frac{kT}{n}\Big)\right] \\ &= \sum_{k=0}^{n-1} U\Big(t_0+T; t_0+\frac{(k+1)T}{n}\Big) \\ &\quad\times\left\{U\Big(t_0+\frac{(k+1)T}{n}; t_0+\frac{kT}{n}\Big) - e^{-iT(H_0+\overline{V})/n}\ \widetilde{u}\Big(t_0+\frac{(k+1)T}{n}; t_0+\frac{kT}{n}\Big)\right\} \\ &\qquad\times\prod_{m=0}^{k-1}\left[e^{-iT(H_0+\overline{V})/n}\ \widetilde{u}\Big(t_0+\frac{(m+1)T}{n}; t_0+\frac{mT}{n}\Big)\right]. \end{aligned}$$
By (4.15) the norm of the difference is bounded by $n[(T/n)\,\|W_0\|\,]^2 = [\,T\,\|W_0\|\,]^2/n$. This shows (4.11). To show (4.12) we repeat the same estimate with $\widetilde{u}$ replaced by $\widetilde{\widetilde{u}}$. There are at most n factors of $\|\widetilde{\widetilde{u}}\|$ which gives $(1+T\|W_0\|/n)^n \le e^{T\|W_0\|}$. With (4.13) we get (4.12). □

5. The limiting time-evolution

In this section we will show that in the limit of rapid rotation the time evolution is dominated by the rotational invariant part of the potential. The contribution from its remaining part disappears as $\omega \to \infty$ by averaging.

We give two different proofs. One is based on a spectral theoretic intuition: on different eigenspaces of the operator J the Hamiltonians H_ω or $H_\omega + \overline{V}$ differ by integer multiples of ω (or $\hbar\omega$ in physical units). As we saw in Lemma 3.1 the effect of W amounts to transitions between different eigenspaces of J. For large ω such transitions are suppressed by the large energy transfer. We study resolvents to make this precise, see Lemma 3.2 and Proposition 5.1.

The other intuition relies on a variant of the Trotter product formula which says that the time evolution is well approximated if one rapidly alternates between the evolutions generated by either H_0 or by $V_{\omega t}$ alone as we saw in Proposition 4.1. A similar, technically more convenient version are the product formulae in Proposition 4.2. In the limit $\omega \to \infty$ the latter evolution depends only on the average $\overline{V}$ of $V_{\omega t}$. This argument is used in the second proof of Theorem 5.2.

PROPOSITION 5.1. *Let H_0 and $V_0 = \overline{V} + W_0$ satisfy the assumptions given in Section 3 (and repeated in Theorem 5.2) and $J\ P_\ell = \ell\ P_\ell$. Then uniformly in $\varphi \in [0, 2\pi]$*

$$\operatorname*{s-lim}_{\omega\to\infty} (H_\omega + \omega\ell + \overline{V} + W_\varphi - i\zeta)^{-1} = (H_0 + \overline{V} - i\zeta)^{-1}\ P_\ell\,. \tag{5.1}$$

PROOF. For $\pm\zeta > \|W_0\| = \|W_\varphi\|$ the sum in the resolvent equation

$$\begin{aligned}(H_\omega + &\omega\ell + \overline{V} + W_\varphi - i\zeta)^{-1}\\ &= (H_\omega + \omega\ell + \overline{V} - i\zeta)^{-1} \sum_{n=0}^{\infty} \left[-W_\varphi\ (H_\omega + \omega\ell + \overline{V} - i\zeta)^{-1}\right]^n\end{aligned}$$

is norm-convergent. For $\varepsilon > 0$ choose $N(\varepsilon)$ such that $\sum_{n>N(\varepsilon)}(\|W_0\|/|\zeta|)^n < \varepsilon$. Finite products of uniformly bounded strongly convergent operators converge as well strongly. To show the uniformity in φ we look at the term with $n = 1$:

$$\begin{aligned}&W_\varphi\ (H_\omega + \omega\ell + \overline{V} - i\zeta)^{-1}\ \Phi\\ &\longrightarrow\ W_\varphi\ (H_0 + \overline{V} - i\zeta)^{-1}\ P_\ell\ \Phi \quad \text{as} \quad \omega \to \infty.\end{aligned}$$

Since W_φ is strongly continuous the set $\{W_\varphi\ \Psi \mid \varphi \in [0, 2\pi]\}$ is precompact for any given vector Ψ (it can be covered by finitely many balls of radius δ for every $\delta > 0$). We can use the strong convergence of the next factor to the left. Similarly for higher, finite n. By Lemma 3.2 we get

$$\begin{aligned}\operatorname*{s-lim}_{\omega\to\infty}\ &(H_\omega + \omega\ell + \overline{V} + W_0 - i\zeta)^{-1}\\ &= (H_0 + \overline{V} - i\zeta)^{-1}\ P_\ell \sum_{n=0}^{\infty} \left[-W_0\ P_\ell\ (H_0 + \overline{V} - i\zeta)^{-1}\ P_\ell\right]^n.\end{aligned}$$

Since $P_\ell\ W_\varphi\ P_\ell = 0$ for all $\ell \in \mathbb{Z}$ only the term with $n = 0$ remains. This shows (5.1). □

Now we turn to the propagator U which solves the time-dependent Schrödinger equation (1.8) in a suitable sense, see the discussion in Section 2. The Schrödinger equation and, consequently, the propagator U depend on the angular velocity ω as a parameter. Analogous results for classical evolutions and scattering by smooth compactly supported potentials have been proved by Schmitz [**7**] using averaging methods.

THEOREM 5.2. *Let* $H_0(\cdot) \in C^1(\mathbb{R}^\nu, \mathbb{R})$ *with* $H_0(\mathbf{p}) = h(|\mathbf{p}|)$ *having unbounded derivative* h'. *When the real valued multiplication operator* $V_0 = \overline{V} + W_0$ *is split according to* (3.1) *we assume that the averaged potential* $\overline{V}$ *satisfies for some* $a < 1$ *and* $b < \infty$*:* $\|\overline{V}\,\Psi\| \le a\,\|H_0\,\Psi\| + b\,\|\Psi\|$ *for all* Ψ *in a domain of essential self-adjointness of* H_0. *Let* W_0 *be bounded. Then for any* $T \in \mathbb{R}$ *(uniformly on compact intervals)*

$$\operatorname*{s-lim}_{\omega\to\infty} U(t_0+T, t_0) = e^{-iT(H_0+\overline{V})} \tag{5.2}$$

uniformly in $t_0 \in \mathbb{R}$.

The uniformity in t_0 is clear because $U(t_0+T, t_0) = R(\omega t_0)\; U(T,0)\; R(\omega t_0)^*$. Since R is strongly continuous and periodic the set $\{R(\varphi)\,\Psi \mid \varphi \in \mathbb{R}\}$ is precompact in $\mathcal{H}$ for any vector Ψ. The right hand side of (5.2) is rotation invariant. Therefore, it is sufficient to treat $t_0 = 0$.

PROOF WITH RESOLVENTS.
We have to adjust the standard proof slightly because we do not have strong resolvent convergence and because we need some uniformity. We take Φ from the total set of vectors with $\Phi = P_\ell\,\Phi \in \mathcal{D}(H_0+\overline{V})$, $\ell \in \mathbb{Z}$, $\|\Phi\| = 1$. It satisfies $R(\omega T)\,\Phi = e^{-iT\omega\ell}\,\Phi$ and $(H_\omega + \omega\ell + \overline{V})\,\Phi = (H_0+\overline{V})\,\Phi$.

By the representation of the propagator according to the last line of (2.1)

$$\begin{aligned} U(T;0)\; &:= e^{-iT(H_\omega+\overline{V}+W_{\omega\ell})}\; R(\omega T)\;\Phi \\ &= e^{-iT(H_\omega+\omega\ell+\overline{V}+W_\varphi)}\;\Phi \end{aligned}$$

for $\varphi = \omega T$. For the family of cutoff functions $g_k(\mu) := \exp(-\mu^2/k)$ we obtain for some $\zeta \in \mathbb{R}\setminus\{0\}$, uniformly in $\omega \in \mathbb{R}$ and $\varphi \in [0, 2\pi]$,

$$\begin{aligned} &\|g_k(H_\omega+\omega\ell+\overline{V}+W_\varphi)\;\Phi - \Phi\| \\ &\le \|\,[\,g_k(H_\omega+\omega\ell+\overline{V}+W_\varphi) - \mathbb{1}]\;(H_\omega+\omega\ell+\overline{V}+W_\varphi - i\zeta)^{-1}\,\| \\ &\qquad \times\; \|(H_\omega+\omega\ell+\overline{V}+W_\varphi - i\zeta)\;\Phi\| \\ &\le \sup_\mu \left|\left(1-e^{-\mu^2/k}\right)(\mu - i\zeta)^{-1}\right| \;\times\; \left(\|(H_0+\overline{V})\,\Phi\| + \|W_0\| + |\zeta|\right). \end{aligned}$$

For given $\varepsilon > 0$ choose $k = k(\zeta, \Phi)$ large enough such that

$$\|g_k(H_\omega+\omega\ell+\overline{V}+W_\varphi)\;\Phi - \Phi\| < \varepsilon/6$$

and keep it fixed in the sequel. For T in a compact interval I the set of functions $\left\{e^{-iT\cdot}\;g_k(\cdot) \mid T \in I\right\}$ is bounded and equicontinuous. By the Arzela-Ascoli Theorem it is precompact in the set of bounded continuous functions tending towards zero at infinity with the supremum norm. By the Stone-Weierstraß Theorem there are finitely many polynomials P_m, $1 \le m \le m_1$, such that

$$\sup_{\mu\in\mathbb{R}} \left|e^{-iT\mu}\;g_k(\mu) - P_m\left((\mu-i\zeta)^{-1}, (\mu+i\zeta)^{-1}\right)\right| < \varepsilon/6$$

for some $m = m(T)$, $T \in I$. Then for this m

$$\begin{aligned} \Big\|P_m\Big(&\left(H_\omega+\omega\ell+\overline{V}+W_\varphi - i\zeta\right)^{-1}, \left(H_\omega+\omega\ell+\overline{V}+W_\varphi + i\zeta\right)^{-1}\Big)\;\Phi \\ &- e^{-iT(H_\omega+\omega\ell+\overline{V}+W_\varphi)}\;\Phi\Big\| < \varepsilon/3 \end{aligned} \tag{5.3}$$

holds uniformly in $\omega \in \mathbb{R}$, $\varphi \in [0, 2\pi]$, including the special case $\omega = 0$, $W = 0$, i.e., functions of $(H_0 + \overline{V})$. Finally, choose $\omega_1(\varepsilon)$ such that for all $\varphi \in [0, 2\pi]$ and $\omega > \omega_1(\varepsilon)$

$$\begin{aligned} \max_{1\le m\le m_1} \Big\| P_m\Big(\big(H_\omega + \omega\ell + \overline{V} + W_\varphi - i\zeta\big)^{-1}, \big(H_\omega + \omega\ell + \overline{V} + W_\varphi + i\zeta\big)^{-1}\Big)\ \Phi \\ - P_m\Big(\big(H_0 + \overline{V} - i\zeta\big)^{-1}, \big(H_0 + \overline{V} + i\zeta\big)^{-1}\Big)\ \Phi \Big\| < \varepsilon/3 \end{aligned} \tag{5.4}$$

which is possible by Proposition 5.1. Combining the estimates (5.3) and (5.4) yields

$$\Big\| U(T;0)\ \Phi - e^{-iT(H_0+\overline{V})}\ \Phi \Big\| < \varepsilon$$

for all $\omega > \omega_1(\varepsilon)$ and $T \in I$. □

PROOF WITH THE PRODUCT FORMULA.

We use the approximation of the propagator as expressed in the product formula (4.12) and we choose for $\varepsilon > 0$ some large fixed n with $n > (T\,\|W_0\|)^2\, e^{(T\,\|W_0\|)}/\varepsilon$. Then

$$\begin{aligned} &\Big\| \Big(U(t_0+T;t_0) - e^{-iT(H_0+\overline{V})}\Big)\ \Phi \Big\| \\ &\le \frac{\varepsilon}{2} + \Bigg\| \Bigg(\prod_{k=0}^{n-1} \Big[e^{-iT(H_0+\overline{V})/n}\ \widetilde{\widetilde{u}}\Big(\frac{(k+1)T}{n}, \frac{kT}{n}\Big)\Big] - e^{-iT(H_0+\overline{V})}\Bigg)\ \Phi \Bigg\| \\ &\le \frac{\varepsilon}{2} + \sum_{k=0}^{n-1} \Big\| \Big\{ \widetilde{\widetilde{u}}\Big(\frac{(k+1)T}{n}, \frac{kT}{n}\Big) - \mathbb{1}\Big\}\ e^{-ikT(H_0+\overline{V})/n}\ \Phi \Big\| \\ &\le \frac{\varepsilon}{2} + \sum_{k=0}^{n-1} \Big\| \Big\{ \int_0^{T/n} ds\ e^{is(H_0+\overline{V})}\ W_{\omega(s+kT/n)}\ e^{-is(H_0+\overline{V})} \Big\}\ e^{-ikT(H_0+\overline{V})/n}\ \Phi \Big\| \end{aligned}$$

Now we fix Φ from the total set of vectors with $\Phi = P_\ell\ \Phi$ for some $\ell \in \mathbb{Z}$. Note that due to strong continuity of $e^{-i\tau(H_0+\overline{V})}$ the set of vectors $\{e^{-i\tau(H_0+\overline{V})}\ \Phi \mid \tau \in I\}$ is precompact for any compact interval I. The same is true when the bounded operator W_0 is applied to this set.

Due to rotational invariance of $(H_0 + \overline{V})$ the projector P_ℓ can be moved to the right of W and we obtain for a summand in the last formula

$$\Big\| \Big\{ \int_0^{T/n} ds\ e^{is(H_0+\overline{V})}\ e^{-i\omega(J-\ell)(s+kT/n)}\ W_0\ P_\ell\ e^{-is(H_0+\overline{V})} \Big\}\ e^{-ikT(H_0+\overline{V})/n}\ \Phi \Big\|$$

By equation (3.8) $W_0\ P_\ell = \sum_{j\in\mathbb{Z}, j\neq\ell} P_j\ W_0\ P_\ell$ and the precompactness implies that only finitely many $j's$ matter. For all $\tau \in I$

$$\Big\| W_0\ P_\ell\ e^{-i\tau(H_0+\overline{V})}\ \Phi - \sum_{j\neq\ell}^{\text{finite}} P_j\ W_0\ P_\ell\ e^{-i\tau(H_0+\overline{V})}\ \Phi \Big\| < \varepsilon/4n.$$

It remains to estimate a finite sum of terms with $j \neq \ell$

$$\Big\| \int_0^{T/n} ds\ e^{-i\omega(j-\ell)s}\ e^{is(H_0+\overline{V})}\ P_j\ W_0\ P_\ell\ e^{-is(H_0+\overline{V})}\ e^{-ikT(H_0+\overline{V})/n}\ \Phi \Big\|.$$

The integrands are bounded continuous vector valued functions of s and, consequently, are integrable when restricted to the interval $[0, T/n]$. By the Riemann-Lebesgue Lemma their

Fourier transform tends to zero as $\omega \to \infty$. There is $\omega_1(\varepsilon)$ such that the sum is bounded by $\varepsilon/4$ for $\omega > \omega_1(\varepsilon)$. This shows that

$$\left\| \left(U(t_0 + T; t_0) - e^{-iT(H_0 + \overline{V})} \right) \Phi \right\| < \varepsilon \qquad \text{for} \quad \omega > \omega_1(\varepsilon).$$

This concludes the second proof of (5.2). □

6. The self-adjoint sum $H_\omega + V_0$

For the special case $\omega = 0$ the self-adjoint operator or form sum $H_0 + V_0$ has been studied extensively, mainly by methods of perturbation theory, see, e.g., [**6**]. Here we consider only the case $\omega \neq 0$ (unless otherwise stated) for H_ω as given in equations (1.9) and (1.10).

Following Tip [**8**] we derived in [**1**, Lemma 3.1] that V_0 is bounded relative to H_ω with bound less than one if $(1 + |\mathbf{x}|^2)\, V_0$ is bounded relative to $H_0 = |\mathbf{p}|^2/2m$ with bound less than one. The decay is important only for singular potentials, an arbitrary bounded part can always be added. In this section we treat as an example the special case of dimension $\nu = 2$ and $H_0(\mathbf{p}) = |\mathbf{p}|^2/2$ (mass $m = 1$ in adjusted units). We will show that even for locally square integrable potentials no decay towards infinity is needed. Higher dimensions and more general free Hamiltonians will be addressed in a forthcoming paper.

While the global properties of H_0 and H_ω differ very much it is easier to control their difference locally. Therefore, we begin with potentials of compact support.

In two dimensions let $\mathbf{x}^\perp = (-x_2, x_1)^{\text{tr}}$. Then $J = \mathbf{x} \wedge \mathbf{p} = \mathbf{x}^\perp \cdot \mathbf{p}$.

LEMMA 6.1. *Let $V \in L^2(\mathbb{R}^2)$ have compact support in the unit square centered at $\overline{\mathbf{x}} \in \mathbb{Z}^2$ and let $\chi \in C_0^\infty(\mathbb{R}^2)$ satisfy $\chi(\mathbf{x} - \overline{\mathbf{x}}) = 1$ in a neighborhood of the support of V. Then for any $a > 0$ there is a $b = b(a) < \infty$ such that for Ψ with $\psi(\mathbf{x}) \in C_0^\infty(\mathbb{R}^2)$*

$$\|V\, \chi(\cdot - \overline{\mathbf{x}})\, \Psi\| \le a\, \|(H_0 - \omega\, \overline{\mathbf{x}}^\perp \mathbf{p})\, \chi(\cdot - \overline{\mathbf{x}})\, \Psi\| + b\, \|\chi(\cdot - \overline{\mathbf{x}})\, \Psi\|, \tag{6.1}$$

$$\|V\, \chi(\cdot - \overline{\mathbf{x}})\, \Psi\| \le a\, \|(H_0 - \omega J)\, \chi(\cdot - \overline{\mathbf{x}})\, \Psi\| + b\, \|\chi(\cdot - \overline{\mathbf{x}})\, \Psi\|. \tag{6.2}$$

The bounds a and b depend on $\|V\|_2$, but they can be chosen independent of $\overline{\mathbf{x}}$.

For fixed $\overline{\mathbf{x}}$ equation (6.1) is well known in any dimension. The uniformity in $\overline{\mathbf{x}}$ is important here.

PROOF. With $\overline{\mathbf{p}} := \omega\, \overline{\mathbf{x}}^\perp$ we have $H_0(\mathbf{p}) - \omega\, \overline{\mathbf{x}}^\perp\, \mathbf{p} = H_0(\mathbf{p} - \overline{\mathbf{p}}) - |\overline{\mathbf{p}}|^2/2$. For any $a > 0$ we estimate the L^2-norm of the following function of $\mathbf{p}$:

$$\begin{aligned}
\left\| \left(a[(\mathbf{p} - \overline{\mathbf{p}})^2 - |\overline{\mathbf{p}}|^2/2] - i/a \right)^{-1} \right\|_2^2 &= \int \frac{d\mathbf{p}}{\{a\,[\,(\mathbf{p} - \overline{\mathbf{p}})^2 - |\overline{\mathbf{p}}|^2/2\,]\,\}^2 + 1/a^2} \\
&= \pi \int_{|\overline{\mathbf{p}}|^2/2}^{\infty} \frac{du}{u^2 + 1} \\
&= \pi\{\pi/2 + \arctan(|\overline{\mathbf{p}}|^2/2)\} \le \pi^2
\end{aligned}$$

where we have used polar coordinates around $\overline{\mathbf{p}}$ and $u = a^2\,[\,(\mathbf{p} - \overline{\mathbf{p}})^2 - \lambda\,]$. This gives the uniformity in $\overline{\mathbf{p}}$, the remaining proof is standard. Denoting by $\widehat{(\chi\psi)}(\mathbf{p})$ the Fourier transform of $(\chi\psi)(\mathbf{x}) := \chi(\mathbf{x} - \overline{\mathbf{x}})\,\psi(\mathbf{x})$ we estimate

$$\begin{aligned}
&\|(\chi\psi)\|_\infty \leq (1/2\pi)\,\|\widehat{(\chi\psi)}\|_1 \\
&\leq \frac{1}{2\pi}\,\left\|\frac{1}{a[(\mathbf{p}-\overline{\mathbf{p}})^2 - |\,\overline{\mathbf{p}}\,|^2/2\,] - i/a}\right\|_2 \left\|(a[(\mathbf{p}-\overline{\mathbf{p}})^2 - |\,\overline{\mathbf{p}}\,|^2/2\,] - i/a)\,\widehat{(\chi\psi)}\right\|_2 \\
&\leq a\|(H_0 - \overline{\mathbf{p}}\,\mathbf{p})\,(\chi\psi)\|_2 + (1/2a)\,\|(\chi\psi)\|_2\,.
\end{aligned}$$

With $\|V\,(\chi\psi)\|_2 \leq \|V\|_2\,\|(\chi\psi)\|_\infty$ this shows the estimate (6.1) uniformly in $\overline{\mathbf{x}}$. Then (6.2) follows easily from the observation that

$$(J - \overline{\mathbf{x}}^\perp\,\mathbf{p})\,\chi(\cdot - \overline{\mathbf{x}}) = (\mathbf{x} - \overline{\mathbf{x}})\cdot\nabla\chi(\cdot - \overline{\mathbf{x}}) + (\mathbf{x} - \overline{\mathbf{x}})\,\chi(\cdot - \overline{\mathbf{x}})\cdot\mathbf{p}$$

with uniformly bounded functions of $\mathbf{x}$. □

Now we split a potential $V \in L^2_{\mathrm{loc}}$ into four parts. The first of them, $V^{(1)}$, has its support only in those unit squares which are centered at those $\overline{\mathbf{x}} \in \mathbb{Z}^2$ which have *even* integers as coordinates. The remaining three parts have both coordinates of the centers odd or one even and the other odd. In each of the four components each unit square which belongs to the support is well separated from all others. Now we choose a decomposition of the identity

$$\sum_{\overline{\mathbf{x}}\in(2\mathbb{Z})^2} [\,\chi(\cdot - \overline{\mathbf{x}})\,]^2 = 1$$

where $\chi \in C_0^\infty(\mathbb{R}^2)$ and $\chi(\mathbf{x}) = 1$ in a neighborhood of the unit square around the origin. This decomposition splits the potential $V^{(1)}$ into pieces which coincide with V in one unit square and are zero outside of it. For the other components of the potential we use decompositions which are shifted by $(0,1)$, $(1,0)$, or $(1,1)$, respectively.

For $V \in L^2_{\mathrm{loc,\,unif}}$ the L^2-norms of the restrictions to arbitrary unit squares are uniformly bounded. This applies, in particular, to all parts of V constructed above.

THEOREM 6.2. *Any $V \in L^2_{\mathrm{loc,\,unif}}(\mathbb{R}^2)$ is bounded relative to H_ω with relative bound zero. In particular, $(H_\omega + V)$ is essentially self-adjoint on any core of H_ω.*

PROOF. Morgan has shown in [**3**, Theorem 2.3] that (6.2) implies

$$\|V^{(1)}\,\Psi\| = \|V^{(1)} \sum_{\overline{\mathbf{x}}\in(2\mathbb{Z})^2} \chi(\cdot - \overline{\mathbf{x}})\,\Psi\| \leq a\,\|H_\omega\Psi\| + b\,\|\Psi\|$$

and analogously for the other three components. □

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