ON NEW RELATIONS BETWEEN SPECTRAL PROPERTIES OF JACOBI MATRICES AND THEIR COEFFICIENTS

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ABSTRACT. We study the spectral properties of Jacobi matrices. By using "higher order" trace formulae we obtain a result relating the properties of the elements of Jacobi matrices and the corresponding spectral measures. Complicated expressions for traces of some operators can be magically simplified allowing us to apply induction arguments. Our theorems are generalizations of a recent result of R. Killip and B. Simon [17].

1. INTRODUCTION

Let S be the shift operator in $l^2(\mathbb{N})$, $\mathbb{N} = \{0, 1, 2, ...\}$, whose action on the canonical orthonormal basis $\{e_n\}_{n=0}^{\infty}$ is given by $Se_n = e_{n+1}$. Let A, B be selfadjoint diagonal operators, $Ae_n = \alpha_n e_n$, $Be_n = \beta_n e_n$, $\alpha_n > -1$, $\beta \in \mathbb{R}$. We study the spectral properties of the operator

$$J = S + S^* + Q, \quad \text{where } Q = SA + AS^* + B.$$

This operator can be identified with the following Jacobi matrix

$$J = \begin{pmatrix} \beta_0 & 1+\alpha_0 & 0 & 0 & \cdots \\ 1+\alpha_0 & \beta_1 & 1+\alpha_1 & 0 & \cdots \\ 0 & 1+\alpha_1 & \beta_2 & 1+\alpha_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.1}$$

If the entries of this matrix are bounded, then J is a bounded operator in $l^2(\mathbb{N})$. To every such operator J we associate the following measure μ given by

$$m_\mu(z) := (e_0, (J-z)^{-1}e_0) = \int \frac{d\mu(t)}{t-z}, \qquad z \in \mathbb{C}. \tag{1.2}$$

The spectral significance of this function can be seen from the equality

$$\mu(\delta) = (E_J(\delta)e_0, e_0), \tag{1.3}$$

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where E_J denotes the spectral measure of J and $\delta \subset \mathbb{R}$ is a Borel set. Obviously,

$$\mu(\mathbb{R}) = \|e_0\|^2 = 1.$$

Conversely, for each probability measure μ there is a standard procedure of constructing a Jacobi matrix via the corresponding orthogonal polynomials (see [1], [24] and also [17] for historical references and bibliography).

Since there is a one-to-one correspondence between Jacobi matrices and probability measures, it is natural to ask how the properties of entries of Jacobi matrices are related to the properties of probability measures. We are interested in a class of matrices J "close" to the "free" matrix J_0 for which $\alpha_n = 0$ and $\beta_n = 0$, $n = 0, 1, \ldots$.

It is convenient to replace m_μ by

$$\begin{aligned} M_\mu(k) &= -m_\mu\big(z(k)\big) = -m_\mu\big(k + k^{-1}\big) \\ &= \int \frac{k\, d\mu(t)}{1 - tk + k^2}, \qquad |k| < 1. \end{aligned} \tag{1.4}$$

It is known (see [25]) that the limit

$$M(e^{i\theta}) = \lim_{r\to 1} M_\mu(re^{i\theta}), \;\; r < 1,$$

exists almost everywhere on the unit circle and that Im $M(e^{i\theta}) \geq 0$ for $\theta \in (0, \pi)$. Moreover, since $M(e^{-i\theta}) = \overline{M(e^{i\theta})}$, we obtain Im $M(e^{i\theta}) \leq 0$ for $\theta \in (-\pi, 0)$.

In order to formulate our main result we denote by $\mathfrak{S}_p$ the standard Shatten classes of compact operators:

$$\mathfrak{S}_p = \{T : \;\; \mathrm{tr}\,(T^*T)^{p/2} < \infty \}.$$

Theorem 1.1. *Let J be a Jacobi matrix and let μ be the corresponding measure* (1.3). *Assume that the operator $Q = J - J_0$ satisfies*

$$\begin{cases} Q \in \mathfrak{S}_3 \;\text{ if }\; \mathrm{rank}\, A = \infty, \\ Q \in \mathfrak{S}_4 \;\text{ if }\; \mathrm{rank}\, A < \infty. \end{cases} \tag{1.5}$$

Then

$$\sum_n (\alpha_n + \cdots + \alpha_{n+m-1})^2 + \sum_n (\beta_n + \cdots + \beta_{n+m-1})^2 < \infty \tag{1.6}$$

if and only if μ satisfies the following three properties:

(1) *The support of μ is $[-2, 2] \cup \{E_j^+\}_{j=1}^{N_+} \cup \{E_j^-\}_{j=1}^{N_-}$, where $\pm E_j^\pm > 2$, $0 \leq N_\pm \leq \infty$.*

(2) (Quasi-Szegö Condition)

$$\int_{-\pi}^{\pi} \log\left[\frac{\sin(\theta)}{\mathrm{Im}\, M(e^{i\theta})}\right] \sin^2 m\theta \, d\theta < \infty.$$

(3) (Lieb–Thirring Bound)

$$\sum_{j=1}^{N_+} |E_j^+ - 2|^{3/2} + \sum_{j=1}^{N_-} |E_j^- + 2|^{3/2} < \infty.$$

It is interesting that with the growth of m the Quasi-Szegö condition (2) becomes weaker and weaker allowing Im M to have exponential zeros at $\frac{\pi n}{m}$, $n = -m, \dots, m-1$. Moreover the condition (1.6) with $m = l$ implies the corresponding condition with $m = 2l$ but not conversely. In particular, this means that there are Jacobi matrices satisfying (1.6) with $m = 2l$ such that the function Im M vanishes at least at one of the points $\frac{\pi(2n+1)}{2l}$, $n = -l, \dots, l-1$.

The main technical parts of the proof of Theorem 1.1 are Lemmas 2.1 and 2.2, see Sections 2.2-2.5. It is really surprising how after some involved calculations one can simplify rather complicated formulae and finally use induction arguments.

Although Theorem 1.1 is a natural generalization of a recent result of Killip-Simon [17], it has some disadvantages. Namely, for a given measure μ we are not able to check in advance whether the conditions (1.5) are fulfilled for the corresponding Jacobi matrix J.

However, in the case $m = 2$ we are able avoid this obstacle and obtain a stronger result where we do have *a priori* the condition (1.5).

Theorem 1.2. *Let J be a Jacobi matrix and let μ be the corresponding measure* (1.3). *Then the conditions*

$$\begin{cases} \sum_n \alpha_n^4 + \sum_n {\beta_n}^4 < \infty \\ \sum_n (\alpha_n + \alpha_{n+1})^2 + \sum_n (\beta_n + \beta_{n+1})^2 < \infty \end{cases} \tag{1.7}$$

hold if and only if μ satisfies the following three properties:

(1) *The support of μ is* $[-2, 2] \cup \{E_j^+\}_{j=1}^{N_+} \cup \{E_j^-\}_{j=1}^{N_-}$, *where* $\pm E_j^\pm > 2$, $0 \le N_\pm \le \infty$.

(2) (Quasi-Szegö Condition)

$$\int_{-\pi}^{\pi} \log\left[\frac{\sin(\theta)}{\text{Im } M(e^{i\theta})}\right] \sin^2 2\theta \, d\theta < \infty.$$

(3) (Lieb–Thirring Bound)

$$\sum_{j=1}^{N_+} |E_j^+ - 2|^{3/2} + \sum_{j=1}^{N_-} |E_j^- + 2|^{3/2} < \infty.$$

This theorem defines a one-to-one correspondence between a class of probability measures and the class of Jacobi matrices satisfying the condition (1.7).

Given a singular measure ρ_s on $[-2, 2]$ with total mass less than one, we are able to construct a Jacobi matrix with the properties (1.7) such that the singular component of the measure μ on this interval coincides with ρ_s. The corresponding fact was noticed in [17] for Jacobi matrices with $Q \in \mathfrak{S}_2$ and in [9] for Schrödinger operators with a class of L^2 potentials.

The next theorem is an immediate corollary of our main results. Let the measure μ be decomposed into the sum $\mu = \mu_{ac} + \mu_{pp} + \mu_{sc}$ of absolutely continuous, pure point and singular continuous components with respect to the Lebesgue measure. Then

$$\text{Im } M(e^{i\theta}) = \frac{d\mu_{ac}(t)}{dt}, \quad t = 2\cos\theta.$$

The conditions (1.5) and (1.6), in particular, imply the quasi-Szegö condition from Theorem 1.1 and therefore $\frac{d\mu_{ac}(t)}{dt} \neq 0$ almost everywhere on $(-2, 2)$. Thus we obtain:

Theorem 1.3. *Let $m > 0$ be an integer number and let $Q = 2\text{Re }(SA) + B$ satisfy the condition* (1.5). *If the operators $\sum_{n=0}^{m} S^n A(S^*)^n$ and $\sum_{n=0}^{m} S^n B(S^*)^n$ are of the Hilbert-Schmidt class, then the spectral measure of the operator J does not vanish on subsets $K \subset \mathbb{R}_+ = (-2, 2)$ whose Lebesgue measure is positive.*

Remark 1. If $m = 2$ the previous theorem can be strengthen. In this case Theorem 1.2 allow us to replace (1.5) and (1.6) by (1.7).

Remark 2. Closely related results can be found in [16], Theorem 4, where the author used a "locally spectral technique" in order to obtain a continuous version of Theorem 1.3.

When proving our main results we use high order trace formulae for Jacobi matrices (Case's sum rules [4], [5]). Note that the usefulness of trace formulae in the study of a.c. properties of the spectrum of discrete Schrödinger operators was first observed by P. Deift and R. Killip [8], where the authors found that the conditions $\{\beta_n\}_{n=0}^{\infty} \in l^2$, $\alpha_n = 0$, $\forall n$, guarantee that a.c. spectrum is essentially supported by $[-2, 2]$. The sharpness of this result is confirmed by examples constructed in B. Simon [27], where the author shows that for each $\varepsilon > 0$ there is a potential B satisfying $|\beta_n| = O(n^{-1/2+\varepsilon})$ and such that the operator $H = S + S^* + B$ has a pure point spectrum. Previously a class of such perturbations were also studied by A. Kiselev in [18]. Some results on spectral properties of a class of operators $S + S^* + 2\text{Re }(SA)$ were obtained in J. Janas and S.N. Naboko [14]. Recently S. Belov and A. Rybkin [2] have considered WKB-asymptotics of generalized eigenfunctions which implies preservation of the a.c. spectrum for the case $\alpha_n, \beta_n = O(n^{-2/3-\varepsilon})$. Notice that the conditions which were imposed on A and B in [2], [14] were much stronger than those in Theorem 1.3. There is also a possibility of investigating the a.c. spectrum with the help of the Gilbert-Pearson [13] theory, see also [15]. It was established in [13] (see also [21]) that the a.c. spectrum is related to the absence of subordinate solutions.

The paper [8] was also a culmination of a long sequence of papers concerning a.c. spectral properties of Schrödinger operators in $L^2(0,\infty)$ (see, for example, [6], [7] and [26]). It was proved in [8] that for the operator $-d^2/dx^2 + V$, the condition $V \in L^2$ suffices for the a.c. spectrum to be essentially supported by $(0,\infty)$. In both discrete and continuous cases P. Deift and R. Killip used trace formulae for Schrödinger operators involving L^2-norms of the corresponding potentials. This result has been recently generalized by S. Molchanov, M. Novitskii and B. Vainberg, [22], where the authors used higher order trace formulae involving first KDV integrals. The structure of trace formulae for Jacobi matrices is somewhat surprising. In contrast to the continuous case where the high order trace formulae involve the derivatives of the potential, the corresponding trace formulae for Jacobi matrices can be rearranged in a such a way that they involve the mean values of its entries.

2. TRACE FORMULAE

1. In this section we assume that S is the shift operator in $l^2(\mathbb{Z})$ whose action on the standard orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$ is given by $Se_n = e_{n+1}$. Then, in particular, $S^* = S^{-1}$. Let A and B be finite rank diagonal operators on $l^2(\mathbb{Z})$. Let

$$H_0 = S + S^* \quad \text{and} \quad H = H_0 + Q, \tag{2.1}$$

on $l^2(\mathbb{Z})$, where $Q = SA + AS^* + B$. Without loss of generality (since one can always pass from Q to $S^m Q S^{-m}$) we can assume that $\alpha_n = \beta_n = 0$ for $n < 0$. For every $k \in \mathbb{C} \setminus \{0\}$ there exists a solution $\psi = \{\psi_n\}_{n=-\infty}^{\infty}$ of the equation

$$(1+\alpha_n)\psi_{n+1} + (1+\alpha_{n-1})\psi_{n-1} + \beta_n\psi_n = (k+1/k)\psi_n, \ n \in \mathbb{Z}, \tag{2.2}$$

such that $\psi_n = k^{-n}$ to the right of the "support" of Q. For $n < 0$ this solution can be written as a linear combination of k^{-n} and k^n

$$\psi_n = a(k)k^{-n} + b(k)k^n, \tag{2.3}$$

where a and b depend analytically on $k \neq 0$. We substitute ψ_n by $\psi_n = \zeta_n\phi_n$, where

$$\zeta_n = \begin{cases} \prod_{j=0}^{n-1}(1+\alpha_j), & \text{if } n \geq 1, \\ 1, & \text{if } n \leq 0. \end{cases}$$

Then

$$(1+\alpha_n)^2\phi_{n+1}+\phi_{n-1}+\beta_n\phi_n=(k+1/k)\phi_n.$$

We can now rewrite this equation as

$$v_n=\tau-\frac{k}{k^2-1}\sum_{m=n}^{\infty}(1-k^{2(n-m)})\Big(\beta_m v_m+\frac{2\alpha_m+\alpha_m^2}{k}v_{m+1}\Big), \tag{2.4}$$

where $v_n=k^n\phi_n$ and $\tau=\prod_0^\infty(1+\alpha_k)^{-1}$.

For $|k|>1$ (2.4) can be solved by repeated substitution (or Neumann series), from which we conclude that

$$a=\lim_{n\to-\infty}v_n=\tau\Big(1-\frac{\operatorname{tr}B}{k}+\frac{1}{k^2}\{\operatorname{tr}(I-(I+A)^2)+$$
$$\sum\sum\frac{(1-k^{2|n-m|})}{2}\beta_m\beta_n\}\Big)+O\Big(\frac{1}{k^3}\Big),\quad \text{as } k\to\infty.$$

Therefore

$$\begin{aligned}\tau^{-1}a=&1-\frac{\operatorname{tr}B}{k}-\frac{1}{k^2}\Big\{\operatorname{tr}((I+A)^2-I)\\&+\frac{(\operatorname{tr}B^2-(\operatorname{tr}B)^2)}{2}\Big\}+O\Big(\frac{1}{k^3}\Big),\end{aligned} \tag{2.5}$$

as $k\to\infty$.

Let us denote

$$D(z)=\det(I+Q(H_0-z)^{-1}),\qquad z=k+k^{-1}.$$

The standard scattering matrix $\sigma(z)$ for the pair of operators H_0 and H can be expressed in terms of the coefficients a and b defined in (2.3)

$$\sigma(z)=\begin{pmatrix}\frac{1}{a(k)} & -\frac{b(1/k)}{a(k)}\\ \frac{b(k)}{a(k)} & \frac{1}{a(k)}\end{pmatrix}.$$

Introducing the Wronskian

$$W_n=W_n(\psi,\overline{\psi})=\psi_n\overline{\psi}_{n-1}-\psi_{n-1}\overline{\psi}_n,$$

we observe that for $k=e^{i\theta},\ \theta\in(0,\pi)$, the relation $\sum_n(W_{n+1}-W_n)=0$ implies

$$|a|^2-|b|^2=1.$$

Considering (2.2) for $k\in\mathbb{S}^1$ we also easily conclude that

$$\begin{cases}\overline{a(e^{i\theta})}=a(e^{-i\theta})\\ \overline{b(e^{i\theta})}=b(e^{-i\theta})\end{cases}.$$

Then, in particular, for $k=e^{i\theta}$, $z\in(-2,2)$, we have

$$\det\sigma(z)=\overline{a(k)}a(k)^{-1}=e^{-2i\arg a(k)}. \tag{2.6}$$

It is well known [29], Section 8.2, that for $\lambda \in (-2,2)$ the limit

$$\lim_{\varepsilon\to+0} \arg D(\lambda+i\varepsilon),$$

exists. The following equality is known as the Birman-Krein formula [3]

$$\log\det\sigma(\lambda) = -2i\arg D(\lambda), \quad \lambda\in(-2,2). \tag{2.7}$$

The latter formula together with (2.6) implies

$$\arg D(2\cos\theta) = \arg a(e^{i\theta}).$$

The zeros of functions $D(k+k^{-1})$ and $a(k)$ coincide and according to (2.5)

$$D(k+k^{-1}) - \tau^{-1}a(k) = O(|k|^{-2}), \quad \text{as } |k|\to\infty.$$

This implies

$$\tau^{-1}a(k) = \det(I+Q(H_0-z)^{-1}), \quad z = k+k^{-1},$$

and thus

$$\log(a(k)) = \log(\tau) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\mathrm{tr}\big(Q(H_0-z)^{-1}\big)^n, \;\; z\to\infty. \tag{2.8}$$

2. The coefficients Λ_n in the following expansion

$$\log(a(k)) = -\sum_{j=0}^{\infty}\Lambda_j k^{-j}, \quad k\to\infty, \tag{2.9}$$

were introduced for example in [12] (see also [5] for Jacobi matrices). The coefficients Λ_n can be obtained by expanding each term of (2.8) into Laurent series in k and could be expressed via Chebyshev polynomials of the first kind (see [17]). For our purposes it is not sufficient to know a finite number of coefficients Λ_n or to have above mentioned implicit representation . We need to study the structure of these coefficients and their dependence on the perturbation. Therefore we establish the following two results.

Lemma 2.1. *Let $A=0$. Then the coefficient Λ_{2m} can be written in the form*

$$\Lambda_{2m} = \frac{1}{2}\mathrm{tr}\,\Big(\sum_{n=0}^{m-1} S^{-n}BS^n\Big)^2 + F_{2m}(B),$$

where F_{2m} is finite when $B\in\mathfrak{S}_4$.

For the general case when $A\neq 0$ we prove:

Lemma 2.2. *Let $Q = SA+AS^{-1}$. Then the coefficient Λ_{2m} admits the representation*

$$\Lambda_{2m} - 2\,\mathrm{tr}\log(I+A) = \frac{1}{2}\mathrm{tr}\,\Big(\sum_{n=0}^{m-1} S^{-n}BS^n\Big)^2 + 2\,\mathrm{tr}\,\Big(\sum_{n=0}^{m-1} S^{-n}AS^n\Big)^2 + \Psi_{2m}(Q),$$

where $\Psi_{2m}(Q)$ is finite when $B, A \in \mathfrak{S}_3$.

Some explicit formulae for coefficients Λ_m, $m \le 4$, can be found in the literature (see, for example, [12], p. 155). Let us introduce the operator

$$L = (I + A)^2 - I.$$

Then

$$\Lambda_0 = \log(\tau), \quad \Lambda_1 = \mathrm{tr}B,$$

$$\Lambda_2 = \frac{1}{2}\mathrm{tr}A_0^2 + \mathrm{tr}(L),$$

$$\Lambda_3 = \frac{1}{3}\mathrm{tr}\, A_0^3 + \mathrm{tr}\,(B + S^*BS)\big(\frac{1}{2}I + L\big),$$

$$\begin{aligned} \Lambda_4 = {} & \frac{1}{4}\mathrm{tr}\,\big((B^2 + L + SLS^*)^2 + 2(B + S^*BS)^2(L+1)\big) \\ & + \frac{1}{4}\mathrm{tr}\,\big((L + SLS^*)^2 - 2(L^2 - 2L)\big). \end{aligned} \tag{2.10}$$

As we shall see later, the analysis of coefficients Λ_m for the Srchrödinger operator will lead to the analysis of similar coefficients for Jacobi matrices and vice versa.

The function $a(k)$ vanishes when $k + 1/k$ is an eigenvalue of (2.2). Let $\{\varkappa_n\}$ be the zeros of $a(k)$ lying in the domain $|k| > 1$. We introduce the Blaschke product

$$G = \prod_n \frac{\varkappa_n - k}{1 - \varkappa_n k}\frac{\varkappa_n}{|\varkappa_n|}, \qquad (\overline{\varkappa_n} = \varkappa_n).$$

Clearly $|G| = 1$ on the unit circle and Re $\big(\log(G/a)\big)$ is an odd function of θ when $k = e^{i\theta}$. Thus, by using (2.9) we find

$$\begin{aligned} \frac{2}{\pi}\int_{-\pi}^{\pi} \log|a|\sin^2(m\theta)d\theta &= \frac{1}{2\pi i}\int_{|k|=1} \log(G/a)\frac{(k^{2m}-1)^2}{k^{2m+1}}\, dk \\ &= 2\log(\tau) + \Lambda_{2m} - \sum_n f(\varkappa_n^{2m}), \end{aligned}$$

where $f(t) = 1/2(t - t^{-1}) - \log(t) > 0$ for $t > 1$.

3. Assume for a moment that

$$Q = B, \;\; B = \mathrm{diag}\{\beta_n\}.$$

The resolvent of the free discrete Schrödinger operator can be written as

$$\begin{aligned} (H_0 - z)^{-1} &= -\frac{1}{k}(I - S/k)^{-1}(I - S^{-1}/k)^{-1} \\ &= -\frac{1}{k}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{S^{j-m}}{k^{j+m}} = -\sum_{p=0}^{\infty}\frac{1}{k^{p+1}}\sum_{m=0}^{p} S^{p-2m}. \end{aligned} \tag{2.11}$$

This implies that the term with $n = 2$ in the decomposition (2.8) has the following representation

$$\operatorname{tr} Q(H_0 - z)^{-1} Q(H_0 - z)^{-1} =$$

$$\sum_{p=0}^{\infty} \frac{1}{k^{p+2}} \operatorname{tr} \Big(\sum_{n=0}^{p} S^{-n} B S^{n} \sum_{m=0}^{n} S^{-2m} B \sum_{j=0}^{p-n} S^{p-2j} \Big) =: \sum_{p=0}^{\infty} \frac{\Gamma_{p+2}}{k^{p+2}}.$$

If p is odd, then $p - 2m$ is also odd and since $Q = B$ is a diagonal perturbation, we obtain $\Gamma_p = 0$. Therefore we shall only study Γ_p with even values of p. Clearly

$$\Gamma_{2(p+1)} = \operatorname{tr} \sum_{n=0}^{2p} S^{-n} B S^{n} \sum_{m=0}^{n} S^{-2m} B \sum_{j=0}^{2p-n} S^{2p-2j}$$

$$= \operatorname{tr} \sum_{n=0}^{2p} S^{-n} B S^{n} \sum_{m=0}^{n} S^{-2m} B \sum_{l=n-p}^{p} S^{2l}. \tag{2.12}$$

If $0 \le n \le p$, then the terms in the last sum of (2.12) with $l \ne m$ cancel. For $p + 1 \le n \le 2p$ we obtain that in the summation with respect to m in (2.12) survives only if $m = l$. This implies

$$\Gamma_{2(p+1)} = \operatorname{tr} \sum_{n=0}^{p} S^{-n} B S^{n} \sum_{m=0}^{n} S^{-2m} B S^{2m} + \operatorname{tr} \sum_{n=p+1}^{2p} S^{-n} B S^{n} \sum_{l=n-p}^{p} S^{-2l} B S^{2l}.$$

On the other hand we notice that the latter two traces are almost equal. Indeed,

$$\operatorname{tr} \sum_{n=p+1}^{2p} S^{-n} B S^{n} \sum_{l=n-p}^{p} S^{-2l} B S^{2l}$$

$$= \operatorname{tr} \sum_{n=p+1}^{2p} S^{-n} B S^{n} \sum_{m=0}^{2p-n} S^{-2(m+n-p)} B S^{2(m+n-p)}$$

$$= \operatorname{tr} \sum_{n=p+1}^{2p} S^{n-2p} B S^{2p-n} \sum_{m=0}^{2p-n} S^{-2m} B S^{2m} = \operatorname{tr} \sum_{n=0}^{p-1} S^{-n} B S^{n} \sum_{m=0}^{n} S^{-2m} B S^{2m}.$$

Therefore (2.12) can be rewritten as

$$\Gamma_{2(p+1)} = \operatorname{tr} \sum_{n=0}^{p} S^{-n} B S^{n} \sum_{m=0}^{n} S^{-2m} B S^{2m} + \operatorname{tr} \sum_{n=0}^{p-1} S^{-n} B S^{n} \sum_{m=0}^{n} S^{-2m} B S^{2m}.$$

It is now easy to express $\Gamma_{2(p+1)}$ via Γ_{2p}

$$\Gamma_{2(p+1)} = \Gamma_{2p} + \operatorname{tr} S^{-p} B S^{p} \sum_{m=0}^{p} S^{-2m} B S^{2m} + \operatorname{tr} S^{-p+1} B S^{p-1} \sum_{m=0}^{p-1} S^{-2m} B S^{2m}$$

$$= \Gamma_{2p} + \mathrm{tr} S^{-p} B S^p \sum_{m=0}^{2p} S^{-m} B S^m,$$

which magically becomes

$$= \Gamma_{2p} + 2\mathrm{tr}\, B \sum_{m=1}^{p} S^{-m} B S^m + \mathrm{tr} B^2.$$

Obviously

$$\Gamma_2 = \mathrm{tr}\, B^2$$

and by using induction we obtain that

$$\Gamma_{2(p+1)} = \mathrm{tr}\, \Big(\sum_{n=0}^{p} S^{-n} B S^n \Big)^2. \tag{2.13}$$

Finally

$$\mathrm{tr}\, Q(H_0 - z)^{-1} Q(H_0 - z)^{-1} = \sum_{p=0}^{\infty} \Gamma_{2(p+1)} k^{-2(p+1)}, \tag{2.14}$$

where $\Gamma_{2(p+1)}$ is given by (2.13).

4. Let us make another temporary assumption, that is

$$Q = SA + AS^{-1}, \;\; A = A^*.$$

Then

$$\mathrm{tr}\, Q(H_0 - z)^{-1} Q(H_0 - z)^{-1} = 2\mathrm{tr}\, A(H_0 - z)^{-1} A(H_0 - z)^{-1} \tag{2.15}$$
$$+ \mathrm{tr}\, SA(H_0 - z)^{-1} SA(H_0 - z)^{-1} + \mathrm{tr}\, AS^{-1}(H_0 - z)^{-1} AS^{-1}(H_0 - z)^{-1}.$$

Consider the last term in the right hand side of the latter equality and apply the identity (2.11) for the resolvent $(H_0 - z)^{-1}$. Then

$$\mathrm{tr}\, AS^{-1}(H_0 - z)^{-1} AS^{-1}(H_0 - z)^{-1}$$
$$= \mathrm{tr}\, SAS^{-1}(H_0 - z)^{-1} AS^{-1}(H_0 - z)^{-1} S^{-1} =: \sum_{p=2}^{\infty} \frac{\Upsilon_p}{k^p},$$

where by using the same arguments as in (2.12) we have

$$\Upsilon_{2(p+1)} = \mathrm{tr}\, \sum_{n=0}^{2p} S^{-n+1} A S^{n-1} \sum_{m=0}^{n} S^{-2m} A \sum_{j=n-p}^{p} S^{2j-2}$$
$$= \mathrm{tr}\, \sum_{n=0}^{2p} S^{-n+1} A S^{n-1} \sum_{m=0}^{n} S^{-2m} A \sum_{j=n-p-1}^{p-1} S^{2j}$$
$$+ \mathrm{tr}\, \sum_{n=0}^{p-1} S^{-n+1} A S^{n-1} \sum_{m=0}^{n} S^{-2m} A S^{2m} + \mathrm{tr}\, \sum_{n=p+1}^{2p} S^{-n+1} A S^{n-1} \sum_{m=n-p-1}^{p-1} S^{-2m} A S^{2m}$$

$$+\mathrm{tr}\, S^{-p+1}AS^{p-1}\sum_{m=0}^{p-1} S^{-2m}AS^{2m}.$$

Let us consider the second trace of the last expression. A simple computation leads us to

$$\mathrm{tr}\sum_{n=p+1}^{2p} S^{-n+1}AS^{n-1}\sum_{m=n-p-1}^{p-1} S^{-2m}AS^{2m}$$

$$= \mathrm{tr}\sum_{n=p+1}^{2p} S^{n-2p-1}AS^{2p-n+1}\sum_{m=0}^{2p-n} S^{-2m}AS^{2m}$$

and substituting $j = 2p - n$ we find

$$= \mathrm{tr}\sum_{j=0}^{p-1} S^{-j-1}AS^{j+1}\sum_{m=0}^{j} S^{-2m}AS^{2m}.$$

Thus,

$$\Upsilon_{2(p+1)} = \Upsilon_{2p} +$$

$$+\mathrm{tr}\, S^{-p+2}AS^{p-2}\sum_{m=0}^{p-1} S^{-2m}AS^{2m} + \mathrm{tr}\, S^{-p}AS^{p}\sum_{m=0}^{p-1} S^{-2m}AS^{2m}$$

$$-\mathrm{tr}\, S^{-p+2}AS^{p-2}\sum_{m=0}^{p-2} S^{-2m}AS^{2m} + \mathrm{tr}\, S^{-p+1}AS^{p-1}\sum_{m=0}^{p-1} S^{-2m}AS^{2m}$$

Clearly

$$\mathrm{tr}\, S^{-p+2}AS^{p-2}\sum_{m=0}^{p-1} S^{-2m}AS^{2m} - \mathrm{tr}\, S^{-p+2}AS^{p-2}\sum_{m=0}^{p-2} S^{-2m}AS^{2m}$$

$$= \mathrm{tr}\, S^{p}AS^{-p}.$$

Moreover

$$\mathrm{tr}\, S^{-p}AS^{p}\sum_{m=0}^{p-1} S^{-2m}AS^{2m} + \mathrm{tr}\, S^{p}AS^{-p} + \mathrm{tr}\, S^{-p+1}AS^{p-1}\sum_{m=0}^{p-1} S^{-2m}AS^{2m}$$

$$= \mathrm{tr}\, S^{-p}AS^{p}\sum_{m=0}^{p} S^{-2m}AS^{2m} + \mathrm{tr}\, S^{-p}AS^{p}\sum_{m=0}^{p-1} S^{-2m-1}AS^{2m+1}$$

$$= \mathrm{tr}\, S^{-p}AS^{p}\sum_{m=0}^{2p} S^{-m}AS^{m}.$$

By using symmetry

$$\operatorname{tr} S^{-p}AS^{p}\sum_{m=0}^{2p} S^{-m}AS^{m} = 2\operatorname{tr} A\sum_{m=1}^{p} S^{-m}AS^{m} + \operatorname{tr} A^{2},$$

which finally gives us

$$\Upsilon_{2(p+1)} = \Upsilon_{2p} + 2\operatorname{tr} A\sum_{m=1}^{p} S^{-m}AS^{m} + \operatorname{tr} A^{2}.$$

Since

$$\Upsilon_{2} = 0 = \operatorname{tr} A^{2} - \operatorname{tr} A^{2},$$

we find that magic works even this time and gives us

$$\Upsilon_{2(p+1)} = \operatorname{tr}\Big(\sum_{n=0}^{p} S^{-n}AS^{n}\Big)^{2} - \operatorname{tr} A^{2}.$$

Similarly we obtain that the term $\operatorname{tr} SA(H_0-z)^{-1}SA(H_0-z)^{-1}$ appearing in (2.15), has exactly the same Laurent series. For the coefficients in the corresponding expansion of the first term in the right hand side of (2.15) we can use (2.14). It implies

$$\operatorname{tr} Q(H_0-z)^{-1}Q(H_0-z)^{-1} = 2\sum_{p=2}^{\infty}\frac{I_p}{k^p},$$

where

$$I_{2(p+1)} = 2\operatorname{tr}\Big(\sum_{n=0}^{p} S^{-n}AS^{n}\Big)^{2} - \operatorname{tr} A^{2}.$$

5. Finally we consider the general case, when $Q = SA + AS^{-1} + B$ where $A \neq 0,\ B \neq 0$. By using (2.11) we find

$$\operatorname{tr} Q(H_0-z)^{-1} = -\operatorname{tr}(SA + AS^{-1} + B)\sum_{p=0}^{\infty}\frac{1}{k^{p+1}}\sum_{m=0}^{p} S^{p-2m}$$

$$= -2\operatorname{tr} A\sum_{m=0}^{\infty}\frac{1}{k^{2m}} - \operatorname{tr} B\sum_{m=0}^{\infty}\frac{1}{k^{2m+1}}.$$

Let us denote by $F_{2m}(Q)$ the coefficient at k^{-2m} in the decomposition of

$$\sum_{n=3}^{2m}\frac{(-1)^{n+1}}{n}\operatorname{tr}\big(Q(H_0-z)^{-1}\big)^{n}.$$

This coefficient is finite if one of the two following conditions hold

$$i)\ B \in \mathfrak{S}_3, \quad A \in \mathfrak{S}_3 \quad \text{or} \quad ii)\ B \in \mathfrak{S}_4, \quad A = 0. \tag{2.16}$$

Therefore the constants Λ_{2m} appearing in (2.9) are equal to

$$\Lambda_{2m} = 2\mathrm{tr}A + \Gamma_{2m}/2 + I_{2m} + F_{2m}(Q). \tag{2.17}$$

3. Proof of Theorem 1.1

1. Let A and B be matrices of finite rank whose elements $\alpha_n = \beta_n = 0$ for $n < 0$ and let us denote by $\Lambda_n(J)$ the coefficients in the expansion for the Fredholm determinant

$$\log\det\big(I + Q(J_0 - z)^{-1}\big) = -\sum_{n=1}^{\infty} \Lambda_n(J)\, k^{-n}, \qquad z = k + k^{-1}.$$

Suppose H and H_0 are the operators defined in (2.1). We would now like to compare the coefficients $\Lambda_n(J)$ with $\Lambda_n = \Lambda_n(H)$ defined by in (2.9) and appearing in the corresponding decomposition for the operators H and H_0. For a fixed m let us introduce

$$R_{2m}(Q) = \Lambda_{2m}(J) - \Lambda_{2m}(H).$$

According to Lemma 2.12, [17], the coefficients $\Lambda_n(J)$ and $\Lambda_n(H)$ coincide with the coefficients $c_n(H, H_0)$ and $c_n(J, J_0)$ given by

$$c_n(K, K_0) = \frac{2}{n}\mathrm{tr}\left(T_n\Big(\frac{1}{2}K\Big) - T_n\Big(\frac{1}{2}K_0\Big)\right),$$

where T_n are Chebyshev polynomials. This implies that $R_{2m}(Q)$ is a polynomial of at most first m elements of the matrices A and B, i.e. is a polynomial of $\beta_0, \beta_1, \dots, \beta_{m-1}$ and $\alpha_0, \alpha_1, \dots, \alpha_{m-1}$. Indeed, this can be seen from the following "splitting argument". If we subtract the operator $W = 2\mathrm{Re}\ (\cdot, e_0)e_{-1}$ from H, then the result is decomposed into the orthogonal sum of two operators $J_- \oplus J$ defined on $l^2(\mathbb{Z}_-)$ and $l^2(\mathbb{N})$ respectively, where $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N}$. Then $\Lambda_n(J_-) = 0$. Therefore

$$\frac{2}{n}\mathrm{tr}\left(T_{2m}\big(\frac{1}{2}(H-W)\big) - T_{2m}\big(\frac{1}{2}(H_0-W)\big)\right) = \Lambda_{2m}(J_-) + \Lambda_{2m}(J) = \Lambda_{2m}(J),$$

and our statement follows from the fact that the difference

$$\mathrm{tr}\left(T_{2m}\big(\frac{1}{2}(H - W)\big) - T_{2m}\big(\frac{1}{2}(H)\big)\right)$$

is a polynomial of at most m elements of A and B.

Let $J^{(N)}$ be an operator which realization in the standard basis $\{e_n\}_{n=0}^{\infty}$ is given by

$$J^{(N)} = \begin{pmatrix} \beta_{N+1} & 1+\alpha_{N+1} & 0 & \dots \\ 1+\alpha_{N+1} & \beta_{N+2} & 1+\alpha_{N+2} & \dots \\ 0 & 1+\alpha_{N+2} & \beta_{N+3} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{3.1}$$

and let

$$P_{N,2m}(Q) := \Lambda_{2m}(J) - \Lambda_{2m}(J^{(N)}) - 2\sum_{k=0}^{N} \log(1+\alpha_k).$$

The "tails" in the sums $\Lambda_{2m}(J)$ and $\Lambda_{2m}(J^{(N)})$ cancel each other, so that the elements of the matrices B and A do not enter in this difference $P_{N,2m}(Q)$ starting from the index $N+m$.

Thus $P_{N,2m}(Q)$ is a continuous function of at most $N+m$ first elements of the matrices B and A and can be extended to arbitrary matrices B, A. Below $P_{N,2m}(Q)$ is extended for any B and A. Denote

$$\Phi_{2m}(\mu) = \frac{1}{\pi}\int_{-\pi}^{\pi} \log\left|\frac{\sin\theta}{\text{Im } M_\mu}\right| \sin^2 m\theta \, d\theta + \sum_n f(\xi_n^{-2m}), \tag{3.2}$$

where ξ_n are the poles of M_μ in $D = \{z : |z| < 1\}$ and $f(t) = 1/2(t - t^{-1}) - \log(t)$. It is important for us that

$$\Phi_{2m}(\mu) - \Phi_{2m}(J^{(N)}) = P_{N,2m}(Q) \tag{3.3}$$

for the function M_μ meromophic in the neighbourhood of the unit disc. The identity (3.3) is valid if Q is of a finite rank. The arbitrary case follows from Proposition 4.3 and Theorem 4.4 [17].

Notice that $y \mapsto -\log(y)$ is convex. Employing Jensen's inequality we find

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \log\left[\frac{\sin(\theta)}{\text{Im } M_\mu(e^{i\theta})}\right] \sin^2 m\theta \, d\theta = -\frac{2}{\pi}\int_0^{\pi} \log\left[\frac{\text{Im } M_\mu}{\sin\theta}\right] \sin^2(m\theta)\, d\theta$$

$$\geq -\log\left[\frac{2}{\pi}\int_0^{\pi} (\text{Im } M_\mu)\sin(\theta)\, d\theta\right] - \frac{2}{\pi}\int_0^{\pi} \log\left[\frac{\sin^2(m\theta)}{\sin^2(\theta)}\right] \sin^2(m\theta)\, d\theta$$

$$= -\log[\mu_{\text{ac}}(-2,2)] - \frac{2}{\pi}\int_0^{\pi} \log\left[\frac{\sin^2(m\theta)}{\sin^2(\theta)}\right] \sin^2(m\theta)\, d\theta \geq$$

$$-\frac{2}{\pi}\int_0^{\pi} \log\left[\frac{\sin^2(m\theta)}{\sin^2(\theta)}\right] \sin^2(m\theta)\, d\theta =: C(m), \tag{3.4}$$

where we use that $\mu_{\text{ac}}(-2,2) \leq 1$.

Formulae (3.3) and (3.4) imply

$$P_{N,2m}(Q) \leq \Phi_{2m}(\mu) - C(m).$$

The latter inequality was obtained for M_μ meromophic in the neighbourhood of the unit disc. However this inequality can also be extended to arbitrary measures μ satisfying Conditions (1)-(3) of Theorem 1.1. We apply here the same argument as in [17], Section 8, repeating the corresponding proof for the sake of completeness.

Let μ be a probability measure obeying Condition (1) of Theorem 1.1 and let

$$\mu \geq \gamma\mu_0, \tag{3.5}$$

where μ_0 is the "free" Jacobi measure (the measure with $M_{\mu_0}(z) = z$), $\gamma > 0$. Then

$$P_{N,2m}(Q) \leq \Phi_{2m}(\mu) - C(m). \tag{3.6}$$

Indeed, given any J and associated to it M-function $M(z)$, there is a natural approximating family of M-functions meromorphic in a neighborhood of the closure of the unit disc $\bar{D}$. The next result is proved in [17], Lemma 8.3.

Lemma 3.1. *Let M_μ be the M-function of a probability measure μ obeying Condition (1) of Theorem 1.1. Define*

$$M^{(r)}(z) = r^{-1}M_\mu(rz) \tag{3.7}$$

for $0 < r < 1$. Then, there is a family of probability measures $\mu^{(r)}$ such that $M^{(r)} = M_{\mu^{(r)}}$.

It is also proved in [17] that

$$\limsup_{r\uparrow 1} \int -\log|\mathrm{Im}\, M_{\mu^{(r)}}(e^{i\theta})|\, d\theta \leq \int -\log|\mathrm{Im}\, M_\mu(e^{i\theta})|\, d\theta. \tag{3.8}$$

The poles of $M_{\mu^{(r)}}$ are given by

$$\xi_j(\mu^{(r)}) = \frac{\xi_j}{r},$$

where we consider only those j for which $|\xi_j| < r$. Thus $\sum f(\xi_j^{-2m}(\mu^{(r)}))$ is monotonically increasing function of r whose limit is equal to $\sum f(\xi_j^{-2m})$, as $r \uparrow 1$.

Let us substitute $\sin^2 m\theta = 1 - \cos^2 m\theta$ in (3.2). Then by using (3.8) and Fatou's lemma we obtain

$$\Phi_{2m}(\mu) \geq \limsup\, \Phi_{2m}(\mu^{(r)}). \tag{3.9}$$

Moreover, the convergence $M_{\mu^{(r)}}(z) \to M_\mu(z)$ is uniform on compact subsets of upper half of D, which means that the coefficients of Jacobi matrices must converge. Thus for any N

$$P_{N,2m}(Q) = \lim_{r\uparrow 1} P_{N,2m}(Q(r))$$
$$\leq \liminf \Phi_{2m}(\mu^{(r)}) - C(m) \leq \Phi_{2m}(\mu) - C(m).$$

For a fixed $\gamma \in (0,1)$ let $\mu_\gamma = (1-\gamma)\mu + \gamma\mu_0$. Since μ_γ obeys (3.5) and Condition (1) of Theorem 1.1, we find

$$P_{N,2m}(Q_\gamma) \leq \Phi_{2m}(\mu_\gamma) - C(m). \tag{3.10}$$

Let $M_\gamma := M_{\mu_\gamma}$ and note that

$$\text{Im } M_\gamma(e^{i\theta}) = (1-\gamma)\text{Im } M(e^{i\theta}) + \gamma \sin\theta.$$

It implies

$$\log|\text{Im } M_\gamma(e^{i\theta})| = \log(1-\gamma) + \log\left|\text{Im } M(e^{i\theta}) + \frac{\gamma}{1-\gamma}\sin\theta\right|.$$

We see that up to the convergent to zero term $\log(1-\gamma)$, the function $\log|\text{Im } M_\gamma(e^{i\theta})|$ is monotone in γ. By using the monotone convergence theorem we then find

$$\Phi_{2m}(\mu) = \lim_{\gamma\downarrow 0} \Phi_{2m}(\mu_\gamma) \tag{3.11}$$

(the eigenvalue term is independent of γ, since the point masses of μ_γ have the same positions as those of μ).

On the other hand, since $\mu_\gamma \to \mu$ weakly,

$$P_{N,2m}(Q) = \lim_{\gamma\downarrow 0} P_{N,2m}(Q_\gamma). \tag{3.12}$$

Finally we observe that according to Lemmas 2.1, 2.2 there exists an independent of N function $Z_{2m}(Q)$ such that

$$\left|P_{N,2m}(Q) - \frac{1}{2}\sum_{j=0}^{N}(\beta_j +, \dots, \beta_{j+m-1})^2 - 2\sum_{j=0}^{N}(\alpha_j + \dots \alpha_{j+m-1})^2\right|$$
$$\leq Z_{2m}(Q)$$

and Z_{2m} is finite if either $A = 0$ and $B \in \mathfrak{S}_4$, or $A = \mathfrak{S}_3$ and $B \in \mathfrak{S}_3$. The latter inequality together with (3.10)–(3.12) imply that if Conditions (1)-(3) of the theorem are satisfied, then

$$\frac{1}{2}\sum_{j=0}^{\infty}(\beta_j + \dots + \beta_{j+m-1})^2 + 2\sum_{j=0}^{\infty}(\alpha_j + \dots + \alpha_{j+m-1})^2 < \infty.$$

Let Π_n be the orthogonal projector onto the span of the vectors $\{e_0, e_1, \dots, e_n\}$ and let $A(n) = \Pi_n A$, $B(n) = \Pi_n B$ and $Q(n) = SA(n) + A(n)S^* + B(n)$. We have shown the following result:

Corollary 3.1. *Let the conditions (1)-(3) of Theorem 1.1 are fulfilled. Then there exists a constant $C = C(m,Q) > 0$ such that*

$$\Lambda_{2m}(Q(n)) - 2\,\mathrm{tr}\,\log(I + A(n)) \leq C, \qquad \forall n.$$

2. Conversely, suppose that the conditions (1.6) and (1.5) are fulfilled. We would like to establish that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \log\left[\frac{\sin(\theta)}{\mathrm{Im}\, M_\mu(e^{i\theta})}\right] \sin^2 m\theta\, d\theta < \infty. \tag{3.13}$$

Definition. Let ν, μ be finite Borel measures on a compact Hausdorff space X. The entropy $S(\nu \mid \mu)$ of ν relative to μ is defined by

$$S(\nu \mid \mu) = \begin{cases} -\infty & \text{if } \nu \text{ is not } \mu\text{-ac}, \\ -\int \log(\frac{d\nu}{d\mu}) d\nu & \text{if } \nu \text{ is } \mu\text{-ac}. \end{cases} \tag{3.14}$$

The following result is proved in the paper of Simon and Killip, [17], Corollary 5.3.

Lemma 3.2. *$S(\nu \mid \mu)$ is weakly upper semicontinuous in μ, that is, if $\mu_n \xrightarrow{w} \mu$, then*

$$S(\nu \mid \mu) \geq \limsup_{n\to\infty} S(\nu \mid \mu_n).$$

Let us use the fact that the trace formulae are valid at least for finite rank operators A, B. Suppose now that A, B are arbitrary compact selfadjoint operators such that (1.5) and (1.6) hold. It is clear then that

$$\Lambda_{2m} + 2\log(\tau) < \infty. \tag{3.15}$$

Note that the sequences of operators $A(n), B(n)$ converge to A and B in $\mathfrak{S}_3$ or $\mathfrak{S}_4$ (depending on which part of the theorem we prove), so that the sequences of operators $\sum_{j=1}^{m-1} S^j A(n)(S^*)^j$ and $\sum_{j=1}^{m-1} S^j B(n)(S^*)^j$ converge in $\mathfrak{S}_2$. Let

$$J_n = S + S^* + Q(n), \quad \text{and} \quad \mu_n(\delta) = (E_{J_n}(\delta)e_0, e_0),$$

where δ is an arbitrary Borel set. We first notice that

$$\Lambda_{2m}(J_n) + 2\Lambda_0(J_n)$$

converges to

$$\Lambda_{2m}(J) + 2\Lambda_0(J), \quad \text{as} \quad n \to \infty.$$

As always we assume that $\alpha_j > -1$, $j \in \mathbb{N}$. Since $(J_n - z)^{-1}$ converges to $(J - z)^{-1}$ uniformly on compact subsets of the upper half-plane we obtain that μ_n is weakly convergent to μ,

$$\mu_n \xrightarrow{w} \mu, \quad \text{as} \quad n \to \infty.$$

Indeed, the difference between the resolvents is the operator

$$(J_n - z)^{-1} - (J - z)^{-1} = (J_n - z)^{-1}(Q - Q(n))(J - z)^{-1},$$

whose norm can be estimated by $C_0||Q - Q(n)||$, where C_0 is independent of n. Therefore m_{μ_n} converges uniformly to m_μ on compact subsets of the upper half plane.

Applying Lemma 3.2 we obtain that if $d\nu = \sin^2(m\theta)d\theta$ and μ is the spectral measure of J, then

$$S(\nu \mid \mu) > -\infty.$$

This is exactly what is needed for (3.13).

In order to complete the proof of Theorem 1.1 we only have to show that (1.6) and (1.5) imply Condition (3). Obviously for finite rank matrices $A(n)$ and $B(n)$, if N is large enough, then (3.2) takes the form

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \log\Big|\frac{\sin\theta}{\text{Im } M_{\mu_n}}\Big| \sin^2 m\theta \, d\theta + \sum_j f([\xi_j(n)]^{-2m})$$
$$= \Lambda_{2m}(J_n) - 2\sum_{j=0}^{\infty} \log(1 + \alpha_j(n)).$$

Let $p \in \mathbb{N}$. Since $f(t) \geq 0$ for $t > 1$ then from (3.4) we obtain

$$\sum_{j=1}^{p} f([\xi_j(n)]^{-2m}) \leq \Lambda_{2m}(J_n) - 2\sum_{j=0}^{\infty} \log(1 + \alpha_j(n)) - C(m),$$

where $\xi_j(n) \in (-1, 1)$ are the poles of M-function of the Jacobi matrix J_n. Now for a fixed finite p we can pass in this inequality to the limit as $n \to \infty$ and obtain

$$\sum_{j=1}^{p} f(\xi_j^{-2m}) \leq \Lambda_{2m}(J) - 2\sum_{j=0}^{\infty} \log(1 + \alpha_j) - C(m).$$

Since the eigenvalues $E_j^{\pm}$ are the points $\xi_i + 1/\xi_i$, this inequality leads to the Lieb-Thirring bound, i.e. Condition (3) of Theorem 1.1.

The proof of Theorem 1.1 is complete.

In the end of this Section we would like give a converse statement to Corollary 3.1.

Corollary 3.2. *Let the operators A and B be compact in $l^2(\mathbb{N})$ and let there exist a constant $C = C(m, Q) > 0$ such that*

$$\Lambda_{2m}(Q(n)) - 2\text{tr } \log(I + A(n)) \leq C, \qquad \forall n. \tag{3.16}$$

Then the conditions (1)-(3) of Theorem 1.1 are satisfied.

Remark. If Conditions (1)-(3) of Theorem 1.1 are satisfied, then by Rakhmanov's theorem [11] both operators A and B are compact. This means that Corollaries 3.1 and 3.2 are converse to each other and therefore estabish a one to one correspondence between classes of measures satisfying Conditions (1)-(3) and operators A and B with properties (3.16).

4. Proof of Theorem 1.2

Let $\lambda_n = (1+\alpha_n)^2 - 1$ be the eigenvalues of the operator $L = (I+A)^2 - I$ and let $\alpha = \{\alpha_n\}_{n=0}^\infty$, $\beta = \{\beta_n\}_{n=0}^\infty$, $\lambda = \{\lambda_n\}_{n=0}^\infty$. By using Rakhmanov's theorem for Jacobi matrices [11], we find that both $\alpha_n, \beta_n \to 0$, as $n \to \infty$. Therefore without loss of generality we can assume that the norms $\|\alpha\|_{l^\infty}$ and $\|\beta\|_{l^\infty}$ are sufficiently small.

Applying (2.10) let us compute the difference

(4.1)
$$\begin{aligned}
&\Lambda_4 - 2\operatorname{tr}\log(I+A)\\
&= \frac{1}{4}\sum_n \Big((\lambda_n + \lambda_{n-1} + {\beta_n}^2)^2 + 2(\beta_n + \beta_{n-1})^2(\lambda_n + 1) + (\lambda_n + \lambda_{n-1})^2\\
&\qquad -\frac{4}{3}\lambda_n^3 + \lambda_n^4 + O(|\lambda_n|^5)\Big)\\
&= \frac{1}{2}\sum_n \Big((\lambda_n + \lambda_{n-1})^2 + \frac{1}{2}({\beta_n}^4 + \lambda_n^4) + (\lambda_n + \lambda_{n-1}){\beta_n}^2 + \lambda_n\lambda_{n-1}(\lambda_n + \lambda_{n-1})\\
&\qquad -\frac{1}{3}(\lambda_n + \lambda_{n-1})^3 + (\beta_n + \beta_{n-1})^2(\lambda_n + 1) + O(|\lambda_n|^5)\Big)\\
&= \frac{1}{2}\sum_n \Big((\lambda_n + \lambda_{n-1})^2 + (\beta_n + \beta_{n-1})^2(\lambda_n + 1) + \frac{1}{2}({\beta_n}^4 + \lambda_n^4)
\end{aligned}$$

(4.2)
$$\begin{aligned}
&+(\lambda_n + \lambda_{n-1})({\beta_n}^2 + \lambda_n(\lambda_n + \lambda_{n-1}) - \lambda_n^2) + O(|\lambda_n|^5 + |\lambda_n + \lambda_{n-1}|^3)\Big)\\
&\geq \frac{1}{2}\sum_n \Big((\lambda_n+\lambda_{n-1})^2 + (\beta_n+\beta_{n-1})^2(\lambda_n+1) + \frac{1}{2}({\beta_n}^4+\lambda_n^4) - \frac{1}{2\varepsilon}(\lambda_n+\lambda_{n-1})^2\\
&\qquad -\frac{\varepsilon}{2}({\beta_n}^2 - \lambda_n^2 + \lambda_n(\lambda_n + \lambda_{n-1}))^2 + O(|\lambda_n|^5 + |\lambda_n + \lambda_{n-1}|^3)\Big),
\end{aligned}$$

as $\|\lambda\|_{l^\infty} \to 0$. Obviously

$$({\beta_n}^2 - \lambda_n^2 + \lambda_n(\lambda_n + \lambda_{n-1}))^2 \leq ({\beta_n}^2 - \lambda_n^2)^2 + o(|\beta_n|^4 + |\lambda_n|^4 + |\lambda_n + \lambda_{n-1}|^2).$$

We now use that $({\beta_n}^2-\lambda_n^2)^2 \le {\beta_n}^4+\lambda_n^4$. Therefore choosing, for example, $\varepsilon = 3/4$ we obtain

$$\Lambda_4 - 2\mathrm{tr}\log(I+A) \ge \sum_n (\frac{1}{6}(\lambda_n+\lambda_{n-1})^2 + \frac{1}{16}({\beta_n}^4+\lambda_n^4))$$

$$+\sum_n (\frac{1}{2}(\beta_n+\beta_{n-1})^2(\lambda_n+1) + o(|\beta_n|^4+|\lambda_n|^4+|\lambda_n+\lambda_{n-1}|^2)).$$

This implies that if $\|\beta_n\|_{l^\infty}$ and $\|\lambda_n\|_{l^\infty}$ are sufficiently small, then $\Lambda_4 - 2\mathrm{tr}\log(I+A)$ can be estimated from below by a constant times

$$\sum_n \{(\lambda_n+\lambda_{n-1})^2 + (\beta_n+\beta_{n-1})^2 + {\beta_n}^4+\lambda_n^4\}. \tag{4.3}$$

The required upper estimate by a constant times (4.3) follows from (4.2). Finally Corollaries 3.1 and 3.2 imply the proof of Theorem 0.2.

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