# Aperiodic Lorentz gas: recurrence and ergodicity

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**Abstract**

We prove that any generic (i.e., possibly aperiodic) Lorenz gas in two dimensions, with finite horizon and non-degenerate geometrical features, is ergodic if it is recurrent. We also give examples of aperiodic recurrent gases.

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## 1 Introduction

By Lorentz gas (LG) we mean the free motion of a point particle in the plane subject to elastic collisions against a fixed array of dispersing scatterers. Each scatterer $\mathcal{O}_\alpha$ is an open, bounded, connected, simply connected, strictly convex domain of $\mathbb{R}^2$, with smooth boundary, and is labeled by the variable $\alpha \in \mathcal{I}$. The scatterers are assumed to be pairwise disjoint. A LG is called *periodic* when $\cup_{\alpha\in\mathcal{I}} \mathcal{O}_\alpha$ is left invariant by $G$, a discrete group of translations in the plane, whose fundamental domain $\mathbb{R}^2/G$ is compact.

Notice the departure from the usual terminology, whereby 'Lorentz gas' means 'periodic Lorentz gas' (PLG). To avoid possible confusions let us also recall that some literature calls 'Lorentz gas' the dynamical system defined on $\mathbb{R}^2/G$, in the periodic case. However, common practice has nowadays christened such systems *Sinai billiards.*

The LG has a long and honorable history. It was first introduced in its periodic and three-dimensional version by Lorentz in 1905, to describe the motion of an electron in a crystal [Lo]. Later in the century it became extremely popular in statistical mechanics and ergodic theory. The scientific community was seeking a rather realistic model whose stochastic properties could be rigorously proven—in particular Boltzmann's Ergodic Hypothesis (namely, ergodicity). It is because of this connection with statistical mechanics that it was renamed 'Lorentz *gas*'. At this point the periodicity condition had lost its *raison d'être*, but still remained in virtually all the publications (at least as far as I know), on grounds of mathematical convenience.

As a matter of fact, most of the massive literature in this area of research is actually concerned with Sinai billiards, and many results on the PLG were generated as corollaries of theorems on the Sinai billiard. It is definitely outside the scope of this introduction to mention all that is known about these systems, but it is worthwhile to recall that they are so termed after Sinai, who in 1970 proved that they possess the $K$-property. The same result was extended to higher dimension by Sinai and Chernov in 1987 [SC]. (A rather complete reference on billiards is [T]. The hasty reader may find a brief history of *dispersing* billiards in the introduction of [L].)

Turning to LGs, it is not surprising that much less work has been done on them: the relevant invariant measure for these systems is infinite, and ordinary ergodic theory does not apply. Still it makes sense to pose the question of the stochastic properties; only, one must utilize a more refined ergodic theory (as found, e.g., in the beautiful book by Aaronson [A]). For instance, the most sound definition of ergodicity in infinite measure is the *indecomposability* of the system modulo the measure. That is, a dynamical system endowed with an invariant measure $\mu$ is ergodic when the only invariant subsets mod $\mu$ have measure zero or are complements of zero-measure sets (cf. Definition 4.1 later on).

Another question that arises immediately for infinite-measure systems is that of recurrence (cf. Definition 2.1), because the Poincaré Recurrence Theorem does not hold. For the PLG this has baffled mathematicians for decades, and was answered in the affirmative only in the last few years, by Schmidt [Sch] and Conze [Co] independently. Both proofs use the Central Limit Theorem (for a suitable class of observables) obtained by Bunimovich and Sinai in 1981 [BS]. All these results assume *finite horizon*: that is, the free path between collisions must be bounded above.

Recurrence implies ergodicity (in the above sense) by a result of Simányi [Si].

We finish this short review of the PLG by mentioning a few other important properties that are not so central in the present work. Again for finite-horizon gases, the dynamics is diffusive and converges to a Brownian motion upon suitable rescaling [BS] (see also [BSC].) The finite-horizon condition appears to be physically relevant, as infinite-horizon PLGs are believed to have a super-diffusive behavior (the mean square displacement at time $t$ increasing like $t \log t$) [B]. A promising generalization of the Central Limit Theorem is contained in a recent work by Szász and Varjú [SzV].

In this paper we treat the generic LG with finite horizon. Subject to very mild conditions on the curvature of the scatterers and their relative position (cf. (2.2)-(2.3)), we prove that recurrence implies ergodicity (Theorem 4.2) and we give several examples of recurrent aperiodic LGs (Propositions 5.3 and 5.4).

The exposition is thus organized: In Section 2 we lay down the necessary definitions and basic facts about billiard dynamics. In Section 3 we show that our systems are hyperbolic. In Section 4 we give the ergodicity result. In Section 5 we prove recurrence for a class of systems.

Throughout the paper we use the theory of hyperbolic systems with singularities without really reviewing it for the reader. Rather, we try to be as precise as possible as to what part of the theory we are applying, or we need to modify, at any given time. Standard references include [CFS, SC, LW].

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# 2 Preliminaries

We study the billiard dynamics on $\mathbb{R}^2 \setminus \cup_{\alpha\in\mathcal{I}} \mathcal{O}_\alpha$ (notation introduced in Section 1): A material point moves with constant velocity until it hits the boundary of its allowed domain. The point is then instantaneously scattered by the obstacle according to the Fresnel law: the angle of incidence equals the angle of reflection. It is clear that in this process the modulus of the velocity never changes (a consequence of the conservation of energy for this singular Hamiltonian system). So this modulus is conventionally fixed to be 1. The dynamical system thus defined is called the *billiard flow* on $\mathbb{R}^2 \setminus \cup_{\alpha\in\mathcal{I}} \mathcal{O}_\alpha$.

Since for most of the time the dynamics is trivial, we take the point of view of discrete dynamical systems, and consider the Poincaré cross-section corresponding to collisions with the obstacles. More precisely, we restrict our phase space to points of the type $x = (q, v)$, with $q \in \partial\mathcal{O}_\alpha$, for some $\alpha$, and $v \in \mathrm{T}_q\mathbb{R}^2 = \mathbb{R}^2$, such that $|v| = 1$ and $v$ points outwardly with respect to $\mathcal{O}_\alpha$. Any such point is also called a *line element* and the whole set is denoted by $\mathcal{M}$. Given an $x \in \mathcal{M}$ we define $Tx =: x_1 =: (q_1, v_1)$ to be first line element, in the forward (flow-)trajectory of $x$, that belongs to $\mathcal{M}$ (see Fig. 1). $T$ is then the Poincaré map induced by the flow on the cross-section $\mathcal{M}$, and the $T$-orbit of $x$ is the sequence of snapshots of the flow-trajectory of $x$, taken immediately after each collision. (In this work I will try to be as consistent as possible and use the term 'orbit' for an orbit of a map, such as $T$; and 'trajectory' for an orbit of the billiard flow.)

A convenient way to parametrize the phase space is to consider, for each $\alpha$, the set $\mathcal{M}_\alpha := S_{L_\alpha} \times [0, \pi]$, where $S_L$ is the circle of circumference $L$ (i.e., the interval $[0, L]$ after identification of the endpoints). A pair $(r, \varphi) \in \mathcal{M}_\alpha$ represents the unit vector, based on $q \in \partial\mathcal{O}_\alpha$, that forms an angle $\varphi$ with the clockwise tangent vector to $\partial\mathcal{O}_\alpha$ in $q$; and $q$ is determined as the point whose clockwise distance along $\partial\mathcal{O}_\alpha$ (from a certain fixed origin) is $r$—see again Fig. 1. $L_\alpha$ is thus the length of the

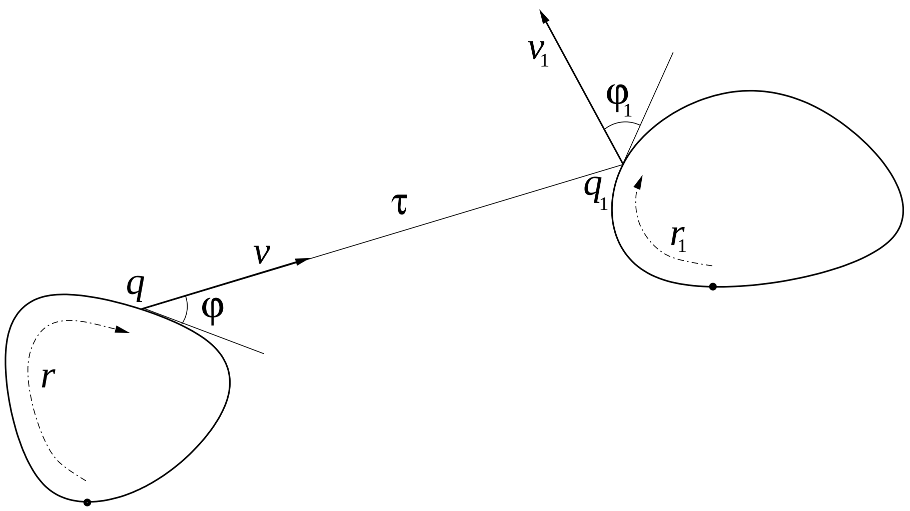


Figure 1: Basic definitions for the billiard map.

closed curve $\partial\mathcal{O}_\alpha$. With a forgivable abuse of notation, then, the phase space is rewritten as $\mathcal{M} = \sqcup_{\alpha\in\mathcal{I}}\mathcal{M}_\alpha$ (the symbol denoting disjoint union).

We endow $\mathcal{M}$ with the measure $\mu$ defined by $d\mu(r,\varphi) = \sin\varphi\, dr d\varphi$. This is the physically relevant measure because it is induced on $\mathcal{M}$ by the Liouville measure for the billiard flow (which, by the way, is the uniform measure both in the position and in the momentum coordinates). Hence $\mu$ is preseved by $T$ [CFS]. Obviously $\mu(\mathcal{M}) = \infty$. From a mathematical point of view, this is perhaps the most significant feature of the Lorentz gas.

It is well known that systems of the billiard type are discontinuous: any line element $x$ whose trajectory is tangent to the next obstacle, say $\mathcal{O}_\beta$, is a point of discontinuity for $T$, because other line elements arbitrarily close to $x$ may hit or miss $\mathcal{O}_\beta$, which causes $Tx$ to end up in completely different regions of the phase space. The discontinuity set is then given by $\mathcal{S} := \mathcal{S}^+ := T^{-1}\partial\mathcal{M}$. Analogously, $\mathcal{S}^- := T\partial\mathcal{M}$ is the discontinuity set of $T^{-1}$. It is not hard to deduce that the differential of $T$ blows up at $\mathcal{S}$ (see below). Therefore $\mathcal{S}$ is usually called the *singularity set*. It is made up of smooth curves (*singularity lines*) with the property that the endpoints of any curve belong to other such curves [S, SC, LW].

For all elements in $\mathcal{M}\setminus\mathcal{S}$, the differential of $T$ is known ([LW, §14], [L, §3]) and, however, easily computed with the help of Fig. 1 and a little patience. Denote by $\tau = \tau(x) = \tau(r,\varphi)$ the *free path* of the line element $x$. Also, if $x \in \mathcal{M}_\alpha$, call $k_\alpha(r)$ the curvature of $\partial\mathcal{O}_\alpha$ in the base-point of $x$: this is a smooth function by hypothesis, and positive by convention. Also set $(r_1,\varphi_1) := x_1 = Tx$; $k = k_\alpha(r)$; $k_1 = k_\beta(r_1)$ (if

$x_1 \in \mathcal{M}_\beta$). Then

$$DT_x = \begin{bmatrix} -\dfrac{\sin\varphi}{\sin\varphi_1} - \dfrac{k\tau}{\sin\varphi_1} & \dfrac{\tau}{\sin\varphi_1} \\ \\ k + k_1\dfrac{\sin\varphi}{\sin\varphi_1} + \dfrac{kk_1\tau}{\sin\varphi_1} & -1 - \dfrac{k_1\tau}{\sin\varphi_1} \end{bmatrix}. \tag{2.1}$$

Before introducing the class of systems that we treat in this work we need a very important, although standard, definition.

**Definition 2.1** *The dynamical system* $(\mathcal{M}, T, \mu)$ *is said to be* recurrent *(in the sense of Poincaré) if, for every measurable set* $A$*, the orbit of* $\mu$*-almost every* $x \in A$ *returns to* $A$ *at least once (and thus infinitely many times, due to the invariance of* $\mu$*).*

We need not require the above property to hold for both the forward and backward orbits because our systems are reversible; i.e., there exists a map $I$ on $\mathcal{M}$ such that $I^2 = id$ and $T^{-1}IT = I$. (This map is of course $I(r, \varphi) = (r, \pi - \varphi)$.)

Since the Poincaré Recurrence Theorem fails here ($\mu(\mathcal{M}) = \infty$), one cannot take Definition 2.1 for granted and indeed there are obvious cases of non-recurrent systems—think of a LG with finitely many scatterers. (In fact these open billiards are usually studied only on the recurrent part of their phase space [LM, St].)

Let us accept the abuse of notation whereby $k(x)$ means $k_\alpha(r)$, for $x = (r, \varphi) \in \mathcal{M}_\alpha$. Then we denote by $\mathcal{X}$ the class of all *recurrent* LGs for which there exist four positive numbers, $k_m, k_M, \tau_m, \tau_M$, such that, $\forall x \in \mathcal{M}$,

$$k_m \le k(x) \le k_M; \tag{2.2}$$

$$\tau_m \le \tau(x) \le \tau_M. \tag{2.3}$$

The first condition is of a physical nature: we want to model a gas, something that does not look much different in different parts of the plane. In fact, this condition and the convexity of the scatterers imply that no $\mathcal{O}_\alpha$ can be too big or too small; more precisely

$$2k_M^{-1} \le diam(\mathcal{O}_\alpha) \le 2k_m^{-1}; \tag{2.4}$$

$$2\pi k_M^{-1} \le L_\alpha \le 2\pi k_m^{-1}. \tag{2.5}$$

Condition (2.3) is primarily dictated by mathematical feasibility, as we will see later. However, the existence of an upper bound for the free path (which goes by the name of *finite-horizon condition*) does make a difference in the dynamics, as recalled in the Introduction.

For a given $x$, define $\mathcal{C}(x) := \{(dr, d\varphi) \in \mathrm{T}_x\mathcal{M} \mid dr\, d\varphi \le 0\}$, that is, the second and fourth quadrant of $\mathrm{T}_x\mathcal{M}$ in the $\{\partial/\partial r, \partial/\partial\varphi\}$ basis. This is usually called the *unstable cone* at $x$ [LW, L]. Take $u = (dr, d\varphi) \in \mathcal{C}(x)$. Any such vector is called *unstable*. The checkered sign configuration of (2.1) implies that $DT_x u =: (dr_1, d\varphi_1) \in \mathcal{C}(Tx)$, and

$$(dr_1)^2 \geq \left(\frac{\sin\varphi}{\sin\varphi_1}\right)^2 \left(1 + \frac{k\tau}{\sin\varphi}\right)^2 dr^2; \tag{2.6}$$

$$(d\varphi_1)^2 \geq \left(1 + \frac{k_1\tau}{\sin\varphi_1}\right)^2 d\varphi^2. \tag{2.7}$$

Therefore, if we define the metric

$$\|u\|_x^2 = \|(dr, d\varphi)\|_{(r,\varphi)}^2 := \sin^2\varphi \, dr^2 + d\varphi^2, \tag{2.8}$$

and set $\lambda := 1 + k_m \tau_m > 1$, we obtain from (2.2)-(2.3):

$$\|DTu\|_{x_1} \geq \lambda \, \|u\|_x. \tag{2.9}$$

We name (2.8) the *increasing norm* for unstable vectors.

# 3 Hyperbolicity

In this section we see how to construct local stable and unstable manifolds (LSUMs) for a.e. point of $\mathcal{M}$, and how to prove that they are absolutely continuous w.r.t. $\mu$. Although the subject is very standard, it is useful for later purposes to state the definition of LSUM.

For $n \in \mathbb{N} \cup \{\infty\}$, let us denote $\mathcal{S}_n^\pm := \bigcup_{i=0}^{n-1} T^{\mp i} \mathcal{S}^\pm$.

**Definition 3.1** *Given a point* $x \in \mathcal{M} \setminus \mathcal{S}_\infty^{+(-)}$, *we define a* local (un)stable manifold $W^{s(u)}$ for $T$ at $x$ *to be a* $C^1$ *curve containing* $x$ *in its (one-dimensional) interior, not intersecting* $\mathcal{S}_\infty^{+(-)}$, *and such that:*

*(a) The tangent spaces to* $W^{s(u)}$ *are included in the pull-backwards (or push-forwards) of all future (past) stable (unstable) cones. (E.g., in the unstable case, this means:* $\forall y \in W^u$, $\mathrm{T}_y W^u \subset \bigcap_{n \leq 0} DT_{T^n y}^{-n} \mathcal{C}(y)$*);*

*(b)* $\forall y \in W^{s(u)}$, $d(T^n y, T^n x) \to 0$, *as* $n \to +\infty(-\infty)$*;*

*(c) If* $W_0^{s(u)}$ *is another such manifold, then so is* $W^{s(u)} \cap W_0^{s(u)}$.

Condition *(c)* is a sort of uniqueness property. For this reason we refer to any LSUM as *the* LSUM at $x$ and denote it by $W^{s(u)}(x)$. One readily sees that $T^{-1}W^u(x)$ is a LUM at $T^{-1}x$ and $TW^s(x)$ is a LSM at $Tx$. This earns the two collections of local stable and unstable manifolds the name of *local invariant foliations*.

Condition *(a)* states that the tangent vectors to, say, $W^u(x)$ are unstable 'at all orders'. This and (2.9) imply that if $y \in W^u(x)$ then, $\forall n \le 0$, $d_{||}(T^n y, T^n x) \le d_{||}(y,x)\lambda^n$, where $d_{||}$ is the distance in the $\| \cdot \|$-norm. We therefore say that $W^u(x)$ is *exponentially unstable* in the increasing metric. We will see later that this property holds for the Riemannian metric too.

We do not recall the other basic notions of hyperbolic theory, such as the definition of absolutely continuous foliation. (The reader who would like to review the fundamentals of Pesin's theory, especially in the context of billiards, can consult [KS, SC] and the references therein.)

Even as concerns proofs, we only show the details of a couple of lemmas that are specific to our system, and simply sketch the remaining customary arguments—mainly to convince the reader that the infiniteness of $\mu$ is no big trouble. It will also be noticed that the recurrence property is *not* needed here.

The first observation one has to make to prove that a certain billiard is hyperbolic is that the singularities are not a big obstruction. The lemma that does the job is the following.

**Lemma 3.2** *For a.e. $x \in \mathcal{M}$, these exists a $C_0 = C_0(x)$ such that, $\forall n \neq 0$,*

$$d_{||}(T^n x, \mathcal{S}^+ \cup \mathcal{S}^- \cup \partial\mathcal{M}) \ge C_0\, |n|^{-4}.$$

This is in turn based on the next lemma, whose formulation needs two extra definitions. Let $\mathcal{S}^\pm_\alpha := \mathcal{S}^\pm \cap \mathcal{M}_\alpha$. Also, for $A \subseteq \mathcal{M}$ and $\varepsilon > 0$, set

$$A_{(\varepsilon)} := \left\{ x \in \mathcal{M} \,\middle|\, d_{||}(x, A) \le \varepsilon \right\}. \tag{3.1}$$

**Lemma 3.3** *There exists a constant $C_1$,* independent of $\alpha \in \mathcal{I}$, *such that, for $\varepsilon > 0$ small enough,*

$$\mu\left( (\mathcal{S}^+_\alpha \cup \mathcal{S}^-_\alpha \cup \partial\mathcal{M}_\alpha)_{(\varepsilon)} \right) \le C_1\, \varepsilon.$$

Proof of Lemma 3.3. Obviously we can treat $(\mathcal{S}^\pm_\alpha)_{(\varepsilon)}$ and $(\mathcal{M}_\alpha)_{(\varepsilon)}$ separately. For the latter case the assertion is immediate since $(\mathcal{M}_\alpha)_{(\varepsilon)} = \{(r,\varphi) \in \mathcal{M}_\alpha \,|\, \varphi \le \varepsilon \text{ or } \varphi \ge \pi - \varepsilon\}$ and $L_\alpha$ is bounded above. (Remember that $\mathcal{M}_\alpha$ is a cylinder based on the circle of length $L_\alpha$.)

As for $\mathcal{S}^+_\alpha$, it is easy to see that it is composed of a finite number of strictly increasing smooth curves $\gamma$, and this number does not depend on $\alpha$. In fact, any scatterer "seen" from $\mathcal{O}_\alpha$ can only generate two tangent elements $(r,\varphi)$ for any given $r$, and, since $\mathcal{O}_\alpha$ is convex, $\varphi$ must grow as $r$ increases (cf. Fig. 1). But the number of accessible scatterers (we will also call them *nearest neighbors*) is bounded above, due to the finite-horizon condition and (2.4). Notice that the length of each $\gamma$ cannot exceed $L_\alpha + \pi$, which is less than some universal constant by (2.5).

Analogous reasoning works for $\mathcal{S}_\alpha^-$, except that the curves are strictly decreasing.

So we only need prove that $\mu(\gamma_{(\varepsilon)}) \le C_2\,\varepsilon$, with $C_2$ independent of $\alpha$. This is essentially contained in [SC, Lemma 2] but since the terminology used there is rather different from that of the present work, we give a simple "analytical" proof here.

First of all, if $\gamma$ keeps away from $\partial\mathcal{M}_\alpha$, the result is obvious as $d_{||}$ is equivalent to the Riemannian distance $d$ there. So it suffices to look at, say, $\gamma \cap \{\varphi \le \delta\}$, for some $\delta > 0$. Since $k_\alpha(r)$ is bounded below, $\gamma$ can only intersect $\{\varphi = 0\}$ transversally. Therefore, without loss of generality, we can focus on the case $\gamma = \{0\} \times [0, \delta]$. Given $x = (r, \varphi) \in \mathcal{M}_\alpha$, the ball in the unstable metric $B_{||}(x, \varepsilon)$ is more or less an ellipse of semiaxes $\varepsilon/\sin\varphi$ and $\varepsilon$; and is certainly contained in the rectangle $[r - \varepsilon/\sin(\varphi-\varepsilon), r + \varepsilon/\sin(\varphi-\varepsilon)] \times [\varphi-\varepsilon, \varphi+\varepsilon]$ (notice that this might exceed $\mathcal{M}_\alpha$).

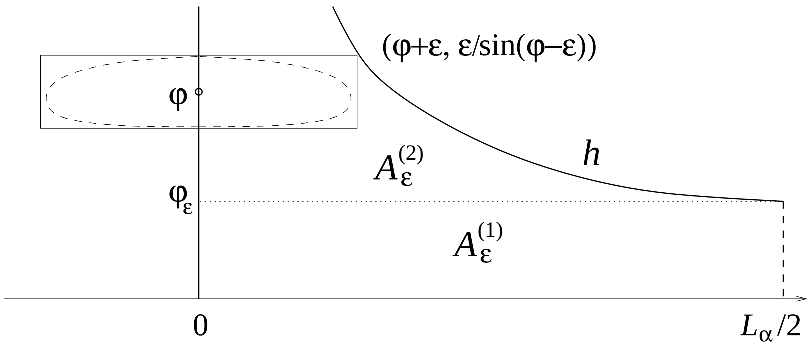


Figure 2: The set $A_\varepsilon$ as in the proof of Lemma 3.3.

Hence, with the help of Fig. 2, we see that the "right" part of $\gamma_{(\varepsilon)}$ is contained in $A_\varepsilon := \{(r, \varphi) \in \mathcal{M}_\alpha \mid 0 \le \varphi \le \delta;\ 0 \le r \le h(\varphi)\}$, with

$$h(\varphi) = \begin{cases} \dfrac{\varepsilon}{\sin(\varphi - 2\varepsilon)} & \varphi \in [2\varepsilon, \delta]; \\ \infty & \varphi \in [0, 2\varepsilon). \end{cases} \tag{3.2}$$

But this is evidently an overestimate, as the right part of $\gamma_{(\varepsilon)}$ only comprises values of $r$ within $[0, L_\alpha/2]$. Set $\varphi_\varepsilon$ so that $h(\varphi_\varepsilon) = L_\alpha/2$. This means that

$$\varphi_\varepsilon = \arcsin\left(\frac{2\varepsilon}{L_\alpha}\right) + 2\varepsilon \simeq 2\varepsilon\left(\frac{1}{L_\alpha} + 1\right). \tag{3.3}$$

(We say that $a \simeq b$ if both quantities depend on $\varepsilon$ and $a/b \to 1$, as $\varepsilon \to 0^+$.) Discarding the exceeding part of $A_\varepsilon$, we split the reminaing part in two pieces:

$$A_\varepsilon^{(1)} = [0, L_\alpha/2] \times [0, \varphi_\varepsilon]; \tag{3.4}$$
$$A_\varepsilon^{(2)} = \{\varphi_\varepsilon \le \varphi \le \delta;\ 0 \le r \le h(\varphi)\}. \tag{3.5}$$

One can estimate $\mu(A_\varepsilon^{(1)})$ from above using the Lebesgue measure; (3.3) and (2.5) ensure that this is less than some $C_3\,\varepsilon$. On the other hand, for $\delta$ small,

$$\begin{aligned} \frac{\mu(A_\varepsilon^{(2)})}{\varepsilon} &= \int_{\varphi_\varepsilon}^{\delta} \frac{\sin\varphi}{\sin(\varphi-2\varepsilon)}\, d\varphi \simeq \int_{\varphi_\varepsilon}^{\delta} \frac{\varphi}{\varphi-2\varepsilon}\, d\varphi = \int_{\varphi_\varepsilon-2\varepsilon}^{\delta-2\varepsilon} \left(1+\frac{2\varepsilon}{y}\right) dy = \\ &= \delta - \varphi_\varepsilon + 2\varepsilon[\log(\delta-2\varepsilon) - \log(\varphi_\varepsilon - 2\varepsilon)] \le 2\delta, \end{aligned} \tag{3.6}$$

as $\varepsilon \to 0^+$, since from (3.3) $\varphi_\varepsilon - 2\varepsilon \simeq 2\varepsilon/L_\alpha$. Q.E.D.

Proof of Lemma 3.2. Without loss of generality we will only consider forward semiorbits. Denote

$$\mathcal{N}_n(\alpha) := \left\{ \beta \in \mathcal{I} \;\middle|\; \exists x \in \mathcal{M}_\alpha,\ k \le n, \text{ such that } T^k x \in \mathcal{M}_\beta \right\}. \tag{3.7}$$

In other words, $\mathcal{N}_n(\alpha)$ is the index set of all $k^{th}$*-nearest neighbors* of $\mathcal{O}_\alpha$, for $k \le n$. By the same line of reasoning as in the previous proof, it is clear that

$$\#\mathcal{N}_n(\alpha) \le C_4\, n^2, \tag{3.8}$$

$C_4$ not depending on $\alpha$. Consider an $x \in \mathcal{M}_\alpha$. If the inequality $d_{||}(T^n x, \mathcal{S}^+ \cup \mathcal{S}^- \cup \partial\mathcal{M}) \le n^{-4}$ is verified only for a finite number of $n$'s, then $C_0(x)$ can be found so that $d_{||}(T^n x, \mathcal{S}^+ \cup \mathcal{S}^- \cup \partial\mathcal{M}) \ge C_0\, n^{-4}$ foa all $n > 0$. This does *not* happen if, and only if, $x \in \mathcal{M}_\alpha$ and

$$T^n x \in \bigcap_{m \ge 1} \bigcup_{n \ge m} \left( \left( \mathcal{S}^+ \cup \mathcal{S}^- \cup \partial\mathcal{M} \right)_{(n^{-4})} \right). \tag{3.9}$$

But the arguments above about the phase-space accessibility of orbits starting in $\mathcal{M}_\alpha$ show that (3.9) is equivalent to

$$x \in \mathcal{M}_\alpha \cap \bigcap_{m \ge 1} \bigcup_{n \ge m} T^{-n} \left( \bigcup_{\beta \in \mathcal{N}_n(\alpha)} \left( \mathcal{S}_\beta^+ \cup \mathcal{S}_\beta^- \cup \partial\mathcal{M}_\beta \right)_{(n^{-4})} \right). \tag{3.10}$$

The measure of the above r.h.s. is zero by Borel-Cantelli, as the measure of the individual terms decreases like $n^{-2}$ (by Lemma 3.3, (3.8), and the invariance of $\mu$ w.r.t. $T$). Taking the union over $\alpha \in \mathcal{I}$ completes the proof. Q.E.D.

The existence and absolute continuity of the local invariant foliations is a local matter. More precisely, for $x \in \mathcal{M}$, it has to do with the behavior of $DT$ on a sequence of balls centered at $T^n x$ and of shrinking radii, as $|n| \to \infty$.

Say we want to consider LUMs. Lemma 3.2 implies that, for some $C_5 > 0$, no element of the sequence $\{B_{||}(T^n x, C_5 \lambda^{-|n|})\}_{n \le 0}$ can intersect the singularities or the boundary of the phase space. So, inside those balls, we have a *bona fide* Pesin's theory. The invariance of the unstable cones ($DT\mathcal{C}(x) \subset \mathcal{C}(Tx)$) and (2.9) provide the streching mechanism for *unstable curves* (curves whose tangent vectors are everywhere unstable). And the fact that $DT^{-n}\mathcal{C}(T^n x)$ collapses to a line (in this case, an easy consequence of (2.9) and the invariance of $\mu$) proves that the process of pushing forward unstable curves "from the past" has a limit, which is precisely the LUM $W^u(x)$. To make things even simpler, we are dealing with a *uniformly hyperbolic* system (w.r.t. $\| \cdot \|$) and standard results [KH, §6] show that $W^u(x)$ is as regular as $T$ is (meaning, away from $\mathcal{S}^+$).

Condensing in the same way the arguments for the absolute continuity is impossible. Suffice it to say that the major task is to control the *distorsion coefficient* of $T$ along the sequence of shrinking balls. The distorsion coefficient is a certain logarithmic derivative of $|DTu|$, for some vector $u$, with $|u| = 1$; and the requirement is that it grows less than exponentially as $n \to -\infty$. General results on finite-horizon billiards show that this quantity can increase at most like a negative power of $\sin \varphi_n$, with $(r_n, \varphi_n) := y_n \in B_{\|}(T^n x, C_5 \lambda^n)$ (we are simplifying a bit here—see [L, §7]). But $\sin \varphi_n \geq C_6 \min\{\varphi_n, \pi - \varphi_n\} = C_6\, d_{\|}(y_n, \partial\mathcal{M})$ and Lemma 3.2 says in particular that this distance does not approach zero faster than $|n|^{-4}$. Which verifies the requirement.

**Remark 3.4** The above also proves that the LSUMs are exponentially (un)stable w.r.t. the Riemannian metric. In fact $| \cdot |_{(r,\varphi)} \leq (\sin\varphi)^{-1} \| \cdot \|_{(r,\varphi)}$ and in Definition 3.1, *(b)* the polynomial growth of $(\sin\varphi)^{-1}$ is tamed by the exponential contraction of $d_{\|}(T^n y, T^n x)$. Hence the convergence rate is any $\lambda' < \lambda$.

None of the above assertions depend on the infiniteness of $\mu$, or the non-compactness of the billiard. To describe it in lay terms, the particle has no way of knowing whether it is traveling in a plane or in a torus, by only looking at small neighborhoods of its orbit.

# 4 Ergodicity

For dynamical systems with an infinite invariant measure, the many available definitions of ergodicity fail to be equivalent. The definition that is usually retained is the following [A]:

**Definition 4.1** *A dynamical system $(\mathcal{M}, T, \mu)$ is ergodic if every $A \subseteq \mathcal{M}$, measurable and invariant mod $\mu$ (i.e., $\mu(T^{-1}A \,\Delta\, A) = 0$), has either zero measure or full measure (i.e., $\mu(\mathcal{M} \setminus A) = 0$).*

Definition 4.1 does not require the system to be recurrent, but it usually makes little sense if there is a dissipative part (thus causing the whole system to be dissipative). We do not have that problem here.

The goal of this section is to prove the next result.

**Theorem 4.2** *Let the Lorentz gas $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{I}}$ belong to $\mathcal{X}$. The associated dynamical system $(\mathcal{M}, T, \mu)$ is ergodic.*

The essential tool to work out Theorem 4.2 is the *local ergodicity theorem*, also known as the *fundamental theorem* for hyperbolic billiards. We state it in a rather technical guise that is convenient for our purposes.

**Theorem 4.3** *Let $A$ be a full-measure subset of $\mathcal{M}$. Then, for any $x_0$ that possesses a semiorbit (i.e., $x_0 \in \mathcal{M} \setminus \mathcal{S}_\infty^+$, or $x_0 \in \mathcal{M} \setminus \mathcal{S}_\infty^-$), there is a neighborhood $U$ of $x_0$ with the following property:*

*A.e. two points $x', x'' \in U$ are connected by a finite alternating sequence of LSUMs, $W^s(x_1), W^u(x_2), ..., W^u(x_m)$, with $x_1 := x'$ and $x_m := x''$. Each LSUM intersects the next transversally in a point of $A$. Also $x_1, ..., x_m \in A$.*

We do not give the complete proof of the local ergodicity theorem. As the name suggests, most of it is strictly local and does not require $\mu$ to be finite [LW, §§8-12]. The technical conditions that are called for are all verified for dispersing billiards; this has been known since [S].

The non-local part in the proof of Theorem 4.3 is in general the most laborious and for this reason has its own name, the *tail bound* [LW, §13]. We state it here in the form of a lemma and prove it in detail.

First we assume that the LSUMs are maximal; i.e., for every $x$ for which a LUM exists, we take $W^u(x)$ to be the union of all the LUMs at $x$. This implies that, for a.e. $x$, $\partial W^u(x) \subset \partial\mathcal{M} \cup \mathcal{S}_\infty^-$. (It might also happen that $\partial W^u(x)$ intersects the closure of $\mathcal{S}_\infty^-$—which is typically the whole $\mathcal{M}$—but standard reasonings show that this cannot occur for more than a null-measure set of $x$'s. See [L, Rk. 8.9].) The analogous statement holds of course for $W^s(x)$.

If $y \in \partial W^u(x) \cap T^m\mathcal{S}^-$, for some $m \geq 0$, we say that $W^u(x)$ is *cut* by $T^m\mathcal{S}^-$ at $y$. The radius of $W^u(x)$ is defined as the minimum distance, *along $W^u(x)$ and in the increasing metric*, between $x$ and the two endpoints of $W^u(x)$.

**Lemma 4.4** *For every $x_0 \in \mathcal{M}$, there is a neighborhood $U_0$ of $x_0$, and a $\delta_0 > 0$ such that, $\forall \eta > 0$, $\exists M$ that verifies*

$$\mu\left(\left\{ x \in U_0 \,\middle|\, W^u(x) \text{ has radius} < \delta \text{ because it is cut by} \bigcup_{m=M+1}^{\infty} T^m\mathcal{S}^- \right\}\right) \leq \eta\delta,$$

*for every $\delta \leq \delta_0$.*

Proof of Lemma 4.4. Let $Y = Y(U_0, \delta, M)$ be the set in the statement of the lemma and, $\forall x \in Y$, set $m(x)$ to be the smallest integer such $T^m\mathcal{S}^-$ cuts $W^u(x)$ within a distance $\delta$ of $x$. If we define $Y_m := \{x \in Y \mid m(x) = m\}$, then

$$\mu(Y) = \sum_{m>M} \mu(Y_m) = \sum_{m>M} \mu(T^{-m} Y_m). \tag{4.1}$$

The first equality holds because the $Y_m$'s are the level sets of the function $m$; the second equality is due to the invariance of $\mu$.

For $x \in Y_m$ denote by $y = y(x) \in \partial W^u(x) \cap T^m \mathcal{S}^-$ the point where the cut occurs. By the exponential contraction of $W^u(x)$, the portion of $T^{-m}W^u(x)$ between $T^{-m}x$ and $T^{-m}y \in \mathcal{S}^-$ has length less than $\delta\lambda^{-m}$ (in the increasing metric). So $d_{\|}(T^{-m}x, \mathcal{S}^-) < \delta\lambda^{-m}$.

Let $\alpha \in \mathcal{I}$ so that $x_0 \in \mathcal{M}_\alpha$, and select any $U_0 \subseteq \mathcal{M}_\alpha$. Following the line of reasoning of Lemma 3.2, $T^{-m}Y_m \subset \cup_{\beta\in\mathcal{N}_m(\alpha)} \mathcal{M}_\beta$. This and the above imply that

$$T^{-m}x \in \bigcup_{\beta\in\mathcal{N}_m(\alpha)} (\mathcal{S}_\beta^-)_{(\delta\lambda^{-m})}, \tag{4.2}$$

whose measure, by Lemma 3.3 and (3.8), is $\le C_1 C_4\, m^2\, \delta\lambda^{-m}$. Therefore, from (4.1),

$$\frac{\mu(Y)}{\delta} \le C_1 C_4 \sum_{m>M} m^2\, \lambda^{-m} \to 0, \tag{4.3}$$

as $M \to +\infty$. Q.E.D.

**Lemma 4.5** *Fixed an $\alpha \in \mathcal{I}$ and denoted by $T_\alpha$ the first return map induced by $T$ on $\mathcal{M}_\alpha$ (this is well defined a.e. by recurrence), $(\mathcal{M}_\alpha, T_\alpha, \mu)$ is ergodic.*

PROOF. We proceed in three steps:

STEP 1: *The LSUMs relative to $T$ are also LSUMs relative to $T_\alpha$.* Let us see to it. In Definition 3.1, part *(c)* has nothing to do with $T$ or $T_\alpha$. Part *(a)* is obvious since $T_\alpha^k y = T^{n_k} y$, for some $n_k < 0$. For part *(b)* it is sufficient to prove that if $y \in W^{s(u)}(x)$ then the (future or past) return times to $\mathcal{M}_a$ are the same for $y$ and $x$. But this follows from *(b)* itself as, by definition, $d_{\|}(\mathcal{M}_\alpha, \mathcal{M}_\beta) = \infty$ when $\alpha \neq \beta$; therefore the sequence of scatterers hit is the same for for $y$ and $x$.

STEP 2: *Any $x_0$ that possesses a semiorbit has a neighborhood $U$ contained in one ergodic component.* This is Hopf's idea. Take a measurable $f$, continuous (and thus uniformly continuous) on $\mathcal{M}_\alpha$, and consider its forward and backward ergodic averages w.r.t. $T_\alpha$:

$$f^\pm(x) := \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} (f \circ T_\alpha^{\pm k})(x). \tag{4.4}$$

By a standard application of Birkhoff's Theorem, the set

$$A := \left\{ x \in \mathcal{M} \;\middle|\; f^+(x), f^-(x) \text{ exist and coincide} \right\} \tag{4.5}$$

has full measure. Let us use it in Theorem 4.3. In the resulting neighborhood $U$, a.e. two points $x', x''$ are connected by a curvilinear polyline made up of LSUMs. Call $y$ the intersection point between $W^s(x_1)$ and $W^u(x_2)$. From Step 1 and the uniform continuity of $f$, $f^+(y) = f^+(x_1)$ and $f^-(y) = f^-(x_2)$. But $f^+(y) = f^-(y)$ because $y \in A$. Proceeding this way we conclude that $f^+(x') = f^+(x'')$; that is, $f^+$

is a.e. constant in $U$. This continues to hold for every $f \in L^1(\mathcal{M})$, via Birkhoff's Theorem and customary density arguments.

STEP 3: *There is only one ergodic component for* $(\mathcal{M}_\alpha, T_\alpha, \mu)$. The set of points that fail to have a semiorbit is $\mathcal{S}_\infty^+ \cap \mathcal{S}_\infty^-$. But, as explained in Section 3, $\mathcal{S}_\infty^+$ and $\mathcal{S}_\infty^-$ are countable unions of smooth curves, respectively strictly increasing and strictly decreasing. Therefore there can be at most one point of intersection for each pair of increasing–decreasing curves; that is, at most countably many points. Hence $\mathcal{M}_\alpha \setminus (\mathcal{S}_\infty^+ \cap \mathcal{S}_\infty^-)$ is path-connected. Q.E.D.

Now it is simple to prove the main result of Section 4.

PROOF OF THEOREM 4.2. We claim that the ergodic decomposition of $T$ is coarser than the partition $\{\mathcal{M}_\alpha\}_{\alpha \in \mathcal{I}}$. In fact the existence of $A, B$, invariant subsets, such that $\mu(A \cap \mathcal{M}_\alpha), \mu(B \cap \mathcal{M}_\alpha) > 0$, for some $\alpha \in \mathcal{I}$, contradicts Lemma 4.5.

Therefore, if there were more than one ergodic component, there would be two nearest neighbors $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ such that $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are contained in two different invariant sets. But this is absurd since one can always find $A \subset \mathcal{M}_\alpha$, $\mu(A) > 0$, such that $T(A) \subset \mathcal{M}_\beta$. Q.E.D.

# 5 Recurrence

We have recalled in the introduction that a finite-horizon periodic Lorentz gas is recurrent (as in Definition 2.1) [Co, Sch] and thus belongs to $\mathcal{X}$ (conditions (2.2)-(2.3) being trivially satisfied). Here we present other examples from the class $\mathcal{X}$.

**Definition 5.1** *The LG* $\{\mathcal{O}_\alpha\}_{\alpha \in \mathcal{I}}$ *is called a* finite modification *of the finite-horizon PLG* $\{\mathcal{O}_\alpha\}_{\alpha \in \mathcal{I}_P}$ *if* $\mathcal{I} = (\mathcal{I}_P \setminus \mathcal{I}_1) \cup \mathcal{I}_2$, *where:*

*(a)* $\mathcal{I}_1$ *is a finite subset of* $\mathcal{I}_P$*;*

*(b)* $\mathcal{I}_2$ *is the index set of a finite LG such that* $d(\mathcal{O}_\alpha, \mathcal{O}_\beta) > 0$ *for any* $\alpha \in \mathcal{I}_2$, $\beta \in \mathcal{I}_P \setminus \mathcal{I}_1$.

**Remark 5.2** Calling the above a 'perturbation' does not seem appropriate, as the modification can be arbitrarily big, provided it is finite.

An example of a finite modification of a PLG is illustrated in Fig. 3. Since it is clear that Definition 5.1 implies (2.2)-(2.3), we focus on recurrence.

**Proposition 5.3** *A finite modification of a PLG is recurrent.*

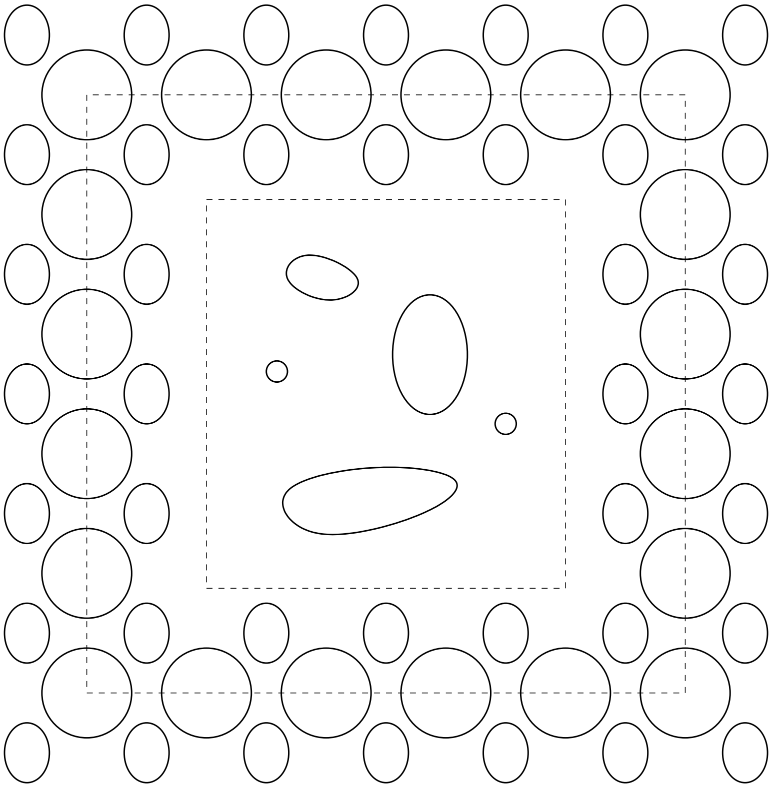

Figure 3: A finite modification of a periodic Lorentz gas. The inner dashed square encloses the scatterers associated to $\mathcal{I}_2$; the outer square corresponds to $\mathcal{J}$.

PROOF. Let us introduce $\mathcal{J} := \{\beta \in \mathcal{I} \mid \beta \in \mathcal{N}_1(\alpha), \text{ for some } \alpha \in \mathcal{I}_2\}$. This is $\mathcal{I}_2$ plus the indices of the nearest neighbors to the scatterers labeled by $\mathcal{I}_2$ (cf. Fig. 3). Also denote $\mathcal{M}_\mathcal{J} := \sqcup_{\alpha\in\mathcal{J}} \mathcal{M}_\alpha$. We proceed in three steps:

STEP 1: *A.e. $x \in \mathcal{M}_\mathcal{J}$ returns to $\mathcal{M}_\mathcal{J}$.* First of all, for *every* $x \in \mathcal{M}_{\mathcal{I}_2}$ (notation understood), $Tx \in \mathcal{M}_\mathcal{J}$. Then consider the set $\big\{x \in \mathcal{M}_{\mathcal{J}\setminus\mathcal{I}_2} \mid T^n x \notin \mathcal{M}_\mathcal{J}, \forall n > 0\big\}$. The forward orbit of every point there lies the "periodic part" of the plane. One uses the recurrence of a PLG to conclude that this set has measure zero.

STEP 2: *Denoted by $T_\mathcal{J}$ the return map to $\mathcal{M}_\mathcal{J}$ (well defined a.e. by Step 1), $T_\mathcal{J}$ is recurrent.* This is the Poincaré Theorem applied to $(\mathcal{M}_\mathcal{J}, T_\mathcal{J}, \mu)$.

STEP 3: *$T$ is recurrent.* It suffices to prove that $\mathcal{M}_\mathcal{J}$ is a global cross-section (mod $\mu$) for $T : \mathcal{M} \longrightarrow \mathcal{M}$. But a positive-measure subset of $\mathcal{M}$ whose orbit never intersects $\mathcal{M}_\mathcal{J}$ would contradict the ergodicity of a PLG. Q.E.D.

If the systems above seem too close to those already known, we provide further examples of LGs whose aperiodicity is not limited to a compact region of $\mathbb{R}^2$.

**Proposition 5.4** *There are LGs in the class $\mathcal{X}$ that are not finite modifications of PLGs.*

PROOF. We present an algorithm to construct such gases explicitly. Start with a PLG $\mathcal{L}_0 := \{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{I}_0}$ and denote by $T_0$ the corresponding billiard map. Furthermore, fix an index $\alpha \in \mathcal{I}_0$ and a point $O \in \mathcal{O}_\alpha$ which will be our origin from now on.

For $x \in \mathcal{M}_\alpha$, denote by $n_1 = n_1(x, T_0) \in \mathbb{N} \cup \{+\infty\}$ the first return time of $x$ to $\mathcal{M}_\alpha$, w.r.t. $T_0$. Indicated by $q(x)$ the point in $\mathbb{R}^2$ corresponding to $x$, we define

$$A_0(R) := \left\{ x \in \mathcal{M}_\alpha \;\middle|\; \sup_{0<k<n_1} dist(q(T_0^k x), O) \le R \right\}, \tag{5.1}$$

where $dist$ is the distance in $\mathbb{R}^2$. Since $T_0$ is recurrent, $A_0(R) \nearrow \mathcal{M}_a$ mod $\mu$, as $R \to +\infty$. Hence there is an $R_0$ such that $\mu(\mathcal{M}_\alpha \setminus A_0(R_0)) \le \varepsilon_0$. Here $\{\varepsilon_k\}$ is a sequence of positive numbers such that $\varepsilon_k \searrow 0$, for $k \to +\infty$.

Now modify $\mathcal{L}_0$ in the annulus of radii $R_0$ and $R_0+\rho$ ($\rho$ is a sufficiently large fixed quantity). This means that one can modify the shape, the position and the number of scatterers within that annulus. Call the resulting LG $\mathcal{L}_1 := \{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{I}_1}$, and the resulting map $T_1$. This is a finite modification of a PLG. Furthermore, denoted by $T_{\alpha,k}$ the first return map to $\mathcal{M}_a$ relative to $T_k$, it is obvious that $(T_{\alpha,1})|_{A_0(R_0)} = (T_{\alpha,0})|_{A_0(R_0)}$.

It is clear that this process can be repeated recursively. Starting with $\mathcal{L}_k$, a finite modification of a PLG, one defines $A_k(R)$ as in (5.1), with $T_k$ in the place of $T_0$. By Proposition 5.3, $A_k(R) \nearrow \mathcal{M}_a$ mod $\mu$, as $R \to +\infty$. One then chooses $R_k > R_{k-1} + \rho$ such that

$$\mu(\mathcal{M}_\alpha \setminus A_k(R_k)) \le \varepsilon_k \tag{5.2}$$

and makes a modification in the annulus of radii $R_k$ and $R_k + \rho$, to obtain $\mathcal{L}_{k+1}$ and $T_{k+1}$. We emphasize that this modification can be made so that the curvature of the scatterers and the free path among them stay bounded above and below, to comply with (2.2)-(2.3). Evidently,

$$A_k(R_k) = A_{k+1}(R_k) \subseteq A_{k+1}(R_{k+1}); \tag{5.3}$$

$$(T_{\alpha,k+1})|_{A_k(R_k)} = (T_{\alpha,k})|_{A_k(R_k)}\,. \tag{5.4}$$

Since we never change the LG twice in the same region of the plane, this modification process has a limit, which we call $\bar{\mathcal{L}}$, with the corresponding maps, $\bar{T}$ for the LG and $\bar{T}_\alpha$ for the returns to $\mathcal{M}_a$. In other words, from (5.2)-(5.4), $\bar{T}_\alpha : \mathcal{M}_\alpha \longrightarrow \mathcal{M}_\alpha$ is well defined a.e. via $\left(\bar{T}_\alpha\right)\big|_{A_k(R_k)} = (T_{\alpha,k})|_{A_k(R_k)}$.

By construction, *all* points of $\cup_k A_k(R_k)$ come back to $\mathcal{M}_a$. So, as far as $\bar{T}$ is concerned, $\mathcal{M}_a$ is contained in $B$, the *conservative* part of $\bar{\mathcal{M}}$ (i.e., the complement of its dissipative part [A]). $B$ is obviously $\bar{T}$-invariant.

We claim that the LSUMs of $\bar{T}$ that intersect $B$ are wholly contained in $B$, mod $\mu$. (The LSUMs exist a.e. and are absolutely continuous by Section 3—no recurrence was used there.) More precisely, this means that there is a set $B_0$, $\mu(B\Delta B_0) = 0$, such that, called $B_1$ the union of all the LSUMs based in $B_0$, we have $\mu(B\Delta B_1) = 0$.

In fact, take $x \in B \cap \mathcal{M}_\beta$, $\beta \in \bar{\mathcal{I}}$; passing maybe to a full-measure subset $B_0$, we can assume that $x$ returns to $\mathcal{M}_\beta$ infinitely many times, both in the past and in the future. Then any $y \in W^s(x)$, say, returns to $\mathcal{M}_\beta$ infinitely many times in the future (Step 1 of Lemma 4.5). Therefore, save for a null-measure set of exceptions, $y$ cannot be part of a wandering set and thus belongs to $B$.

At this point, Theorem 4.3 proves that the decomposition $\{B, \bar{\mathcal{M}} \setminus B\}$ is coarser than $\{\mathcal{M}_\alpha\}_{\alpha\in\bar{\mathcal{I}}}$. That $B = \bar{\mathcal{M}} \bmod \mu$ follows from the invariance of $B$ as in the proof of Theorem 4.2. Q.E.D.

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