HAUSDORFF DIMENSION OF SPECTRUM OF ONE-DIMENSIONAL SCHRÖDINGER OPERATOR WITH STURMIAN POTENTIALS

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ABSTRACT. Let $\beta \in (0,1)$ be an irrational, and $[a_1, a_2, \cdots]$ the continued fraction expansion of β. Let H_β be the one-dimensional Schrödinger operator with Sturmian potentials. We prove that if the potential strength $V > 20$, then the Hausdorff dimension of the spectrum $\sigma(H_\beta)$ is strictly great than zero for any irrational β, and is strictly less than 1 if and only if $\liminf_{k\to\infty}(a_1 a_2 \cdots a_k)^{1/k} < \infty$.

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1. Introduction

Since the discovery of quasi-crystal by Schechtman et al. ([13]), the one dimensional discrete Schrödinger operators with Sturmian potentials have largely been studied, see [2, 14, 6, 12] and references therein. Given $0 < \beta < 1$, let $s_n(\beta) := [(n+1)\beta] - [n\beta] (n \in \mathbb{Z})$ be the Sturmian sequence associated with β, and let H_β be the one dimensional discrete Schrödinger operator on l^2,

$$(H_\beta \psi)_n = \psi_{n-1} + \psi_{n+1} + v_n \psi_n, \ \forall n \in \mathbb{Z}, \ \psi = \{\psi_n\}_{n\in\mathbb{Z}} \in l^2, \tag{1.1}$$

where $v_n = V s_n$ is the potential at site n and $V > 0$ is the potential strength. Let $\sigma(H_\beta)$ denote the spectrum of the operator H_β. It is well known that if β is rational, then $\sigma(H_\beta)$ is a set of union of finite intervals, thus we are interested in the case β being irrational.

It is proved by Bellissard, Iochum, Scoppola and Testart, [2](1989) that if β is irrational then $\sigma(H_\beta)$ is a Cantor set of Lebesgue measure zero. (On the other direction, in stead of Sturmian potential, some authors considered the primitive substitutive potential, it is proved that in this case, the spectrum is also a Cantor set of Lebesgue measure zero. Moreover, Axel and Peyriere gave a estimation of the Hausdorff dimension for the Thue-Mores potential. For more details, we refer to [1],[11].)

Since then, whether the Hausdorff dimension of $\sigma(H_\beta)$ is strictly less than 1 and strictly greater than 0 have absorbed a lots of attentions. Raymond [12](1997) studied this problem under the restriction $V > 4$. Connected with the continued fraction expansion of β, he exhibited an interesting recurrent structure of the spectrums. For $\beta = \frac{\sqrt{5}-1}{2}$, i.e., the golden mean, he give an upper bound of the Hausdorff dimension of the corresponding

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spectrum, which is strictly less than 1. He also gave a conjecture on the lower bound of Hausdorff dimension.

Damanik, Killip and Lenz [3](2000) prove that if β has bounded density (this means if $[a_1, a_2, \cdots]$ is the continued fraction expansion of β, then $\limsup_{k\to\infty} \frac{1}{k}\sum_{i=1}^k a_i < \infty$), then the Hausdorff dimension of the spectral measure of H_β is strictly greater than 0. Since the spectral measure is supported by the spectrum $\sigma(H_\beta)$, the Hausdorff dimension of the spectrum has also strictly positive lower bound. In their estimating, they applied the theory developed by del Rio, Jitomirskaya, Last and Simon, see [4],[8],[9](1995-1999). Note that the set of irrational numbers with bounded density has Lebesgue measure zero, so their result is only valid for a set of β's of Lebesgue measure zero.

To estimate the Hausdorff dimension of the spectrum of H_β, one of the key steps is to estimate the length of spectral generating bands. Raymond [12] has treated the case of β being golden mean with $V > 4$, and he pointed that it is technically difficult to treat the general cases.

In this paper, by studying more precisely the recurrent structure of the spectrum $\sigma(H_\beta)$ observed in [12], and generalizing some techniques analogous to the studies of Moran structure in [5, 7], we will estimate the Hausdorff dimension of the spectrum for general β, and we get the following main result of this paper.

Theorem 1. *Let $[a_1, a_2, \cdots]$ be the continued fraction expansion of irrational $\beta \in (0,1)$ and*

$$M(\beta) := \liminf_{k\to\infty}(a_1 a_2 \cdots a_k)^{1/k}.$$

Let $V > 20$ and $t_1 = \frac{3}{V-8}$, $t_2 = \frac{1}{4(V+8)}$, then

(1) *If $M(\beta) < \infty$, then*

$$\max\{\frac{\ln 2}{10\ln 2 - 3\ln t_2}, \frac{\ln M - \ln 3}{\ln M - \ln(t_2/3)}\} \le \dim_H(\sigma(H_\beta)) \le \frac{2\ln M + \ln 3}{2\ln M - \ln t_1};$$

(2) *If $M(\beta) = \infty$, then*

$$\dim_H(\sigma(H_\beta)) = 1.$$

From above theorem, we get immediately

Corollary 2. *If $V > 20$, then for any irrational $\beta \in (0,1)$,*

(i) $\dim_H(\sigma(H_\beta)) > 0$;

(ii) *Let $[a_1, a_2, \cdots]$ be the continued fraction expansion of β, then $\dim_H(\sigma(H_\beta)) < 1$ if and only if*

$$\liminf_{k\to\infty}(a_1 a_2 \cdots a_k)^{1/k} < \infty.$$

Remark 1. *Let*

$$\Gamma := \{\beta = [a_1, a_2, \cdots] \in (0,1);\ \liminf_{k\to\infty}(a_1 a_2 \cdots a_k)^{1/k} < \infty, \}$$

then by the metrical theory of continued fractions, the set Γ has Lebesgue measure one ([10]). Consequently, if $V > 20$ then $\dim_H(\sigma(H_\beta)) > 0$ for all $\beta \in (0,1)$, and $1 > \dim_H(\sigma(H_\beta)) > 0$ for almost all $\beta \in (0,1)$.

Remark 2. *Here is an example such that the Hausdorff dimension of the spectrum is* 1*: Take* $\beta = 1/(e-1)$*, it is known that* $\beta = [1,1,2,1,1,4,\cdots,1,1,2k,\cdots]$*, thus* $\dim_H \sigma(H_\beta) = 1$ *for* $V > 20$ *by Theorem* 1.(2).

Remark 3. *Let* $M(\beta) < \infty$*, then for any* $\epsilon > 0$*, there exist* V *large enough, such that* $\dim_H \sigma(H_\beta) < \epsilon$.

2. Spectral structure

In this section, we give some facts on the structure of the spectrum of H_β.

Let $\beta = [a_1, a_2, \cdots, a_i, \cdots] \in (0,1)$ be an irrational, let $p_k/q_k (k > 0)$ be the k-th asymptotic fractions of β defined by:

$$\begin{array}{lll} p_{-1} = 1, & p_0 = 0, & p_{k+1} = a_{k+1}p_k + p_{k-1},\ k \geq 0, \\ q_{-1} = 0, & q_0 = 1, & q_{k+1} = a_{k+1}q_k + q_{k-1},\ k \geq 0. \end{array}$$

Let $k \geq 1$ and $x \in \mathbb{R}$, we define the transfer matrix $M_k(x)$ over q_k sites by

$$\mathbf{M}_k(x) := \begin{bmatrix} x - v_{q_k} & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x - v_{q_k-1} & -1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} x - v_2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x - v_1 & -1 \\ 1 & 0 \end{bmatrix},$$

by convention, take

$$\mathbf{M}_{-1}(x) = \begin{bmatrix} 1 & -V \\ 0 & 1 \end{bmatrix}, \quad \mathbf{M}_0(x) = \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix}.$$

Let $t_{(k,p)}(x) = \operatorname{tr} \mathbf{M}_{k-1}(x)\mathbf{M}_k^p(x)$ and $\sigma_{(k,p)} = \{x \in \mathbb{R} : |t_{(k,p)}(x)| \leq 2\}$, where trM stands for the trace of the matrix M. With these notations, we collect some known facts that will be used later, for more details, we refer to [2, 12, 14, 15].

(A) Renormalization relation: for any $k \geq 0$,

$$\mathbf{M}_{k+1}(x) = \mathbf{M}_{k-1}(x)(\mathbf{M}_k(x))^{a_{k+1}}, \quad \text{so, } t_{(k+2,0)}(x) = t_{(k,a_{k+1})}(x). \tag{2.1}$$

(B) Trace relation: for any $k \geq 0, p \geq -1$,

$$t^2_{(k+1,0)}(x) + t^2_{(k,p)}(x) + t^2_{(k,p+1)}(x) - t_{(k+1,0)}(x)t_{(k,p)}(x)t_{(k,p+1)}(x) = V^2 + 4. \tag{2.2}$$

(C) Let $\deg t_{(k,p)}(x)$ denote the degree of the polynomial $t_{(k,p)}(x)$, then for $V > 4$, $\sigma_{(k,p)}$ is composed of $\deg t_{(k,p)}(x)$ pairwise disjoint closed intervals which we call the *bands* of $\sigma_{(k,p)}$(we call them also the constructive intervals of $\sigma_{(k,p)}$), we note often $B_{(k,p)}$ these bands if no confusion happens. Moreover $t_{(k,p)}(x)$ is monotone on each band $B_{(k,p)}$.

Proposition A1. ([2, 12]) *With the notations above, we have*

(1) $\sigma_{(k+2,0)} \cup \sigma_{(k+1,0)} \subset \sigma_{(k+1,0)} \cup \sigma_{(k,0)}$;

(2) $\sigma(H_\beta) = \bigcap_{k \geq 0} (\sigma_{(k+1,0)} \cup \sigma_{(k,0)})$;

(3) *For any* $k \in \mathbb{N}$ *and* $p \geq -1$, $\sigma_{(k,p+1)} \subset \sigma_{(k+1,0)} \cup \sigma_{(k,p)}$;

(4) *For any* $k \in \mathbb{N}$, $p \geq 0$ *and* $V > 4$, $\sigma_{(k+1,0)} \cap \sigma_{(k,p)} \cap \sigma_{(k,p-1)} = \emptyset$.

By Proposition A1.(3, 4) and the fact that $\sigma_{(k,p)}$ is a finite union of bands, we have that

Corollary 3. *For any* $k \in \mathbb{N}$ *and* $p \geq -1$*,any band of* $\sigma_{(k,p+1)}$ *is contained entirely either in a band of* $\sigma_{(k+1,0)}$ *or in a band of* $\sigma_{(k,p)}$.

Definition 1. ([12]) *Let $k \in \mathbb{N}$, we define three types of bands as follows:*
(k, I)*-type band: a band of* $\sigma_{(k,1)}$ *contained in a band of* $\sigma_{(k,0)}$*;*
(k, II)*-type band: a band of* $\sigma_{(k+1,0)}$ *contained in a band of* $\sigma_{(k,-1)}$*;*
(k, III)*-type band: a band of* $\sigma_{(k+1,0)}$ *contained in a band of* $\sigma_{(k,0)}$*.*

Due to Proposition A1.(3) and (4), the three kinds of types of bands are well defined([12]), we call these bands *spectral generating bands of order k* (the type I band is called the type I gap in [12]). Denoted by $\mathscr{G}_k$ the set of all spectral generating bands of order k.

Note that for $k = 0$, there is only one $(0, \mathrm{I})$-type band $\sigma_{(0,1)} = [V - 2, V + 2]$ and only one $(0, \mathrm{III})$ type band $\sigma_{(1,0)} = [-2, 2]$. Notice that all bands of $\sigma_{(k+1,0)}$ is either (k, II) or (k, III)-type (see Lemma 4), but there exist some bands of $\sigma_{(k,1)}$ which are not generating bands.

The following lemma shows us some recurrent relation between spectral generating bands of order k and order $k + 1$.

Lemma A2. ([12]) *Let $k \in \mathbb{N}$,*
(1) *A* (k, I)*-type band contains a unique band of* $\sigma_{(k+2,0)}$ *which is a* $(k+1, \mathrm{II})$*-type band;*
(2) *A* (k, II)*-type band contains* $a_{k+1} + 1$ *bands of* $\sigma_{(k+1,1)}$ *which are of* $(k + 1, \mathrm{I})$*-type bands; and contains* a_{k+1} *bands of* $\sigma_{(k+2,0)}$ *which are of* $(k + 1, \mathrm{III})$*-type bands;*
(3) *A* (k, III) *type band contains* a_{k+1} *bands of* $\sigma_{(k+1,1)}$ *which are of* $(k+1, \mathrm{I})$*-type bands, and contains* $a_{k+1} - 1$ *bands of* $\sigma_{(k+2,0)}$ *which are of* $(k + 1, \mathrm{III})$*-type bands.*

Lemma 2 concludes any spectral generating bands of order k contains certainly some spectral generating bands of order $k+1$, but it doesn't tell us if it is contained in a spectral band of order $k - 1$, the following lemma determines completely the recurrent relation of successive spectral generating bands and establishes the relation with the spectrum.

Lemma 4. (a) *Given $k \geq 0$, let $G_k = \bigcup_{B \in \mathscr{G}_k} B$ be the union of all the spectral generating bands of order k, then $\sigma(H_\beta) = \bigcap_{k \geq 0} G_k$.*
(b) *Let $k > 0$, then any spectral generating band of order $k+1$ is contained in a spectral generating band of order k.*

Proof. (a). By the definition of spectral generating bands, $G_k \subset \sigma_{(k,0)} \cup \sigma_{(k+1,0)}$. Hence by Proposition A1.(1) and (2), to prove the conclusion (a), we only need to show that $\sigma_{(k+1,0)} \cup \sigma_{(k+2,0)} \subset G_k$. Suppose first $x \in \sigma_{(k+1,0)}$, by renormalization relation (2), $\sigma_{(k+1,0)} = \sigma_{(k-1,a_k)}$ and $\sigma_{(k,-1)} = \sigma_{(k-1,a_k-1)}$, thus by Proposition A1.(3), we have

$$\sigma_{(k+1,0)} \subset \sigma_{(k,0)} \cup \sigma_{(k-1,a_k-1)} = \sigma_{(k,0)} \cup \sigma_{(k,-1)}. \tag{2.3}$$

Hence Corollary 3 implies that any band of $\sigma_{(k+1,0)}$ is either of (k, II)-type or of (k, III)-type, that is, either a band of (k, II)-type or a band of (k, III)-type contains the point x.

Now suppose $x \in \sigma_{(k+2,0)} \backslash \sigma_{(k+1,0)}$, without loose of generality, suppose the band $B_{(k+2,0)}$ containing x. Since by renormalization relation (2.1), $t_{(k+2,0)} = t_{(k,a_{k+1})}$, then $\sigma_{(k+2,0)} = \sigma_{(k,a_{k+1})}$. Notice that $x \notin \sigma_{(k+1,0)}$, by applying Corollary 3 repeatedly, there exists bands $B_{(k,p)} \subset \sigma_{(k,p)} (p = 0, 1, \cdots, a_{k+1})$ such that

$$x \in B_{(k+2,0)} = B_{(k,a_{k+1})} \subset B_{(k,a_{k+1}-1)} \subset \cdots \subset B_{(k,1)} \subset B_{(k,0)},$$

so x is contained in a (k,I)-type band. We thus prove $\sigma_{(k+1,0)}\cup\sigma_{(k+2,0)}\subset G_k$.

(b). By renormalization relation (2.1), $\sigma_{(k+1,-1)}=\sigma_{(k,a_{k+1}-1)}$, so by Corollary 3 any $(k+1,\mathrm{II})$ type band is a band of $\sigma_{(k+2,0)}\backslash\sigma_{(k+1,0)}=\sigma_{(k,a_{k+1})}\backslash\sigma_{(k+1,0)}$, then the analysis in (a) implies also that any $(k+1,\mathrm{II})$ type band is contained in a (k,I) type band.

Now we consider $(k+1,\mathrm{I})$ and $(k+1,\mathrm{III})$ type bands. Notice that by definition, any such band is contained in a band $B_{(k+1,0)}$ of $\sigma_{(k+1,0)}$, so by (2.3), such bands is contained either in a (k,II) or in a (k,III)-type band. □

3. Recurrent Moran structure

Let $\mathscr{G}=\bigcup_{k\geq 0}\mathscr{G}_k$, then $\mathscr{G}$ is the set of all spectral generating bands. From the discussions of the last section, we see that the generating structure of $\mathscr{G}$ looks like(in fact much complex) the structure of a kind of fractals, *the Moran sets*, that has been discussed in [5, 7]. In spite of it, we introduce a kind of matrices for describing the structure of $\mathscr{G}$.

Let $\mathscr{T}=\{\mathbf{T}_n\}_{n\geq 1}$ be a sequence of non-negative 3×3 matrices

$$\mathbf{T}_n:=(t_{ij}(n))_{3\times 3}:=\begin{bmatrix}0 & 1 & 0\\ a_n+1 & 0 & a_n\\ a_n & 0 & a_n-1\end{bmatrix},\quad n\geq 1. \tag{3.1}$$

where a_n is the n-th partial quotient of β. Let $\mathscr{A}=\{\mathrm{I},\mathrm{II},\mathrm{III}\}$ be an alphabet.

Intuitively, $t_{ij}(n)$ represent a $(n-1,i)$-type band generates $t_{ij}(n)$ bands of (n,j)-type, where $i,j\in\mathscr{A}$.

Let $k\geq 0$ be an integer, then from Lemma 4, for any spectral generating band $B\in\mathscr{G}_k$, there exists a unique word $i_0i_1i_2\cdots i_k\in\mathscr{A}^{k+1}$, such that B is a (k,i_k) type band which is contained in a (l,i_l) type band for any $0\leq l<k$, we call $i_0i_1i_2\cdots i_k$ the type index of B. With these notations, we see that

$$D_k:=\{i_0i_1i_2\cdots i_k\mid i_0,i_1,\cdots,i_k\in\mathscr{A},\ i_0=\mathrm{I}\text{ or III},\ \mathrm{t_{i_{j-1}i_j}(j)>0,1\leq j\leq k}\},$$

is the set of all possible type index of bands in $\mathscr{G}_k$. Set $D=\cup_{k\geq 0}D_k$, by convention we assume $D_0=\{\mathrm{I},\mathrm{III}\}$. Notice that two different bands of order k may have the same type index. In fact, for any $k>0$ and $\tau=i_0i_1\cdots i_k\in D$, let

$$K_\tau=t_{i_0i_1}(1)t_{i_1i_2}(2)\cdots t_{i_{k-1}i_k}(k), \tag{3.2}$$

then there are just K_τ bands in $\mathscr{G}_k$ with type index τ.

$\mathscr{G}$ has the following recurrent Moran structure: for any $B\in\mathscr{G}$, there exists $k\geq 0$ such that $B\in\mathscr{G}_k$, let $\tau=i_0i_1\cdots i_k$ be the type index of B, then by Lemma A2, for any $s\in\mathscr{A}$, B contains $t_{i_ks}(k+1)$ bands in $\mathscr{G}_{k+1}$ with type index $\tau*s\in D_{k+1}$ (where '$*$' stands for the concatenation of words), and all the spectral generating bands of order $k+1$ contained in B are mutually disjoint.

For estimating the Hausdorff dimension of the recurrent Moran structure, we need to estimate the length of bands in $\mathscr{G}$(it is difficult in general, in fact, even the bands in $\mathscr{G}_k$ with the same type index may have different length), so we introduce the following two auxiliary sequences of matrices $\mathscr{P}=\{\mathbf{P}_n\}_{n\geq 1}$ and $\mathscr{Q}=\{\mathbf{Q}_n\}_{n\geq 1}$ as follows:

$$\mathbf{P}_n := (p_{ij}(n))_{3\times 3} := \begin{bmatrix} 0 & t_1^{a_n-1} & 0 \\ t_1/a_n & 0 & t_1/a_n \\ t_1/a_n & 0 & t_1/a_n \end{bmatrix}, \tag{3.3}$$

$$\mathbf{Q}_n := (q_{ij})_{3\times 3} := \begin{bmatrix} 0 & t_3^{a_n-1} & 0 \\ t_3(a_n+2)^{-3} & 0 & t_3(a_n+2)^{-3} \\ t_3(a_n+2)^{-3} & 0 & t_3(a_n+2)^{-3} \end{bmatrix}, \tag{3.4}$$

where $t_1 = \frac{3}{V-8}, t_3 = \frac{1}{V+5}$.

Let $\mathscr{W} = \{\mathbf{W}_n = (w_{ij}(n))_{3\times 3}\}_{n\geq 1}$ be a sequence of matrices with $w_{ij}(n) \geq 0 (1 \leq i, j \leq 3, n \geq 1)$, then for any $\tau = i_0 i_1 \cdots i_k$, define

$$L_\tau(\mathscr{W}) = w_{i_0 i_1}(1) w_{i_1 i_2}(2) \cdots w_{\tau_{k-1}\tau_k}(k). \tag{3.5}$$

Proposition 5. *Let $\beta = [a_1, a_2, \cdots] \in (0,1)$ be an irrational and H_β the one dimensional Schrödinger operator defined in* (1.1) *with $V > 20$.*

Then for any spectral generating band $B \in \mathscr{G}$ with type index τ,

$$4L_\tau(\mathscr{Q}) \leq |B| \leq 4L_\tau(\mathscr{P}) \tag{3.6}$$

Now we introduce another two auxiliary sequence of matrices $\mathscr{T}' = \{\mathbf{T}'_n\}_{n\geq 1} = (t'_{ij}(n))_{3\times 3}$ and $\mathscr{Q}' = \{\mathbf{Q}'_n\}_{n\geq 1} = \{(q'_{ij}(n))_{3\times 3}\}_{n\geq 1}$ as follows.

If $a_n < 3$, define

$$\mathbf{T}'_n = \mathbf{T}_n, \quad \mathbf{Q}'_n = \begin{bmatrix} 0 & t_2(a_n+2)^{-1} & 0 \\ t_2(a_n+2)^{-1} & 0 & t_2(a_n+2)^{-1} \\ t_2(a_n+2)^{-1} & 0 & t_2(a_n+2)^{-1} \end{bmatrix}; \tag{3.7}$$

and if $a_n \geq 3$, define

$$\mathbf{T}'_n = \begin{bmatrix} 0 & 1 & 0 \\ b_n & 0 & b_n \\ b_n & 0 & b_n \end{bmatrix}, \quad \mathbf{Q}'_n = \begin{bmatrix} 0 & t_2^{a_n-1} & 0 \\ t_2(a_n+2)^{-1} & 0 & t_2(a_n+2)^{-1} \\ t_2(a_n+2)^{-1} & 0 & t_2(a_n+2)^{-1} \end{bmatrix}, \tag{3.8}$$

where $t_2 = \frac{1}{4(V+8)}$ and $b_n = \max\{2, [(a_n-1)/2]\}$.

Proposition 6. *With the notations and the hypotheses above, there exists $\mathscr{G}' \subset \mathscr{G}$ such that $\mathscr{G}' \bigcap \mathscr{G}_0 = \mathscr{G}_0$, for each band $B \in \mathscr{G}'$ of type index $\tau = i_0 i_1 \cdots i_k (k \geq 1)$ contains at least $t'_{i_k s}(k+1)$ bands in $\mathscr{G}' \cap \mathscr{G}_{k+1}$ of type index $\tau * s$, and*

$$|B| \geq 4L_\tau(\mathscr{Q}'). \tag{3.9}$$

Propositions 5 and 6 will be proved in the next section.

4. Estimating of the length of spectral generating bands

This section is devoted to the proofs of Propositions 5 and 6.

Let $S_p(x)$ be the Chebischev polynomial defined by

$$\begin{aligned} &S_0(x) \equiv 0, \quad S_1(x) \equiv 1, \\ &S_{l+1}(x) = xS_l(x) - S_{l-1}(x), \quad l \geq 1. \end{aligned}$$

For any integer $p \geq 0$, we get immediately by induction

$$S_p(2\cos\theta) = \frac{\sin p\theta}{\sin\theta}, \quad \theta \in \mathbb{R}. \tag{4.1}$$

Before our estimating, we recall some facts obtained in ([2, 12]):

$$t_{(k,p)} = t_{(k,0)}S_{p+1}(t_{(k+1,0)}) - t_{(k,-1)}S_p(t_{(k+1,0)}). \tag{4.2}$$

$$t_{(k,p+1)} = t_{(k,1)}S_{p+1}(t_{(k+1,0)}) - t_{(k,0)}S_p(t_{(k+1,0)}) \tag{4.3}$$

$$t_{(k,p+1)} = t_{(k+1,0)}t_{(k,p)} - t_{(k,p-1)} \tag{4.4}$$

Consider the equation

$$x^2 + y^2 + z^2 - xyz - (V^2 + 4) = 0,$$

then

$$z_\pm(x, y, V) = \frac{xy}{2} \pm \frac{1}{2}\sqrt{4V^2 + (4 - x^2)(4 - y^2)}. \tag{4.5}$$

For two branches $z = z_+$ or $z = z_-$, let $z_1 := z_1(x, y, V) := \frac{\partial z(x,y,V)}{\partial x}$ and $z_2 := z_2(x, y, V) := \frac{\partial z(x,y,V)}{\partial y}$ be the partial derivatives of z with respect to x and y respectively, then for any $|x| \leq 2$, $|y| \leq 2$, and $V \geq 4$,

$$|z_1(x, y, V)| \leq 1, \quad |z_2(x, y, V)| \leq 1. \tag{4.6}$$

Fix any (k, II)-type band $B_{(k+1,0)}$, then by definition, $B_{(k+1,0)} \subset \sigma_{(k,-1)}$, thus for any $x \in B_{(k+1,0)}$, we have $|t_{(k+1,0)}(x)| \leq 2$, $|t_{(k,-1)}(x)| \leq 2$. Moreover, by (2.2), $t_{(k,0)}(x) = z_\pm(t_{(k+1,0)}(x), t_{(k,-1)}(x), V)$, so $V - 2 \leq |t_{(k,0)}(x)| \leq V + 2$.

Summarize above discussion, we see that for any $x \in B_{(k+1,0)}$,

$$|t_{(k+1,0)}(x)| \leq 2, \quad |t_{(k,-1)}(x)| \leq 2, \quad V - 2 \leq |t_{(k,0)}(x)| \leq V + 2. \tag{4.7}$$

Take $p \geq 1$, consider all the bands $B_{(k,p)}$ contained in the band $B_{(k+1,0)}$. Assume $x_p^{(i)} (i = 1, \cdots, p)$ are p points in $B_{(k+1,0)}$ satisfying

$$t_{(k+1,0)}(x_p^{(i)}) = 2\cos\frac{i\pi}{p+1}, \tag{4.8}$$

where every $x_p^{(i)}$ is uniquely determined since $t_{(k+1,0)}(x)$ is monotone on the interval $B_{(k+1,0)}$. Thus from (4.1), $S_{p+1}(t_{(k+1,0)}(x_p^{(i)})) = 0$ and $S_p(t_{(k+1,0)}(x_p^{(i)})) = \pm 1$, so by (4.2) and (4.7), $|t_{(k,p)}(x_p^{(i)})| \leq 2$. This means, the points $x_p^{(i)}$ are located in $\sigma_{(k,p)}$. Moreover there are just p bands of $B^{(i)}_{(k,p)} (1 \leq i \leq p)$ contained in $B_{(k+1,0)}$, and the bands $B^{(i)}_{(k,p)}$ contains the point $x_p^{(i)} (i = 1, \cdots, p)$, see [12].

Proposition 7. *Let $V > 20$ and $B_{(k+1,0)}$ a band of (k,II)-type. If for any $x \in B_{(k+1,0)}$, $|t'_{(k,-1)}(x)|/|t'_{(k+1,0)}(x)| \leq 1/3$, then for any $p \geq 1$ and $x \in B^{(i)}_{(k,p)} (1 \leq i \leq p)$, we have*

$$3 < \frac{V-8}{3}(p+1) \leq \left|\frac{t'_{(k,p)}(x)}{t'_{(k+1,0)}(x)}\right| \leq \frac{V+8}{4}(p+1)^3.$$

In particular,
(i) *If $p \leq 4$, then for any $x \in B^{(i)}_{(k,p)} (1 \leq i \leq p)$,*

$$\left|\frac{t'_{(k,p)}(x)}{t'_{(k+1,0)}(x)}\right| \leq 4(V+8)(p+1);$$

(ii) *If $p \geq 2$, then there are at least $\max\{2,[p/2]\}$ bands of $\{B^{(i)}_{(k,p)}\}_{1 \leq i \leq p}$, such that for any x in these bands,*

$$\left|\frac{t'_{(k,p)}(x)}{t'_{(k+1,0)}(x)}\right| \leq 4(V+8)(p+1).$$

Proof We proceed our proof by the following four steps.

1. Introduce parameter c.

From (4.1), we get

$$\begin{aligned} S_p(2\cos\theta) &= \frac{\sin(p+1-1)\theta}{\sin\theta} \\ &= \frac{\sin(p+1)\theta\cos\theta - \cos(p+1)\theta\sin\theta}{\sin\theta} \\ &= S_{p+1}(2\cos\theta)\cos\theta - \cos(p+1)\theta, \end{aligned}$$

which yields

$$|S_p(2\cos\theta)| \leq |S_{p+1}(2\cos\theta)| + 1. \tag{4.9}$$

Fix any band $B^{(i)}_{(k,p)}$ and let $x = 2\cos\theta$, then for any $x \in B^{(i)}_{(k,p)}$, we get by (4.2), (4.7), (4.9),

$$\begin{aligned} 2 &\geq |t_{(k,p)}(x)| \\ &\geq |t_{(k,0)}(x)S_{p+1}(t_{(k+1,0)}(x))| - |t_{(k,-1)}(x)S_p(t_{(k+1,0)}(x))| \\ &\geq (V-2)|S_{p+1}(t_{(k+1,0)}(x))| - (|S_{p+1}(t_{(k+1,0)}(x))|+1)|t_{(k,-1)}(x)| \\ &= (V-4)|S_{p+1}(t_{(k+1,0)}(x))| - 2, \end{aligned}$$

which follows that $|S_{p+1}(t_{(k+1,0)}(x))| \leq \frac{4}{V-4}$, hence if $V > 20$, we have $|S_{p+1}(t_{(k+1,0)}(x))| \leq \frac{1}{4}$, then by (4.9), $|S_p(t_{(k+1,0)}(x))| \leq \frac{5}{4}$.

For any $x \in B^{(i)}_{(k,p)}$, by (4.8) and monotonicity of $t_{(k+1,0)}(x)$ on $B_{(k+1,0)}$, there exists a unique c with $|c| < 1$ such that $t_{(k+1,0)}(x) = 2\cos(\frac{i\pi}{p+1} + \frac{c\pi}{p+1})$. Since

$$S_{p+1}(2\cos(\frac{i\pi}{p+1} + \frac{c\pi}{p+1})) = \frac{\sin(i+c)\pi}{\sin\frac{i+c}{p+1}\pi} = \frac{(-1)^i\sin c\pi}{\sin\frac{i+c}{p+1}\pi},$$

$|S_{p+1}(t_{(k+1,0)}(x))| \leq \frac{1}{4}$ implies $|\sin c\pi| \leq \frac{1}{4}$, so $|\tan c\pi| \leq \frac{2}{7}$ and $|c| \leq \frac{1}{10}$.

From above discussion, we have for any $V > 20$, $x \in B^{(i)}_{(k,p)}$ and c defined as above,

$$\begin{aligned} &|S_{p+1}(t_{(k+1,0)}(x))| \leq \frac{1}{4}, \quad |S_p(t_{(k+1,0)}(x))| \leq \frac{5}{4}, \\ &|\sin c\pi| \leq \frac{1}{4}, \quad |\tan c\pi| \leq \frac{2}{7}, \quad |c| \leq \frac{1}{10}. \end{aligned} \tag{4.10}$$

2. Estimating $S'_{p+1}(x)$, the derivation of $S_{p+1}(x)$.
From (4.1),

$$S'_p(2\cos\theta) = \frac{\sin p\theta \cos\theta - p\cos p\theta \sin\theta}{2\sin^3\theta} = \frac{\sin(p+1)\theta - (p+1)\cos p\theta \sin\theta}{2\sin^3\theta},$$

then

$$\begin{aligned} & S'_{p+1}(2\cos\tfrac{i+c}{p+1}\pi) \\ = & \frac{\sin(i+c)\pi\cos\frac{i+c}{p+1}\pi - (p+1)\cos(i+c)\pi\sin\frac{i+c}{p+1}\pi}{2\sin^3\frac{i+c}{p+1}\pi} \\ = & \frac{(-1)^{i+1}(p+1)\cos c\pi}{2\sin^2\frac{i+c}{p+1}\pi}\left[1 - \frac{\tan c\pi}{(p+1)\tan\frac{i+c}{p+1}\pi}\right], \end{aligned} \tag{4.11}$$

which yields for any $x \in B^{(i)}_{(k,p)}$ and the corresponding parameter c,

$$|S'_{p+1}(t_{(k+1,0)}(x))| \geq \frac{(p+1)\sqrt{3}}{4}(1 - \frac{2}{7}\frac{1}{(i+c)\pi}) \geq \frac{(p+1)\sqrt{3}}{4}(1 - \frac{4}{7\pi}) \geq \frac{(p+1)}{3}, \tag{4.12}$$

on the other hand, since for $p \geq 1$, $i = 1, \cdots, p$,

$$\sin(\frac{i+c}{p+1}\pi) \geq \sin\frac{\pi}{2(p+1)} \geq (\frac{\sqrt{2}}{2}/\frac{\pi}{4})\frac{\pi}{2(p+1)} = \frac{\sqrt{2}}{p+1},$$

we get by (4.11),

$$|S'_{p+1}(t_{(k+1,0)}(x))| \leq \frac{(p+1)}{2\sin^2\frac{\pi}{2(p+1)}} \leq \frac{(p+1)}{2(\frac{\sqrt{2}}{p+1})^2} \leq \frac{(p+1)^3}{4}. \tag{4.13}$$

In particular, if $p \leq 4$, then by (4.11),

$$|S'_{p+1}(t_{(k+1,0)}(x))| \leq \frac{p+1}{2\sin^2\frac{\pi}{10}} \leq 4(p+1). \tag{4.14}$$

If $p \geq 2$, $p/4 \leq i \leq (3p+4)/4$, then $|\sin\frac{(i+c)\pi}{p+1}| \geq \frac{\sqrt{2}}{4}$. So for any $x \in B^{(i)}_{(k,p)}$ and the corresponding parameter c, still by (4.11),

$$|S'_{p+1}(t_{(k+1,0)}(x))| \leq \frac{(p+1)}{2\sin^2\frac{(i-c)\pi}{p+1}} \leq 4(p+1). \tag{4.15}$$

3. Relation between S'_{p+1} and S'_p.

From (4.1),

$$S_p'(2\cos\tfrac{i+c}{p+1}\pi) = \frac{\sin(i+c)\pi-(p+1)\cos\frac{p(i+c)}{p+1}\pi\sin\frac{i+c}{p+1}\pi}{2\sin^3\frac{i+c}{p+1}\pi}$$

$$= \frac{(-1)^{i+1}(p+1)\cos c\pi}{2\sin^2\frac{i+c}{p+1}\pi}\left[\cos\tfrac{i+c}{p+1}\pi+\tan c\pi\sin\tfrac{i+c}{p+1}\pi-\frac{\tan c\pi}{(p+1)\sin\frac{i+c}{p+1}\pi}\right],$$

we get by associating with (4.11),

$$\begin{aligned}
& 2|S_{p+1}'(2\cos\tfrac{i+c}{p+1}\pi)|-|S_p'(2\cos\tfrac{i+c}{p+1}\pi)| \\
\geq\ & \frac{(p+1)\cos c\pi}{2\sin^2\frac{i+c}{p+1}\pi}\left[2-\frac{2|\tan c\pi|}{(p+1)\tan\frac{i+c}{p+1}\pi}-|\cos\tfrac{i+c}{p+1}\pi|-|\tan c\pi|-\frac{|\tan c\pi|}{(p+1)\sin\frac{i+c}{p+1}\pi}\right] \\
\geq\ & \frac{(p+1)\cos c\pi}{2\sin^2\frac{i+c}{p+1}\pi}\left(2-2\frac{2}{7}\frac{2}{\pi}-1-\frac{2}{7}-\frac{2}{7}\frac{1}{\sqrt{2}}\right)\ \geq 0.
\end{aligned}$$

Thus for any $x\in B_{(k,p)}^{(i)}$,

$$|S_p'(x)|\leq 2|S_{p+1}'(x)|. \tag{4.16}$$

4. Estimate $|t_{(k,p)}'(x)/t_{(k+1,0)}'(x)|$.
By (4.2),

$$\begin{aligned}
\frac{t_{(k,p)}'(x)}{t_{(k+1,0)}'(x)} &= S_{p+1}'(t_{(k+1,0)}(x))t_{(k,0)}-S_p'(t_{(k+1,0)}(x))t_{(k,-1)}+ \\
& S_{p+1}(t_{(k+1,0)}(x))\frac{t_{(k,0)}'(x)}{t_{(k+1,0)}'(x)}-S_p(t_{(k+1,0)}(x))\frac{t_{(k,-1)}'(x)}{t_{(k+1,0)}'(x)}.
\end{aligned} \tag{4.17}$$

From (2.2) and (4.5), we have,

$$t_{(k,0)}(x)=z(t_{(k+1,0)}(x),t_{(k,-1)}(x),V),$$

so

$$\frac{t_{(k,0)}'(x)}{t_{(k+1,0)}'(x)}=z_1(t_{(k+1,0)}(x),t_{(k,-1)}(x),V)+z_2(t_{(k+1,0)}(x),t_{(k,-1)}(x),V)\frac{t_{(k,-1)}'(x)}{t_{(k+1,0)}'(x)}. \tag{4.18}$$

From (4.6), (4.10), (4.12), (4.16), (4.17), (4.18), we see that if $|t_{(k,-1)}'(x)/t_{(k+1,0)}'(x)|\leq\frac{1}{3}$, then

$$\begin{aligned}
& |t_{(k,p)}'(x)/t_{(k+1,0)}'(x)| \\
\geq\ & |S_{p+1}'(t_{(k+1,0)}(x))|(|t_{(k,0)}|-2|t_{(k,-1)}|)-\frac{1}{4}-(\frac{1}{4}+\frac{5}{4})\left|\frac{t_{(k,-1)}'(x)}{t_{(k+1,0)}'(x)}\right| \\
>\ & \frac{p+1}{3}(V-6)-1\geq\frac{p+1}{3}(V-8)
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
&|t'_{(k,p)}(x)/t'_{(k+1,0)}(x)| \\
\leq\ & |S'_{p+1}(t_{(k+1,0)}(x))|(|t_{(k,0)}|+2|t_{(k,-1)}|)+\frac{1}{4}+(\frac{1}{4}+\frac{5}{4})\left|\frac{t'_{(k,-1)}(x)}{t'_{(k+1,0)}(x)}\right| \qquad (4.20)\\
\leq\ & |S'_{p+1}(t_{(k+1,0)}(x))|(V+6)+1.
\end{aligned}$$

So by (4.13) and (4.20), for any $x \in B^{(i)}_{(k,p)}$, $i=1,2,\cdots,p$,

$$\left|\frac{t'_{(k,p)}(x)}{t'_{(k+1,0)}(x)}\right| \leq \frac{(p+1)^3}{4}(V+6)+1 \leq \frac{V+8}{4}(p+1)^3. \qquad (4.21)$$

By (4.14) and (4.20), if $p \leq 4$, then for any $x \in B^{(i)}_{(k,p)}$, $i=1,2,\cdots,p$,

$$\left|\frac{t'_{(k,p)}(x)}{t'_{(k+1,0)}(x)}\right| \leq 4(p+1)(V+6)+1 \leq 4(p+1)(V+8), \qquad (4.22)$$

and by (4.15) and (4.20), if $p \geq 2$, then for any $x \in B^{(i)}_{(k,p)}$, $p/4 \leq i \leq (3p+4)/4$,

$$\left|\frac{t'_{(k,p)}(x)}{t'_{(k+1,0)}(x)}\right| \leq 4(p+1)(V+6)+1 \leq 4(p+1)(V+8). \qquad (4.23)$$

Note that the number of integer i satisfy $p/4 \leq i \leq (3p+4)/4$ is larger than $\max\{2,[p/2]\}$.

From (4.19), (4.21), (4.22), (4.23), we prove the conclusion of the proposition. $\square$

Fix any (k,III)-type band $B_{(k+1,0)}$, then by definition, $B_{(k+1,0)} \subset \sigma_{(k,0)}$. As same as the discussion before last proposition, we see that for any $x \in B_{(k+1,0)}$,

$$|t_{(k+1,0)}(x)| \leq 2, \quad |t_{(k,0)}(x)| \leq 2, \quad V-2 \leq |t_{(k,1)}(x)| \leq V+2. \qquad (4.24)$$

Now fix any $p \geq 1$, consider all the bands $B_{(k,p+1)}$ contained in the band $B_{(k+1,0)}$. Still denote $x^{(i)}_p (i=1,\cdots,p)$ as p points in $B_{(k+1,0)}$ satisfying (4.8) (notice that the point $x^{(i)}_p$ here are located in type III bands). Thus $S_{p+1}(t_{(k+1,0)}(x^{(i)}_p))=0$ and $S_p(t_{(k+1,0)}(x^{(i)}_p))=\pm1$, so by (4.3) and (4.24), $|t_{(k,p+1)}(x^{(i)}_p)| \leq 2$. This means, the points $x^{(i)}_p$ are located in $\sigma_{(k,p+1)}$. Moreover there are just p bands of $B^{(i)}_{(k,p+1)}(1 \leq i \leq p)$ contained in $B_{(k+1,0)}$, and the bands $B^{(i)}_{(k,p+1)}$ contains the point $x^{(i)}_p (i=1,\cdots,p)$, see [12].

Proposition 8. *Let $V>20$ and $B_{(k+1,0)}$ be a (k,III)-type band. If for any $x \in B_{(k+1,0)}$, $|t'_{(k,0)}(x)/t'_{(k+1,0)}(x)| \leq \frac{1}{3}$, then for any $p \geq 1$, any $x \in B^{(i)}_{(k,p+1)}(1 \leq i \leq p)$, we have*

$$3 \leq \frac{V-8}{3}(p+1) \leq \left|\frac{t'_{(k,p+1)}(x)}{t'_{(k+1,0)}(x)}\right| \leq \frac{V+8}{4}(p+1)^3.$$

In particular, we have

(i) *If $p \leq 4$, then for any $x \in B^{(i)}_{(k,p+1)}(1 \leq i \leq p)$,*

$$|t'_{(k,p+1)}(x)/t'_{(k+1,0)}(x)| \leq 4(V+8)(p+1);$$

(ii) *If $p \geq 2$, there are at least* $\max\{2, [p/2]\}$ *bands in* $\{B^{(i)}_{(k,p+1)}\}_{1\leq i\leq p}$ *contained in the* $B_{(k+1,0)}$ *such that for any x in these bands,*

$$|t'_{(k,p+1)}(x)/t'_{(k+1,0)}(x)| \leq 4(V+8)(p+1).$$

Proof. The first three steps of proof are entirely same as the proof of Proposition 7. Now we begin from step 4.

4. Estimate $|t'_{(k,p+1)}(x)/t'_{(k+1,0)}(x)|$.

By (4.3),

$$\begin{aligned}\frac{t'_{(k,p+1)}(x)}{t'_{(k+1,0)}(x)} &= S'_{p+1}(t_{(k+1,0)}(x))t_{(k,1)} - S'_p(t_{(k+1,0)}(x))t_{(k,0)}+ \\ &S_{p+1}(t_{(k+1,0)}(x))\frac{t'_{(k,1)}(x)}{t'_{(k+1,0)}(x)} - S_p(t_{(k+1,0)}(x))\frac{t'_{(k,0)}(x)}{t'_{(k+1,0)}(x)}.\end{aligned} \tag{4.25}$$

From (2.2) and (4.5), we have,

$$t_{(k,1)}(x) = z(t_{(k+1,0)}(x), t_{(k,0)}(x), V),$$

so

$$\frac{t'_{(k,1)}(x)}{t'_{(k+1,0)}(x)} = z_1(t_{(k+1,0)}(x), t_{(k,0)}(x), V) + z_2(t_{(k+1,0)}(x), t_{(k,0)}(x), V)\frac{t'_{(k,0)}(x)}{t'_{(k+1,0)}(x)}. \tag{4.26}$$

As similar as (4.19), from (4.6), (4.10), (4.12), (4.16), (4.25), (4.26), we see that if $|t'_{(k,0)}(x)/t'_{(k+1,0)}(x)| \leq \frac{1}{3}$, then

$$\begin{aligned}&|t'_{(k,p+1)}(x)/t'_{(k+1,0)}(x)| \\ \geq\ & |S'_{p+1}(t_{(k+1,0)}(x))|(|t_{(k,1)}| - 2|t_{(k,0)}|) - \frac{1}{4} - (\frac{1}{4}+\frac{5}{4})\left|\frac{t'_{(k,0)}(x)}{t'_{(k+1,0)}(x)}\right| \\ >\ & \frac{p+1}{3}(V-6) - 1 \geq \frac{p+1}{3}(V-8)\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}&|t'_{(k,p+1)}(x)/t'_{(k+1,0)}(x)| \\ \leq\ & |S'_{p+1}(t_{(k+1,0)}(x))|(|t_{(k,1)}| + 2|t_{(k,0)}|) + \frac{1}{4} + (\frac{1}{4}+\frac{5}{4})\left|\frac{t'_{(k,0)}(x)}{t'_{(k+1,0)}(x)}\right| \\ \leq\ & |S'_{p+1}(t_{(k+1,0)}(x))|(V+6) + 1.\end{aligned} \tag{4.28}$$

So by (4.13) and (4.28), for any $x \in B^{(i)}_{(k,p+1)}$, $i = 1, 2, \cdots, p$,

$$\left|\frac{t'_{(k,p+1)}(x)}{t'_{(k+1,0)}(x)}\right| \leq \frac{(p+1)^3}{4}(V+6) + 1 \leq \frac{V+8}{4}(p+1)^3. \tag{4.29}$$

By (4.14) and (4.28), if $p \leq 4$, then for any $x \in B^{(i)}_{(k,p+1)}$, $i = 1, 2, \cdots, p$,

$$\left|\frac{t'_{(k,p+1)}(x)}{t'_{(k+1,0)}(x)}\right| \leq 4(p+1)(V+6) + 1 \leq 4(p+1)(V+8), \tag{4.30}$$

and by (4.15) and (4.28), if $p \geq 2$, then for any $x \in B^{(i)}_{(k,p+1)}$, $p/4 \leq i \leq (3p+4)/4$,

$$\left|\frac{t'_{(k,p+1)}(x)}{t'_{(k+1,0)}(x)}\right| \leq 4(p+1)(V+6)+1 \leq 4(p+1)(V+8). \tag{4.31}$$

From (4.27), (4.29), (4.30), (4.31), we prove the conclusion of the proposition. □

Now we are going to estimate (k,I)-type bands.

Proposition 9. *Let $B_{(k,1)}$ be any (k,I)-type band and $V \geq 8$. For any $p \geq 1$, let $B_{(k,p)}$ be the unique band of $\sigma_{(k,p)}$ contained in $B_{(k,1)}$. If for any $x \in B_{(k,1)}$, $|t'_{(k,0)}(x)/t'_{(k,1)}(x)| \leq \dfrac{1}{3}$, then for any $x \in B_{(k,p)}$,*

$$\begin{aligned} 3 \leq V-5 \leq &\left|\frac{t'_{(k,p)}(x)}{t'_{(k,p-1)}(x)}\right| \leq V+5, \quad p>1,\\ (V-5)^{p-1} \leq &\left|\frac{t'_{(k,p)}(x)}{t'_{(k,1)}(x)}\right| \leq (V+5)^{p-1}. \end{aligned}$$

Proof. Similarly to (4.7), for any $x \in B_{(k,p)}$,

$$|t_{(k,p)}(x)| \leq 2, \quad |t_{(k,p-1)}(x)| \leq 2, \quad V-2 \leq |t_{(k+1,0)}(x)| \leq V+2. \tag{4.32}$$

By (4.4), we have

$$\frac{t'_{(k,p+1)}}{t'_{(k,p)}} = t_{(k+1,0)} + t_{(k,p)}\frac{t'_{(k+1,0)}}{t'_{(k,p)}} - \frac{t'_{(k,p-1)}}{t'_{(k,p)}}.$$

By (2.2) and (4.5)), we have

$$t_{(k+1,0)} = z(t_{(k,p-1)}, t_{(k,p)}, V),$$

so

$$\frac{t'_{(k,p+1)}}{t'_{(k,p)}} = t_{(k+1,0)} + t_{(k,p)}z_2(t_{(k,p-1)}, t_{(k,p)}, V) - [1 - t_{(k,p)}z_1(t_{(k,p-1)}, t_{(k,p)}, V)]\frac{t'_{(k,p-1)}}{t'_{(k,p)}}.$$

By (4.32) and (4.6), for $p \geq 1$, if for any $x \in B_{(k,p+1)}$,

$$|t'_{(k,p-1)}(x)/t'_{(k,p)}(x)| \leq \frac{1}{3}, \tag{4.33}$$

then

$$3 \leq V-5 \leq |t'_{(k,p+1)}(x)/t'_{(k,p)}(x)| \leq V+5. \tag{4.34}$$

For $p=1$ the condition (4.33) is a hypothesis of the proposition. Since $B_{(k,1)} \supset B_{(k,2)} \supset \cdots B_{(k,p)}$ (see [12] for detail), for $p = l > 1$ the condition (4.33) is implied by (4.34) in case of $p = l-1$. This prove the proposition. □

Proposition 10. *Assume that $V > 20$ and $k \geq 0$. Let $B^{(\mathrm{I})}_{(k,1)}$, $B^{(\mathrm{II})}_{(k+1,0)}$ and $B^{(\mathrm{III})}_{(k+1,0)}$ are (k,I)-type, (k,II)-type and (k,III)-type bands respectively, then for any $x_1 \in B^{(\mathrm{I})}_{(k,1)}$, $x_2 \in B^{(\mathrm{II})}_{(k+0,1)}$ and $x_3 \in B^{(\mathrm{III})}_{(k+1,0)}$,*

$$\left|\frac{t'_{(k,0)}(x_1)}{t'_{(k,1)}(x_1)}\right| \leq \frac{1}{3}, \quad \left|\frac{t'_{(k,-1)}(x_2)}{t'_{(k+1,0)}(x_2)}\right| \leq \frac{1}{3}, \quad \left|\frac{t'_{(k,0)}(x_3)}{t'_{(k+1,0)}(x_3)}\right| \leq \frac{1}{3}. \tag{4.35}$$

Proof. We proceed the proof by induction on the order of the bands.

For $k=0$, there are only two bands, i.e., $(0,\mathrm{I})$-type band $[V-2,V+2]$ and $(0,\mathrm{III})$-type band $[-2,2]$. Since $t'_{(0,0)} \equiv 0$, and $t'_{(0,1)} = t'_{(1,0)} \equiv 1$,

$$\frac{t'_{(0,0)}(x)}{t'_{(0,1)}(x)} = \frac{t'_{(0,0)}(x)}{t'_{(1,0)}(x)} = 0 < \frac{1}{3}.$$

Suppose that the conclusion is true for $k=n$, that is, the inequalities (4.35) hold, we are going to verify (4.35) for the order $k=n+1$.

Consider a $(n+1,\mathrm{I})$-type band $B^{(\mathrm{I})}_{(n+1,1)}$. Suppose firstly it is contained in a (n,II)-type band $B^{(\mathrm{II})}_{(n+1,0)}$. By induction hypothesis, for any $x \in B^{(\mathrm{II})}_{(n+1,0)}$, $\left|\frac{t'_{(n,-1)}(x)}{t'_{(n+1,0)}(x)}\right| \leq \frac{1}{3}$. Since By renormalization relation (2.1), $t_{(n+1,1)} = t_{(n,a_n+1)}$, setting $p=a_n+1$, applying Propositions 7, we have for any $x \in B^{(I)}_{(n+1,1)}$

$$\left|\frac{t'_{(n+1,1)}(x)}{t'_{(n+1,0)}(x)}\right| = \left|\frac{t'_{(n,a_n+1)}(x)}{t'_{(n+1,0)}(x)}\right| \geq \frac{V-8}{3}(p+1) > 3,$$

which implies $\left|\frac{t'_{(n+1,0)}(x)}{t'_{(n+1,1)}(x)}\right| < \frac{1}{3}$.

Suppose now $B^{(\mathrm{I})}_{(n+1,1)}$ is contained in a (n,III)-type band $B^{(\mathrm{III})}_{(n+1,0)}$. By induction hypothesis, for any $x \in B^{(\mathrm{II})}_{(n+1,0)}$, $\left|\frac{t'_{(n,0)}(x)}{t'_{(n+1,0)}(x)}\right| \leq \frac{1}{3}$. Since $t_{(n+1,1)} = t_{(n,a_n+1)}$, setting $p=a_n$ and applying Propositions 8, we have for any $x \in B^{(\mathrm{I})}_{(n+1,1)}$

$$\left|\frac{t'_{(n+1,1)}(x)}{t'_{(n+1,0)}(x)}\right| = \left|\frac{t'_{(n,a_n+1)}(x)}{t'_{(n+1,0)}(x)}\right| \geq \frac{V-8}{3}(p+1) > 3,$$

which implies again $\left|\frac{t'_{(n+1,0)}(x)}{t'_{(n+1,1)}(x)}\right| < \frac{1}{3}$.

Similarly we have for any $(n+1,\mathrm{III})$-type band $B^{(\mathrm{III})}_{(n+2,0)}$ and $x \in B^{(\mathrm{III})}_{(n+2,0)}$, $\left|\frac{t'_{(n+1,0)}(x)}{t'_{(n+2,0)}(x)}\right| < \frac{1}{3}$.

Let $B_{(n+2,0)}$ be any $(n+1,\mathrm{II})$-type band and suppose it is contained in an (n,I)-type band $B_{(n,1)}$, then by hypothesis of the induction, for any $x \in B_{(n,1)}$, $\left|\frac{t'_{(n,0)}(x)}{t'_{(n,1)}(x)}\right| \leq \frac{1}{3}$. By renormalization relation (2.1), $t_{(n,a_{n+1}-1)} = t_{(n+1,-1)}$ and $t_{(n,a_{n+1})} = t_{(n+2,0)}$, taking $p=a_{n+1}$, applying Proposition 9, for any x in $B_{(n+2,0)}$,

$$\left|\frac{t'_{(n+1,-1)}(x)}{t'_{(n+2,0)}(x)}\right| = \left|\frac{t'_{(n,a_{n+1}-1)}(x)}{t'_{(k,a_{n+1})}(x)}\right| \leq \frac{1}{3}.$$

This prove the inequalities (4.35) for $k = n+1$. □

Proof of Proposition 5.

For convenient for any $k \geq 1$, let

$$s_{(k,\mathrm{I})} = t_{(k,1)}, \quad s_{(k,\mathrm{II})} = t_{(k+1,0)}, \quad s_{(k,\mathrm{III})} = t_{(k+1,0)}. \tag{4.36}$$

Now fix $k \geq 0$.

1). Consider any $(k+1,\mathrm{I})$-type band $B_{(k+1,1)}$. By (2.1), $t_{(k+1,1)} = t_{(k,a_{k+1}+1)}$, thus by Lemma A2 and Lemma 4, $B_{(k+1,1)}$ is contained in a (k,II)-type band or a (k,III)-type band.

Suppose $B_{(k+1,1)}$ is contained in a (k,II)-type band $B_{(k+1,0)}$. Let $p = a_{k+1}+1$, then by Proposition 7 and Proposition 10, for any $x \in B_{(k+1,1)}$,

$$\frac{a_{k+1}}{t_1} < \frac{V-8}{3}(a_{k+1}+2) \leq \left|\frac{t'_{(k+1,1)}(x)}{t'_{(k+1,0)}(x)}\right| \leq \frac{V+8}{4}(a_{k+1}+2)^3 < \frac{(a_{k+1}+2)^3}{t_3},$$

which yields by the definitions of t_1, t_3, $p_{ij}(n)$ and $q_{ij}(n)$ in (3.3) and (3.4),

$$\frac{1}{p_{21}(k+1)} \leq \left|\frac{s'_{(k+1,\mathrm{I})}(x)}{s'_{(k,\mathrm{II})}(x)}\right| \leq \frac{1}{q_{21}(k+1)}. \tag{4.37}$$

Now suppose $B_{(k+1,1)}$ is contained in a (k,III)-type band $B_{(k+1,0)}$. Let $p = a_{k+1}$, by Proposition 8 and Proposition 10, for any $x \in B_{(k+1,1)}$,

$$\frac{a_{k+1}}{t_1} < \frac{V-8}{3}(a_{k+1}+1) \leq \left|\frac{t'_{(k+1,1)}(x)}{t'_{(k+1,0)}(x)}\right| \leq \frac{V+8}{4}(a_{k+1}+1)^3 < \frac{(a_{k+1}+2)^3}{t_3},$$

which gives

$$\frac{1}{p_{31}(k+1)} \leq \left|\frac{s'_{(k+1,\mathrm{I})}(x)}{s'_{(k,\mathrm{III})}(x)}\right| \leq \frac{1}{q_{31}(k+1)}. \tag{4.38}$$

2). Let $B_{(k+2,0)}$ be a $(k+1,\mathrm{III})$-type band . Since $t_{(k+2,0)} = t_{(k,a_{k+1})}$, by Lemma A2 and Lemma 4, $B_{(k+2,0)}$ is contained either in a (k,II)-type band or in a (k,III)-type band.

If $B_{(k+2,0)}$ is contained in a (k,II)-type band $B_{(k+1,0)}$. Let $p = a_{k+1}$, then by Proposition 7 and Proposition 10, for any $x \in B_{(k+2,0)}$,

$$\frac{a_{k+1}}{t_1} < \frac{V-8}{3}(a_{k+1}+1) \leq \left|\frac{t'_{(k+2,0)}(x)}{t'_{(k+1,0)}(x)}\right| \leq \frac{V+8}{4}(a_{k+1}+1)^3 < \frac{(a_{k+1}+2)^3}{t_3},$$

which implies

$$\frac{1}{p_{23}(k+1)} \leq \left|\frac{s'_{(k+1,\mathrm{III})}(x)}{s'_{(k,\mathrm{II})}(x)}\right| \leq \frac{1}{q_{23}(k+1)}. \tag{4.39}$$

If $B_{(k+2,0)}$ is contained in a (k,III)-type band $B_{(k+1,0)}$. Let $p = a_{k+1}-1(\geq 1)$, then by Proposition 8 and Proposition 10, for any $x \in B_{(k+2,0)}$,

$$\frac{a_{k+1}}{t_1} = \frac{V-8}{3}a_{k+1} \leq \left|\frac{t'_{(k+2,0)}(x)}{t'_{(k+1,0)}(x)}\right| \leq \frac{V+8}{4}a_{k+1}^3 < \frac{(a_{k+1}+2)^3}{t_3},$$

which yields

$$\frac{1}{p_{33}(k+1)} \le \left|\frac{s'_{(k+1,\mathrm{III})}(x)}{s'_{(k,\mathrm{III})}(x)}\right| \le \frac{1}{q_{33}(k+1)}. \tag{4.40}$$

3). Let $B_{(k+2,0)}$ be any $(k+1,\mathrm{II})$-type band , then by Lemma A2 and Lemma 4, it is contained in an (k,I)-type band $B_{(k,1)}$. Then by $t_{(k+2,0)} = t_{(k,a_{k+1})}$, Propositions 9 and 10, we get for any $x \in B_{(k+2,0)}$

$$t_1^{1-a_{k+1}} < (V-5)^{a_{k+1}-1} \le \left|\frac{t'_{(k+2,0)}(x)}{t'_{(k,1)}(x)}\right| \le (V+5)^{a_{k+1}-1} = t_3^{1-a_{k+1}},$$

which gives

$$\frac{1}{p_{12}(k+1)} \le \left|\frac{s'_{(k+1,\mathrm{II})}(x)}{s'_{(k,\mathrm{I})}(x)}\right| \le \frac{1}{q_{12}(k+1)}. \tag{4.41}$$

4) There are only two spectral generating bands of order 0. For any x in $(0,\mathrm{I})$-type band $B_{(0,1)} = [V-2, V+2]$,

$$s'_{(0,\mathrm{I})}(x) = t'_{(0,1)}(x) = (x-V)' = 1.$$

For any x in $(0,\mathrm{III})$-type band $B_{(1,0)} = [-2,2]$,

$$s'_{(0,\mathrm{III})}(x) = t'_{(1,0)}(x) = x' = 1.$$

Now we are going to prove the inequality (3.6). Let B be any spectral generating band with type index $\tau = \tau_1 \cdots \tau_k$, then

$$s'_{(k,\tau_k)} = \frac{s'_{(k,\tau_k)}}{s'_{(0,\tau_0)}} = \frac{s'_{(1,\tau_1)}}{s'_{(0,\tau_0)}} \frac{s'_{(2,\tau_2)}}{s'_{(1,\tau_1)}} \cdots \frac{s'_{(k,\tau_k)}}{s'_{(k-1,\tau_{k-1})}}.$$

From the definition $L_\tau(\cdot)$ by (3.5), we get by inequalities $(4.37)-(4.41)$, for any $x \in B$,

$$1/L_\tau(\mathscr{P}) \le |s'_{(k,\tau_k)}(x)| \le 1/L_\tau(\mathscr{Q}). \tag{4.42}$$

Notice that $s_{(k,\tau_k)}(x)$ is monotone on the band B and $s_{(k,\tau_k)}(B) = [-2,2]$, so by medium value theorem and (4.42)

$$4L_\tau(\mathscr{Q}) \le |B| \le 4L_\tau(\mathscr{P}),$$

which completes the proof of the proposition. □

Proof of Proposition 6.

Let $\mathscr{G}'_0 = \mathscr{G}_0$. Suppose $\mathscr{G}'_k (\subset \mathscr{G}_k)$ is defined, we will define $\mathscr{G}'_{k+1} (\subset \mathscr{G}_{k+1})$.

We distinguish two cases $a_{k+1} \ge 3$ and $a_{k+1} < 3$.

If $a_{k+1} < 3$, then take $\mathscr{G}'_{k+1}$ to be the set of bands of $\mathscr{G}_{k+1}$ which is contained in some band of $\mathscr{G}'_k$.

Now consider the case of $a_{k+1} \ge 3$, we construct $\mathscr{G}'_{k+1}$ from $\mathscr{G}_{k+1}$ by distinguishing the following five cases.

1). Let $p = a_{k+1} + 1(> 2)$, by Proposition 7.(ii) and Proposition 10, any (k, II)-type band $B_{(k+1,0)}$ contains at least $\max\{2, [(a_{k+1}+1)/2]\}(\geq b_{k+1})$ bands of $(k+1, \mathrm{I})$-type, such that for any x in these bands,

$$\left|\frac{s'_{(k+1,\mathrm{I})}(x)}{s'_{(k,\mathrm{II})}(x)}\right| = \left|\frac{t'_{(k+1,1)}(x)}{t'_{(k+1,0)}(x)}\right| \leq 4(V+8)(a_{k+1}+2) = \frac{1}{q'_{21}(k+1)}, \tag{4.43}$$

where $q'_{ij}(n)$ has been defined in (3.7)-(3.8).

2). Let $p = a_{k+1}(> 2)$, by Proposition 7.(ii) and Proposition 10, any (k, II)-type band $B_{(k+1,0)}$ contains at least $\max\{2, [a_{k+1}/2]\}(\geq b_{k+1})$ bands $(k+1, \mathrm{III})$-type, such that for any x in these bands,

$$\left|\frac{s'_{(k+1,\mathrm{III})}(x)}{s'_{(k,\mathrm{II})}(x)}\right| = \left|\frac{t'_{(k+2,0)}(x)}{t'_{(k+1,0)}(x)}\right| \leq 4(V+8)(a_{k+1}+1) < \frac{1}{q'_{23}(k+1)}. \tag{4.44}$$

3). Let $p = a_{k+1}(> 2)$, by Proposition 8.(ii) and Proposition 10, any (k, III)-type band $B_{(k+1,0)}$ contains at least $\max\{2, [a_{k+1}/2]\}(\geq b_{k+1})$ bands of $(k+1, \mathrm{I})$-type, such that for any x in these bands,

$$\left|\frac{s'_{(k+1,\mathrm{I})}(x)}{s'_{(k,\mathrm{III})}(x)}\right| = \left|\frac{t'_{(k+1,1)}(x)}{t'_{(k+1,0)}(x)}\right| \leq 4(V+8)(a_{k+1}+1) < \frac{1}{q'_{31}(k+1)}. \tag{4.45}$$

4). Let $p = a_{k+1} - 1(\geq 2)$, by Proposition 7.(ii) and Proposition 10, any (k, III)-type band $B_{(k+1,0)}$ contains at least $\max\{2, [(a_{k+1}-1)/2]\}(= b_{k+1})$ bands of $(k+1, \mathrm{III})$-type, such that for any x in these bands,

$$\left|\frac{s'_{(k+1,\mathrm{III})}(x)}{s'_{(k,\mathrm{III})}(x)}\right| = \left|\frac{t'_{(k+2,0)}(x)}{t'_{(k+1,0)}(x)}\right| \leq 4(V+8)a_{k+1} < \frac{1}{q'_{33}(k+1)}. \tag{4.46}$$

5). Any $(k+1, \mathrm{II})$-type band which is contained in some band of $\mathscr{G}'_k$.

Define $\mathscr{G}'_{k+1}$ the set of the bands chosen by the above five manners respectively and let $\mathscr{G}' = \bigcup_{k\geq 0} \mathscr{G}'_k$.

Similarly, if $a_{k+1} < 3$, by Proposition 7.(i), 8.(i) and Proposition 10 we can get (4.43)-(4.46) respectively for $(k+1, \mathrm{I})$ and $(k+1, \mathrm{III})$-type bands in $\mathscr{G}'_{k+1}$.

For any band $B \in \mathscr{G}'$ with type index $\tau = i_0 i_1 \cdots i_k$, from (4.43)-(4.46) and (4.41) (notice that $q'_{12}(n) = q_{12}(n)$ for $a_k \geq 3$ and $q'_{12}(n) < q_{12}(n)$ for $a_k < 3$), we obtain by the same discussion as in the proof of Proposition 5,

$$|B| \geq 4L_\tau(\mathscr{Q}'). \qquad \square$$

5. UPPER BOUND

This section is devoted to the estimation of the upper bound of the Hausdorff dimension of the spectrum $\sigma(H_\beta)$.

Keep the notations above, let $d_k = \max_{\tau\in D_k} L_\tau(\mathscr{P})$, and

$$\alpha^* := \inf\{0 \leq \alpha \leq 1 : \liminf_{k\to\infty} \sum_{\tau\in D_k} K_\tau(L_\tau(\mathscr{P}))^\alpha < \infty\}, \tag{5.1}$$

then we have

Proposition 11. *If* $\lim_{k\to\infty} d_k = 0$, *then* $\dim_H \sigma(H_\beta) \le \alpha^*$.

Proof. Suppose $1 \ge \alpha > \alpha^*$, then by the definition of α^*, there exist $M > 0$ and a sequence $k_n \subset \mathbb{N}$, such that for any $n \ge 1$

$$\sum_{\tau\in D_{k_n}} K_\tau(L_\tau(\mathscr{P}))^\alpha < M.$$

For any k, $\{B: B \in \mathscr{G}_k\}$ is a covering of H_β, since $\lim_{k\to\infty} d_k = 0$, we have by Proposition 5

$$\begin{aligned}\mathscr{H}^\alpha(\sigma(H_\beta)) &\le \liminf_{n\to\infty} \sum_{B\in\mathscr{G}_k} |B|^\alpha \\ &\le \liminf_{n\to\infty} 4^\alpha \sum_{\tau\in D_{k_n}} K_\tau(L_\tau(\mathscr{P}))^\alpha \\ &\le 4^\alpha M,\end{aligned}$$

which yields $\dim_H \sigma(H_\beta) \le \alpha$, since $\alpha > \alpha*$ is arbitrary, we get $\dim_H \sigma(H_\beta) \le \alpha^*$. $\square$

Consider the sequence of matrices

$$\mathscr{R} := \{\mathbf{R}(\alpha, n)\}_{n\ge1} := \{(r_{ij}(n))_{3\times3}\}_{n\ge1}$$

with $r_{ij}(n) := (t_{ij}(n)\cdot(p_{ij}(n))^\alpha)$. Let $\mathbf{v} = (1, 0, 1)$ be the 3-dimensional row vector and $\mathbb{1} = (1, 1, 1)'$ the transposition of the vector $(1, 1, 1,)$. Assume $\tau \in D$, $s \in \mathscr{A}$, if the last letter of τ is s, then we denote $\tau \succ s$. By the definitions of $\mathscr{R}$, $\mathscr{P}$, K_τ and L_τ, we have

$$\begin{aligned}&\textstyle\sum_{\tau\in D_k} K_\tau(L_\tau(\mathscr{P}))^\alpha \\ =\ & \Big(\sum_{\tau\in D_k,\tau\succ I} K_\tau(L_\tau(\mathscr{P}))^\alpha, \cdots, \sum_{\tau\in D_k,\tau\succ III} K_\sigma(L_\tau(\mathscr{P}))^\alpha\Big)\cdot\mathbb{1} \\ =\ & \Big(\sum_{s\in\mathscr{A}}\sum_{\tau\in D_{k-1},\tau\succ s} K_\tau(L_\tau(\mathscr{P}))^\alpha t_{s*I}(k)(p_{s*I}(k))^\alpha, \cdots, \\ & \quad \sum_{s\in\mathscr{A}}\sum_{\tau\in D_{k-1},\tau\succ s} K_\tau(L_\tau(\mathscr{P}))^\alpha t_{s*III}(k)(p_{s*III}(k))^\alpha\Big)\cdot\mathbb{1} \\ =\ & \Big(\sum_{\tau\in D_{k-1},\tau\succ I} K_\tau(L_\tau(\mathscr{P}))^\alpha, \cdots, \sum_{\tau\in D_{k-1},\tau\succ III} K_\sigma(L_\tau(\mathscr{P}))^\alpha\Big)\cdot\mathbf{R}(\alpha,k)\cdot\mathbb{1} \\ =\ & \mathbf{v}\cdot\mathbf{R}(\alpha,1)\cdot\mathbf{R}(\alpha,2)\cdot\cdots\cdot\mathbf{R}(\alpha,k)\cdot\mathbb{1},\end{aligned} \tag{5.2}$$

by associating with (5.1), we obtain another expression of α^*:

$$\alpha^* := \inf\{0 \le \alpha \le 1 :\ \liminf_{k\to\infty} \mathbf{v}\cdot\prod_{n=1}^k \mathbf{R}(\alpha, n)\cdot\mathbb{1} < \infty\}.$$

For any matrix $\mathbf{M} = (m_{ij})_{l\times l}$, consider the following norm of $\mathbf{M}$,

$$\|\mathbf{M}\| := \max_{1\le i\le l} \sum_{j=1}^l |m_{ij}|.$$

Since

$$\mathbf{v}\cdot\prod_{n=1}^k \mathbf{R}(\alpha, n)\cdot\mathbb{1} \le 2\left\|\prod_{n=1}^k \mathbf{R}(\alpha, n)\right\|, \tag{5.3}$$

by Proposition 11 and (5.3),

$$\dim_H \sigma(H_\beta) \le \inf\{0 \le \alpha \le 1 : \liminf_{k\to\infty} \left\| \prod_{n=1}^{k} \mathbf{R}(\alpha, n) \right\| < \infty\} \tag{5.4}$$

Proposition 12. *Let* $[a_1, a_2, \cdots]$ *be the continued fraction expansion of* β*, and*

$$M = \liminf_{k\to\infty}(a_1 a_2 \cdots a_k)^{1/k}.$$

Let $V > 20$ *and* $t_1 = \frac{3}{V-8}$*. Then*

$$\dim_H \sigma(H_\beta) \le \frac{2\ln M + \ln 3}{2\ln M - \ln t_1}.$$

Proof. From the definition of M, there exists a sequence $\{i_k\}_{k\ge1}$ such that

$$\lim_{k\to\infty}(a_1 a_2 \cdots a_{i_k})^{1/i_k} = M,$$

thus for any $\varepsilon > 0$, there exists $N > 0$ such that for any $k > N$,

$$(a_1 a_2 \cdots a_{i_k})^{1/i_k} < M + \varepsilon.$$

Define

$$\tilde{\mathbf{P}}_n := \{\tilde{p}_{ij}(n)\}_{3\times3} := \begin{bmatrix} 0 & a_n^{-1} & 0 \\ t_1(a_n)^{-1} & 0 & t_1(a_n)^{-1} \\ t_1(a_n)^{-1} & 0 & t_1(a_n)^{-1} \end{bmatrix},$$

since for $V > 20$, $a_n^{-1} \ge t_1^{a_n-1}$, we have for any $n \ge 1$, $\tilde{\mathbf{P}}_n \ge \mathbf{P}_n$. Thus

$$\tilde{\mathbf{R}}(\alpha, n) := (t_{ij}(n)(\tilde{p}_{ij}(n)^\alpha)_{3\times3} = \left(\frac{t_1}{a_n}\right)^\alpha \begin{bmatrix} 0 & t_1^{-\alpha} & 0 \\ a_n+1 & 0 & a_n \\ a_n & 0 & a_n-1 \end{bmatrix}$$

It is difficult to estimate directly the norm of the product of above matrices, but if we consider the product of the product of two consecutive matrices, we can estimate it. Now we proceed the estimation.

Notice that

$$\begin{aligned} & \left\|(\tilde{\mathbf{R}}(\alpha, n)\cdot\tilde{\mathbf{R}}(\alpha, n+1))\right\| \\ = & \left\| \left(\frac{t_1^2}{a_n a_{n+1}}\right)^\alpha \begin{bmatrix} (a_{n+1}+1)t_1^{-\alpha} & 0 & a_{n+1}t_1^{-\alpha} \\ a_n a_{n+1} & (a_n+1)t_1^{-\alpha} & a_n(a_{n+1}-1) \\ (a_n-1)a_{n+1} & a_n t_1^{-\alpha} & (a_n-1)(a_{n+1}-1) \end{bmatrix} \right\| \\ \le & \; 3t_1^\alpha(a_n a_{n+1})^{1-\alpha}, \end{aligned}$$

the last inequality can be obtained by distinguishing two cases $a_{n+1} > 1$ and $a_{n+1} = 1$. Therefore

$$\begin{aligned} & \left\| \prod_{j=i}^{i+2k-1} \tilde{\mathbf{R}}(\alpha, j) \right\| \\ \le & \left\|\tilde{\mathbf{R}}(\alpha, i)\tilde{\mathbf{R}}(\alpha, i+1)\right\| \cdots \left\|\tilde{\mathbf{R}}(\alpha, i+2k-2)\tilde{\mathbf{R}}(\alpha, i+2k-1)\right\| \\ \le & \; 3^k t_1^{k\alpha}(a_i a_{i+1}\cdots a_{i+2k-1})^{1-\alpha}. \end{aligned}$$

Hence if $1 \geq \alpha \geq \dfrac{2\ln(M+\varepsilon)+\ln 3}{2\ln(M+\varepsilon)-\ln t_1}$, then for any $k > N$

$$\begin{aligned}\left\|\prod_{j=1}^{i_k}\mathbf{R}(\alpha,j)\right\| &\leq \left\|\prod_{j=1}^{i_k}\tilde{\mathbf{R}}(\alpha,j)\right\| \\ &\leq \max\{1,\left\|\tilde{\mathbf{R}}(\alpha,1)\right\|\}3^{[i_k/2]}t_1^{[i_k/2]\alpha}(a_1a_2\cdots a_{i_k})^{1-\alpha} \\ &\leq C3^{i_k/2}t_1^{\alpha i_k/2}(M+\varepsilon)^{i_k(1-\alpha)} \\ &\leq C<\infty,\end{aligned}$$

which yields that $\dim_H \sigma(H_\beta) \leq \alpha$ by (5.4), so $\dim_H \sigma(H_\beta) \leq \dfrac{2\ln(M+\varepsilon)+\ln 3}{2\ln(M+\varepsilon)-\ln t_1}$.

Let $\epsilon \to 0$, we get finally

$$\dim_H \sigma(H_\beta) \leq \frac{2\ln M+\ln 3}{2\ln M-\ln t_1}. \qquad \square$$

6. LOWER BOUND

In this section, we will estimate the lower bound of the spectrum by using the idea for estimating the Homogeneous Moran sets([5, 7]).

Proposition 13. *Let $\{n_k\}_{k\geq 1}$ be a sequence of positive integers and $\{c_k\}_{k\geq 1}$ a sequence of positive numbers with $0 < c_k \leq 1$ for any $k \geq 1$ and $c_1c_2\cdots c_k \to 0$ if $k \to \infty$. Suppose $\mathscr{F}$ is a subset of $\mathscr{G}$ such that for any $k \geq 1$ and $B \in \mathscr{F}\bigcap\mathscr{G}_k$,*

1). B contains n_{k+1} bands in $\mathscr{F}\bigcap\mathscr{G}_{k+1}$;

2). $|B| \geq 4c_1c_2\cdots c_k$.

Then

$$\dim_H \sigma(H_\beta) \geq \liminf_{k\to\infty}\frac{\ln n_1n_2\cdots n_k}{-\ln c_1c_2\cdots c_{k+1}n_{k+1}}.$$

Proof. Let $E = \bigcap_{k\geq 1}\bigcup_{B\in\mathscr{F}\cap\mathscr{G}_k} B$. Since $\sigma(H_\beta) = \bigcap_{k\geq 1}\mathscr{G}_k$ by Lemma 4, $E \subset \sigma(H_\beta)$.

Let $t_* = \liminf\limits_{k\to\infty}\frac{\ln n_1n_2\cdots n_k}{-\ln c_1c_2\cdots c_kc_{k+1}n_{k+1}}$ and $\delta_k = c_1c_2\cdots c_k$, then for any $\alpha < t_*$, there exists integer $k_0 > 0$ such that for any $k \geq k_0$,

$$n_1n_2\cdots n_k(\delta_{k+1}n_{k+1})^\alpha \geq 1.$$

Suppose μ is the probability measure on E such that for any band $B \in \mathscr{F}$ of order k,

$$\mu(B) = (n_1\cdots n_k)^{-1}.$$

From Frostman Lemma, we only need to prove that for any closed interval U with $|U| < 4\delta_{k_0}$, $\mu(U) \leq c|U|^\alpha$, where c is a positive constant independent of U.

Now given $|U| < 4\delta_{k_0}$, then there exists $k \geq k_0$ such that $4\delta_{k+1} \leq |U| < 4\delta_k$, so there are at most two bands in $\mathscr{F}\bigcap\mathscr{G}_k$ intersect with U, and hence at most $2n_{k+1}$ bands in $\mathscr{F}$ of order $k+1$ intersect U. On the other hand, U intersect at most $2(\frac{|U|}{4\delta_{k+1}}+1) \leq \frac{|U|}{\delta_{k+1}}$ bands in $\mathscr{F}$ of order $k+1$. Since $\min\{a,b\} \leq a^tb^{1-t}$ for any $a,b > 0$ and $0 \leq t \leq 1$, we have

$$\begin{aligned}\mu(U) &\leq \frac{1}{n_1\cdots n_{k+1}}\min\{\frac{|U|}{\delta_{k+1}}, 2n_{k+1}\}\\ &\leq \frac{1}{n_1\cdots n_{k+1}}\left(\frac{|U|}{\delta_{k+1}}\right)^{\alpha}(2n_{k+1})^{1-\alpha}\\ &\leq \frac{2^{1-\alpha}|U|^{\alpha}}{n_1\cdots n_k(n_{k+1}\delta_{k+1})^{\alpha}}\\ &\leq 2^{1-\alpha}|U|^{\alpha},\end{aligned}$$

which yields the conclusion of the proposition. □

Proposition 14. *Let $[a_1, a_2, \cdots]$ be the continued fraction expansion of β and $M = \liminf_{k\to\infty}(a_1a_2\cdots a_k)^{1/k}$. Let $V > 20$ and $t_2 = \frac{1}{4(V+8)}$, then*

1). If $M < \infty$, we have

$$\dim_H \sigma(H_\beta) \geq \max\{\frac{\ln 2}{10\ln 2 - 3\ln t_2}, \frac{\ln M - \ln 3}{\ln M - \ln t_2/3}\}.$$

2). If $M = \infty$, we have

$$\dim_H \sigma(H_\beta) = 1.$$

Proof. We first construct a subset $\mathscr{F}$ of $\mathscr{G}'$(which has been constructed in Proposition 6) by the following way.

Let $m_0 = \inf\{n \geq 1 : a_n \geq 3\}$, by convention define $m_0 = \infty$ if for any n, $a_n < 3$.

If $a_1 \geq 3$, then let $\mathscr{F}_0 = [-2, 2]$, which is a $(0, \mathrm{III})$-type band.

Consider now the case of $a_1 < 3$ and we distinguish three different cases: If m_0 is odd, take $\mathscr{F}_0 = [V-2, V+2]$, which is a $(0, \mathrm{I})$-type band; if m_0 is even, take $\mathscr{F}_0 = [-2, 2]$; if m_0 is infinity, then we choose $\mathscr{F}_0$ to be one of the above two bands arbitrarily. Set $i_0 = \mathrm{I}$ or III according to the type of band we choose above.

Suppose $\mathscr{F}_{k-1}$ has been chosen, then all bands of $\mathscr{F}_{k-1}$ have the same type index $i_0i_1\cdots i_{k-1}$. Now we define $\mathscr{F}_k$, let $B \in \mathscr{F}_{k-1}$, we choose the bands of order k generated by B by the following rule:

1). Suppose $a_k < 3$. If B is of type $(k-1, \mathrm{I})$, then only one (k, II)-type band will be generated by B that we choose as the element of $\mathscr{F}_k$, in this case, $n_k = 1$; if B is of type $(k-1, \mathrm{II})$, then we take all $a_k + 1$ bands of (k, I)-type generated by B as the elements of $\mathscr{F}_k$, and $n_k = a_k + 1$; if B is type $(k-1, \mathrm{III})$, then we take all a_k bands of (k, I)-type bands generated by B as the elements of $\mathscr{F}_k$, and $n_k = a_k$. Take $i_k = \mathrm{I}$ or II according to the type of the band we choose.

2). Suppose $m_0 < \infty$, we only need to consider the case $a_k \geq 3$.

i). If B is a $(k-1, \mathrm{II})$-type band or a $(k-1, \mathrm{III})$-type band, then by Proposition 6, it contains at least $b_k(= \max\{2, [\frac{a_k-1}{2}]\})$ bands of (k, I)-type and b_k bands of (k, III)-type which satisfy (3.9), we only take b_k bands of (k, I)-type or b_k bands of (k, III)-type bands(i.e., $n_k = b_k$) from $\mathscr{G}'_k$ by the following way: if $a_{k+1} \geq 3$, then take b_k bands of (k, III)-type; if $a_{k+1} < 3$, we distinguish further two different cases, let $m_k = \inf\{n > k : a_n < 3\} - k$, then take b_k bands of (k, I)-type if m_k is odd, and b_k bands of (k, III)-type if m_k is even. Take $i_k = \mathrm{I}$ or III according our choice for the bands.

ii). By the discussions above, the case B is $(k-1, \mathrm{I})$-type band will not happen.

Let $\mathscr{F}_k$ be the set of the spectral generating bands of order k chosen by the above way and $\mathscr{F} := \bigcup_{k\geq 0}\mathscr{F}_k$, let $\tau = i_0 i_1\cdots i_k\cdots$ and $\tau|_k = i_0 i_1\cdots i_k$. Then for any $k\geq 0$, $\mathscr{F}_k\subset\mathscr{G}'_k$ so $\mathscr{F}\subset\mathscr{G}'$, and every band of $\mathscr{F}_k$ has type index $\tau|_k$. By the above discussions, we have

(a1). For any $B\in\mathscr{F}_k$, B contains n_{k+1} bands in $\mathscr{F}_{k+1}$,

(a2). Let $c_k = t_2(a_k+2)^{-1}$, then by Proposition 6, $|B|\geq 4\mathscr{L}_{\tau|_k}(\mathscr{Q}) = 4c_1c_2\cdots c_k$.

(a3). For any $k\geq 1$, $n_k\leq a_k+2\leq 6n_k$, $n_k > a_k/3$;

(a4). For any $j\geq 1$, at least one of n_j, n_{j+1}, n_{j+2} is larger than 2, so for any $k\geq 1$, $n_1n_2\cdots n_k\geq 2^{k/3-1}$.

(a5). Let $E := \cap_{k\geq 1}\mathscr{G}_k$, then $E\subset\sigma(H_\beta)$.

By Proposition 13 and properties (a1)-(a5), we have

$$\begin{aligned}
\dim_H\sigma(H_\beta) &\geq \dim_H E\geq \liminf_{k\to\infty}\frac{\ln n_1n_2\cdots n_k}{-\ln c_1c_2\cdots c_{k+1}n_{k+1}}\\
&= \liminf_{k\to\infty}\frac{\ln n_1n_2\cdots n_k}{-(k+1)\ln t_2+\ln[(a_1+2)\cdots(a_{k+1}+2)/n_{k+1}]}\\
&\geq \liminf_{k\to\infty}\frac{\ln(n_1n_2\cdots n_k)^{1/k}}{\ln(n_1n_2\cdots n_k)^{1/k}-(\ln t_2)(k+1)/k+(\ln 6)(k+1)/k}\\
&\geq \frac{\ln 2^{1/3}}{\ln 6+\ln 2^{1/3}-\ln t_2}\geq\frac{\ln 2}{10\ln 2-3\ln t_2},
\end{aligned}$$

and on the other hand

$$\begin{aligned}
\dim_H\sigma(H_\beta) &\geq \dim_H(E)\geq \liminf_{k\to\infty}\frac{\ln n_1n_2\cdots n_k}{-\ln c_1c_2\cdots c_{k+1}n_{k+1}}\\
&\geq \liminf_{k\to\infty}\frac{\ln(a_1/3)(a_2/3)\cdots(a_k/3)}{-(k+1)\ln t_2+\ln(a_1+2)(a_2+2)\cdots(a_k+2)(a_{k+1}+2)/n_{k+1}}\\
&\geq \liminf_{k\to\infty}\frac{\ln(a_1a_2\cdots a_k)^{1/k}-\ln 3}{-(\ln t_2)(k+1)/k+\ln 3(a_1a_2\cdots a_k)^{1/k}+(\ln 6)/k}\\
&\geq \frac{\liminf_{k\to\infty}\ln(a_1a_2\cdots a_k)^{1/k}-\ln 3}{\liminf_{k\to\infty}\ln(a_1a_2\cdots a_k)^{1/k}+\ln 3-\ln t_2},
\end{aligned}$$

which yields $\dim_H\sigma(H_\beta)\geq\frac{\ln M-\ln 3}{\ln M-\ln(t_2/3)}$ if $M<\infty$ and $\dim_H\sigma(H_\beta)\geq 1$ if $M=\infty$.

We prove thus the conclusion of the proposition. □

Proof of Theorem 1.

The proof follows from directly Propositions 13 and 14. □

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