SPECTRAL GAPS FOR PERIODIC ELLIPTIC OPERATORS WITH HIGH CONTRAST: AN OVERVIEW

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Abstract. We discuss the band-gap structure and the integrated density of states for periodic elliptic operators in the Hilbert space $L_2(\mathbb{R}^m)$, for $m \geq 2$. We specifically consider situations where high contrast in the coefficients leads to weak coupling between the period cells. Weak coupling of periodic systems frequently produces spectral gaps or spectral concentration. Our examples include Schrödinger operators, elliptic operators in divergence form, Laplace-Beltrami-operators, Schrödinger and Pauli operators with periodic magnetic fields. There are corresponding applications in heat and wave propagation, quantum mechanics, and photonic crystals.

Introduction

We consider periodic elliptic partial differential operators of the second order with high contrast in (some of) the coefficients. The classes of operators discussed here include Schrödinger operators, magnetic Schrödinger and Pauli operators, the acoustic operator, Laplace-Beltrami operators, and Maxwell operators. We are mainly interested in the question whether high contrast in the coefficients leads to weak coupling between the period cells and to spectral concentration and/or spectral gaps. For most of the paper, we assume a simple geometry like the one shown in Figure 1, for which the strongest results are obtained. We include some remarks on more general situations, like sponges in dimensions 3 and higher. Our main theme is to show that simple topological assumptions lead to spectral consequences for a wide class of—rather different—elliptic operators. We do not discuss the question of absolute continuity here.

The paper is organized as follows. Section 1 contains some basic definitions, assumptions and background results concerning monotone convergence of quadratic forms. In Section 2, we briefly report on some "classical" results on the convergence of Schrödinger operators with high barriers. Similar results for strong magnetic barriers in the Schrödinger and Pauli case are discussed in Section 3. Section 4 is devoted to divergence type operators $-\nabla \cdot (1 + \lambda\chi_\Omega)\nabla$, where Ω is a periodic, open subset of $\mathbb{R}^m$, with $m \geq 2$. There is some superficial similarity between divergence type operators and Laplace-Beltrami operators. The existence of spectral gaps for periodic Laplace-Beltrami operators with a conformal

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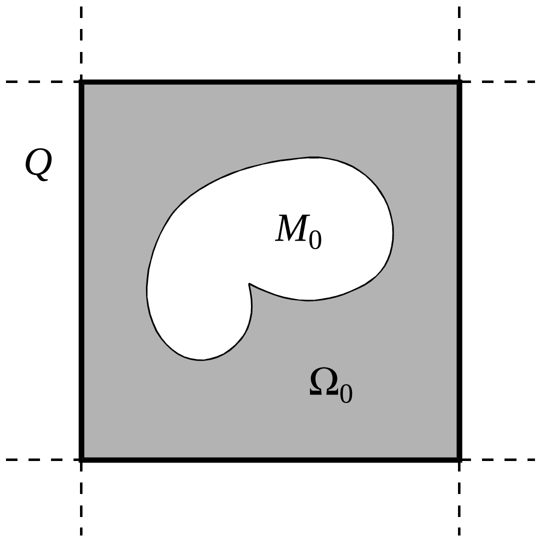

FIGURE 1. The fundamental period cell Q and the set $M_0 = Q \cap M$.

factor in the metric tensor is discussed in Section 5. We also refer to constructions of a purely "geometric" nature that yield Laplace-Beltrami-operators with gaps. We conclude in Section 6 with some references to the work of A. Figotin, P. Kuchment and others on the Maxwell operator and applications to photonic crystals.

1. Preliminaries and basic assumptions

Throughout the present paper, we will always assume that $\Omega \subset \mathbb{R}^m$ is open and periodic w.r.t. some discrete lattice Γ; for most purposes it is enough to consider $\Gamma = \mathbb{Z}^m$. Q denotes a fundamental period cell of Γ; we may take $Q = (0, 1]^m$ if $\Gamma = \mathbb{Z}^m$. The complement of Ω is denoted as M, a closed and periodic set. We will study two-component media with material parameter $\lambda = 1$ on M, and $\lambda \gg 1$ on Ω. The strongest results are obtained in the "model situation" of Figure 1 where Ω contains the boundary of the period cell Q.

Our basic Hilbert spaces are $\mathcal{H} = L_2(\mathbb{R}^m)$ and $\mathcal{H}^1(\mathbb{R}^m)$, the Sobolev space of L_2-functions that have weak derivatives in L_2.

All our operators are semi-bounded, self-adjoint operators T acting in some L_2-space; they are obtained from some closed, semi-bounded quadratic form $\mathfrak{t}$ via the usual representation theorem ([K, RS-I]) and satisfy $D(T) \subset D(\mathfrak{t})$ and the relation

$$\mathfrak{t}[u, v] = \langle Tu, v \rangle, \qquad u \in D(T), \quad v \in D(\mathfrak{t}). \tag{1.1}$$

As usual, we say $T_2 \geq T_1$ iff the associated quadratic forms satisfy $D(\mathfrak{t}_2) \subset D(\mathfrak{t}_1)$ and $\mathfrak{t}_2[u, u] \geq \mathfrak{t}_1[u, u]$, for all $u \in D(\mathfrak{t}_2)$. We also write $\mathfrak{t}[u] := \mathfrak{t}[u, u]$ in the sequel.

For a sequence of non-negative, self-adjoint operators T_n in $\mathcal{H}$ we say that T_n converges to a limit T_∞ in the *strong resolvent sense* iff

$$(T_n + I)^{-1} f \to (T_\infty + I)^{-1} f, \qquad f \in \mathcal{H}; \tag{1.2}$$

for convergence in the *norm resolvent sense* we require uniform convergence for all $f \in \mathcal{H}$ s.th. $\|f\| \leq 1$.

Most of our results hinge upon the following convergence theorem for quadratic forms ([RS-I], [S]):

1.1. **Theorem.** *Let (T_n) be a monotonic sequence of non-negative, self-adjoint operators. Then there exists a self-adjoint operator $T_\infty \geq 0$ s.th. $T_n \to T_\infty$ in strong resolvent sense.*

For an *increasing* sequence of quadratic forms $0 \leq \mathfrak{t}_n \leq \mathfrak{t}_{n+1}$ with domains $\mathcal{Q}_n$ the limiting form domain is given by $\mathcal{Q}_\infty = \{u \in \cap_{n=1}^\infty \mathcal{Q}_n; \sup \mathfrak{t}_n[u] < \infty\}$. Here it may happen that T_∞ acts in a strictly smaller Hilbert space than the T_n. In this case, $(T_\infty + I)^{-1}$ has to be complemented by the zero operator on a suitable subspace of $\mathcal{H}$. For a *decreasing* sequence it is shown in [RS-I, S] that there exists a largest closable form that is smaller than $\mathfrak{t}_n$, for all n.

In a periodic situation, strong resolvent convergence often implies convergence of spectral densities. For the definition of the *integrated density of states* (i.d.s.) we refer to [RS-IV], [PF] or [DIT]. In our context, the i.d.s. is a monotonically increasing function $F : \mathbb{R} \to \mathbb{R}$ that tends to zero at $-\infty$. Such functions have at most a countable number of discontinuities. The associated Borel measure μ is then called the *density of states measure.* We will say that a sequence (μ_n) of density of states measures converges weakly to μ_∞ if $F_n(t) \to F_\infty(t)$ at all points of continuity t of F_∞. The proof of the subsequent proposition is elementary if one uses the definition of [RS-IV] for the i.d.s.

1.2. **Proposition.** *In addition to the assumptions of Theorem 1.1, suppose that the operators T_n are periodic (with respect to the the same lattice Γ), and that a (finitely-valued) i.d.s. F_n exist for T_n as well as for T_∞. Then the corresponding measures μ_n converge weakly to μ_∞.*

It is well-known that *norm resolvent convergence* implies spectral convergence (cf. [RS-I; Ch. VIII]) on any compact interval of the real line. In particular, we have the following simple consequence for the existence of spectral gaps: suppose $T_n \to T_\infty$ in norm-resolvent sense. If T_∞ has a gap around $E \in \mathbb{R}$, then the T_n will also have a gap around E, for large n. More precisely, suppose that $(a, b) \cap \sigma(T_\infty) = \emptyset$ and let $\varepsilon > 0$. Then $(a + \varepsilon, b - \varepsilon) \cap \sigma(T_n) = \emptyset$, for all $n \geq n_\varepsilon$.

2. Schrödinger operators

As a warm-up, we illustrate the above convergence theorems by some simple results on Schrödinger operators of the form $-\Delta + \lambda V(x)$, where V is non-negative. It has been a "folk theorem" for quite some time that $-\Delta + \lambda\chi_B$ converges in strong resolvent sense to the Dirichlet Laplacian on the complement $\mathbb{R}^m \setminus B$, under suitable assumptions on B.

Suppose $V(x) \geq 0$ is a periodic, real-valued (continuous or measurable and bounded) potential s.th. $\{x \in \mathbb{R}^m \mid V(x) \neq 0\} = \Omega$. We let

$$H_\lambda = -\Delta + \lambda V(x), \qquad \lambda \geq 0, \tag{2.1}$$

with associated quadratic forms $\mathfrak{h}_\lambda[u] = \|\nabla u\|^2 + \lambda \int V|u|^2$, for $u \in \mathcal{H}^1$.

To describe the limit, we will need here (and in the following sections) the *Dirichlet Laplacian* on the closed set $M = \mathbb{R}^m \setminus \Omega$, denoted as $-\Delta_M$ ([Hed, HZ]). The operator $-\Delta_M$ is constructed from the quadratic form $\|\nabla u\|^2$ with domain

$$\tilde{\mathcal{H}}^1_0(M) := \{u \in \mathcal{H}^1(\mathbb{R}^m) \mid u(x) = 0 \text{ a.e. in } M^C\}. \tag{2.3}$$

For sufficiently regular boundary of M, $-\Delta_M$ agrees with the usual Dirichlet Laplacian. In the general case, it is shown in [HZ] that there exists a Borel set M^*, $M^\circ \subset M^* \subset M$, such that $-\Delta_M$ acts as a self-adjoint operator in the Hilbert-space $L_2(M^*)$. It is now easy to see that

$$\{u \in \mathcal{H}^1(\mathbb{R}^m) \mid \sup_{\lambda \geq 0} \mathfrak{h}_\lambda[u] < \infty\} = \tilde{\mathcal{H}}^1_0(M), \tag{2.4}$$

and so Theorem 1.1 implies that H_λ converges in strong resolvent sense to $-\Delta_M$. (To be more precise, the resolvent of $-\Delta_M$ acts in $L_2(M^*)$ and has to be complemented by the zero operator on $L_2(\mathbb{R}^m \setminus M^*)$ in Eqn. (1.1).) Up to this point, periodicity was not required. If we now assume that Ω is periodic, then for all operators involved the i.d.s. is well-defined, and we may also conclude that the i.d.s. converges.

As was observed in [HH-1], one can actually do better and upgrade to norm resolvent convergence. The proof combines uniform exponential decay estimates for the resolvent kernels with Schur's Lemma and local compactness of $-\Delta$.

The above convergence results do not require Ω to be connected. In our "model case" of Figure 1 where Ω contains a neighborhood of ∂Q, $-\Delta_M$ is a countable direct sum of copies of $-\Delta_{M \cap Q}$. Therefore, the spectrum of $-\Delta_M$ is a discrete set with each point in the spectrum an eigenvalue of infinite multiplicity. We therefore see that the spectrum of H_λ concentrates at a discrete set of points, and an arbitrarily large number of spectral gaps opens up, as $\lambda \to \infty$.

3. Magnetic Schrödinger and Pauli operators

Here we briefly mention some results of [HH-1,2], [HN], [B-1,2] on strongly coupled periodic magnetic fields.

(1) Let $\vec{a} \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ be a magnetic vector potential and $B = d\vec{a}$ the associated magnetic field. We assume that B is periodic w.r.t. the lattice Γ, and that $B(x) = 0$ for all $x \in M$, while $B(x) \neq 0$ for all $x \in \Omega$. Note that periodicity of $\vec{a}$ is *not* required. The magnetic Schrödinger operator is then given as [CFKS]

$$H(\lambda\vec{a}) = (-\mathrm{i}\nabla - \lambda\vec{a})^2, \tag{3.1}$$

and obtained via the closure of the forms

$$\langle H(\lambda\vec{a})u, u\rangle = \|(-\mathrm{i}\nabla - \lambda\vec{a}(x))u\|^2, \qquad u \in C_c^\infty(\mathbb{R}^m).$$

Evidently, there is *no* monotonicity w.r.t. λ. However, in any open subset U of Ω where a fixed entry $B_{ij}(x)$ of the magnetic tensor is bounded away from zero, the Avron-Herbst-Simon bound [AHS] gives a *local* lower bound for the quadratic form, expressing the barrier-like effect of a strong magnetic field. If the vector potential $\vec{a}$ vanishes on M, the above results for $-\Delta + \mu\chi_U$, $\mu \to \infty$, can be combined with the Feynman-Kac-Itô-formula to obtain strong resolvent convergence of $H(\lambda\vec{a})$ to $-\Delta_M$, as $\lambda \to \infty$ ([HH-1]). For given magnetic field B, it depends on the topology of the set Ω and on flux conditions whether a gauge exists s.th. $d\vec{a} = B$ and $\vec{a}(x) = 0$ a.e. on M.

Under the above conditions, we also obtain an upgrade to norm-resolvent convergence, as before, and get the same spectral consequences as for $-\Delta + \lambda\chi_\Omega$.

(2) The situation where $\vec{a}(x)$ is allowed to have non-zero values on M has been analyzed in detail in [HN], using some basic homology theory and de Rham's Theorem. It can be shown that, under suitable assumptions, the norm difference of the resolvents of $H(\lambda\vec{a})$ and $H_M(\lambda\vec{a})$ tends to zero, as $\lambda \to \infty$, where $H_M(\lambda\vec{a}) = (-\mathrm{i}\nabla - \lambda\vec{a})^2$, acting in $L_2(M^*)$ with Dirichlet boundary conditions. As a consequence, one can conclude that the spectrum of $H(\lambda\vec{a})$ approaches a periodic or a quasi-periodic function of λ, as $\lambda \to \infty$. The precise result involves the flux of B through the connected components of $(\overline{M^\circ})^C$, the complement of the closure of the interior of M.

(3) For the **Pauli operator** in $\mathbb{R}^2$, we have the pair of operators

$$H_\pm(\lambda\vec{a}) = H(\lambda\vec{a}) \mp B, \tag{3.2}$$

both acting in $L_2(\mathbb{R}^2)$. Here supersymmetry (cf. [CFKS]) implies

$$\sigma(H_+(\lambda\vec{a})) \setminus \{0\} = \sigma(H_-(\lambda\vec{a})) \setminus \{0\}. \tag{3.3}$$

Note that the results in (1) and (2) remain unchanged if we include an electric background potential V (e.g., V bounded and measurable), while supersymmetry is destroyed upon addition of an electric potential.

Under suitable assumptions, one can show ([B-1,2]) that the operators $H_\pm(\lambda\vec{a})$ have a common gap: in the simplest case, we assume $B \geq 0$. A more delicate result requires that B is positive in a neighborhood of ∂Q and that there exists a periodic gauge s.th. $d\vec{a} = B$. (By the divergence theorem, this gauge condition is equivalent to the *flux condition* $\int_Q B(x)dx = 0$).

4. Periodic divergence type operators

Here we report on some results of [HL-1,2] concerning periodic two-component media with high contrast. In the following we always assume $m \geq 2$. The operators dicussed below occur in acoustics, heat conduction, and propagation of electro-magnetic waves in photonic crystals.

The quadratic form is given by the *Dirichlet integral*

$$\mathfrak{t}_\lambda[u] := \int (1 + \lambda\chi_\Omega(x))|\nabla u(x)|^2 dx, \qquad u \in \mathcal{H}^1, \quad \lambda \geq 0; \tag{4.1}$$

this defines a monotonically increasing family of forms. The associated self-adjoint operators T_λ can formally be written as $T_\lambda = -\nabla \cdot (1 + \lambda\chi_\Omega)\nabla$.

It is clear that only such functions u from the form domain survive taking $\lambda \to \infty$ that are constant on each connected component of Ω. If Ω is connected, this implies that only such u's will remain in the limiting form domain $\mathcal{Q}_\infty$ that are identically zero on Ω. We have:

4.1. **Theorem.** [HL-1] *Suppose Ω is open, periodic, and connected. Then T_λ converges to the Dirichlet-Laplacian $-\Delta_M$ on $M = \mathbb{R}^m \setminus \Omega$ in strong resolvent sense. Furthermore, the associated density of states measures converge weakly.*

The standard Floquet-decomposition gives a more detailed picture. Let $T_\lambda^{(\theta)}$ denote the operator $-\nabla \cdot (1 + \lambda\chi_\Omega)\nabla$, acting in $L_2(Q)$, with θ-periodic boundary conditions on ∂Q, for $\theta \in (-\pi, \pi]^m$; cf. [RS-IV, Ku]. Then Theorem 1.1 can be applied in each fiber, i.e., for each fixed θ. As each $T_\lambda^{(\theta)}$ has compact resolvent, we even obtain norm resolvent convergence in the fibers. The limiting operators $T_\infty^{(\theta)}$ have form domains $\mathcal{Q}_\infty^{(\theta)}$ consisting of those functions $u \in \mathcal{H}^1(Q)$ that are constant in each connected component of $\Omega \cap Q$ and satisfy θ-periodic boundary conditions. If Ω is connected and intersects each face of the period cell Q, it follows that such u must vanish on $\Omega \cap Q$, provided $\theta \neq (0, \ldots, 0)$. Pursuing this observation in detail, one obtains

4.2. **Theorem.** ([HL-1]) *Suppose $\Omega \subset \mathbb{R}^m$ is open, periodic, connected and contains ∂Q. We also assume that M^* (defined in Section 2) is not of measure zero. Let $0 < \delta_1 \leq \delta_2 \leq \ldots$ denote the eigenvalues of $-\Delta_{M\cap Q}$. We then have:*

(i) The first spectral band of T_λ converges to the interval $[0, \delta_1]$.

(ii) For λ large, T_λ has a spectral gap following the first band.

(iii) The i.d.s. measure of T_λ concentrates at the set $\{\delta_k; k \in \mathbb{N}\}$, as $\lambda \to \infty$.

It follows from (i) that there is *no* norm resolvent convergence of the family (T_λ) (The only possible candidate for the limit is $-\Delta_M$ which has no spectrum below $\delta_1 > 0$. But norm resolvent convergence implies spectral convergence.)

For smoothly bounded M a much more detailed picture of the process of spectral concentration is given in [F].

4.3. *Remark.* More general results can be found in [HL]. First of all, without additional effort one can handle elliptic operators of the type $-\nabla \cdot (\mathbf{a}(x) + \lambda\mathbf{b}(x))\nabla$, under suitable assumptions on the (symmetric) matrices $\mathbf{a}(x)$ and $\mathbf{b}(x)$, like $a_{ij}, b_{ij} \in L_\infty(\mathbb{R}^m)$, $\mathbf{a}(x)$ uniformly positive definite, $\mathbf{b}(x) = \mathbf{0}$ on M, and positive definite for $x \in \Omega$.

Second, [HL] also gives criteria for the existence of higher gaps: if the eigenspace of the k-th Dirichlet eigenvalue δ_k of $-\Delta_{M\cap Q}$ contains an eigenfunction d_k s.th.

$\int d_k \neq 0$ then $\sigma(T_\lambda)$ will have a gap for λ large that converges to (δ_k, ν_k), for some $\nu_k > \delta_k$.

4.4. *Remark.* There are natural and simple situations where one is led to consider period cells that are more general than $(0,1]^m$. In fact, all of the above works if Γ is a discrete sublattice of $\mathbb{R}^m$ that spans $\mathbb{R}^m$ and if Q is a contractible subset of $\mathbb{R}^m$ with piecewise smooth boundary tessalating $\mathbb{R}^m$ via the translations $\gamma \in \Gamma$. Here we think, in particular, of the class of results that depend on the assumption that Ω contains the boundary of the period cell.

4.5. *Remark.* In $\mathbb{R}^m$, $m \geq 3$, it is possible to have both Ω and M connected. A *sponge* is a typical example (albeit, in reality, not periodic but random!). Here our analysis would yield that, for high contrast, the i.d.s. is approximately given by the i.d.s. of T_∞. The latter can be further analyzed by Floquet-decomposition if we assume that Ω intersects each face of Q (think of the case where M is connected from one cell to the neighboring cells through thin filaments only).

4.6. *Remark.* It is natural to ask what happens for a *decreasing* sequence of operators like

$$S_\lambda = -\nabla \cdot \frac{1}{1+\lambda\chi_M}\nabla, \qquad 0 \leq \lambda \uparrow \infty. \tag{4.2}$$

The associated quadratic forms $\mathfrak{s}_\lambda$ are defined by $\mathfrak{s}_\lambda[u] = \int (1+\lambda\chi_M)^{-1}|\nabla u|^2 dx$ on $\mathcal{H}^1(\mathbb{R}^m)$. Evidently, the largest form that is smaller than $\mathfrak{s}_\lambda$, for all $\lambda \geq 0$, has domain $\mathcal{H}^1(\mathbb{R}^m)$ and takes the values $\int_\Omega |\nabla u|^2 dx$. Let us call this form $\mathfrak{s}_\infty$. It is easy to see that $\mathfrak{s}_\infty$ is closable. Let S_∞ denote the self-adjoint operator associated with the closure of $\mathfrak{s}_\infty$. It follows from Simon's theorem [RS-I; Thm. S. 16] that the operators S_λ converge to S_∞ in strong resolvent sense.

Let us assume, for the moment, that any $u \in \mathcal{H}^1(\Omega)$ can be extended to a function in $\mathcal{H}^1(\mathbb{R}^m)$ and that $C_c^\infty(M^\circ)$ is dense in $L_2(M)$. Then it is not difficult to show that the closure of the form $\mathfrak{s}_\infty$ has domain $\mathcal{H}^1(\Omega) \oplus L_2(M)$, and takes the values $\mathfrak{s}_\infty[u] = \int_\Omega |\nabla u(x)|^2 dx$. Under these assumptions S_∞ is the direct sum of the Neumann Laplacian on Ω and the zero operator on $L_2(M)$. In particular, the i.d.s. of S_∞ would be infinite on the positive real line, and it seems rather hopeless to extract any useful spectral information on S_λ. (Of course, the rôles of M and Ω are interchangeable here if the boundary of M is smooth enough.)

4.7. *Remark.* In the theory of photonic crystals, one is led to study the "high-contrast/thin-wall limit" of the acoustic operator in $\mathbb{R}^2$. In typical applications, the optically dense medium occupies a connected, periodic set M consisting of thin walls, cf. [FK-1,2]. In our notation, we would then consider the limit of $T_\lambda = -\nabla(1+\lambda\chi_\Omega)\nabla$, as $\lambda \to \infty$. Note that now Ω is *not* connected. Our results yield a strong resolvent limit and a Floquet-decomposition analogous to what we have discussed above. The limiting operators $T_\infty^{(\theta)}$ in the fibers now have a form domain consisting of functions that are constant on $\Omega \cap Q$, where the constant value will be non-zero, in general. Therefore, it is much harder to determine the spectrum of the $T_\infty^{(\theta)}$ than in the case of Eqn. (4.1). If $\Omega \cap Q$ exhausts "most of" Q (thin wall asymptotics), then the analysis can proceed again [FK-1,2], [KK].

4.8. *Remark.* Random Media and the convergence of the i.d.s. for the operators of Eqn. (4.1) with a random geometry have been studied by K. Lienau [L]. Since we need a connected Ω for our approach to work well, the ergodic process that generates the random set M has to respect this property. Here various methods from stochastic geometry ([SKM]) can be employed.

5. Periodic Laplace-Beltrami operators

In this section we report on some related results on periodic Laplace-Beltrami operators, cf. [P-1,2]. Let $\mathcal{M}$ be a non-compact Riemannian manifold of dimension $d \geq 2$. We call $\mathcal{M}$ a *periodic* or *covering manifold* iff $\Gamma = \mathbb{Z}^m$ acts properly discontinuously and isometrically on $\mathcal{M}$. Under these conditions, the quotient $\mathcal{M}/\Gamma$ is again a Riemannian manifold which we assume to be compact.

We call Q a *period cell* or *fundamental domain* of $\mathcal{M}$ iff Q is open, Q does not intersect any other translate $\gamma + Q$ except for $\gamma = 0$, and the union of all translates of $\overline{Q}$ covers the whole manifold $\mathcal{M}$. Note that we assume here that Q is an open set. One can apply Floquet decomposition in the same way as before; here, ϑ-periodicity means that $u(\gamma + x) = e^{i\vartheta\cdot\gamma}u(x)$ for all $x \in \overline{Q}$ and $\gamma \in \Gamma$ such that $\gamma + x \in \overline{Q}$ (plus an analogous condition for the first derivatives).

Denote by (g_{ij}) the metric tensor of $\mathcal{M}$ in some chart. We consider conformal deformations of the given periodic metric, i.e., we set

$$g_{ij}(x;\lambda) = \varrho_\lambda^2(x)\, g_{ij}(x).$$

Here, ϱ_λ is a strictly positive, periodic, smooth function converging pointwise to the indicator function of a closed periodic set M as $\lambda \to \infty$. In particular, we assume that

(1) $M_0 := M \cap Q$ is compactly contained in Q with smooth ∂M;
(2) $\varrho_\lambda(x) = 1$ for all $x \in M$;
(3) $\varrho_\lambda(x) = \frac{1}{\lambda}$, for all $x \in \Omega$ with $\operatorname{dist}(x, M) \geq \lambda^{-m}$.

To simplify the notation, we restrict ourselves to the conformally flat case, i.e., we assume that $\mathcal{M} = \mathbb{R}^m$ with the Euclidean metric tensor $g_{ij}(x) = \delta_{ij}$, defined on $\mathbb{R}^m$. We assume that the boundary ∂M has a tubular neighborhood where we can introduce local coordinates (r, y) given by the distance r from ∂M and $y \in \partial M$. For technical reasons we need the additional assumption that these coordinates can be extended to $\Omega \cap Q$; e.g., think of a ball M_0 centered in the period cell $Q = (0,1)^m$. The Laplace-Beltrami-operator B_λ corresponding to the deformed metric is given by

$$B_\lambda = -\frac{1}{\sqrt{g_\lambda}}\partial_j\left(g^{ij}(\cdot;\lambda)\sqrt{g_\lambda}\right)\partial_i$$

(using the summation convention), where $g_\lambda = \det g_{ij}(\cdot, \lambda) = \rho_\lambda^{2m}$ denotes the square of the volume density of the deformed metric tensor. This operator is defined as a non-negative, self-adjoint operator via the quadratic form

$$\mathfrak{b}_\lambda[u] := \int g^{ij}(\cdot, \lambda)\partial_i u\, \partial_j \overline{u}\sqrt{g_\lambda}\, dx = \int |\nabla u|^2\, \varrho_\lambda^{m-2}\, dx, \qquad u \in \mathcal{H}^1(\mathbb{R}^m) \quad (5.1)$$

in the Hilbert space $\mathcal{H}_\lambda := L_2(\mathbb{R}^m)$ with inner product $\int u\overline{v}\sqrt{g_\lambda}dx$. Note that, in general, both the Dirichlet integral *and* the volume form depend on the parameter λ. The fundamental difference to the case of divergence type operators lies in the dependence of the inner product on λ. In particular, for $m \geq 3$, there is no monotonicity.

Floquet-decomposition implies that it is enough to study ϑ-periodic b.v.p.'s on Q. Furthermore, Dirichlet-Neumann-bracketing yields enclosures of bands by Neumann and Dirichlet eigenvalues $E_k^{N/D}(\lambda)$ (ordered by min-max and repeated according to multiplicities). Similarly, denote by E_k^N the Neumann eigenvalues of $M_0 = M \cap Q$.

Since the norm of the Hilbert space depends on λ, we cannot apply Theorem 1.1. Instead, the convergence of the eigenvalues on Q to the Neumann eigenvalues on M_0 is obtained directly through an application of the min-max-principle and a careful comparison of the corresponding Raleigh quotients.

In particular, we show that E_k^N is (approximately) a lower bound for $E_k^N(\lambda)$ (using the restriction $u|_{M_0}$ as transition operator from $\mathcal{H}^1(Q)$ to $\mathcal{H}^1(M_0)$). In the same way, we prove that E_k^N is also an approximate upper bound for $E_k^D(\lambda)$ provided $m \geq 3$ (using an extension operator from $\mathcal{H}^1(M_0)$ to $\mathcal{H}_0^1(Q)$, the $\mathcal{H}^1$-closure of $C_c^\infty(Q)$, as transition operator). Therefore, we have

5.1**. Theorem.** ([P-1,2]) *Let $m \geq 3$. Then $\sigma(B_\lambda)$ converges to the spectrum of the Neumann Laplacian on M, as $\lambda \to \infty$. In particular, gaps open up, as $\lambda \to \infty$.*

5.2. *Remark.* The case $m = 2$ is special. Here, only the inner product depends on λ. Inverting the rôle of the norm and the quadratic form yields a limiting operators in each fiber which, however, depend on θ. In the special case of a 2-dimensional cylinder one can nevertheless prove the existence of spectral gaps by direct calculations (separation of variables), cf. [P-1,2].

5.3. *Remark.* In contrast to the Euclidean case, we may choose the dimension d of the manifold $\mathcal{M}$ independently of the dimension of periodicity m: think of the surface of a cylinder ($d = 2, m = 1$) or the surface of a periodic jungle gym ($d = 2, m = 3$). Furthermore, one can allow any Abelian finitely generated group Γ of infinite order, i.e., products of $\mathbb{Z}$ and cyclic groups.

5.4. *Remark.* In some sense, the case of Laplace-Beltrami operators is the opposite of the case of divergence type operators: while the divergence type operator needs the factor $(1 + \lambda\chi_\Omega)$ to be *large* on Ω to produce spectral gaps, the Laplace-Beltrami operator needs the factor ϱ_λ^{m-2} to be *small* on Ω.

5.5. *Remark.* Davies and Harrell [DH] proved the existence of at least one gap in the periodic conformally flat case (transforming the conformal Laplacian into a corresponding Schrödinger operator). This result is a special case of our result. Using similar methods as [DH], Green [G] obtained examples with a finite number of gaps in the 2-dimensional conformally flat case. Recently, Yoshitomi [Y] proved the existence of spectral gaps for the (Dirichlet) Laplacian on periodically curved quantum wave guides. In [DH, G, Y], the existence of gaps is established by analyzing a one-dimensional problem. Here, in contrast, we directly study the multi-dimensional problem.

5.6. *Remark.* More "geometric" constructions of periodic manifolds with spectral gaps can be found in [P-1,2], think for example of an infinite number of copies of a compact manifold, joined by thin cylinders. If these cylinders are small or thin enough, one again obtains spectral gaps of the corresponding Laplace-Beltrami operator. These results are closely related to earlier works of Chavel and Feldman [CF] and Anné [A].

6. Maxwell operators

Another important elliptic operator is the *Maxwell operator* given as

$$M = -\mathrm{curl}\frac{1}{\varepsilon(x)}\mathrm{curl}, \tag{6.1}$$

acting on solenoïdal vector fields in $\mathbb{R}^2$ or in $\mathbb{R}^3$, where $\varepsilon(x)$ denotes the dielectricity of the medium.

The existence of gaps for periodic structures is of fundamental importance for photonic crystals. In many cases, one is satisfied to find intervals of low spectral density (quasi-gaps). This class of problems has been studied in depth in a series of papers by A. Figotin, P. Kuchment and others (cf., e.g., [FK-1,2], [KK]) where a complete analysis for high-contrast and/or thin-wall asymptotics is given.

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