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# Proving The Ergodic Hypothesis for Billiards With Disjoint Cylindric Scatterers

**Nándor Simányi**[1]

University of Alabama at Birmingham
Department of Mathematics
Campbell Hall, Birmingham, AL 35294 U.S.A.
E-mail: simanyi@math.uab.edu

**Abstract.** In this paper we study the ergodic properties of mathematical billiards describing the uniform motion of a point in a flat torus from which finitely many, pairwise disjoint, tubular neighborhoods of translated subtori (the so called cylindric scatterers) have been removed. We prove that every such system is ergodic (actually, a Bernoulli flow), unless a simple geometric obstacle for the ergodicity is present.

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## §1. Introduction

Non-uniformly hyperbolic systems (possibly, with singularities) play a pivotal role in the ergodic theory of dynamical systems. Their systematic study started several decades ago, and it is not our goal here to provide the reader with a comprehensive

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review of the history of these investigations but, instead, we opt for presenting in nutshell a cross section of a few selected results.

In 1939 G. A. Hedlund and E. Hopf [He(1939)], [Ho(1939)], proved the hyperbolic ergodicity of geodesic flows on closed, compact surfaces with constant negative curvature by inventing the famous method of "Hopf chains" constituted by local stable and unstable invariant manifolds.

In 1963 Ya. G. Sinai [Sin(1963)] formulated a modern version of Boltzmann's ergodic hypothesis, what we call now the "Boltzmann-Sinai ergodic hypothesis": the billiard system of $N$ ($\geq 2$) hard spheres of unit mass moving in the flat torus $\mathbb{T}^\nu = \mathbb{R}^\nu/\mathbb{Z}^\nu$ ($\nu \geq 2$) is ergodic after we make the standard reductions by fixing the values of the trivial invariant quantities. It took seven years until he proved this conjecture for the case $N = 2$, $\nu = 2$ in [Sin(1970)]. Another 17 years later N. I. Chernov and Ya. G. Sinai [S-Ch(1987)] proved the hypothesis for the case $N = 2$, $\nu \geq 2$ by also proving a powerful and very useful theorem on local ergodicity.

In the meantime, in 1977, Ya. Pesin [P(1977)] laid down the foundations of his theory on the ergodic properties of smooth, hyperbolic dynamical systems. Later on this theory (nowadays called Pesin theory) was significantly extended by A. Katok and J-M. Strelcyn [K-S(1986)] to hyperbolic systems with singularities. That theory is already applicable for billiard systems, too.

Until the end of the seventies the phenomenon of hyperbolicity (exponential instability of the trajectories) was almost exclusively attributed to some direct geometric scattering effect, like negative curvature of space, or strict convexity of the scatterers. This explains the profound shock that was caused by the discovery of L. A. Bunimovich [B(1979)]: certain focusing billiard tables (like the celebrated stadium) can also produce complete hyperbolicity and, in that way, ergodicity. It was partly this result that led to Wojtkowski's theory of invariant cone fields, [W(1985)], [W(1986)].

The big difference between the system of two spheres in $\mathbb{T}^\nu$ ($\nu \geq 2$, [S-Ch(1987)]) and the system of $N$ ($\geq 3$) spheres in $\mathbb{T}^\nu$ is that the latter one is merely a so called semi-dispersive billiard system (the scatterers are convex but not strictly convex sets, namely cylinders), while the former one is strictly dispersive (the scatterers are strictly convex sets). This fact makes the proof of ergodicity (mixing properties) much more complicated. In our series of papers jointly written with A. Krámli and D. Szász [K-S-Sz(1990)], [K-S-Sz(1991)], and [K-S-Sz(1992)], we managed to prove the (hyperbolic) ergodicity of three and four billiard spheres in the toroidal container $\mathbb{T}^\nu$. By inventing new topological methods and the Connecting Path Formula (CPF), in my two-part paper [Sim(1992)] I proved the (hyperbolic) ergodicity of $N$ hard spheres in $\mathbb{T}^\nu$, provided that $N \leq \nu$.

The common feature of hard sphere systems is — as D. Szász pointed this out first in [Sz(1993)] and [Sz(1994)] — that all of them belong to the family of so called cylindric billiards, the definition of which can be found later in this paragraph. However, the first appearance of a special, 3-D cylindric billiard system took place in [K-S-Sz(1989)], where we proved the ergodicity of a 3-D billiard flow with two orthogonal cylindric scatterers. Later D. Szász [Sz(1994)] presented a complete picture (as far as ergodicity is concerned) of cylindric billiards with cylinders whose generator subspaces are spanned by mutually orthogonal coordinate axes. The task of proving ergodicity for the first non-trivial, non-orthogonal cylindric billiard system was taken up in [S-Sz(1994)].

Finally, in our joint venture with D. Szász [S-Sz(1999)] we managed to prove the complete hyperbolicity of *typical* hard sphere systems.

**Cylindric billiards.** Consider the $d$-dimensional ($d \geq 2$) flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathcal{L}$ supplied with the usual Riemannian inner product $\langle\,.\,,\,.\,\rangle$ inherited from the standard inner product of the universal covering space $\mathbb{R}^d$. Here $\mathcal{L} \subset \mathbb{R}^d$ is supposed to be a lattice, i. e. a discrete subgroup of the additive group $\mathbb{R}^d$ with $\text{rank}(\mathcal{L}) = d$. The reason why we want to allow general lattices, other than just the integer lattice $\mathbb{Z}^d$, is that otherwise the hard sphere systems would not be covered. The geometry of the structure lattice $\mathcal{L}$ in the case of a hard sphere system is significantly different from the geometry of the standard lattice $\mathbb{Z}^d$ in the standard Euclidean space $\mathbb{R}^d$, see subsection 2.4 of [Sim(2002)].

The configuration space of a cylindric billiard is $\mathbf{Q} = \mathbb{T}^d \setminus (C_1 \cup \cdots \cup C_k)$, where the cylindric scatterers $C_i$ ($i = 1, \dots, k$) are defined as follows:

Let $A_i \subset \mathbb{R}^d$ be a so called lattice subspace of $\mathbb{R}^d$, which means that $\text{rank}(A_i \cap \mathcal{L}) = \dim A_i$. In this case the factor $A_i/(A_i \cap \mathcal{L})$ is a subtorus in $\mathbb{T}^d = \mathbb{R}^d/\mathcal{L}$ which will be taken as the generator of the cylinder $C_i \subset \mathbb{T}^d$, $i = 1, \dots, k$. Denote by $L_i = A_i^\perp$ the orthocomplement of $A_i$ in $\mathbb{R}^d$. Throughout this paper we will always assume that $\dim L_i \geq 2$. Let, furthermore, the numbers $r_i > 0$ (the radii of the spherical cylinders $C_i$) and some translation vectors $t_i \in \mathbb{T}^d = \mathbb{R}^d/\mathcal{L}$ be given. The translation vectors $t_i$ play a crucial role in positioning the cylinders $C_i$ in the ambient torus $\mathbb{T}^d$. Set

$$C_i = \left\{ x \in \mathbb{T}^d :\ \text{dist}\,(x - t_i, A_i/(A_i \cap \mathcal{L})) < r_i \right\}.$$

In order to avoid further unnecessary complications, we always assume that the interior of the configuration space $\mathbf{Q} = \mathbb{T}^d \setminus (C_1 \cup \cdots \cup C_k)$ is connected. The phase space $\mathbf{M}$ of our cylindric billiard flow will be the unit tangent bundle of $\mathbf{Q}$ (modulo some natural identification at its boundary), i. e. $\mathbf{M} = \mathbf{Q} \times \mathbb{S}^{d-1}$. (Here $\mathbb{S}^{d-1}$ denotes the unit sphere of $\mathbb{R}^d$.)

The dynamical system $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$, where $S^t$ ($t \in \mathbb{R}$) is the dynamics defined by uniform motion inside the domain $\mathbf{Q}$ and specular reflections at its boundary (at the scatterers), and $\mu$ is the Liouville measure, is called the cylindric billiard flow we want to investigate. (As to the notions and notations in connection with semi-dispersive billiards, the reader is kindly recommended to consult the work [K-S-Sz(1990)].)

**Transitive cylindric billiards.**

The main conjecture concerning the (hyperbolic) ergodicity of cylindric billiards is the "Erdőtarcsa conjecture" (named after the picturesque village in rural Hungary where it was originally formulated) that appeared as Conjecture 1 in Section 3 of [S-Sz(2000)].

**Transitivity.** Let $L_1, \dots, L_k \subset \mathbb{R}^d$ be subspaces, $\dim L_i \geq 2$, $A_i = L_i^\perp$, $i = 1, \dots, k$. Set

$$\mathcal{G}_i = \{U \in \mathrm{SO}(d) : U|A_i = \mathrm{Id}_{A_i}\},$$

and let $\mathcal{G} = \langle \mathcal{G}_1, \dots, \mathcal{G}_k \rangle \subset \mathrm{SO}(d)$ be the algebraic generate of the compact, connected Lie subgroups $\mathcal{G}_i$ in $\mathrm{SO}(d)$. The following notions appeared in Section 3 of [S-Sz(2000)].

**Definition.** We say that the system of base spaces $\{L_1, \dots, L_k\}$ (or, equivalently, the cylindric billiard system defined by them) is *transitive* if and only if the group $\mathcal{G}$ acts transitively on the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$.

**Definition.** We say that the system of subspaces $\{L_1, \dots, L_k\}$ has the Orthogonal Non-splitting Property (ONSP) if there is no non-trivial orthogonal splitting $\mathbb{R}^d = B_1 \oplus B_2$ of $\mathbb{R}^d$ with the property that for every index $i$ ($1 \leq i \leq k$) $L_i \subset B_1$ or $L_i \subset B_2$.

The next result can be found in Section 3 of [S-Sz(2000)] (see 3.1–3.6 thereof):

**Proposition.** For the system of subspaces $\{L_1, \dots, L_k\}$ the following three properties are equivalent:

(1) $\{L_1, \dots, L_k\}$ is transitive;

(2) the system of subspaces $\{L_1, \dots, L_k\}$ has the ONSP;

(3) the natural representation of $\mathcal{G}$ in $\mathbb{R}^d$ is irreducible.

**The Erdőtarcsa conjecture.** A cylindric billiard flow is ergodic if and only if it is transitive. In that case the cylindric billiard system is actually a completely hyperbolic Bernoulli flow, see [C-H(1996)] and [O-W(1998)].

In order to avoid unnecessary complications, throughout the paper we always assume that

$$\operatorname{int}\mathbf{Q} \text{ is connected, and} \tag{1.1}$$

$$\begin{gathered}\text{the } d\text{-dim spatial angle } \alpha(q) \text{ subtended by } \mathbf{Q} \\ \text{at any of its boundary points } q \in \partial\mathbf{Q} \text{ is positive.}\end{gathered} \tag{1.2}$$

The Erdőtarcsa Conjecture has not been proved so far in full generality. Certain partial results, however, exist. Without pursuing the goal of achieving completeness, here we cite just two of such results:

**Theorem of [Sim(2001)].** Almost every hard disk system (i. e. hard sphere system in a 2-D torus) is hyperbolic and ergodic. (Here "almost every" is meant with respect to the outer geometric parameters $(r;\, m_1, \dots, m_N)$, where $r > 0$ is the common radius of the disks, while $m_1, \dots, m_N$ are the masses.)

**Theorem of [Sim(2002)].** Every hard sphere system is completely hyperbolic, i. e. all of its relevant Lyapunov exponents are nonzero almost everywhere.

In this paper we are mainly interested in understanding the ergodic properties of cylindric billiard flows $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ in which the closures $\bar{C}_i$ of the scattering cylinders $C_i$ are pairwise disjoint, i. e.

$$\bar{C}_i \cap \bar{C}_j = \emptyset \text{ for } 1 \le i < j \le k. \tag{1.3}$$

Elementary linear algebra shows that for such a (disjoint) cylindric billiard system it is true that $\operatorname{span}\{A_i, A_j\} \neq \mathbb{R}^d$ for $1 \le i,\, j \le k$ or, equivalently,

$$L_i \cap L_j \neq \{0\} \text{ for } 1 \le i,\, j \le k. \tag{1.4}$$

From now on we drop the disjointness condition (1.3) by only keeping the somewhat relaxed condition (1.4) above. The first, very simple, question that arises here is to characterize the transitivity of the $\mathcal{G}$-action on $\mathbb{S}^{d-1}$ under the condition (1.4) above. The following proposition immediately follows from (1.4) and the characterization (2) of the transitivity above:

**Proposition 1.5.** A cylindric billiard system with the additional property (1.4) is transitive (that is, the $\mathcal{G}$-action on the velocity sphere $\mathbb{S}^{d-1}$ is transitive) if and only if $\operatorname{span}\{L_1, \dots, L_k\} = \mathbb{R}^d$ or, equivalently, $\bigcap_{i=1}^k A_i = \{0\}$. $\square$

Now we are able to put forward the result of this paper:

**Theorem.** Assume that the cylindric billiard flow $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ enjoys the geometric properties (1.1), (1.2), and (1.4) above. Then the transitivity condition $\bigcap_{i=1}^k A_i = \{0\}$ implies that the flow $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ is completely hyperbolic and ergodic.

**Remark.** In the case of a hard sphere system with masses $m_1, m_2, \dots, m_N$ the base space $L_{ij}$ of the cylinder $C_{ij}$ (describing the collision between the spheres labelled by $i$ and $j$) is obviously the set

$$\left\{(\delta q_1, \dots, \delta q_N) \in \mathcal{T}\mathbf{Q} \,\middle|\, \delta q_k = 0 \text{ for } k \notin \{i,\, j\}\right\}.$$

(See also (4.4) in [S-Sz(2000)].) Therefore, in such systems the intersection of any two base spaces is zero. This shows that our present result is complementary to any possible past and future result about hard sphere systems.

**Organizing the Paper.** Section 2 contains the indispensable technical preparations, definitions, and notations. §3 is devoted to proving that if a non-singular orbit segment $S^{[a,b]}x$ of the flow $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ has a combinatorially rich symbolic collision sequence (in a well defined sense) then $S^{[a,b]}x$ is sufficient (geometrically hyperbolic) modulo a codimension-2 algebraic subset of the phase space. Finally, the closing Section 4 contains the inductive proof of

(H1) the so called "Chernov–Sinai Ansatz" for the flow $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$, i. e. that — informally speaking — for almost every singular phase point $x \in \mathbf{M}$ the forward semi-trajectory after the singularity is sufficient

and

(H2) outside of a slim subset $S \subset \mathbf{M}$ (for the notion of slimness, please see §2 below) it is true that
(i) $S^{(-\infty,\infty)}x$ has at most one singularity;
(ii) $S^{(-\infty,\infty)}x$ is sufficient.

Section 4 concludes with putting together all the above results and applying the Theorem on Local Ergodicity for algebraic semi-dispersive billiards [B-Ch-Sz-T(2001)] to complete the proof of our Theorem.

**Remark.** In order to simplify the notations, throughout the paper we will assume that the fundamental lattice $\mathcal{L} \subset \mathbb{R}^d$ of the factorization $\mathbb{T}^d = \mathbb{R}^d/\mathcal{L}$ is the standard integer lattice $\mathbb{Z}^d$. This is not a significant restriction of generality, for the entire proof of the theorem carries over to the general case easily by an almost word-by-word translation.

## §2. Prerequisites

**Trajectory Branches.** We are going to briefly describe the discontinuity of the flow $\{S^t\}$ caused by a multiple collision at time $t_0$. Assume first that the pre–collision velocities of the particles are given. What can we say about the possible post–collision velocities? Let us perturb the pre–collision phase point (at time $t_0 - 0$) infinitesimally, so that the collisions at $\sim t_0$ occur at infinitesimally different moments. By applying the collision laws to the arising finite sequence of collisions, we see that the post-collision velocities are fully determined by the time– ordering of the considered collisions. Therefore, the collection of all possible time-orderings of these collisions gives rise to a finite family of continuations of the trajectory beyond $t_0$. They are called the **trajectory branches**. It is quite clear that similar statements can be said regarding the evolution of a trajectory through a multiple collision **in reverse time**. Furthermore, it is also obvious that for any given phase point $x_0 \in \mathbf{M}$ there are two, $\omega$-high trees $\mathcal{T}_+$ and $\mathcal{T}_-$ such that $\mathcal{T}_+$ ($\mathcal{T}_-$) describes all the possible continuations of the positive (negative) trajectory $S^{[0,\infty)}x_0$ ($S^{(-\infty,0]}x_0$). (For the definitions of trees and for some of their applications to billiards, cf. the beginning of §5 in [K-S-Sz(1992)].) It is also clear that all possible continuations (branches) of the whole trajectory $S^{(-\infty,\infty)}x_0$ can be uniquely described by all possible pairs $(B_-, B_+)$ of $\omega$-high branches of the trees $\mathcal{T}_-$ and $\mathcal{T}_+$ ($B_- \subset \mathcal{T}_-, B_+ \subset \mathcal{T}_+$).

Finally, we note that the trajectory of the phase point $x_0$ has exactly two branches, provided that $S^t x_0$ hits a singularity for a single value $t = t_0$, and the phase point $S^{t_0}x_0$ does not lie on the intersection of more than one singularity manifolds. (In this case we say that the trajectory of $x_0$ has a "simple singularity".)

**Neutral Subspaces, Advance, and Sufficiency.** Consider a **nonsingular** trajectory segment $S^{[a,b]}x$. Suppose that $a$ and $b$ are **not moments of collision**.

**Definition 2.1.** *The neutral space $\mathcal{N}_0(S^{[a,b]}x)$ of the trajectory segment $S^{[a,b]}x$ at time zero ($a < 0 < b$) is defined by the following formula:*

$$\begin{aligned}\mathcal{N}_0(S^{[a,b]}x) = \big\{W \in \mathcal{Z}\colon\ \exists(\delta > 0)\ s.\ t.\ \forall\alpha \in (-\delta, \delta)\\ V\left(S^a\left(Q(x) + \alpha W, V(x)\right)\right) = V(S^a x)\ \text{and}\ V\left(S^b\left(Q(x) + \alpha W, V(x)\right)\right) = V(S^b x)\big\}.\end{aligned}$$

($\mathcal{Z}$ is the common tangent space $\mathcal{T}_q\mathbf{Q}$ of the parallelizable manifold $\mathbf{Q}$ at any of its points $q$, while $V(x)$ is the velocity component of the phase point $x = (Q(x),\ V(x))$.)

It is known (see (3) in §3 of [S-Ch (1987)]) that $\mathcal{N}_0(S^{[a,b]}x)$ is a linear subspace of $\mathcal{Z}$ indeed, and $V(x) \in \mathcal{N}_0(S^{[a,b]}x)$. The neutral space $\mathcal{N}_t(S^{[a,b]}x)$ of the segment $S^{[a,b]}x$ at time $t \in [a, b]$ is defined as follows:

$$\mathcal{N}_t(S^{[a,b]}x) = \mathcal{N}_0\left(S^{[a-t,b-t]}(S^t x)\right).$$

It is clear that the neutral space $\mathcal{N}_t(S^{[a,b]}x)$ can be canonically identified with $\mathcal{N}_0(S^{[a,b]}x)$ by the usual identification of the tangent spaces of $\mathbf{Q}$ along the trajectory $S^{(-\infty,\infty)}x$ (see, for instance, §2 of [K-S-Sz(1990)]).

Our next definition is that of the **advance**. Consider a non-singular orbit segment $S^{[a,b]}x$ with the symbolic collision sequence $\Sigma = (\sigma_1, \dots, \sigma_n)$ ($n \geq 1$), meaning that $S^{[a,b]}x$ has exactly $n$ collisions with $\partial\mathbf{Q}$, and the $i$-th collision ($1 \leq i \leq n$) takes place at the boundary of the cylinder $C_{\sigma_i}$. For $x = (Q, V) \in \mathbf{M}$ and $W \in \mathcal{Z}$, $\|W\|$ sufficiently small, denote $T_W(Q, V) := (Q + W, V)$.

**Definition 2.2.** *For any $1 \le k \le n$ and $t \in [a,b]$, the advance*

$$\alpha(\sigma_k)\colon\, \mathcal{N}_t(S^{[a,b]}x) \to \mathbb{R}$$

*of the collision $\sigma_k$ is the unique linear extension of the linear functional $\alpha(\sigma_k)$ defined in a sufficiently small neighborhood of the origin of $\mathcal{N}_t(S^{[a,b]}x)$ in the following way:*

$$\alpha(\sigma_k)(W) := t_k(x) - t_k(S^{-t}T_W S^t x).$$

Here $t_k = t_k(x)$ is the time moment of the $k$-th collision $\sigma_k$ on the trajectory of $x$ after time $t = a$. The above formula and the notion of the advance functional

$$\alpha_k = \alpha(\sigma_k) :\, \mathcal{N}_t\left(S^{[a,b]}x\right) \to \mathbb{R}$$

has two important features:

(i) If the spatial translation $(Q,V) \mapsto (Q+W,V)$ is carried out at time $t$, then $t_k$ changes linearly in $W$, and it takes place just $\alpha_k(W)$ units of time earlier. (This is why it is called “advance”.)

(ii) If the considered reference time $t$ is somewhere between $t_{k-1}$ and $t_k$, then the neutrality of $W$ precisely means that

$$W - \alpha_k(W) \cdot V(x) \in A_{\sigma_k},$$

i. e. a neutral (with respect to the collision $\sigma_k$) spatial translation $W$ with the advance $\alpha_k(W) = 0$ means that the vector $W$ belongs to the generator space $A_{\sigma_k}$ of the cylinder $C_{\sigma_k}$.

It is now time to bring up the basic notion of **sufficiency** (or, sometimes it is also called **geometric hyperbolicity**) of a trajectory (segment). This is the utmost important necessary condition for the proof of the fundamental theorem for semi-dispersive billiards, see Condition (ii) of Theorem 3.6 and Definition 2.12 in [K-S-Sz(1990)].

**Definition 2.3.**

(1) *The nonsingular trajectory segment $S^{[a,b]}x$ ($a$ and $b$ are supposed not to be moments of collision) is said to be* **sufficient** *if and only if the dimension of $\mathcal{N}_t(S^{[a,b]}x)$ ($t \in [a,b]$) is minimal, i.e. dim $\mathcal{N}_t(S^{[a,b]}x) = 1$.*

(2) *The trajectory segment $S^{[a,b]}x$ containing exactly one singularity (a so called “simple singularity”, see above) is said to be* **sufficient** *if and only if both branches of this trajectory segment are sufficient.*

**Definition 2.4.** *The phase point $x \in \mathbf{M}$ with at most one singularity is said to be sufficient if and only if its whole trajectory $S^{(-\infty,\infty)}x$ is sufficient, which means, by definition, that some of its bounded segments $S^{[a,b]}x$ are sufficient.*

In the case of an orbit $S^{(-\infty,\infty)}x$ with a simple singularity, sufficiency means that both branches of $S^{(-\infty,\infty)}x$ are sufficient.

**No accumulation (of collisions) in finite time.** By the results of Vaserstein [V(1979)], Galperin [G(1981)] and Burago-Ferleger-Kononenko [B-F-K(1998)], in a semi-dispersive billiard flow there can only be finitely many collisions in finite time intervals, see Theorem 1 in [B-F-K(1998)]. Thus, the dynamics is well defined as long as the trajectory does not hit more than one boundary components at the same time.

**Slim sets.** We are going to summarize the basic properties of codimension-two subsets $A$ of a smooth manifold $M$. Since these subsets $A$ are just those negligible in our dynamical discussions, we shall call them **slim**. As to a broader exposition of the issues, see [E(1978)] or §2 of [K-S-Sz(1991)].

Note that the dimension $\dim A$ of a separable metric space $A$ is one of the three classical notions of topological dimension: the covering (Čech-Lebesgue), the small inductive (Menger-Urysohn), or the large inductive (Brouwer-Čech) dimension. As it is known from general general topology, all of them are the same for separable metric spaces.

**Definition 2.5.** *A subset $A$ of $M$ is called slim if and only if $A$ can be covered by a countable family of codimension-two (i. e. at least two) closed sets of $\mu$–measure zero, where $\mu$ is a smooth measure on $M$. (Cf. Definition 2.12 of [K-S-Sz(1991)].)*

**Property 2.6.** *The collection of all slim subsets of $M$ is a $\sigma$-ideal, that is, countable unions of slim sets and arbitrary subsets of slim sets are also slim.*

**Proposition 2.7 (Locality).** *A subset $A \subset M$ is slim if and only if for every $x \in A$ there exists an open neighborhood $U$ of $x$ in $M$ such that $U \cap A$ is slim. (Cf. Lemma 2.14 of [K-S-Sz(1991)].)*

**Property 2.8.** *A closed subset $A \subset M$ is slim if and only if $\mu(A) = 0$ and $\dim A \leq \dim M - 2$.*

**Property 2.9 (Integrability).** *If $A \subset M_1 \times M_2$ is a closed subset of the product of two manifolds, and for every $x \in M_1$ the set*

$$A_x = \{y \in M_2\colon\ (x, y) \in A\}$$

*is slim in $M_2$, then $A$ is slim in $M_1 \times M_2$.*

The following propositions characterize the codimension-one and codimension-two sets.

**Proposition 2.10.** *For any closed subset $S \subset M$ the following three conditions are equivalent:*

(i) $\dim S \le \dim M - 2$;
(ii) *$intS = \emptyset$ and for every open connected set $G \subset M$ the difference set $G \setminus S$ is also connected;*
(iii) *$intS = \emptyset$ and for every point $x \in M$ and for any open neighborhood $V$ of $x$ in $M$ there exists a smaller open neighborhood $W \subset V$ of the point $x$ such that for every pair of points $y, z \in W \setminus S$ there is a continuous curve $\gamma$ in the set $V \setminus S$ connecting the points $y$ and $z$.*

(See Theorem 1.8.13 and Problem 1.8.E of [E(1978)].)

**Proposition 2.11.** *For any subset $S \subset M$ the condition $\dim S \le \dim M - 1$ is equivalent to $intS = \emptyset$. (See Theorem 1.8.10 of [E(1978)].)*

We recall an elementary, but important lemma (Lemma 4.15 of [K-S-Sz(1991)]). Let $R_2$ be the set of phase points $x \in \mathbf{M} \setminus \partial\mathbf{M}$ such that the trajectory $S^{(-\infty,\infty)}x$ has more than one singularities.

**Proposition 2.12.** *The set $R_2$ is a countable union of codimension-two smooth sub-manifolds of $M$ and, being such, it is slim.*

The next lemma establishes the most important property of slim sets which gives us the fundamental geometric tool to connect the open ergodic components of billiard flows.

**Proposition 2.13.** *If $M$ is connected, then the complement $M \setminus A$ of a slim set $A \subset M$ necessarily contains an arc-wise connected, $G_\delta$ set of full measure. (See Property 3 of §4.1 in [K-S-Sz(1989)]. The $G_\delta$ sets are, by definition, the countable intersections of open sets.)*

**The subsets $\mathbf{M}^0$ and $\mathbf{M}^\#$.** Denote by $\mathbf{M}^\#$ the set of all phase points $x \in \mathbf{M}$ for which the trajectory of $x$ encounters infinitely many non-tangential collisions in both time directions. The trajectories of the points $x \in \mathbf{M} \setminus \mathbf{M}^\#$ are lines: the motion is linear and uniform, see the appendix of [Sz(1994)]. It is proven in lemmas A.2.1 and A.2.2 of [Sz(1994)] that the closed set $\mathbf{M} \setminus \mathbf{M}^\#$ is a finite union of hyperplanes. It is also proven in [Sz(1994)] that, locally, the two sides of a hyperplanar component of $\mathbf{M} \setminus \mathbf{M}^\#$ can be connected by a positively measured beam of trajectories, hence, from the point of view of ergodicity, in this paper it is enough to show that the connected components of $\mathbf{M}^\#$ entirely belong to one ergodic component. This is what we are going to do in this paper.

Denote by $\mathbf{M}^0$ the set of all phase points $x \in \mathbf{M}^\#$ the trajectory of which does not hit any singularity, and use the notation $\mathbf{M}^1$ for the set of all phase points $x \in \mathbf{M}^\#$ whose orbit contains exactly one, simple singularity. According to Proposition

2.12, the set $\mathbf{M}^{\#} \setminus (\mathbf{M}^0 \cup \mathbf{M}^1)$ is a countable union of smooth, codimension-two ($\geq 2$) submanifolds of $\mathbf{M}$, and, therefore, this set may be discarded in our study of ergodicity, please see also the properties of slim sets above. Thus, we will restrict our attention to the phase points $x \in \mathbf{M}^0 \cup \mathbf{M}^1$.

**The "Chernov-Sinai Ansatz".** An essential precondition for the Theorem on Local Ergodicity by Bálint–Chernov–Szász–Tóth is the so called "Chernov-Sinai Ansatz" which we are going to formulate below. Denote by $\mathcal{SR}^+ \subset \partial\mathbf{M}$ the set of all phase points $x_0 = (q_0, v_0) \in \partial\mathbf{M}$ corresponding to singular reflections (a tangential or a double collision at time zero) supplied with the post-collision (outgoing) velocity $v_0$. It is well known that $\mathcal{SR}^+$ is a compact cell complex with dimension $2d-3 = \dim\mathbf{M} - 2$. It is also known (see Lemma 4.1 in [K-S-Sz(1990)]) that for $\nu$-almost every phase point $x_0 \in \mathcal{SR}^+$ (Here $\nu$ is the Riemannian volume of $\mathcal{SR}^+$ induced by the restriction of the natural Riemannian metric of $\mathbf{M}$.) the forward orbit $S^{(0,\infty)}x_0$ does not hit any further singularity. The Chernov-Sinai Ansatz postulates that for $\nu$-almost every $x_0 \in \mathcal{SR}^+$ the forward orbit $S^{(0,\infty)}x_0$ is sufficient (geometrically hyperbolic).

**The Theorem on Local Ergodicity.** The Theorem on Local Ergodicity by Bálint–Chernov–Szász–Tóth (Theorem 4.4 of [B-Ch-Sz-T(2001)]) claims the following: Let $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ be a semi-dispersive billiard flow with the properties (1.1)–(1.2) and such that the smooth components of the boundary $\partial\mathbf{Q}$ of the configuration space are algebraic hypersurfaces. (The cylindric billiards with (1.1)–(1.2) automatically fulfill this algebraicity condition.) Assume – further – that the Chernov-Sinai Ansatz holds true, and a phase point $x_0 \in \mathbf{M} \setminus \partial\mathbf{M}$ is given with the properties

(i) $S^{(-\infty,\infty)}x$ has at most one singularity,

and

(ii) $S^{(-\infty,\infty)}x$ is sufficient. (In the case of a singular obit $S^{(-\infty,\infty)}x$ this means that both branches of $S^{(-\infty,\infty)}x$ are sufficient.)

Then some open neighborhood $U_0 \subset \mathbf{M}$ of $x_0$ belongs to a single ergodic component of the flow $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$. (Modulo the zero sets, of course.)

## §3. Geometric Considerations

Consider a non-singular trajectory segment $S^{[a,b]}x_0 = \left\{x_t = S^t x_0 \,\middle|\, a \leq t \leq b\right\}$ of the cylindric billiard flow $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ with the symbolic collision sequence $\Sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$, meaning that there are time moments $a < t_1 < t_2 < \dots < t_n < b$ such that $S^t x \notin \partial\mathbf{M}$ for $t \in [a,b] \setminus \{t_1, \dots, t_n\}$, and $Q\left(S^{t_i}x\right) \in \partial C_{\sigma_i}$, $i = 1, \dots, n$. We assume that

(1) $\dim\left(L_{\sigma_i} \cap L_{\sigma_j}\right) \geq 2$ for $1 \leq i,\, j \leq n$
(the so called "codimension-two condition" imposed on $\Sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$), and

(2) $\operatorname{span}\{L_{\sigma_1}, \dots, L_{\sigma_n}\} = \mathbb{R}^d$, i. e. the system of cylinders $C_{\sigma_1}, \dots, C_{\sigma_n}$ is transitive, see also §1.

The first result of this section is

**Proposition 3.1.** Under the above conditions (1)–(2) the non-singular orbit segment $S^{[a,b]}x_0$ is hyperbolic (sufficient, cf. §2) modulo some codimension-two (i. e. at least two) submanifolds of the phase space.

**Proof.** The proof is based upon the following, simple lemma:

**Lemma 3.2.** Let $n = 2$, i. e. $\Sigma\left(S^{[a,b]}x_0\right) = (\sigma_1,\, \sigma_2)$. Then the advance functionals (cf. §2) $\alpha_1,\, \alpha_2 :\, \mathcal{N}_0\left(S^{[a,b]}x_0\right) \to \mathbb{R}$ corresponding to $\sigma_1$ and $\sigma_2$ are the same, unless the phase point $x_0$ belongs to some codimension-two ($\geq 2$) submanifold.

**Proof.** We may assume that the reference time $t = 0$ is between the collisions $\sigma_1$ and $\sigma_2$, i. e. $t_1 = t(\sigma_1) < 0 < t_2 = t(\sigma_2)$. Consider an arbitrary neutral vector $\delta q \in \mathcal{N}_0\left(S^{[a,b]}x_0\right)$. The neutrality of $\delta q$ with the advances $\alpha_i = \alpha_i(\delta q)$ $(i = 1, 2)$ means that

$$\delta q - \alpha_i v_0 \in A_{\sigma_i} \quad (i = 1, 2), \tag{3.3}$$

where $v_0 = v(x_0)$, $x_0 = (q_0,\, v_0)$. If $\alpha_1 = \alpha_1(\delta q)$ happens to be different from $\alpha_2 = \alpha_2(\delta q)$, then the equations in (3.3) yield that $(\alpha_1 - \alpha_2)v_0 \in \operatorname{span}\{A_{\sigma_1},\, A_{\sigma_2}\}$, i. e. $v_0 \in \operatorname{span}\{A_{\sigma_1},\, A_{\sigma_2}\}$. However,

$$c := \operatorname{codim}\left(\operatorname{span}\{A_{\sigma_1},\, A_{\sigma_2}\}\right) = \dim\left(L_{\sigma_1} \cap L_{\sigma_2}\right) \geq 2$$

(by our assumption (1)), and the event $v_0 \in \operatorname{span}\{A_{\sigma_1},\, A_{\sigma_2}\}$ is clearly described by a submanifold of codimension $c$. $\square$

Finishing the proof of the proposition:

According to the lemma, apart from a codimension-two ($\geq 2$) exceptional set $E \subset \mathbf{M}$ it is true that all advance functionals $\alpha_i = \alpha_{\sigma_i} :\, \mathcal{N}\left(S^{[a,b]}x_0\right) \to \mathbb{R}$ coincide. Assume that $x_0 \notin E$ and the reference time $t = 0$ is chosen to be right before the first collision $\sigma_1$ of $\Sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$. Consider an arbitrary neutral vector $w = \delta q \in \mathcal{N}_0\left(S^{[a,b]}x_0\right)$. By replacing $w$ with $w - \alpha v_0$ ($x_0 = (q_0, v_0)$, $\alpha = \alpha_i(w)$ is the common value of the advances $\alpha_i(w)$) we easily achieve that $\alpha_i(w) = 0$ for $i = 1, 2, \dots, n$. The relation $\alpha_1(w) = 0$ means that $w \in A_{\sigma_1}$ and $DS^{t_1^*}\left((w, 0)\right) = (w, 0)$, where $t(\sigma_1) < t_1^* < t(\sigma_2)$. Similarly, $\alpha_2(w) = 0$ means that $w \in A_{\sigma_2}$ and $DS^{t_2^*}\left((w, 0)\right) = (w, 0)$, where $t(\sigma_2) < t_2^* < t(\sigma_3)$, etc. We get that

$$w \in \bigcap_{i=1}^{n} A_{\sigma_i} = \operatorname{span}\left\{L_{\sigma_i} \middle|\, 1 \le i \le n\right\}^{\perp} = \{0\},$$

thus $w = 0$. This shows that the original neutral vector was indeed a scalar multiple of the velocity $v_0$, so $\dim\mathcal{N}_0\left(S^{[a,b]}x_0\right) = 1$ whenever $x_0 \notin E$. $\square$

**Remark 3.4.** It is clear from the above proof that without the assumption (1) of the proposition we obtain a codimension-one exceptional set $E \subset \mathbf{M}$ outside of which the statement holds true. Indeed, the overall assumption on the geometry of our cylindric billiard system is that $L_i \cap L_j \neq \{0\}$ for any pair of base spaces $L_i$ and $L_j$.

## Some Observations Concerning Codimension-one Exceptional Manifolds $J \subset \mathbf{M}$

The last remaining question of this section is this: In the original set-up (i. e. when only $\dim\left(L_{\sigma_i} \cap L_{\sigma_j}\right) \ge 1$ is assumed in (1) of Proposition 3.1) how an enhanced version of Proposition 3.1 excludes the existence of a codimension-one, smooth sub-manifold $J \subset \mathbf{M}$ separating different ergodic components of the flow $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$?

Given a codimension-one, flow-invariant, smooth sub-manifold $J \subset \mathbf{M}$, consider a normal vector $n_0 = (z, w)$ $(\neq 0)$ of $J$ at the phase point $y \in J$, i. e. for any tangent vector $(\delta q,\, \delta v) \in \mathcal{T}_y\mathbf{M}$ the relation $(\delta q,\, \delta v) \in \mathcal{T}_y J$ is true if and only if $\langle \delta q, z\rangle + \langle \delta v, w\rangle = 0$. Here $\langle\, .\, ,\, .\, \rangle$ is the Euclidean inner product of the tangent space $\mathbb{R}^d$ of $\mathbb{T}^d$ at every point $q \in \mathbb{T}^d$. Let us determine first the time-evolution $n_0 \longmapsto n_t$ $(t > 0)$ of this normal vector as time $t$ elapses. If there is no collision on the orbit segment $S^{[0,t]}y$, then the relationship between $(\delta q,\, \delta v) \in \mathcal{T}_y\mathbf{M}$ and $(\delta q',\, \delta v') = (DS^t)\,(\delta q,\, \delta v)$ is obviously

$$\begin{aligned} \delta v' &= \delta v, \\ \delta q' &= \delta q + t\delta v, \end{aligned} \tag{3.5}$$

from which we obtain that

$$\begin{aligned} (\delta q',\, \delta v') \in \mathcal{T}_{y'} J &\Leftrightarrow \langle \delta q' - t\delta v',\, z\rangle + \langle \delta v',\, w\rangle = 0 \\ &\Leftrightarrow \langle \delta q',\, z\rangle + \langle \delta v',\, w - tz\rangle = 0. \end{aligned}$$

This means that $n_t = (z,\, w - tz)$. It is always very useful to consider the quadratic form $Q(n) = Q((z, w)) =: \langle z, w\rangle$ associated with the normal vector $n = (z, w) \in$

$\mathcal{T}_y\mathbf{M}$ of $J$ at $y$. $Q(n)$ is the so called "infinitesimal Lyapunov function", see [K-B(1994)] or part A.4 of the Appendix in [Ch(1994)]. For a detailed exposition of the relationship between the quadratic form $Q$, the relevant symplectic geometry and the dynamics, please see [L-W(1995)].

**Remark.** Since the normal vector $n = (z, w)$ of $J$ is only determined up to a nonzero scalar multiplier, the value $Q(n)$ is only determined up to a positive multiplier. However, this means that the sign of $Q(n)$ (which is the utmost important thing for us) is uniquely determined. This remark will gain a particular importance in the near future.

From the above calculations we get that

$$Q(n_t) = Q(n_0) - t||z||^2 \le Q(n_0). \tag{3.6}$$

The next question is how the normal vector $n$ of $J$ gets transformed $n^- \mapsto n^+$ through a collision (reflection) at time $t = 0$? Elementary geometric considerations show (see Lemma 2 of [Sin(1979)], or formula (2) in §3 of [S-Ch(1987)]) that the linearization of the flow

$$\left(DS^t\right)\Big|_{t=0} :\ (\delta q^-,\, \delta v^-) \longmapsto (\delta q^+,\, \delta v^+)$$

is given by the formulae

$$\begin{aligned} \delta q^+ &= R\delta q^-, \\ \delta v^+ &= R\delta v^- + 2\cos\phi R V^* K V \delta q^-, \end{aligned} \tag{3.7}$$

where the operator $R:\ \mathcal{T}_q\mathbf{Q} \to \mathcal{T}_q\mathbf{Q}$ is the orthogonal reflection across the tangent hyperplane $\mathcal{T}_q\partial\mathbf{Q}$ of $\partial\mathbf{Q}$ at $q \in \partial\mathbf{Q}$ ($y^- = (q, v^-) \in \partial\mathbf{M}$, $y^+ = (q, v^+) \in \partial\mathbf{M}$), $V:\ (v^-)^\perp \to \mathcal{T}_q\partial\mathbf{Q}$ is the $v^-$-parallel projection of the orthocomplement hyperplane $(v^-)^\perp$ onto $\mathcal{T}_q\partial\mathbf{Q}$, $V^*:\ \mathcal{T}_q\partial\mathbf{Q} \to (v^-)^\perp$ is the adjoint of $V$, i. e. it is the projection of $\mathcal{T}_q\partial\mathbf{Q}$ onto $(v^-)^\perp$ being parallel to the normal vector $\nu(q)$ of $\partial\mathbf{Q}$ at $q \in \partial\mathbf{Q}$, $K:\ \mathcal{T}_q\partial\mathbf{Q} \to \mathcal{T}_q\partial\mathbf{Q}$ is the second fundamental form of $\partial\mathbf{Q}$ at $q$ and, finally, $\cos\phi = \langle \nu(q),\, v^+\rangle$ is the cosine of the angle $\phi$ subtended by $v^+$ and the normal vector $\nu(q)$. For the formula (3.7), please also see the last displayed formula of §1 in [S-Ch(1982)], or (i) and (ii) of Proposition 2.3 in [K-S-Sz(1990)]. We note that it is enough to deal with the tangent vectors $(\delta q^-,\, \delta v^-) \in (v^-)^\perp \times (v^-)^\perp$ ($(\delta q^+,\, \delta v^+) \in (v^+)^\perp \times (v^+)^\perp$), for the manifold $J$ under investigation is supposed to be flow-invariant, so any vector $(\delta q,\, \delta v) = (\alpha v,\, 0)$ ($\alpha \in \mathbb{R}$) is automatically inside $\mathcal{T}_y J$. The backward version (inverse)

$$\left(DS^t\right)\Big|_{t=0} :\ (\delta q^+,\, \delta v^+) \mapsto (\delta q^-,\, \delta v^-)$$

can be deduced easily from (3.7):

$$\begin{aligned} \delta q^- &= R\delta q^+, \\ \delta v^- &= R\delta v^+ - 2\cos\phi R V_1^* K V_1 \delta q^+, \end{aligned} \tag{3.8}$$

where $V_1 : (v^+)^\perp \to \mathcal{T}_q \partial\mathbf{Q}$ is the $v^+$-parallel projection of $(v^+)^\perp$ onto $\mathcal{T}_q\partial\mathbf{Q}$. By using formula (3.8), one easily computes the time-evolution $n^- \longmapsto n^+$ of a normal vector $n^- = (z, w) \in \mathcal{T}_{y^-}\mathbf{M}$ of $J$ if a collision $y^- \longmapsto y^+$ takes place at time $t = 0$:

$$\begin{aligned} (\delta q^+, \delta v^+) \in \mathcal{T}_{y^+}J &\Leftrightarrow \langle R\delta q^+, z\rangle + \langle R\delta v^+ - 2\cos\phi R V_1^* K V_1 \delta q^+, w\rangle = 0 \\ &\Leftrightarrow \langle \delta q^+, Rz - 2\cos\phi V_1^* K V_1 R w\rangle + \langle \delta v^+, Rw\rangle = 0. \end{aligned}$$

This means that

$$n^+ = (Rz - 2\cos\phi V_1^* K V_1 R w, Rw) \tag{3.9}$$

if $n^- = (z, w)$. It follows that

$$\begin{aligned} Q(n^+) &= Q(n^-) - 2\cos\phi \langle V_1^* K V_1 R w, Rw\rangle \\ &= Q(n^-) - 2\cos\phi \langle K V_1 R w, V_1 R w\rangle \le Q(n^-). \end{aligned} \tag{3.10}$$

Here we used the fact that the second fundamental form $K$ of $\partial\mathbf{Q}$ at $q$ is positive semi-definite, which just means that the billiard system is semi-dispersive.

The last simple observation on the quadratic form $Q(n)$ regards the involution $I : \mathbf{M} \to \mathbf{M}$, $I(q, v) = (q, -v)$ corresponding to the time reversal. If $n = (z, w)$ is a normal vector of $J$ at $y$, then, obviously, $I(n) = (z, -w)$ is a normal vector of $I(J)$ at $I(y)$ and

$$Q\left(I(n)\right) = -Q(n). \tag{3.11}$$

By switching — if necessary — from the separating manifold $J$ to $I(J)$, and by taking a suitable remote image $S^t(J)$ $(t >> 1)$, in the spirit of (3.6), (3.10)–(3.11) we can assume that

$$Q(n) < 0 \tag{3.12}$$

for every *unit* normal vector $n \in \mathcal{T}_y\mathbf{M}$ of $J$ near a phase point $y \in J$.

**Remark 3.13.** There could be, however, a little difficulty in achieving the inequality $Q(n) < 0$, i. e. (3.12). Namely, it may happen that $Q(n_t) = 0$ for every $t \in \mathbb{R}$. According to (3.6), the equation $Q(n_t) = 0$ $(\forall t \in \mathbb{R})$ implies that $n_t =: (z_t, w_t) = (0, w_t)$ for all $t \in \mathbb{R}$ and, moreover, in the view of (3.9), $w_t^+ = Rw_t^-$ is the transformation law at any collision $y_t = (q_t, v_t) \in \partial\mathbf{M}$. Furthermore, at every collision $y_t = (q_t, v_t) \in \partial\mathbf{M}$ the projected tangent vector $V_1 Rw_t^- = V_1 w_t^+$ lies in the null space of the operator $K$ (see also (3.9)), and this means that $w_0$ is a neutral vector for the entire trajectory $S^{\mathbb{R}}y$, i. e. $w_0 \in \mathcal{N}\left(S^{\mathbb{R}}y\right)$. (For the notion of neutral vectors and $\mathcal{N}\left(S^{\mathbb{R}}y\right)$, cf. §2 above.) On the other hand, this is impossible for the following reason: Any tangent vector $(\delta q, \delta v)$ from the space $\mathcal{N}\left(S^{\mathbb{R}}y\right) \times \mathcal{N}\left(S^{\mathbb{R}}y\right)$ is automatically tangent to the separating manifold $J$, thus for any normal vector $n = (z, w) \in \mathcal{T}_y\mathbf{M}$ of a separating manifold $J$ one has

$$(z,\, w) \in \mathcal{N}\left(S^{\mathbb{R}}y\right)^{\perp} \times \mathcal{N}\left(S^{\mathbb{R}}y\right)^{\perp}. \tag{3.14}$$

(As a direct inspection shows. We always tacitly assume that the exceptional manifold $J$ is locally defined by the equation $J = \left\{x \in U_0 \middle|\ \dim\mathcal{N}\left(S^{[a,b]}x\right) > 1\right\}$ with orbit segments $S^{[a,b]}x$ whose symbolic sequence is combinatorially rich, i. e. it typically provides sufficient phase points.) The membership in (3.14) is, however, impossible with a nonzero vector $w \in \mathcal{N}\left(S^{\mathbb{R}}y\right)$. $\square$

**Singularities.**

Consider a smooth, connected piece $\mathcal{S} \subset \mathbf{M}$ of a singularity manifold corresponding to a singular (tangential or double) reflection *in the past*. Such a manifold $\mathcal{S}$ is locally flow-invariant and has one codimension, so we can speak about its normal vectors $n$ and the uniquely determined sign of $Q(n)$ for $0 \neq n \in \mathcal{T}_y\mathbf{M}$, $y \in \mathcal{S}$, $n \perp \mathcal{S}$ (depending on the foot point, of course). Consider first a phase point $y^+ \in \partial\mathbf{M}$ right after the singular reflection that is described by $\mathcal{S}$. It follows from the proof of Lemma 4.1 of [K-S-Sz(1990)] and Sub-lemma 4.4 therein that at $y^+ = (q, v^+) \in \partial\mathbf{M}$ any tangent vector $(0, \delta v) \in \mathcal{T}_{y^+}\mathbf{M}$ lies actually in $\mathcal{T}_{y^+}\mathcal{S}$ and, consequently, the normal vector $n = (z, w) \in \mathcal{T}_{y^+}\mathbf{M}$ of $\mathcal{S}$ at $y^+$ necessarily has the form $n = (z, 0)$, i. e. $w = 0$. Thus $Q(n) = 0$ for any normal vector $n \in \mathcal{T}_{y^+}\mathbf{M}$ of $\mathcal{S}$. According to the monotonicity inequalities (3.6) and (3.10) above,

$$Q(n) < 0 \tag{3.15}$$

for any phase point $y \in \mathcal{S}$ of a past singularity manifold $\mathcal{S}$.

The above observations lead to the following conclusion:

**Proposition 3.16.** Assume that the separating manifold $J \subset \mathbf{M}$ ($J$ is smooth, connected, $\operatorname{codim}(J) = 1$) is selected in such a way that $Q(n_y) < 0$ for all normal vectors $0 \neq n_y \in \mathcal{T}_y\mathbf{M}$ of $J$ at any point $y \in J$, see above. Suppose further that the non-singular orbit segments $S^{[a,b]}y$ ($y \in B_0$, $B_0$ is a small open ball, $0 < a < b$ fixed) have the common symbolic collision sequence $\Sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ with the relaxed properties

(1)' $\dim\left(L_{\sigma_i} \cap L_{\sigma_j}\right) \geq 1$ ($1 \leq i,\, j \leq n$), and

(2)' (the same as (2)) $\operatorname{span}\{L_{\sigma_1}, \dots, L_{\sigma_n}\} = \mathbb{R}^d$.

We claim that for almost every phase point $y \in J \cap B_0$ (i. e. apart from an algebraic variety $E' \subset J \cap B_0$ with $\dim(E') < \dim(J)$) the orbit segment $S^{[a,b]}y$ is hyperbolic (sufficient).

**Proof.** Consider the algebraic variety $E \subset B_0$ of exceptional phase points characterized by the proof of Proposition 3.1, i. e. let

$$E = \left\{ y \in B_0 \middle|\ S^{[a,b]}y \text{ is not hyperbolic} \right\}.$$

The mentioned proof (in particular, the proof of Lemma 3.2) shows that the only way to have a codimension-one, smooth component in the variety $E$ is to have a submanifold defined by the relation

$$v_t \in \operatorname{span}\left\{A_{\sigma_i},\, A_{\sigma_j}\right\} \tag{3.17}$$

for some $i \in \{1, 2, \dots, n-1\}$, $t(\sigma_i) < t < t(\sigma_{i+1})$ ($v_t$ is the velocity at time $t$) with $\operatorname{dimspan}\left\{A_{\sigma_i},\, A_{\sigma_j}\right\} = d - 1$. However, the normal vector $n_t = (z_t, w_t) \in \mathcal{T}_{y_t}\mathbf{M}$ of the manifold defined by (3.17) at the point $y_t = S^t y = (q_t, v_t)$ obviously has the form $(z_t, w_t) = (0, w_t)$, thus $Q(n_t) = 0$. By the assumption of this proposition (and by the monotone non-increasing property of $Q(n_t)$ in $t$, see the inequalities (3.6) and (3.10)) we get that any codimension-one, smooth component of $E$ is transversal to $J$, thus proving the proposition. $\square$

In view of the inequality (3.15) (valid for past-singularity manifolds $\mathcal{S}$), the exceptional manifold $J \subset \mathbf{M}$ featuring Proposition 3.16 may be replaced by any past-singularity manifold $\mathcal{S}$ without hurting the proof of the proposition. Thus, we obtain

**Corollary 3.18.** Let $\mathcal{S}$ be a smooth component of a past-singularity set, $y_0 \in \mathcal{S}$, $0 < a < b$, and assume that the non-singular orbit segment $S^{[a,b]}y_0$ has the symbolic collision sequence $\Sigma = \Sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ fulfilling (1)'–(2)' of Proposition 3.16. Then there is an open neighborhood $B_0$ of $y_0$ in $\mathbf{M}$ such that for almost every phase point $y \in \mathcal{S} \cap B_0$ (with respect to the induced hypersurface measure of $\mathcal{S} \cap B_0$) the symbolic collision sequence $S^{[a,b]}y$ is still the same $\Sigma = \Sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$, and $S^{[a,b]}y$ is hyperbolic. $\square$

(Note that in this result, as usual, the phrase "almost every" may be replaced by saying that "apart from a countable family of smooth, proper sub-manifolds".)

### Yet Another Corollary of (3.6), (3.9), and (3.10)

**Corollary 3.19.** Assume that the orbit segment $S^{[a,b]}y_0$ is not singular, $\Sigma\left(S^{[a,b]}y_0\right) = \Sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ fulfills the relaxed conditions (1)'–(2)' of Proposition 3.16 and, finally, there is a codimension-one, flow-invariant, smooth submanifold $E \ni y_0$ in $\mathbf{M}$ such that

(i) $\Sigma\left(S^{[a,b]}y\right) = \Sigma\left(S^{[a,b]}y_0\right)$ for all $y \in E$, and
(ii) $S^{[a,b]}y$ is not hyperbolic for all $y \in E$.

Let $0 \neq n_t = (z_t, w_t)$ be a normal vector of $S^t(E)$ at the point $y_t = (q_t, v_t)$. We claim that $Q(n_t) < 0$ for all $t > b$.

**Proof.** As we have seen before, the manifold $E$ is defined by the relation $v_t \in \operatorname{span}\left\{A_{\sigma_i},\, A_{\sigma_{i+1}}\right\}$ with some $i \in \{1, 2, \dots, n-1\}$, $\dim\left(\operatorname{span}\left\{A_{\sigma_i},\, A_{\sigma_{i+1}}\right\}\right) = d-1$, $t(\sigma_i) < t < t(\sigma_{i+1})$. Thus $n_t = (0, \bar{w})$, $0 \neq \bar{w} \perp \operatorname{span}\left\{A_{\sigma_i},\, A_{\sigma_{i+1}}\right\}$, $t(\sigma_i) < t < t(\sigma_{i+1})$, meaning also that $Q(n_t) = 0$. Assume, to the contrary of the assertion of this corollary, that $Q(n_\tau) = 0$ for some $\tau > b$. Then, by the non-increasing property of $Q(n_\tau)$ in $\tau$, there is a small $\epsilon > 0$ such that $Q(n_\tau) = 0$ for all $\tau$, $t(\sigma_{i+1}) < \tau < t(\sigma_{i+1}) + \epsilon$. By (3.10) this means that the vector $V_1 R\bar{w} = V\bar{w}$ is in the null space of the operator $K$, i. e. $V\bar{w} \in A_{\sigma_{i+1}}$. This means, in particular, that $\bar{w} \in \operatorname{span}\left\{v_t,\, A_{\sigma_{i+1}}\right\}$ for $t(\sigma_i) < t < t(\sigma_{i+1})$. On the other hand, $\operatorname{span}\left\{v_t,\, A_{\sigma_{i+1}}\right\} \subset \operatorname{span}\left\{A_{\sigma_i},\, A_{\sigma_{i+1}}\right\}$, and $\bar{w} \perp \operatorname{span}\left\{A_{\sigma_i},\, A_{\sigma_{i+1}}\right\}$, a contradiction. $\square$

An almost immediate consequence of the above corollary and the proof of Proposition 3.16 is

**Corollary 3.20.** Assume that the non-singular orbit segments $S^{[a,b]}y$ ($y \in B_0$, $B_0$ is a small, open ball) have the common symbolic collision sequence

$$\left(\Sigma^{(1)},\, \Sigma^{(2)}\right) = \left(\sigma_1^{(1)}, \dots, \sigma_m^{(1)};\, \sigma_1^{(2)}, \dots, \sigma_n^{(2)}\right)$$

such that both $\Sigma^{(j)}$ are *combinatorially rich*, i. e.

$$\operatorname{span}\left\{L_{\sigma_1^{(1)}},\, L_{\sigma_2^{(1)}},\, \dots,\, L_{\sigma_m^{(1)}}\right\} = \operatorname{span}\left\{L_{\sigma_1^{(2)}},\, L_{\sigma_2^{(2)}},\, \dots,\, L_{\sigma_n^{(2)}}\right\} = \mathbb{R}^d.$$

Then the exceptional set

$$E = \left\{y \in B_0 \middle|\, S^{[a,b]}y \text{ is not hyperbolic}\right\}$$

has codimension at least two.

**Proof.** Let $E^{(j)}$ be a smooth, codimension-one exceptional manifold for $\Sigma^{(j)}$, $j = 1, 2$. (The word "exceptional" refers to the fact that these manifolds consist of atypical phase points for which the corresponding $\Sigma^{(j)}$-part of the orbit is not hyperbolic, despite the assumed combinatorial richness of $\Sigma^{(j)}$.) Let $t(\sigma_m^{(1)}) < t < t(\sigma_1^{(2)})$. By the previous corollary, the manifold $S^t\left(E^{(1)}\right)$ has a normal vector $n_t^{(1)}$ with $Q\left(n_t^{(1)}\right) < 0$ at any point $y_t = S^t y$ ($y \in E^{(1)}$), while, by the same corollary again (applied in reverse time), at any phase point $y_t = S^t y$, $y \in E^{(2)}$, the manifold $S^t\left(E^{(2)}\right)$ has a normal vector $n_t^{(2)}$ with $Q\left(n_t^{(2)}\right) > 0$, i. e. $E^{(1)}$ and $E^{(2)}$ are transversal at any point of their intersection. This finishes the proof of the corollary. $\square$

## §4. Hyperbolicity Is Abundant<br>The Inductive Proof

Below we present the inductive proof of the Theorem of this paper. The induction will be performed with respect to the number of cylinders $k$.

Beside the ergodicity (and, therefore, the Bernoulli property, see [C-H(1996)]) and [O-W(1998)]) we will prove (and use as the induction hypothesis!) a few technical properties listed below:

(H1) The Chernov–Sinai Ansatz (see §2 above) holds true for the cylindric billiard flow $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$;

(H2) There exists a slim subset $S \subset \mathbf{M}$ (see §2 for the concept of "slimness") such that for all $x \in \mathbf{M} \setminus S$

(i) $S^{(-\infty,\infty)}x$ has at most one singularity and

(ii) $S^{(-\infty,\infty)}x$ is hyperbolic (in the singular case both branches of $S^{(-\infty,\infty)}x$ are supposed to be hyperbolic, see §2 above).

Consequently, according to the Fundamental Theorem for algebraic, semi-dispersive billiards (Theorem 4.4 of [B-Ch-Sz-T(2001)]),

(H3) For every $x \in \mathbf{M}\setminus S$ the assertion of Theorem 4.4 of [B-Ch-Sz-T(2001)] holds true in some open neighborhood $U_0$ of $x$ in $\mathbf{M}$, in particular, $x$ is a so called "zig-zag point", see Definition 5.1 in [Sz(2000)]. Consequently, since the complementer set $\mathbf{M} \setminus S$ is known to contain a connected set of full measure (see §2) and the open neighborhood $U_0$ of $x$ belongs to a single ergodic component, we get that

(H4) $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ is ergodic, hence it is a Bernoulli flow by [C-H(1996)] and [O-W(1998)].

The above properties (H1)—(H2) will serve for us as the induction hypothesis.

## 1. The base of the induction: $k = 1$

In this case, necessarily, $L_1 = \mathbb{R}^d$ and $A_1 = \{0\}$, so the cylindric billiard system is actually a genuine, $d$-dimensional Sinai–billiard with a single spherical scatterer which has been well known to enjoy the properties (H1)—(H2) since the seminal work [S-Ch(1987)].

## 2. The induction step: $< k \longrightarrow k$, $k \geq 2$.

Let $k \geq 2$, $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ be a cylindric billiard flow fulfilling all the hypotheses of our Theorem, and suppose that the induction hypotheses have been successfully proven for every system (within the framework of the Theorem) with less than $k$ cylindric scatterers.

First we prove (H1) for $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$. The upcoming proof of the Chernov-Sinai Ansatz is going to be a local argument by nature, for stating that "the forward orbit of almost every phase point $x$ on a past-singularity manifold is hyperbolic" is a local assertion.

Let $\mathcal{S}_0 \subset \mathbf{M}^\# \setminus \partial\mathbf{M}$ (For the definition of $\mathbf{M}^\#$, please see §2.) be a small piece of a past-singularity manifold with the following properties:

(1) $\mathcal{S}_0$ is smooth (analytic) and diffeomorphic to $\mathbb{R}^{2d-2}$;

(2) For every phase point $x \in \mathcal{S}_0$ the last collision on the backward orbit $S^{(-\infty,0)}x$ is a singular collision taking place at time $\tau(x) < 0$ so that the collision at $S^{\tau(x)}x$ is a simple singularity, see §2. Consequently, the type of this singularity (see §2) is the same for all $x \in \mathcal{S}_0$.

We will measure the size of the subsets $A \subset \mathcal{S}_0$ by using the hypersurface measure $\nu$ induced on $\mathcal{S}_0$.

First of all, we restrict our attention to the subset

$$A = \left\{x \in \mathcal{S}_0 \middle|\, S^{(0,\infty)}x \text{ is non-singular}\right\} \tag{4.1}$$

of $\mathcal{S}_0$, being a $G_\delta$ subset of $\mathcal{S}_0$ with full measure, $\nu(\mathcal{S}_0 \setminus A) = 0$, for the set $\mathcal{S}_0 \setminus A$ is a countable union of smooth, proper submanifolds of $\mathcal{S}_0$, see Lemma 4.1 in [K-S-Sz(1990)]. With each phase point $x \in A$ we associate the infinite symbolic collision sequence

$$\Sigma(x) = (\sigma_1(x),\, \sigma_2(x), \dots)$$

of the forward orbit $S^{(0,\infty)}x$, the set of cylinders

$$C(x) = \{\sigma_n(x) | \quad n = 1, 2, \dots\}, \tag{4.2}$$

and the linear subspace $L(x)$ of $\mathbb{R}^d$ spanned by the base spaces of these cylinders:

$$L(x) = \text{span}\left\{L_i \big| \; i \in C(x)\right\}. \tag{4.3}$$

Due to the non-singularity of $S^{(0,\infty)}x$ ($x \in A$), all these functions $\Sigma(x)$, $C(x)$, and $L(x)$ are locally constant on the $G_\delta$ set $A$ of $\mathcal{S}_0$ with full measure.

In view of Corollary 3.18, for $\nu$-almost every phase point $x$ of the open-and-closed subset

$$A_0 = \left\{x \in A \big| \; L(x) = \mathbb{R}^d\right\} \tag{4.4}$$

of $A$ the forward orbit $S^{(0,\infty)}x$ is hyperbolic (sufficient). Thus, in order to prove the Ansatz it is enough to show that $A_0 = A$.

We argue by contradiction. Assume that the open-and-closed subset

$$A_1 = \left\{x \in A \big| \; C(x) = \mathcal{C}_0\right\} \neq \emptyset \tag{4.5}$$

for some $\mathcal{C}_0 \subset \{1, 2, \dots, k\}$ with

$$L^* = \text{span}\left\{L_i \big| \; i \in \mathcal{C}_0\right\} \neq \mathbb{R}^d.$$

We will get a contradiction by applying (essentially) the invariant manifold construction ideas borrowed from the proof of Theorem 6.1 of [Sim(1992-A)]. Indeed, we pay attention to the forward orbits $S^{(0,\infty)}x$ of the points $x \in A_1$ governed solely by the sub-billiard dynamics defined by the cylinders $C_i$ with $i \in \mathcal{C}_0$ in $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. This sub-billiard dynamics obviously does not fulfill the transitivity condition, for it is invariant under all spatial translations by the elements of the subtorus $\tilde{A} = A^*/\left(A^* \cap \mathbb{Z}^d\right)$, where $A^* = (L^*)^\perp$ is the orthogonal complement of $L^*$ in $\mathbb{R}^d$. (Recall that the subspaces $L^* = \text{span}\left\{L_i \big| \; i \in \mathcal{C}_0\right\}$ and $A^*$ are lattice subspaces, as elementary linear algebra shows.) The orthogonal direct sum

$$\tilde{L} \oplus \tilde{A} = L^*/\left(L^* \cap \mathbb{Z}^d\right) \oplus A^*/\left(A^* \cap \mathbb{Z}^d\right)$$

of the sub-tori $\tilde{L}$ and $\tilde{A}$ provides a finite covering of $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Therefore, the sub-billiard dynamics $\left\{S^t_{\mathcal{C}_0}\right\}$ defined by the cylinders with index in $\mathcal{C}_0$ is finitely covered by the direct product flow $\{S^t_* \times T^t_*\}$, where $\{S^t_*\}$ is the (transitive) cylindric billiard flow in the torus $\tilde{L} = L^*/\left(L^* \cap \mathbb{Z}^d\right)$ defined by the intersections of the cylinders $C_i$ ($i \in \mathcal{C}_0$) with the sub-torus $\tilde{L}$, while $\{T^t_*\}$ is the almost periodic (uniform) motion in $\tilde{A} = A^*/\left(A^* \cap \mathbb{Z}^d\right)$: More precisely, the phase point $x = (q, v) \in \mathbb{T}^d \times \mathbb{R}^d$ ($x \in A_1$) is decomposed locally as $q = q_1 + q_2$, $q_1 \in \tilde{L}$, $q_2 \in \tilde{A}$, $v = v_1 + v_2$, $v_1 \in L^*$,

$v_2 \in A^* = (L^*)^\perp$, $S^t\left((q,v)\right) = (q(t), v(t))$, $q(t) = q_1(t) + q_2(t)$, $v(t) = v_1(t) + v_2(t)$, $(q_1(t), v_1(t)) = S_*^t(q_1, v_1)$, $v_2(t) = v_2(0) = v_2$, $q_2(t) = q_2 + tv_2 = q_2(0) + tv_2(0)$. We are about to construct the local, weakly stable manifolds $\gamma^{ws}(x)$ for $x = (q, v) = (q_1 + q_2,\, v_1 + v_2) \in A_1$ in the following way:

$$
\begin{aligned}
\gamma^{ws}(x) = \Big\{ y = (q_1 + \delta q_1 + q_2 + \delta q_2,\ v_1 + \delta v_1 + v_2) \Big| \\
\operatorname{dist}\left(S_*^t(q_1, v_1),\ S_*^t(q_1 + \delta q_1, v_1 + \delta v_1)\right) \to 0 \\
\text{exp. fast as } t \to \infty, \text{ and } ||\delta q_1|| + ||\delta v_1|| + ||\delta q_2|| < \epsilon_0 \Big\}.
\end{aligned}
\tag{4.6}
$$

We see that $\gamma^{ws}(x)$ (if it exists as a manifold containing $x$ in its interior) is indeed the weakly stable manifold of the phase point $x$ corresponding to the artificially defined dynamics $S_*^t \times T_*^t$ for $t > 0$. There are two important facts here:

(A) The weakly stable manifolds $\gamma^{ws}(x)$ (yet to be constructed for typical $x \in A_1$) are concave, local orthogonal sub-manifolds (see the "Invariant Manifolds" part of §2 in [K-S-Sz(1990)]) and, as such, they are uniformly transversal to the manifold $\mathcal{S}_0$, see Sub-lemma 4.2 in [K-S-Sz(1990)];

(B) The exponentially stable part

$$
\begin{aligned}
\gamma^{es}(x) = \Big\{ y = (q_1 + \delta q_1 + q_2,\ v_1 + \delta v_1 + v_2) \Big| \\
\operatorname{dist}\left(S_*^t(q_1, v_1),\ S_*^t(q_1 + \delta q_1, v_1 + \delta v_1)\right) \to 0 \\
\text{exp. fast as } t \to \infty, \text{ and } ||\delta q_1|| + ||\delta v_1|| < \epsilon_0 \Big\}
\end{aligned}
\tag{4.6/a}
$$

of $\gamma^{ws}(x)$ $(x = (q_1 + q_2, v_1 + v_2) \in A_1)$ is to be constructed by using the statement of the Fundamental Theorem (Theorem 4.4 of [B-Ch-Sz-T(2001)]) for the $\mathcal{C}_0$-sub-billiard system $\{S_*^t\}$. This statement can be used, for the $\nu$-typical phase points $x = (q, v) = (q_1 + q_2, v_1 + v_2)$ of $A_1$ have the property that the $S_*$-part $\{S_*^t(q_1, v_1)\}$ of their forward orbit is hyperbolic with respect to the sub-billiard system defined by the cylinders $C_i$, $i \in \mathcal{C}_0$, see Corollary 3.18.

According to the above points (A) and (B), there exists a measurable subset $A_2 \subset A_1$ with $\nu(A_2) > 0$ and a number $\delta_0 > 0$ such that for every $x \in A_2$ the manifold $\gamma^{ws}(x)$ exists and its boundary is at least at the distance $\delta_0$ from $x$ (these distances are now measured by using the induced Riemannian metric on $\gamma^{ws}(x)$). Then, by the absolute continuity of the foliation, see Theorem 4.1 in [K-S(1986)], the union

$$B_2 = \bigcup_{x \in A_2} \gamma^{ws}(x) \subset \mathbf{M}$$

has a positive $\mu$-measure in the phase space $\mathbf{M}$.

Finally, the genuine forward orbits $S^{(0,\infty)}x$ of all points $x \in A_2$ avoid a fixed open ball $B_{r_0}$ of radius $r_0 > 0$. (For example: We may take any open ball $B_{r_0}$ inside the interior of any avoided cylinder $C_j$, $j \notin \mathcal{C}_0$.) Therefore, the forward orbit in the direct product dynamics $(S_*^t \times T_*^t)(y)$ of any point $y \in B_2$ ($y \in \gamma^{ws}(x)$, $x \in A_2$) avoids a slightly shrunk open ball $B_{r_0-\delta_0}$ of reduced radius $r_0 - \delta_0$. However, this is clearly impossible, for the following reason: For $y = (q_1 + q_2, v_1 + v_2)$ ($q_1 \in \tilde{L}$, $q_2 \in \tilde{A}$, $v_1 \in L^*$, $v_2 \in A^*$) the $v_2$ component is left invariant by the product flow $S_*^t \times T_*^t$, and for almost every fixed value $v_2 \in A^*$ (namely, for those vectors $v_2$ for which the orbit $tv_2/\left(A^* \cap \mathbb{Z}^d\right)$ ($t \in \mathbb{R}$) is dense in the torus $\tilde{A} = A^*/\left(A^* \cap \mathbb{Z}^d\right)$) the product flow $S_*^t \times T_*^t$ is ergodic on the corresponding level set, since it is the product of a mixing and an ergodic flow. The obtained contradiction finishes the indirect proof of the Chernov-Sinai Ansatz, that is, (H1). $\square$

**Corollary 4.7.** The set

$$NH(\mathcal{S}_0) = \left\{x \in \mathcal{S}_0 \middle|\; S^{(0,\infty)}x \text{ is not hyperbolic}\right\}$$

is a slim set. (In the case of a singular forward orbit non-hyperbolicity of $S^{(0,\infty)}x$ is meant that at least one branch of $S^{(0,\infty)}x$ is not hyperbolic, see §2.)

**Proof.** Since the complement set $\mathcal{S}_0 \setminus A = \mathcal{S}_0 \setminus A_0$ is a countable union of smooth, proper sub-manifolds of $\mathcal{S}_0$, the set $\mathcal{S}_0 \setminus A_0$ is slim. Therefore, it is enough to prove that the intersection $NH(\mathcal{S}_0) \cap A_0$ is slim. However, according to Corollary 3.18, the forward orbit $S^{(0,\infty)}x$ of every $x \in A_0$ is hyperbolic, unless $x$ belongs to a countable union of smooth, proper sub-manifolds of $\mathcal{S}_0$. Thus $NH(\mathcal{S}_0) \cap A_0$ is slim. $\square$

In view of Lemma 4.1 of [K-S-Sz(1990)], the set $R_2$ of phase points with more than one singularity on their orbit is slim, see also §2. Therefore, the final step in proving the remaining unproven induction hypothesis (i. e. (H2)) for our considered model $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ with $k$ ($\geq 2$) cylindric scatterers is to show that the set

$$D = \left\{x \in \mathbf{M}^0 \setminus \partial\mathbf{M} \middle|\; S^{(-\infty,\infty)}x \text{ is not hyperbolic}\right\} \tag{4.8}$$

is slim, i. e. it can be covered by a countable collection of closed subsets $F \subset \mathbf{M}$ with $\mu(F) = 0$ and $\dim F \leq \dim \mathbf{M} - 2$. By the locality of slimness, see §2 above, it is enough to prove that for every element $x \in D$ the point $x$ has an open neighborhood $U$ (in $\mathbf{M}$) such that the set $U \cap D$ is slim. We want to classify the phase points $x \in D$.

Consider, therefore, an arbitrary phase point $x=(q,v)\in D$. Denote the doubly infinite, symbolic collision sequence of $S^{(-\infty,\infty)}x$ by $\Sigma=(\dots,\sigma_{-2},\sigma_{-1},\,\sigma_1,\sigma_2,\dots)$ so that $\sigma_1$ is the first collision in positive time. (The index 0 is not used.) We distinguish between two cases:

**Case I.** $L^*=\operatorname{span}\{L_{\sigma_i}|\ i\in\mathbb{Z}\setminus\{0\}\}\neq\mathbb{R}^d$.

In this case, as we have seen before, the dynamics of $S^{(-\infty,\infty)}x$ is finitely covered by the direct product flow $\{S_*^t\times T_*^t\}$, where $\{S_*^t\}$ is the cylindric billiard flow in the sub-torus $\tilde{L}=L^*/\left(L^*\cap\mathbb{Z}^d\right)$ with the scatterers $C_{\sigma_i}\cap\tilde{L}$, while $\{T_*^t\}$ is the almost periodic (uniform) motion in the orthocomplement torus $\tilde{A}=A^*/\left(A^*\cap\mathbb{Z}^d\right)$, $A^*=(L^*)^\perp$. Now the point is that for the cylindric billiard flow $(\tilde{L},\{S_*^t\},\mu_{\tilde{L}})$ both of the induction hypotheses (H1)–(H2) and, consequently, Theorem 5.2 of [Sz(2000)] apply. For the phase point $x\in D$ the direct product flow $(S_*^t\times T_*^t)(x)$ avoids an open ball, namely any open ball in the interior of any avoided cylinder $C_j$ with

$$j\notin\{\sigma_i|\ i\in\mathbb{Z}\setminus\{0\}\}\,.$$

Consequently, for each component $(q_2,v_2)\in\tilde{A}\times A^*$ of the canonical decomposition of $x=(q,v)=(q_1+q_2,v_1+v_2)$, $q_1\in\tilde{L}$, $v_1\in L^*$, $q_2\in\tilde{A}$, $v_2\in A^*$ it is true that the $\tilde{L}$-orbit $S_*^t(q_1,v_1)$ of $(q_1,v_1)$ avoids an open set $\emptyset\neq B\subset\tilde{L}$ on a doubly unbounded set $H$ of time moments, $\inf H=-\infty$, $\sup H=+\infty$. Therefore, in view of Theorem 5.2 of [Sz(2000)], the $(q_1,v_1)$-part of the phase point $x=(q_1+q_2,v_1+v_2)$ belongs to a slim subset $S_1$ of the phase space $\tilde{L}\times L^*$. According to the integrability property of closed slim sets (cf. Property 4 in §4.1 of [K-S-Sz(1989)]), even the closure $\bar{D}_1$ of the set

$$D_1=\left\{x\in D\middle|\ \operatorname{span}\left\{L_{\sigma_i(x)}|\ i\in\mathbb{Z}\setminus\{0\}\right\}\neq\mathbb{R}^d\right\}\tag{4.9}$$

(covered by Case I) is a slim subset of the phase space $\mathbf{M}$. We note that the set $\bar{D}_1$ is contained in the closed zero-set

$$\begin{aligned}K=\big\{x\in\mathbf{M}^\#\big|\ &x\text{ has a trajectory branch with a symbolic sequence}\\&(\dots,\,\sigma_{-1},\,\sigma_1,\,\dots)\text{ such that }\operatorname{span}\{L_{\sigma_i}|\ i\in\mathbb{Z}\setminus\{0\}\}\neq\mathbb{R}^d\big\},\end{aligned}$$

and the argument with "integrating up" the closed slim sets (by using Property 4 in §4.1 of [K-S-Sz(1989)]) is applied to the closed set $K$.

**Case II.** $L^*=\operatorname{span}\{L_{\sigma_i}|\ i\in\mathbb{Z}\setminus\{0\}\}=\mathbb{R}^d$.

Select a vector $0\neq w\in\mathcal{N}\left(S^{(-\infty,\infty)}x\right)$, $w\perp v$, from the neutral space

$$\mathcal{N}\left(S^{(-\infty,\infty)}x\right)=\mathcal{N}(x)$$

of the considered phase point $x \in D$. For $i \in \mathbb{Z} \setminus \{0\}$ denote by $\alpha_i = \alpha_i(w)$ the "advance" of the collision $\sigma_i$ corresponding to the neutral vector $w$, see §2. Since $w$ is not parallel to $v$, at least two advances with neighboring indices are unequal; we may assume that $\alpha_{-1} \neq \alpha_1$.

It follows from the proof of Lemma 3.2 that the event $\alpha_{-1} \neq \alpha_1$ can only occur if

$$v = v_0 \in \operatorname{span}\left\{A_{\sigma_{-1}},\, A_{\sigma_1}\right\}. \tag{4.10}$$

If the event $\alpha_k \neq \alpha_{k+1}$ $(k \neq -1,\, 0)$ took place for another pair of neighboring advances as well, then, again by the proof of Lemma 3.2, we would have

$$v_t \in \operatorname{span}\left\{A_{\sigma_k},\, A_{\sigma_{k+1}}\right\} \quad (t_k < t < t_{k+1}). \tag{4.11}$$

If at least one of the two subspaces on the right-hand-sides of (4.10) and (4.11) is of codimension higher than one, then the corresponding event alone ensures that the studied phase point $x \in D$ belongs to some codimension-two (i. e. at least two), smooth submanifold of the phase space, and such phase points obviously constitute a slim set, therefore they may be discarded.

Thus, we may assume that

$$\begin{aligned} &\dim\left(\operatorname{span}\left\{A_{\sigma_{-1}},\, A_{\sigma_1}\right\}\right) \\ &= \dim\left(\operatorname{span}\left\{A_{\sigma_k},\, A_{\sigma_{k+1}}\right\}\right) = d-1. \end{aligned} \tag{4.12}$$

Denote by $n_\tau$ a (unit) normal vector of the $S^\tau$-image of the manifold

$$v_0 \in \operatorname{span}\left\{A_{\sigma_{-1}},\, A_{\sigma_1}\right\}$$

at the phase point $S^\tau x = x_\tau$, and by $\tilde{n}_\tau$ a (unit) normal vector of the $S^\tau$-image of the manifold $v_t \in \operatorname{span}\left\{A_{\sigma_k},\, A_{\sigma_{k+1}}\right\}$ $(t_k < t < t_{k+1})$ at the phase point $S^\tau x = x_\tau$. It follows from the proof of Corollary 3.19 that $Q(n_\tau) = 0$ for $t_{-1} < \tau < t_1$, $Q(n_\tau) < 0$ for $\tau > t_1$, $Q(n_\tau) > 0$ for $\tau < t_{-1}$, $Q(\tilde{n}_\tau) = 0$ for $t_k < \tau < t_{k+1}$, $Q(\tilde{n}_\tau) < 0$ for $\tau > t_{k+1}$, and $Q(\tilde{n}_\tau) > 0$ for $\tau < t_k$. Therefore, the two codimension-one sub-manifolds defined by (4.10) and (4.11) are transversal, so the simultaneous validity of (4.10)–(4.11) again results in an event for $x \in D$ showing that $x$ belongs to a slim subset of $\mathbf{M}$.

Thus we may assume that

$$\alpha_k = \alpha_{-1} \neq \alpha_1 = \alpha_l$$

for all $k \le -1$, $l \ge 1$. By adding a suitable, scalar multiple of the velocity $v = v_0$ to the neutral vector $w$, we can achieve that

$$\alpha_k = \alpha_{-1} \ne 0 = \alpha_l \tag{4.13}$$

for all $k \le -1$, $l \ge 1$. The equalities $\alpha_l = 0$ $(l \ge 1)$ mean that

$$w \in \bigcap_{l>0} A_{\sigma_l} = \left(\operatorname{span}\left\{L_{\sigma_l}\,\middle|\, l > 0\right\}\right)^{\perp}, \tag{4.14}$$

see also the closing part of the proof of Proposition 3.1. An analogous argument shows that

$$w - \alpha_{-1} v \in \bigcap_{k<0} A_{\sigma_k} = \left(\operatorname{span}\left\{L_{\sigma_k}\,\middle|\, k < 0\right\}\right)^{\perp}. \tag{4.15}$$

The equations (4.14)–(4.15) and $\alpha_{-1} \ne 0$ imply that

$$\begin{aligned} v \in \operatorname{span} &\left\{\bigcap_{k<0} A_{\sigma_k},\ \bigcap_{l>0} A_{\sigma_l}\right\} \\ &= \bigcap_{k<0} A_{\sigma_k} + \bigcap_{l>0} A_{\sigma_l} := H. \end{aligned} \tag{4.16}$$

Recall that $\bigcap_{n\ne 0} A_{\sigma_n} = \{0\}$ in the actual Case II, and $H \ne \mathbb{R}^d$, since span $\{A_{\sigma_k},\ A_{\sigma_l}\} \ne \mathbb{R}^d$ for $k < 0 < l$. We can assume that the linear direct sum on the right-hand-side of (4.16) is a subspace with codimension one, otherwise (just as many times in the past) the containment in (4.16) would be a codimension-two condition on the initial velocity $v = v_0$, and all such phase points $x = (q, v)$ can be discarded. Following the tradition, denote by $J$ the codimension-one sub-manifold of $\mathbf{M}$ defined by (4.16). The proof of the Ansatz in the case of (4.5) can now be repeated almost word-by-word. Indeed, the phase points

$$\begin{aligned} x = (q,\, v) \in \bar{D} = \bar{D}(\mathcal{A}, \mathcal{B}) = \big\{x \in D \big|\ & v \in H, \\ \{\sigma_k(x) |\ k < 0\} = \mathcal{A},\ \{\sigma_l(x) |\ l > 0\} &= \mathcal{B}\big\} \end{aligned} \tag{4.17}$$

(with given $\mathcal{A}$, $\mathcal{B} \subset \{1, 2, \dots, k\}$ such that span $\{L_j |\ j \in \mathcal{A} \cup \mathcal{B}\} = \mathbb{R}^d$) of the considered type again decompose as $(q, v) = (q_1 + q_2,\ v_1 + v_2)$, $v_1 \in L^* = \operatorname{span}\{L_{\sigma_l} |\ l > 0\}$, $v_2 \in A^* = (L^*)^{\perp} = \bigcap_{l>0} A_{\sigma_l}$, $q_1 \in \tilde{L} = L^* / \left(L^* \cap \mathbb{Z}^d\right)$, $q_2 \in \tilde{A} = A^* / \left(A^* \cap \mathbb{Z}^d\right)$, and the forward orbit $S^{(0,\infty)}x$ of our considered phase point $x \in D \cap J$ (fulfilling all of the mentioned assumptions) is essentially (up-to a finite covering) is governed by the product flow $(q_1(t), v_1(t)) = S_*^t(q_1, v_1)$, $(q_2(t), v_2(t)) = T_*^t(q_2, v_2) = (q_2 + t v_2, v_2)$, where (as said before) $S_*^t$ is the sub-billiard flow in $\tilde{L}$ defined by the intersections of the cylinders $\{C_{\sigma_l} |\ l > 0\}$ with the torus $\tilde{L}$.

**Lemma 4.18.** The exponentially stable component $\gamma^{es}(x)$ of $\gamma^{ws}(x)$ $(x \in D \cap J)$ defined by (4.6/a) is transversal to the codimension-one manifold $J$ described by the membership in (4.16).

**Proof.** Argue by contradiction. Assume that $\mathcal{T}_x\gamma^{es}(x) \subset \mathcal{T}_x J$. The tangent space $\mathcal{T}_x J$ is obviously given by the simple formula

$$\mathcal{T}_x J = \left\{ (\delta q, \delta v) \in \mathcal{T}_x \mathbf{M} \middle|\ \delta v \in H \right\}. \tag{4.19}$$

The second fundamental form $B\left(\gamma^{es}(x)\right)$ of $\gamma^{es}(x)$ at the phase point $x = (q_1 + q_2, v_1 + v_2)$ is known to be negative definite, so its range is the entire orthocomplement $(v_1)^{\perp}$ of $v_1$ in the space

$$L^* = \operatorname{span}\left\{ L_{\sigma_l} \middle|\ l > 0 \right\} = \left( \bigcap_{l>0} A_{\sigma_l} \right)^{\perp}.$$

On the other hand, since

$$v = v_1 + v_2 \in H = \bigcap_{k<0} A_{\sigma_k} + \bigcap_{l>0} A_{\sigma_l}$$

and $v_2 \in A^* = \bigcap_{l>0} A_{\sigma_l} \subset H$, from the assumed relation $\mathcal{T}_x\gamma^{es}(x) \subset \mathcal{T}_x J$ and from $v_1 \in H$ we get that $L^* \subset H$. Since $A^* = (L^*)^{\perp} \subset H$, this means that $H = \mathbb{R}^d$, contradicting $\dim H = d - 1$. This finishes the proof of the lemma. $\square$

Finally, the slimness of the set $D$ in (4.8) will be proven in Case II as soon as we show that $\nu_J(\bar{D}) = 0$, where $\bar{D} = \bar{D}(\mathcal{A}, \mathcal{B})$ is defined in (4.17). This is, however, obtained the same way as the relation $\nu(A_2) = 0$ at the end of the proof of the Ansatz. Indeed, in the case $\nu_J(\bar{D}) > 0$ the union

$$\tilde{D} := \bigcup_{x \in \bar{D}} \gamma^{es}(x)$$

would have a positive $\mu$-measure in $\mathbf{M}$ (by the transversality proved above and by the absolute continuity of the $\gamma^{es}(\,.\,)$ foliation, see Theorem 4.1 in [K-S(1986)]), but this is impossible, for all forward orbits $S^{(0,\infty)}y$ of the points $y \in \tilde{D}$ would avoid a common open ball that can be obtained by slightly shrinking any open ball inside the interior of any avoided cylinder $C_j$ with $j \notin \mathcal{B}$, see also the closing part of the proof of $\nu(A_2) = 0$ above.

This finishes the proof of the fact that the set $D$ in (4.8) is indeed slim. From this, from the proved Chernov-Sinai Ansatz, and from the quoted slimness of the set $R_2$ of phase points with more than one singularities on their orbit we obtain the validity of the induction hypotheses (H1)—(H2) (and therefore (H3)—(H4), as well) for the considered cylindric billiard flow $(\mathbf{M}, \{S^t\}_{t\in\mathbb{R}}, \mu)$ with $k$ cylinders. This finishes the inductive proof of the Theorem. $\square$

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