Einstein's lifts and topologies: topological investigations on the Principle of Equivalence

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Abstract

The gedanken-experiment of Einstein's lift is analyzed in order of determining whether the free-falling observer inside the lift can detect the eventual topological non-triviality of space-time, as it would seem considering a non-globally-hamiltonian action of the symmetry group of the observer's action (that, unfortunately, can be obtained only submitting the lift also to a suitable electromagnetic field) and considering that the observer can locally detect the topological alteration of the constants-of-motion's algebra.

It follows that a problem exists in formalizing the Principle of Equivalence, owing to its indetermination as to the topology of the reference's flat space-time defining the special relativistic laws to which, up to first order terms in the normal coordinates of the lift's Lorentz moving inertial frame, all the non-gravitational Laws of Physics have to collapse.

It is then shown how the problem may be avoided getting rid of the Principle of Equivalence following the Hawking-Ellis' axiomatization of General Relativity purely based on the assumption of the Einstein-Hilbert's action.

Connes' axiomatization of General Relativity having as only dynamical variable the spectrum of Dirac's operator is then used to discuss the initial topological question concerning Einstein's lift in the language of Spectral Geometry, explicitly showing its inter-relation with the celebrated Marc Kac's issue whether one can hear the shape of a drum, and showing how Index Theory is the natural framework in which some partial answer may be obtained.

The whole issue is then analyzed in Connes' Quantum Gravity, suggesting how Noncommutative Geometry allows, through Noncommutative Index Theory, to get some insight along the footsteps followed in the commutative case.

Some attempt of relating the issue to Anandan's remark on the difference among the holonomies of General Relativity and the holonomies of Yang-Mills' theories is finally reported.

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I. LOCAL SIGNATURE OF THE GLOBAL TOPOLOGY

Let us consider a particle of mass m free-falling in the space-time $(M \, , \, g)$: it is described by a classical dynamical system, that I will denote as $FREE - FALL_m(M \, , \, g)$, determined by the action functional:

$$S[q^\mu(\lambda)] \; := \; - m \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{ - g_{\mu \, \nu}(q(\lambda)) \, \frac{dq^\mu(\lambda)}{d\lambda} \frac{dq^\nu(\lambda)}{d\lambda} } \tag{1.1}$$

The reparametrization invariance of the paths in eq.1.1 generates the existence of a primary first class constraint [1], [2] consisting in the on-mass-shell-condition, expressed, adopting Penrose's abstract index convention [3], as:

$$H \; = \; p_a g^{a \, b} p_b \, + \, m^2 \; \approx \; 0 \tag{1.2}$$

individuating the coisotropic submanifold [4]:

$$C \; := \; \{ \, (p \, , \, q) \; :\in \; T^\star M \; : \; p_a g^{a \, b} p_b \, + \, m^2 \; = \; 0 \} \tag{1.3}$$

of the phase space $\Gamma \; := \; (T^\star M \, , \, \omega_{st})$ of $FREE - FALL_m(M \, , \, g)$, where ω_{st} is the standard symplectic form of M's cotangent bundle.

Denoted by σ the 2-form induced by ω_{st} on the constraint's submanifold C:

$$\sigma \; := \; i^\star \omega_{st} \tag{1.4}$$

($i \; : \; C \; \mapsto \; M$ being the inclusion), Weinstein's reduction of the pre-symplectic manifold $(C \, , \, \sigma)$ individuates the reduced phase space:

$$\Gamma_{RED} \; := \; (\frac{C}{K} \, , \, \omega) \tag{1.5}$$

where $K \; := \; TC^\perp$ is C's characteristic distribution and ω is the induced symplectic form on $\frac{C}{K}$.

Let us now consider the time-like geodesic γ solution of the Cauchy's problem:

$$\frac{\delta S[q(\tau)]}{\delta q(\tau)} \; = \; 0 \tag{1.6}$$

$$q(0) \; = \; q_1 \tag{1.7}$$

$$q(0) = q_2 \tag{1.8}$$

where I have imposed the parametrization through proper-time (proper-time gauge fixing) and where q_1 and q_2 belong to a geodesically convex open U of M.

Referring to the celebrated *gedanken experiment* of Einstein's lift, let us suppose our particle to be an observer closed inside the walls of a lift, itself free-falling.

Let us now analyze M's topology:

if we assume Penrose's strong-form of the Cosmic Censorship Conjecture stating the global hyperbolicity of (M , g), i.e. stating the existence in (M , g) of Cauchy-surfaces, the topology of M is of the form $\mathbb{R} \times \Sigma$, Σ being any Cauchy 3-surface of (M , g).

The problem of classifying the homeomorhism's classes of topological-3-manifolds is (as I know) still open; this is partially owed to the fact that Poincaré's conjecture (that every 3-homotopic-sphere $S_h^{(3)}$ is homeomorphic to $S^{(3)}$) is still open [1].

Let us observe that, fortunately, the fact that a Cauchy 3-surface Σ is not simply a tridimensional topological manifold but is endowed with a differentiable structure doesn't change the game since every topological 3-manifold admits one and only one differentiable structure [6].

Contrary, if we don't assume Penrose's strong-form of the Cosmic Censorship Conjecture we have no restrictions on the topology of M.

The problem of classifying 4-topological manifolds was proved by Markov to be recursively-unsolvable still in 1958, the undecidability deriving, essentially, from the recursive-unsolvability of group-theoretic problems regarding the fundamental groups of the 4-topological manifolds (cfr. the 7^{th} chapter "Decision Problems" of [7])

The problem of classifying the simply-connected 4-topological-manifolds has been "almost solved" in 1982 by M.H. Freedman [8] through a theorem stating that:

- the simply-connected 4-topological-manifolds with even intersection form [6], [9] are all homeomorphic
- the simply-connected 4-topological-manifolds with odd intersection form [6], [9] are divided in two equivalence-classes of homeomorhisms

[1] I consider it open until the Clay Mathematical Institute certifies that the recent proof by M.J. Dunwoody [5] is the correct answer to http://www.claymath.org/prizeproblems/poincare.htm

Let us observe, furthermore, that in a space-time $(M\,,\,g)$ M is a differentiable manifold; this complicates the game since, in four dimensions, the topological-manifolds and the differentiable-manifolds are no more bijective:

- Donaldson [10] has proved that, given a simply-connected 4-differentiable-manifold with positive-definite intersection form, such an intersection form is diagonalizable on the integers; an immediate corollary of Donaldson's Theorem is that a simply-connected 4-topological-manifold having even and positive-definite intersection form doesn't admit differentiable structures.

- it is possible, anyway, to determine suitable hypothesis' under which a 4-topological manifold admits differentiable structures. E.g. Quinn [11] has proved that it is sufficient to require that M is not compact.

If a topological manifold of dimension greater than three admits differentiable structures it doesn't imply, anyway, that it admits only one differentiable structure. The dimension four is, as to the classification of differentiable structures, a case for its own: the existence, in certain cases, of a finite, discrete number of exotic structures for compact topological manifolds of dimension different from four (e.g. the 27 exotic-spheres discovered by Milnor [12] in 1956) is a result of Obstruction's Theory, i.e. it is corresponding to the non-triviality of some characteristic class; the (eventual) existence of exotic structure in four dimensions has different origins and may also lead to an uncountable infinity of this kind of structures.

E.g., as it has been proved by Gomft [13], $\mathbb{R}^4$ admits an uncountable infinity of differentiable structures.

The problem of classifying the topologies and the differentiable structures of 4-topological-manifolds is, indeed, one of the most fascinating research fields of contemporary Mathematical Physics, owing to:

- its link [14] with the structure of the Moduli-space Mod_k of the instantons of classical SU(2)-Yang-Mills-theories built on principal bundles with second Chern's class k (already displayed by the proof of Donaldson's Theorem)

- the introduction by Donaldson [15], [16] of a class of powerful topological invariants for simply connected 4-topological-manifolds, namely the celebrated Donaldson's polynomials (certain integer symmetric polynomials in the second integer homology)

- Witten's discovery [17], [18] that Donaldson's polynomials may be obtained as correlation functions of a suitable BRST-supersymmetric topological field theory
- the introduction by Seiberg nd Witten [18] of a simpler approach based on coupled equations for a section of a linear bundle and a connection on an auxiliary fibre bundle with abelian gauge group (precisely U(1)).

In these note, anyway, I won't enter into the mathematical sophistications concerning the classification of the topologies of 4-differentiable- manifolds.

I will instead concentrate all the attention on two basic physical questions:

1. **QUESTION I.1**

   QUESTION ON THE LOCAL OBSERVABILITY OF THE GLOBAL TOPOLOGY

   can the observer in the lift understand, during the proper-time interval $[0\,,\,T]$*, if the topology of* M *is trivial ?*

2. **QUESTION I.2**

   QUESTION ON THE CORRECT FORMULATION OF THE PRINCIPLE OF EQUIVALENCE

   supposing that the question I.1 has affirmative answer, can this fact be formalized saying that Mechanics of the dynamical system $FREE-FALL_m(M\,,\,g)$ *is obtained applying the Principle of Equivalence by taking as reference space-time not the topologically trivial minkowskian space-time* $(\mathbb{R}^4\,,\,\eta := -dx^0 \bigotimes dx^0 + dx^i \bigotimes dx^i)$*, but a topologically non-trivial modification of it, i.e. a space-time of the form* $(X\,,\,\eta := -dx^0 \bigotimes dx^0 + dx^i \bigotimes dx^i)$ *where* X *is a topologically non-trivial 4-differentiable-manifold ?*

II. NON-GLOBALLY-HAMILTONIAN ACTIONS OF THE ISOMETRIES ON THE REDUCED PHASE SPACE

In this section I will try to discuss questionI.1 using a celebrated result of Analitical Mechanics: the cohomological interpretation of the existence of actions by symplectic diffeomorphisms of symmetry groups on the phase space of a classical dynamical system such that they are not globally-hamiltonian, i.e. they haven't an equivariant momentum map [2].

Given the classical dynamical system $FREE-FALL_m(M\,,\,g)$ let us consider an action-from-left Φ through symplectic diffeomorphisms of the isometries's group G of $(M\,,\,g)$ on the reduced phase-space $\Gamma_{RED}\ := (\frac{C}{K}\,,\,\omega)$, i.e.:

$$\Phi\,:\,G\times\Gamma_{RED}\,\mapsto\,\Gamma_{RED}\,:\,(\Phi_g\in Diff(\Gamma_{RED})\,and\,(\Phi_g)^\star\omega\,=\,0)\ \forall g\in G \tag{2.1}$$

Obviously G is a symmetry group of $FREE-FALL_m(M\,,\,g)$, i.e. its motion-equation is invariant under Φ.

A moment map of Φ:

$$\vec{J}\,:\,\Gamma_{RED}\,\mapsto\,L^\star(G)\,:<\vec{J}(x)\,,\,A>=\,J_A(x) \tag{2.2}$$

(where $J_A\in C^\infty(\Gamma_{RED})$ is the classical observable generating the one-parameter group of symplectic diffeomorphisms $\{\Phi_{\exp(t\,A)}\}$) is equivariant, i.e. such that:

$$\{J_A\,,\,J_B\}\;=\;J_{[A\,,\,B]}\ \forall\,A,B\,\in\,L(G) \tag{2.3}$$

if and only if the 2-cocycle $\Sigma\in Z_2[L(G)]$:

$$\Sigma(A\,,\,B)\;:=\;T_e\sigma_B(A)\ \forall\,A\,,\,B\in L(G) \tag{2.4}$$

is identically null, where the map $\sigma_A\,:\,G\,\mapsto\,\mathbb{R}$:

$$\sigma_A(g)\;:=\;\sigma(g)\ \forall\,A\,\in\,L(G) \tag{2.5}$$

is defined through the map $\sigma\,:\,G\,\mapsto\,Hom(L(G)\,,\,\mathbb{R})$:

$$\sigma(g)\,\cdot\,A\;:=\;P_{A,g} \tag{2.6}$$

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[2] unfortunately not all the authors follow the terminology by Marsden and Ratiu that I have adopted [19]: e.g. Marsden and Ratiu's notion of globally-hamiltonianity is called simply hamiltonianity by Mc Duff and Salamon [4] while it is called poissonianity by Arnold [20]

defined, in its turn, though the family of maps $\{P_{A,g} : \Gamma_{RED} \mapsto \mathbb{R} , \, A \in L(G) \, , \, g \in G\}$:

$$P_{A,g}(x) \; := < \vec{J}(\Phi_g(x)) \, , \, A > \; - \; < Ad^{\star}_{g^{-1}} \, , \, A > \tag{2.7}$$

where $Ad^{\star}$ is the co-adjoint representation of G.

The cohomology class $[\Sigma] \in H^2[L(G)]$ is univocally determined by the action Φ.

Let us suppose that Σ is not null.

If $[\Sigma] \; = \; [0]$ it is possible, anyway, "to repair" (through a suitable re-definition of the $\{J_A \, , \, A \in L(G)\}$ corresponding to the addition to Σ of a 2-coborder) the momentum map $\vec{J}$ in order to obtain a different momentum map having the equivariance property; the action is, conseguentially, globally hamiltonian.

If $[\Sigma] \; \neq \; [0]$, contrary, no "repairing" of $\vec{J}$ allows to result in a new equivariant momentum map. Such a situation may be described as a classical anomaly: an equivariant momentum map may be obtained only at the price of substituting G with a central extension of its in complete analogy with the situation occurring in quantum anomalies [21], where the impossibility of representing a symmetry group in a non-projective way, again owed to the non-triviality of a suitable 2-cocyle, may be "repaired" only constructing an ordinary representation of a central extension.

When $H^2[L(G)] \; = \; 0$, as it happens, for example, if $(M \, , \, g) \; = \; (\mathbb{R}^4 \, , \, \eta := \; -dx^0 \bigotimes dx^0 + dx^i \bigotimes dx^i)$, every action Φ of G on the reduced phase space is globally hamiltonian.

We can then ask ourselves the following:

QUESTION II.1

QUESTION ON THE LOCAL OBSERVABILITY OF THE NON-EQUIVARIANT MAP

can the observer in the lift notice the alteration of the Lie algebra of the motion's constants owed to a non-equivariant momentum map ?

If the answer to questionII.1 is affirmative, it is then evident the importance, for the resolution of the questionI.1, of the following:

QUESTION II.2

QUESTION ON THE POSSIBILITY OF INFERRING THE TOPOLOGICAL NON-TRIVIALITY OF A SPACE-TIME FROM THE COHOMOLOGICAL-NON-TRIVIALITY OF THE ALGEBRA OF ITS KILLING VECTOR FIELDS

is it possible, from the observation of the topological non-triviality of $H^2[L(G)]$ to infer the topological non-triviality of M ?

Indeed if the answers to both questionII.1 and questionII.2 were positive, this would imply that the answer to questionI.1 would be positive too.

III. CORRECT FORMALIZATION OF THE PRINCIPLE OF EQUIVALENCE

In this paragraph I will investigate which implication an eventual positive answer to questionII.1 and questionII.2 would have as to questionI.1, as to questionI.2 and, ultimativelly, on the issue concerning the correct formalization of the Principle of Equivalence.

I will, consequentially, assume that the observer closed in the lift may, through a measurement of the constants of motion and of the algebraic relations they satisfy, notify a suitable topological non-triviality of M during the short proper-time's interval $[0\,,\,T]$.

Let us consider a Cartan's gauge [22], i.e. a local section of the general frame bundle $GLM(M\,,\,GL(4,\mathbb{R}))$ and let us assume that:

- it is adapted to the timelike geodesic arc γ, defined by the Cauchy problem of eq.1.6, eq.1.7 and eq.1.8
- all the frames in the set:

$$s_{\gamma} \;:=\; \{(e_0(x)\,,\,\cdots\,,e_3(x))\;:\;x\in\gamma\bigcap M^s\}\;\subset\;s \tag{3.1}$$

  (where M^s denotes the definition's domain of s) are orthonormal
- all the frames' elements $e_0(x)\,,\,x\in\gamma$ are equal to the tangent vectors of the geodesic for a suitable choice, let us call it λ, of the affine parameter

Such a Cartan's gauge is sometimes called [23], [24] a Lorentz-moving-inertial-frame for γ.

Let us now construct the normal coordinates associated to s; this may be done by the following steps:

1. let us pose $X^0(\lambda)\;=\;\lambda$ in all the points along γ
2. for every point $x\in\gamma$ let us draw all space-like geodesics with tangent vectors:

$$X\;=\;X^i e_i(x)\;\in\;T_x M \tag{3.2}$$

   in a fixed parametrization
3. in a suitable neighborhood of $x(\lambda)\;\in\;\gamma$ let us assign to every point y belonging to one of these geodesics at unitary parametric distance from $x(\lambda)$ the coordinates $(X^0\,,\,X^1,\,,\,X^2\,,\,X^3)$

4. making this operation for every point of γ we obtain a coordinates' system $(X^0, X^1, , X^2, X^3)$, defined in a suitable tube around γ, that we define to constitute the normal coordinates associated to the Lorentz moving inertial frame s for γ.

We can now formalize the Principle of Equivalence in the following way [3] :

PRINCIPLE OF EQUIVALENCE:

for every Lorentz moving inertial frame s free-falling along a time-like geodesic γ all the non-gravitational laws of Physics, expressed in the normal coordinates associated to s, have in each point of γ to be equal, up to first-order in these coordinates, to the corresponding special-relativistic laws expressed in the coordinates associated to the respective Lorentz frames

Let us now consider our observer who, in the assumed hypotheses, is able to detect, during the short proper-time interval $[0, T]$ and without looking out of the lift, the topological non-triviality of M.

Using the Principle of Equivalence in the way opposite to the one usually adopted, i.e. utilizing the knowledge of a non-gravitational law on (M, g) to infer the form of the corresponding special-relativistic law, we may infer that the algebraic relations defining the constants-of-motion's Lie algebra in the special-relativistic dynamical system corresponding to $FREE - FALL_m(M, g)$ must themselves show the same alteration that, for $FREE - FALL_m(M, g)$, shows the topological non-triviality of M.

This fact may be rephrased saying that the special relativistic dynamical system corresponding to $FREE - FALL_m(M, g)$ is not the dynamical system $FREE - FALL_m(\mathbb{R}^4, \eta := -dx^0 \bigotimes dx^0 + dx^i \bigotimes dx^i)$ but a special-relativistic dynamical system of the form $FREE - FALL_m(X, \eta := -dx^0 \bigotimes dx^0 + dx^i \bigotimes dx^i)$, where X is some topologically non-trivial 4-differentiable-manifold.

Under the assumed hypotheses this could be at its turn rephrased saying that **the reference-spacetime to use to generalize the non-gravitational special-relativistic laws to the topologically non-trivial space-time (M, g) is not the Minkowskian**

[3] Though not agreeing on Prugovecki's ideas on Quantum Gravity, I adopt here his formulation of the Principle of Equivalence strongly demanding to [23], [24] for all the underlying conceptual sophistications such as as Friedman's distinction between first-order-laws and second-order-laws [25] and its rule as to the Factor-Ordering Problem (cfr. the section 16.3 "The Factor-Ordering Problems in the Principle of Equivalence" of [26])

space time $(\mathbb{R}^4 , \eta := -dx^0 \bigotimes dx^0 + dx^i \bigotimes dx^i)$ **but a flat space-time** $(X , \eta := -dx^0 \bigotimes dx^0 + dx^i \bigotimes dx^i)$ **where X is some topologically non-trivial 4-differentiable-manifold**.

We have, up to this point, assumed that it may be the case that the momentum map for the action of the isometries' group on the reduced phase is not equivariant; unfortunately this is not the case, requiring a slight technical modification of the previous analysis.

The key point consists in the Theorem of Global Hamiltonianity for Cotangent Lifts [19] whose formulation requires the introduction of some preliminary notion.

Given two differentiable-manifold Q_1 and Q_2 and a diffeomorphism $f : Q_1 \mapsto Q_2$, the cotangent lift of f is the map $T^\star f : T^\star Q_1 \mapsto T^\star Q_2$ defined as:

$$< T^\star f(\alpha_s) , v > := < \alpha_s , f_\star v > \quad \alpha_s \in T^\star_s Q_1 , v \in T_q Q_2 , s = f(q) \tag{3.3}$$

Let us now consider a differentiable-manifold Q on which a left-action Φ of a Lie group G by diffeomorphisms is defined.

This action may then be associated with an a action $\Phi_\star$ of G on the symplectic manifold $(T^\star Q , \omega_{st})$, called the cotangent-lift (left) action of Φ, defined as:

$$(\Phi_\star)_g := T^\star_{g \cdot q}(\Phi_{g^{-1}}) \tag{3.4}$$

The Theorem of Global Hamiltonianity for Cotangent Lifts states that any cotangent-lift (left)-action is globally hamiltonian.

It should be clear what a calamity this theorem is for our purposes: it implies that the action of the isometries'-group of a space-time (M , g) on the reduced phase-space Γ_{RED} has an equivariant momentum map, so that it cannot be useful as to our search of an example allowing to give a positive answer to questionII.1.

Fortunately there exist a way of bypassing this obstacle, consisting in considering not a free-particle, but a particle minimally-coupled with an instanton ∇, belonging to the moduli space with topological charge Mod_k, of a Yang-Mills theory on (M , g) with gauge group G (assumed to be a compact and connected Lie group) [27], [6], [9].

If $k = 0$ the principal bundle $P(M , G)$ of the Yang-Mills theory is trivial, i.e. $P = M \times G$, and hence admits a global fibre chart. Conseguentially ∇ admits a global potential A. In this case we can easily infer that, again, the reparametrization invariance of our particle's action leads to the existence of a primary first class constraint stating the weak-vanishing of the hamiltonian:

$$H \, := \, (p_{\mu} \, - \, A_{\mu}) g^{\mu \nu} (p_{\nu} \, - \, A_{\nu}) \, \approx \, 0 \tag{3.5}$$

If, contrary, $k \, \neq \, 0$, $P(M \, , \, G)$ is non-trivial and, conseguentially, it doesn't admit a global fibre chart so, that the instanton ∇ cannot be described by a single global potential.

Let us then take in account a family of fibre-charts $\{ (U_{i} \, , \, \varphi_{i}) \}$ such that $\{ U_{i} \}$ is a contractible open covering of M, and the associated family of local one-potentials $\{ A_{i} \}$. For every chart we may consider the local hamiltonian:

$$H_{i} \, := \, (p_{\mu} \, - \, A_{\mu}^{(i)}) g^{\mu \nu} (p_{\nu} \, - \, A_{\nu}^{(i)}) \tag{3.6}$$

undefined outside U_{i}.

These considerations would then lead to infer that, for $k \, \neq \, 0$, it is not possible to obtain a single global hamiltonian. This is indeed true until we insist in requiring that the phase-space's symplectic structure is the standard one.

Altering the symplectic form in order to absorb the interaction with the Yang-Mills field into the phase-space's geometric structure, it is, anyway, possible to define the particle's classical dynamical system in a way completely independent on the potentials of ∇.

Demanding to the literature [28] for more detailed informations, it will be enough here to mention that it is possible to define a functional $T - Mod_{k} \, : \, Mod_{k} \, \mapsto \, \Omega^{2}(T^{\star} M)$, that I will call a modular-term, with the following properties:

1. $\omega_{st} \, + \, T - Mod_{k}$ is a symplectic form over $T^{\star} M$
2. the dynamical system of our particle minimally-couple with the instanton ∇ may be defined as the classical dynamical system with phase-space $(T^{\star} M \, , \, \omega_{st} \, + \, T - Mod_{k})$ and hamiltonian $H \, = \, p_{a} g^{ab} p_{b} \, + \, m^{2}$.

In the electromagnetic case $G \, = \, U(1)$ a particular modular term is given by:

$$T - Mod_{k} \, := \, \pi^{\star} F_{\nabla} \tag{3.7}$$

where $F_{\nabla} \, := \, \nabla \circ \nabla$ is the curvature of the instanton ∇.

Let us now observe that the alteration of the phase-space's symplectic structure "neutralizes" the Theorem of Global Hamiltonianity for Cotangent Lifts.

This implies that, by adding the minimal interaction with the fixed electromagnetic field, it is possible to generate non-equivariant momentum maps.

IV. THE INDETERMINATION OF THE PRINCIPLE OF EQUIVALENCE AND THE POSSIBILITY OF AVOIDING IT IN AXIOMATIZING GENERAL RELATIVITY

The trial of furnishing an axiomatization of General Relativity has engaged a lot of people for eigthy years.

General Relativity was founded by Einstein on two principles [29]:

1. the Principle of Equivalence
2. the Principle of General Covariance

Yet in 1917, anyway, E. Kretschmann [30] objected that the Principle of General Covariance is tautological, because any putative physical law may be written in a way that make it hold good for all system of coordinates. Following this observation, most of the following axiomatizations descarded the Principle of General Covariance and founded the whole theory on the Principle of Equivalence alone; this is, more or less explicitely , the attitude of almost all the more popular manuals on General Relativity, such as that by Weinberg [31], that by Misner, Thorne and Wheeler [26], and that by Wald [3] [4].

As I have shown in the last section, anyway, the correct formalization of the Principle of Equivalence is highly not-determined, not specifying the topology of the reference-flat lorentzian space-time $(X\,,\,\eta := -dx^0 \bigotimes dx^0 + dx^i \bigotimes dx^i)$ involved as far as *"the corresponding special relativistic laws"* are concerned.

Fortunately, it is possible to get rid of such an ambiguity corncerning the Principle of Equivalence, adopting a different axiomatization of General Relativity in which the Principle of Equivalence takes no parts: the axiomatization by Hawking and Ellis [32] based on:

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[4] Weinberg introduces the Principle of General Covariance in the section 4.1 of [31] just to simplify the analysis, but states that it is a consequence of the Principle of Equivalence, reports Kretschmann's observation on its tautological nature. Misner, Thorne and Wheeler discuss its rule in the section 17.6 of [26] assertin that Mathematics was not sufficientely refined in 1917 to distinguish among the demand for "no prior geometry" and the demand for a "geometric, coordinate-independent formulation of physics, encoding both in the condition of "General Covariance whose ambiguous nature is seen as the source of all the confusions concerned with its discussions from Kretschmann and beyond. Wald introduces it in the section 4.1 of [3] stating it as the condition that there are no preferred vector fields or preferred bases of vector fields pertaining only to the structure of space which appear in any law of physics. He then remarks its vagueness owed to the fact that the phrase "pertaining to space" does not have a precise meaning.

1. the Principle of Local Causality

2. the Principle of Local Energy Conservation

3. the Principle of the Einstein-Hilbert Action

While the Principle of Local Causality and the Principle of Local Energy Conservation pose some constraint on the energy-momentum tensor of matter-fields and hence on the action $S_{matter}[\phi \, , \, g_{ab}]$ describing them, the Principle of the Einstein-Hilbert Action states that the action describing the gravitational field is the functional $S_{gravity} \; : \; Lor_4(X) \; \mapsto \; \mathbb{R}$:

$$S_{gravity}[g_{ab}] \; := \; N_{EH} \int d\mu[g_{a\,b}] \, R[g_{ab}] \tag{4.1}$$

where N_{EH} is a g_{ab}-independent normalization factor, X is a 4-differentiable-manifold and $Lor_4(X)$ is the set of all 4-lorentzian metrics on it.

General Covariance, formalized in a suitable way, may then be seen as a theorem deriving from the Hawking-Ellis' axioms, as we will discuss more extensively in sectionVII

The problems concerning the topological indetermination of the reference space-time X in the Principle of Equivalence becomes, in this framework, completely enclosed in the issue concerning the boundary conditions in the Cauchy Problem for the minimum-action's principle for the whole action:

$$S[g_{ab} \, , \, \phi] \; := \; S_{gravity}[g_{ab}] \, + \, S_{matter}[\phi \, , \, g_{ab}] \tag{4.2}$$

V. CAN THE OBSERVER INSIDE EINSTEIN'S LIFT HEAR THE TOPOLOGY OF SPACE-TIME ?

In a celebrated article in 1966 [33] Marc Kac formulated the following classical problem of Spectral Geometry that I report in the Protter's formulation cited by Gilkey [34] (a detailed exposition of Kac's article is available in the fifth chapter "Spectral Geometry with operators of Laplace type" of [35]):

> *"Suppose a drum is being placed in one room and a person with perfect pitch hears but cannot see the drum. Is it possible for her to deduce the precise shape of the drum just from hearing the fundamental tone and all the overtones ?* (cited by the section 4.2 "Isospectral manifolds" of [34])

I will now show how Kac's question is similar to our questionI.1:

Let us observe, first of all, that the observer closed inside Einstein's Lift has access only to local information exactly as the observer of Kac's problem.

And he tries to use such local information to infer the global geometrical structure of the space(-time) in which he lives.

But as we will see the analogy goes further, since the same kind of local information is of the the same type: the spectrum of a suitable operator.

At this purpose let us observe that, under the mild assumption that the first two Stiefel-Whitney classes $w_1(TX)$, $w_2(TX)$ of the 4-differentiable-manifold X vanish, it is possible to give an alternative axiomatization of General Relativity in which the Principle of the Einstein-Hilbert Action is replaced by the Principle of the Connes Action [36], [37], [38], [39], [40]. stating that the action describing the gravitational field is the functional $S_{gravity}$: $SPIN - SPECTRA \mapsto \mathbb{R}$:

$$S_{gravity}[Sp(D_g)\,,\,\Lambda] \;:=\; N_C\,Tr(\chi(\frac{D^2}{\Lambda^2})) \tag{5.1}$$

(N_C being a $Sp(D_g)$-independent normalization factor), where:

$$SPIN - SPECTRA \;:=\; \{Sp(D_g)\,,\, g \in Riem(X)\} \tag{5.2}$$

where D_g is the Dirac operator of the spin-manifold $(X\,,\,g)$, $Riem(X)$ is the set of all the riemannian metrics over X, Λ is a cut-off and χ is a suitable cut-off function throwing away the contribution of all the eigenvalues of D_g greater than Λ.

It is important to remark that the key point giving foundation to the substitution of the Principle of the Einstein-Hilbert Action with the Principle of the Connes' Action as to the axiomatization of General Relativity is the theorem stating that the category having as objects the closed finite-dimensional spin-manifolds and as morphisms their diffeomorphisms is equivalent to the category having as objects the abelian spectral triples and as morphisms their automorphisms.

It is useful, at this point, to introduce the following terminology:

given a riemannian manifold (X, g) let us define its spectrum as the spectrum of the Laplace-Beltrami operator on it; furthermore, given a spin-manifold (X, g), let us define its spin-spectrum as the spectrum of the Dirac operator on it.

We can now precisely state Kac's question in the following way :

QUESTION V.1

QUESTION IF A RIEMANNIAN MANIFOLD IS DETERMINED BY ITS SPECTRUM

is a riemannian manifold (X, g) determined by its spectrum ?

We can then state the same question as to the Dirac operator:

QUESTION V.2

QUESTION IF A SPIN-MANIFOLD IS DETERMINED BY ITS SPIN-SPECTRUM

is a spin-manifold (X, g) determined by its spin-spectrum ?

That one cannot hear the shape of a drum, i.e. that the answer to both questionV.1 and questionV.2 is negative, was proved by the determination of different isospectral riemannian-manifolds and spin-manifolds.

It must be observed, anyway, that our questionI.1 is much less ambitious, asking only if:

QUESTION V.3

QUESTION IF THE TOPOLOGY OF A SPIN-MANIFOLD IS DETERMINED BY ITS SPIN-SPECTRUM

is the topology of a spin-manifold (X, g) determined by its spin-spectrum ?

The natural framework in which to discuss questionV.3 is Index Theory:

by the Atiyah-Singer Index Theorem the index of the Dirac operator D_g over the spin-manifold $(X\,,\,g)$:

$$Index(\,D_g)\;:=\;dim(Ker(D_g))\;-\;dim(Coker(D_g)) \tag{5.3}$$

may be expressed as:

$$Index(D_g)\;=\;\int_M \hat{A}(X) \tag{5.4}$$

where:

$$\hat{A}(X)\;=\;\prod_{i=1}^{4}\frac{\frac{x_i}{2}}{\sinh(\frac{x_i}{2})} \tag{5.5}$$

is the $\hat{A}$-genus of X expressed in terms of the eigenvalues $(\,x_1\,,\,x_2\,,\,x_3\,,\,x_4\,)$ of the block-form diagonalization of the curvature 2-form [41]:

$$\mathcal{R}_{ab}\;=\;\begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_4 & 0 \end{pmatrix} \tag{5.6}$$

Since the $\hat{A}$-genus of X is a characteristic class, eq.5.4 implies that the index of the Dirac operator is a topological invariant.

Given two spin-manifolds $(\,X_1\,,\,g_1\,)$ and $(\,X_2\,,\,g_2\,)$ the fact that:

$$X_1\,\sim\,X_2\;\Rightarrow\;Index(D_{g_1})\;=\;Index(D_{g_2}) \tag{5.7}$$

doesn't unfortunately imply the converse:

$$Index(D_{g_1})\;=\;Index(D_{g_2})\;\not\Rightarrow\;X_1\,\sim\,X_2 \tag{5.8}$$

doesn't allowing to infer completely the space-time's topology from the dimensions of Dirac-operator's kernel and cokernel.

Though not determining it, anyway, eq.5.4 shows how Index Theory allows to extract information concerning the space-time topology from the spectrum of Dirac's operator, that, as we have seen, may be considered the only dynamical variable of General Relativity.

VI. NONCOMMUTATIVE EINSTEIN'S LIFTS AND NONCOMMUTATIVE TOPOLOGIES

In this section I will briefly sketch how all the analysis' made in the previous sections may be generalized to the quantum case.

Since, according to Noncommutative Geometry [36], [37], [38], [39], [40], a quantum spacetime is nothing but a noncommutative spectral triple $(A\,,\,\mathcal{H}\,,\,D)$ we can introduce the noncommutative analogue of Einstein's-lift's gedanken-experiment (that I will denote briefly as the noncommutative-lift's gedanken-experiment) and state the noncommutative analogue of questionI.1:

QUESTION VI.1

QUESTION ON THE LOCAL OBSERVABILITY OF THE GLOBAL NONCOMMUTATIVE TOPOLOGY

can the observer in the noncommutative lift understand, during the proper-time interval $[0\,,\,T]$, *if the noncommutative topology of* $(A\,,\,\mathcal{H}\,,\,D)$ *is trivial ?*

trying to discuss it in terms of the noncommutative analogue of questionV.3, i.e. of the following:

QUESTION VI.2

QUESTION IF THE TOPOLOGY OF A NONCOMMUTATIVE SPACE-TIME IS DETERMINED BY ITS SPIN-SPECTRUM

is the topology of a noncommutative space-time $(A\,,\,\mathcal{H}\,,\,D)$ *determined by its spin-spectrum ?*

whose precise meaning lies on the two basic theorems of Noncommutative Topology:

1. the Gelfand-Naimark's Theorem stating the equivalence among the category of the compact Hausdorff topological spaces and the category of the abelian $C^{\star}$-algebras

2. the Serre-Swan's Theorem stating the equivalence among the category of vector bundles over a compact topological space M and the category of the finitely generated projective modules over C(M)

The analysis of the previous section strongly suggest that the natural framework in which one can try to get some insight into questionVI.2 is Noncommutative Index Theory.

Now, despite some non-rigorous, naive folklore concerning noncommutative-coordinates at Planck's scale [42], [43] and some even more muddling folklore [44] discussing Quantum Index Theory in the wrong framework of Algebraic Quantum Field Theory, Noncommutative Index Theory is a well defined subject of Noncommutative Geometry with well-established theorems such as the Noncommutative Atiyah-Jänich's Theorem or the Connes-Moscovici's Local Index Formula [40].

VII. ON THE LOCALITY OF GENERAL RELATIVITY AND OF GAUGE THEORIES

On discussing the issue concerning the compatibility among the locality of General Relativity and the claimed non-locality of Quantum Mechanics [5] Jeeva Anandan has recently returned to develop an issue he studied years before [48]: the fact that that the locality of General Relativity lies on the fact that its similarity with the non-local Yang- Mills' theories is only partial.

The discussion of Anandan's claim may be useful to clarify better how the the Hawking-Ellis' axiomatization of General Relativity

- allows, through the Principle of Local Causality, to give up the topological indetermination I have shown to infect the Principle of Equivalence
- is on no way in contradiction with the possibility of locally hearing some (partial) information about the topology of space-time, revealing how the "global versus local issues" are subtle when both K-theory and the correct formulation of Principle of Local Causality are taken into account

The starting point of Anandan's analysis is Trautman's discussion of the following:

QUESTION VII.1

QUESTION IF GENERAL RELATIVITY IS A GAUGE THEORY

Is General Relativity a Yang-Mills' theory of a suitable principal bundle P(M, G), for a suitable choice of the base space M and the structure group G ?

The first attempts to give a positive answer to questionVII.1 were made by Einstein and Dirac trough their investigations in the forthies on the possibility of formulating General

[5] I don't agree both on the often claimed fact (assumed by Anandan) that the Aharonov-Bohm effect proves the non-locality of Quantum Mechanics [45] (the Aharonov-Bohm effect being simply a particular application of Chern-Simons' topological quantum field theory [46]) and on the even more often claimed statement that the quantum mechanical violation of Bell's inequalities is a proof of the (claimed) quantum non-locality: the quantum violation of Bell's inequalities is nothing but an immediate consequence of the fact that Noncommutative Measure Theory, the ground floor of Noncommutative Geometry, is irreducible to Commutative Measure Theory [36], [47]

Relativity in a way such that the dynamical variable is a connection , as in Yang-Mills' theories, and not a metric [49].

The natural starting point appeared, with this respect, to be the substitution of the Einstein-Hilbert's action of eq.4.1 with the Palatini's action:

$$S_{gravity}[g_{ab}\, ,\, \nabla_a] \ := \ N_{EH} \int d\mu[g_{a\,b}]\, R_{ab}[\nabla_a] g^{ab} \tag{7.1}$$

in which the connection ∇_a is taken as an independent dynamical variable, fixed to be the Levi-Civita connection of g_{ab} by the dynamical equation obtained, together with Einstein's equation, varying eq.7.1 w.r.t. to both ∇_a and g_{ab}.

This would lead to suspect that General Relativity is nothing but the Yang Mills theory w.r.t. a principal bundle $P(X, SO(1,3))$, where X is some 4-differential manifold.

I will know show, anyway, that such a suspicion is wrong.

The group of gauge transformations for an $SO(1\,,\,3)$-Yang Mills theory on X is defined as the group of vertical automorphisms of the underlying principal bundle $P(X, SO(1,3))$ given by::

$$\mathcal{G}_{YM} \ := \ \Gamma(X\, ,\, Ad\, P) \tag{7.2}$$

where:

$$Ad\, P \ := \ P \times_{AdSO(1,3)} SO(1,3) \tag{7.3}$$

is the associated adjoint bundle of $P(X, SO(1,3))$.

The group of gauge transformation for General Relativity is, contrary, the group of automorphims of GLM preserving its **soldering form** θ [50], [23]:

$$\mathcal{G}_{GR} \ = \ \{\alpha \in Aut(GLM) \ : \ \alpha \circ \theta \ = \ \theta\} \tag{7.4}$$

containing no other vertical automorphism than the identity:

$$\mathcal{G}_{GR} \bigcap \mathcal{G}_{YM} \ = \ \mathbb{I} \tag{7.5}$$

Since:

$$\mathcal{G}_{GR} \ = \ Diff(M) \tag{7.6}$$

eq.7.4 is nothing but the Theorem of General Covariance, namely the correct mathematical formalization of the Principle of General Covariance, avoiding the tautological bug of its naive formulation shown by Kretchmann that we discussed in sectionIV.

It is importance to remark again that General Covariance is a theorem and not a principle of General Relativity, being implied by the Hawking-Ellis' axiomatization we adopted:

eq.7.4 is implied by the Diff(M)-invariance of eq.4.1 and, hence, by the Principle of the Einstein-Hilbert's Action.

As a consequence a space-time is represented mathematically not by a 4 lorentzian manifold $(M \, , \, g_{ab})$ but by an element of the quotient space $\frac{Lor(M)}{Diff(M)}$, automatically getting rid of all the confusions concerning the *"hole-argument"* of the 1913 Einstein-Grossmann's paper [51], [52], [51], [23], [24].

The negative answer to questionVII.1 resulting from Trautman's analysis [6] reflects, according to Anandan, the following radical structural difference:

- Gauge theories are nonlocal, since its holonomies are nonlocal objects, invariant under change of the enclosed "flux" by one "quantum"
- General Relativity is local since, owing to the peculiar rule of the **soldering form** having no analogous in gauge theories, its holonomies are local objects, noninvariant under change of the enclosed "flux" by one "quantum"

Though intuitivelly suspecting that Anandan's remark could be stricly connected with the issue discussed in the sections I, II, IV, V I have not succeeded yet in formalizing such a link.

A first step in this direction would consist in analyzing Anandan's issue in the Ashtekar's formulation of General Relativity [53], [49] whose phase space and canonical variables are exactly those of a (complex) SU(2)-Yang-Mills theory, while its reduced-space is a subspace of the reduced phase-space of a (complex) SU(2) Yang-Mills theory (obtained taking the quotient w.r.t. the Gauss Law) owing to the existence of four further constraints.

[6] It must be remarked, with this respect, that, defining a a gauge theory as a generic physical theory having as dynamical variable a connection on a principal bundle, Trautman's conclusion is that, though not being a Yang-Mills' theory, General Relativity is a gauge theory; we are using here, anyway, a more restrictive definition of a gauge theory as a synonimous of a Yang-Mills' theory

As to the quantum case discussed in sectionVI the natural framework to analyze its interrelation with Anandan's remark would consist in comparing the loop representations of a quantum complex SU(2) theory and Loop Quantum Gravity [54], [49] based on the loop representation of Ashtekar's formalization of General Relativity as to the rule of the soldering form and the locality of holonomies [7].

[7] The fact that Loop Quantum Gravity differs from Connes' Quantum Gravity and that both these theories differ from String Quantum Gravity shouldn't, in my modest opinion, be dramatized: different approaches to Quantum Gravity should be seen as attempts of climbing a mountain from different faces

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