REAL-VALUED ALGEBRO-GEOMETRIC SOLUTIONS OF THE CAMASSA–HOLM HIERARCHY

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Abstract. We provide a treatment of real-valued, smooth, and bounded algebro-geometric solutions of the Camassa–Holm (CH) hierarchy and describe the associated isospectral torus. We also discuss real-valued algebro-geometric solutions with a cusp behavior.

1. Introduction

Recently, we provided a detailed treatment of the Camassa–Holm (CH) hierarchy with special emphasis on its algebro-geometric solutions. The first nonlinear partial differential equation of this hierarchy, the Camassa–Holm equation, also known as the dispersive shallow water equation [5] is given by

$$4u_t - u_{xxt} - 2uu_{xxx} - 4u_x u_{xx} + 24uu_x = 0, \quad (x,t) \in \mathbb{R}^2 \tag{1.1}$$

(choosing a convenient scaling of x, t). For various aspects of local and global existence, and uniqueness of solutions of (1.1), wave breaking phenomena, soliton-type solutions ("peakons"), complete integrability aspects such as infinitely many conservation laws, (bi-)Hamiltonian formalism, Bäcklund transformations, infinite dimensional symmetry groups, etc., we refer to the literature provided in [12] (see also [13, Ch. 5]). The case of spatially periodic solutions, the corresponding inverse spectral problem, isospectral classes of solutions, and quasi-periodicity of solutions with respect to time are discussed in [7]–[11]. Moreover, algebro-geometric solutions of (1.1) and their properties are studied in [1]–[3], and [12] (see also [13, Ch. 5]).

In Section 2 we recall the basic polynomial recursion formalism that defines the CH hierarchy using a zero-curvature approach. Section 3 recalls the stationary CH hierarchy and the associated algebro-geometric formalism. Section 4 provides a brief summary of self-adjoint canonical systems as needed in this paper, and Section 5 finally discusses the principal result of this paper, the class of real-valued, smooth, and bounded algebro-geometric solutions of the CH hierarchy and the associated isospectral torus.

This paper should be viewed as a companion to our paper [12] (see also [13, Ch. 5]) on the CH hierarchy and we refer to it for background material and pertinent references on the subject.

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2. The CH hierarchy, recursion relations, and hyperelliptic curves

In this section we review the basic construction of the Camassa–Holm hierarchy using a zero-curvature approach following [12] (see also [13, Ch. 5]).

Throughout this section we will suppose the following hypothesis ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$).

Hypothesis 2.1. *In the stationary case we assume that*

$$u \in C^\infty(\mathbb{R}), \ \frac{d^m u}{dx^m} \in L^\infty(\mathbb{R}), \ m \in \mathbb{N}_0. \tag{2.1}$$

In the time-dependent case (*cf.* (2.28)–(2.35)) *we suppose*

$$\begin{aligned}
&u(\,\cdot\,,t) \in C^\infty(\mathbb{R}), \ \frac{\partial^m u}{\partial x^m}(\,\cdot\,,t) \in L^\infty(\mathbb{R}), \ m \in \mathbb{N}_0, \, t \in \mathbb{R}, \\
&u(x,\,\cdot\,), u_{xx}(x,\,\cdot\,) \in C^1(\mathbb{R}), \ x \in \mathbb{R}.
\end{aligned} \tag{2.2}$$

We start by formulating the basic polynomial setup. One defines $\{f_\ell\}_{\ell\in\mathbb{N}_0}$ recursively by

$$f_0 = 1, \quad f_{\ell,x} = -2\mathcal{G}\big(2(4u - u_{xx})f_{\ell-1,x} + (4u_x - u_{xxx})f_{\ell-1}\big), \quad \ell \in \mathbb{N}, \tag{2.3}$$

where $\mathcal{G}$ is given by

$$\mathcal{G}\colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}), \quad (\mathcal{G}v)(x) = \frac{1}{4}\int_{\mathbb{R}} dy\, e^{-2|x-y|}v(y), \quad x \in \mathbb{R}, \ v \in L^\infty(\mathbb{R}). \tag{2.4}$$

One observes that $\mathcal{G}$ is the resolvent of minus the one-dimensional Laplacian at energy parameter equal to -4, that is,

$$\mathcal{G} = \Big(-\frac{d^2}{dx^2} + 4\Big)^{-1}. \tag{2.5}$$

The first coefficient reads

$$f_1 = -2u + c_1, \tag{2.6}$$

where c_1 is an integration constant. Subsequent coefficients are non-local with respect to u. At each level a new integration constant, denoted by c_ℓ, is introduced. Moreover, we introduce coefficients $\{g_\ell\}_{\ell\in\mathbb{N}_0}$ and $\{h_\ell\}_{\ell\in\mathbb{N}_0}$ by

$$g_\ell = f_\ell + \frac{1}{2}f_{\ell,x}, \quad h_\ell = (4u - u_{xx})f_\ell - g_{\ell+1,x}, \quad \ell \in \mathbb{N}_0. \tag{2.7}$$

Explicitly, one computes

$$\begin{aligned}
&f_0 = 1, \quad f_1 = -2u + c_1, \quad f_2 = 2u^2 + 2\mathcal{G}\big(u_x^2 + 8u^2\big) + c_1(-2u) + c_2, \\
&g_0 = 1, \quad g_1 = -2u - u_x + c_1, \\
&g_2 = 2u^2 + 2uu_x + 2\mathcal{G}\big(u_x^2 + u_x u_{xx} + 8uu_x + 8u^2\big) + c_1(-2u - u_x) + c_2, \\
&h_0 = 4u + 2u_x, \\
&h_1 = -2u_x^2 - 4uu_x - 8u^2 \\
&\qquad - 2\mathcal{G}\big(u_x u_{xxx} + u_{xx}^2 + 2u_x u_{xx} + 8uu_{xx} + 8u_x^2 + 16uu_x\big) \\
&\qquad + c_1(4u + 2u_x), \ \text{etc.}
\end{aligned} \tag{2.8}$$

Given Hypothesis 2.1, one introduces the 2×2 matrix U by

$$U(z,x) = \begin{pmatrix} -1 & 1 \\ z^{-1}(4u(x) - u_{xx}(x)) & 1 \end{pmatrix}, \quad x \in \mathbb{R}, \tag{2.9}$$

and for each $n \in \mathbb{N}_0$ the following 2×2 matrix V_n by

$$V_n(z,x) = \begin{pmatrix} -G_n(z,x) & F_n(z,x) \\ z^{-1}H_n(z,x) & G_n(z,x) \end{pmatrix}, \quad n \in \mathbb{N}_0,\ z \in \mathbb{C}\backslash\{0\},\ x \in \mathbb{R}, \tag{2.10}$$

assuming F_n, G_n, and H_n to be polynomials of degree n with respect to z and C^∞ in x. Postulating the zero-curvature condition

$$-V_{n,x}(z,x) + [U(z,x), V_n(z,x)] = 0, \tag{2.11}$$

one finds

$$F_{n,x}(z,x) = 2G_n(z,x) - 2F_n(z,x), \tag{2.12}$$

$$zG_{n,x}(z,x) = (4u(x) - u_{xx}(x))F_n(z,x) - H_n(z,x), \tag{2.13}$$

$$H_{n,x}(z,x) = 2H_n(z,x) - 2(4u(x) - u_{xx}(x))G_n(z,x). \tag{2.14}$$

From (2.12)–(2.14) one infers that

$$\frac{d}{dx}\det(V_n(z,x)) = -\frac{1}{z}\frac{d}{dx}\Big(zG_n(z,x)^2 + F_n(z,x)H_n(z,x)\Big) = 0, \tag{2.15}$$

and hence

$$z^2G_n(z,x)^2 + zF_n(z,x)H_n(z,x) = R_{2n+2}(z), \tag{2.16}$$

where the polynomial R_{2n+2} of degree $2n+2$ is x-independent,

$$R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z - E_m), \quad E_0, E_1, \dots, E_{2n} \in \mathbb{C},\ E_{2n+1} = 0. \tag{2.17}$$

Next one makes the ansatz that F_n, H_n, and G_n are polynomials of degree n, related to the coefficients f_ℓ, h_ℓ, and g_ℓ by

$$F_n(z,x) = \sum_{\ell=0}^{n} f_{n-\ell}(x)z^\ell, \quad G_n(z,x) = \sum_{\ell=0}^{n} g_{n-\ell}(x)z^\ell, \quad H_n(z,x) = \sum_{\ell=0}^{n} h_{n-\ell}(x)z^\ell. \tag{2.18}$$

Insertion of (2.18) into (2.12)–(2.14) then yields the recursion relations (2.3), (2.4) and (2.7) for f_ℓ and g_ℓ for $\ell = 0, \dots, n$. For fixed $n \in \mathbb{N}$ we obtain the recursion (2.7) for h_ℓ for $\ell = 0, \dots, n-1$ and

$$h_n = (4u - u_{xx})f_n. \tag{2.19}$$

(When $n = 0$ one directly gets $h_0 = (4u - u_{xx})$.) Moreover, taking $z = 0$ in (2.16) yields

$$f_n(x)h_n(x) = -\prod_{m=0}^{2n} E_m. \tag{2.20}$$

In addition, one finds

$$h_{n,x}(x) - 2h_n(x) + 2(4u(x) - u_{xx}(x))g_n(x) = 0, \quad n \in \mathbb{N}_0. \tag{2.21}$$

Using the relations (2.19) and (2.7) permits one to write (2.21) as

$$\text{s-CH}_n(u) = (u_{xxx} - 4u_x)f_n - 2(4u - u_{xx})f_{n,x} = 0, \quad n \in \mathbb{N}_0. \tag{2.22}$$

Varying $n \in \mathbb{N}_0$ in (2.22) then defines the stationary CH hierarchy. We record the first few equations explicitly,

$$\text{s-CH}_0(u) = u_{xxx} - 4u_x = 0,$$

$$\begin{aligned}
\text{s-CH}_1(u) &= -2uu_{xxx} - 4u_x u_{xx} + 24uu_x + c_1(u_{xxx} - 4u_x) = 0, \qquad (2.23)\\
\text{s-CH}_2(u) &= 2u^2 u_{xxx} - 8uu_x u_{xx} - 40u^2 u_x + 2(u_{xxx} - 4u_x)\mathcal{G}\big(u_x^2 + 8u^2\big)\\
&\quad - 8(4u - u_{xx})\mathcal{G}\big(u_x u_{xx} + 8uu_x\big)\\
&\quad + c_1(-2uu_{xxx} - 4u_x u_{xx} + 24uu_x) + c_2(u_{xxx} - 4u_x) = 0, \text{ etc.}
\end{aligned}$$

By definition, the set of solutions of (2.22), with n ranging in $\mathbb{N}_0$, represents the class of algebro-geometric CH solutions. If u satisfies one of the stationary CH equations in (2.22) for a particular value of n, then it satisfies infinitely many such equations of order higher than n for certain choices of integration constants c_ℓ. At times it will be convenient to abbreviate (algebro-geometric) stationary CH solutions u simply as CH *potentials*.

Using equations (2.12)–(2.14) one can also derive individual differential equations for F_n and H_n. Focusing on F_n only, one obtains

$$\begin{aligned}
&F_{n,xxx}(z,x) - 4\big(z^{-1}(4u(x) - u_{xx}(x)) + 1\big)F_{n,x}(z,x)\\
&\quad - 2z^{-1}(4u_x(x) - u_{xxx}(x))F_n(z,x) = 0 \qquad (2.24)
\end{aligned}$$

and

$$\begin{aligned}
&-(z^2/2)F_{n,xx}(z,x)F_n(z,x) + (z^2/4)F_{n,x}(z,x)^2 + z^2F_n(z,x)^2\\
&\quad + z(4u(x) - u_{xx}(x))F_n(z,x)^2 = R_{2n+2}(z). \qquad (2.25)
\end{aligned}$$

Equation (2.25) leads to an explicit determination of the integration constants $c_1, \dots, c_n$ in the stationary CH equations (2.22) in terms of the zeros $E_0 = 0$, $E_1, \dots, E_{2n+1}$ of the associated polynomial R_{2n+2} in (2.17). In fact, one can prove

$$c_\ell = c_\ell(\underline{E}), \quad \ell = 0, \dots, n, \qquad (2.26)$$

where

$$\begin{aligned}
c_0(\underline{E}) &= 1,\\
c_k(\underline{E}) &= -\sum_{\substack{j_1,\dots,j_{2n+1}=0\\ j_1+\dots+j_{2n+1}=k}}^{k} \frac{(2j_1)!\cdots(2j_{2n+1})!}{2^{2k}(j_1!)^2\cdots(j_{2n+1}!)^2(2j_1-1)\cdots(2j_{2n+1}-1)}\\
&\qquad \times E_1^{j_1}\cdots E_{2n+1}^{j_{2n+1}}, \quad k \in \mathbb{N}. \qquad (2.27)
\end{aligned}$$

Next, we turn to the time-dependent CH hierarchy. Introducing a deformation parameter $t_n \in \mathbb{R}$ into u (i.e., replacing $u(x)$ by $u(x,t_n)$), the definitions (2.9), (2.10), and (2.18) of U, V_n, and F_n, G_n, and H_n, respectively, still apply. The corresponding zero-curvature relation reads

$$U_{t_n}(z,x,t_n) - V_{n,x}(z,x,t_n) + [U(z,x,t_n), V_n(z,x,t_n)] = 0, \quad n \in \mathbb{N}_0, \qquad (2.28)$$

which results in the following set of time-dependent equations

$$\begin{aligned}
&4u_{t_n}(x,t_n) - u_{xxt_n}(x,t_n) - H_{n,x}(z,x,t_n) + 2H_n(z,x,t_n)\\
&\quad - 2(4u(x,t_n) - u_{xx}(x,t_n))G_n(z,x,t_n) = 0, \qquad (2.29)\\
&F_{n,x}(z,x,t_n) = 2G_n(z,x,t_n) - 2F_n(z,x,t_n), \qquad (2.30)\\
&zG_{n,x}(z,x,t_n) = (4u(x,t_n) - u_{xx}(x,t_n))F_n(z,x,t_n) - H_n(z,x,t_n). \qquad (2.31)
\end{aligned}$$

Inserting the polynomial expressions for F_n, H_n, and G_n into (2.30) and (2.31), respectively, first yields recursion relations (2.3) and (2.7) for f_ℓ and g_ℓ for $\ell =$

$0,\dots,n$. For fixed $n\in\mathbb{N}$ we obtain from (2.29) the recursion for h_ℓ for $\ell=0,\dots,n-1$ and

$$h_n=(4u-u_{xx})f_n. \tag{2.32}$$

(When $n=0$ one directly gets $h_0=(4u-u_{xx})$.) In addition, one finds

$$\begin{aligned}&4u_{t_n}(x,t_n)-u_{xxt_n}(x,t_n)-h_{n,x}(x,t_n)+2h_n(x,t_n)\\&\quad-2(4u(x,t_n)-u_{xx}(x,t_n))g_n(x,t_n)=0,\quad n\in\mathbb{N}_0.\end{aligned} \tag{2.33}$$

Using relations (2.19) and (2.32) permits one to write (2.33) as

$$\mathrm{CH}_n(u)=4u_{t_n}-u_{xxt_n}+(u_{xxx}-4u_x)f_n-2(4u-u_{xx})f_{n,x}=0,\quad n\in\mathbb{N}_0. \tag{2.34}$$

Varying $n\in\mathbb{N}_0$ in (2.34) then defines the time-dependent CH hierarchy. We record the first few equations explicitly,

$$\begin{aligned}\mathrm{CH}_0(u)&=4u_{t_0}-u_{xxt_0}+u_{xxx}-4u_x=0,\\ \mathrm{CH}_1(u)&=4u_{t_1}-u_{xxt_1}-2uu_{xxx}-4u_xu_{xx}+24uu_x+c_1(u_{xxx}-4u_x)=0,\\ \mathrm{CH}_2(u)&=4u_{t_2}-u_{xxt_2}+2u^2u_{xxx}-8uu_xu_{xx}-40u^2u_x\\&\quad+2(u_{xxx}-4u_x)\mathcal{G}\big(u_x^2+8u^2\big)-8(4u-u_{xx})\mathcal{G}\big(u_xu_{xx}+8uu_x\big)\\&\quad+c_1(-2uu_{xxx}-4u_xu_{xx}+24uu_x)+c_2(u_{xxx}-4u_x)=0,\ \text{etc.}\end{aligned} \tag{2.35}$$

Up to an inessential scaling of the (x,t_1) variables, $\mathrm{CH}_1(u)=0$ with $c_1=0$ represents *the Camassa–Holm* equation as discussed in [5].

3. The algebro-geometric CH formalism

This section is devoted to a quick review of the stationary CH hierarchies and the corresponding algebro-geometric formalism as derived in [12] (cf. also [13, Ch. 5]).

We start with the stationary hierarchy and suppose

$$u\in C^\infty(\mathbb{R}),\ \frac{d^mu}{dx^m}\in L^\infty(\mathbb{R}),\ m\in\mathbb{N}_0, \tag{3.1}$$

and assume (2.3), (2.7), (2.9)–(2.11), (2.16)–(2.22), keeping $n\in\mathbb{N}_0$ fixed.

Recalling (2.17),

$$R_{2n+2}(z)=\prod_{m=0}^{2n+1}(z-E_m),\quad E_0,E_1,\dots,E_{2n}\in\mathbb{C},\ E_{2n+1}=0, \tag{3.2}$$

we introduce the (possibly singular) hyperelliptic curve $\mathcal{K}_n$ of arithmetic genus n defined by

$$\mathcal{K}_n\colon \mathcal{F}_n(z,y)=y^2-R_{2n+2}(z)=0. \tag{3.3}$$

In the following we will occasionally impose further constraints on the zeros E_m of R_{2n+2} introduced in (3.2) and assume that

$$E_1,\dots,E_{2n}\in\mathbb{C}\backslash\{0\},\ E_{2n+1}=0. \tag{3.4}$$

We compactify $\mathcal{K}_n$ by adding two points at infinity, P_{∞_+}, P_{∞_-}, with $P_{\infty_+}\neq P_{\infty_-}$, still denoting its projective closure by $\mathcal{K}_n$. Hence $\mathcal{K}_n$ becomes a two-sheeted Riemann surface of arithmetic genus n. Points P on $\mathcal{K}_n\backslash\{P_{\infty_\pm}\}$ are denoted by $P=(z,y)$, where $y(\,\cdot\,)$ denotes the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z,y)=0$.

For notational simplicity we will usually tacitly assume that $n \in \mathbb{N}$ (the case $n=0$ being trivial).

In the following the roots of the polynomials F_n and H_n will play a special role and hence we introduce on $\mathbb{C} \times \mathbb{R}$

$$F_n(z,x) = \prod_{j=1}^{n} (z-\mu_j(x)), \quad H_n(z,x) = h_0(x) \prod_{j=1}^{n} (z-\nu_j(x)), \tag{3.5}$$

temporarily assuming

$$h_0(x) \neq 0, \quad x \in \mathbb{R}. \tag{3.6}$$

Moreover, we introduce

$$\hat{\mu}_j(x) = (\mu_j(x), -\mu_j(x) G_n(\mu_j(x),x)) \in \mathcal{K}_n, \quad j=1,\dots,n, \ x \in \mathbb{R}, \tag{3.7}$$

$$\hat{\nu}_j(x) = (\nu_j(x), \nu_j(x) G_n(\nu_j(x),x)) \in \mathcal{K}_n, \quad j=1,\dots,n, \ x \in \mathbb{R}, \tag{3.8}$$

and

$$P_0 = (0,0). \tag{3.9}$$

The branch of $y(\,\cdot\,)$ near $P_{\infty_\pm}$ is fixed according to

$$\lim_{\substack{|z(P)| \to \infty \\ P \to P_{\infty_\pm}}} \frac{y(P)}{z(P) G_n(z(P),x)} = \mp 1. \tag{3.10}$$

Due to assumption (3.1), u is smooth and bounded, and hence $F_n(z,\,\cdot\,)$ and $H_n(z,\,\cdot\,)$ share the same property. Thus, one concludes

$$\mu_j, \nu_k \in C(\mathbb{R}), \ j,k = 1,\dots,n, \tag{3.11}$$

taking multiplicities (and appropriate reordering) of the zeros of F_n and H_n into account.

Next, we introduce the fundamental meromorphic function $\phi(\,\cdot\,,x)$ on $\mathcal{K}_n$ by

$$\phi(P,x) = \frac{y - z G_n(z,x)}{F_n(z,x)} = \frac{z H_n(z,x)}{y + z G_n(z,x)}, \quad P=(z,y) \in \mathcal{K}_n, \, x \in \mathbb{R}. \tag{3.12}$$

Assuming (3.4) and (3.6), the divisor $(\phi(\,\cdot\,,x))$ of $\phi(\,\cdot\,,x)$ is given by

$$(\phi(\,\cdot\,,x)) = \mathcal{D}_{P_0 \underline{\hat{\nu}}(x)} - \mathcal{D}_{P_{\infty_+} \underline{\hat{\mu}}(x)}, \tag{3.13}$$

taking into account (3.10). Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}, \, \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \sigma^n \mathcal{K}_n, \tag{3.14}$$

where $\sigma^m \mathcal{K}_n$, $m \in \mathbb{N}$, denotes the mth symmetric product of $\mathcal{K}_n$. If h_0 is permitted to vanish at a point $x_1 \in \mathbb{R}$, then for $x = x_1$, the polynomial $H_n(\,\cdot\,,x_1)$ is at most of degree $n-1$ (cf. (2.18)) and (3.13) is altered to

$$(\phi(\,\cdot\,,x_1)) = \mathcal{D}_{P_0 P_{\infty_-} \hat{\nu}_1(x_1),\dots,\hat{\nu}_{n-1}(x_1)} - \mathcal{D}_{P_{\infty_+} \underline{\hat{\mu}}(x)}, \tag{3.15}$$

that is, one of the $\hat{\nu}_j(x)$ tends to P_{∞_-} as $x \to x_1$ (cf. also (3.36)). Analogously one can discuss the case of several $\hat{\nu}_j$ approaching P_{∞_-}. Since this can be viewed as a limiting case of (3.13), we will henceforth not particularly distinguish the case $h_0 \neq 0$ from the more general situation where h_0 is permitted to vanish.

Given $\phi(\,\cdot\,,x)$, one defines the associated Baker-Akhiezer vector $\Psi(\,\cdot\,,x,x_0)$ on $\mathcal{K}_n\backslash\{P_{\infty_+},P_{\infty_-},P_0\}$ by

$$\Psi(P,x,x_0)=\begin{pmatrix}\psi_1(P,x,x_0)\\ \psi_2(P,x,x_0)\end{pmatrix}, \quad P\in\mathcal{K}_n\backslash\{P_{\infty_+},P_{\infty_-},P_0\},\ (x,x_0)\in\mathbb{R}^2, \tag{3.16}$$

where

$$\psi_1(P,x,x_0)=\exp\left(-(1/z)\int_{x_0}^{x}dx'\,\phi(P,x')-(x-x_0)\right), \tag{3.17}$$

$$\psi_2(P,x,x_0)=-\psi_1(P,x,x_0)\phi(P,x)/z. \tag{3.18}$$

Although Ψ is formally the analog of the Baker–Akhiezer vector of the stationary CH hierarchy when compared to analogous definitions in the context of the KdV or AKNS hierarchies, its actual properties in a neighborhood of its essential singularity feature characteristic differences to standard Baker–Akhiezer vectors as discussed in [13, Ch. 5] and [12].

The basic properties of ϕ and Ψ then read as follows.

Lemma 3.1. *Suppose* (3.1)*, assume the nth stationary CH equation* (2.22) *holds, and let $P=(z,y)\in\mathcal{K}_n\backslash\{P_{\infty_+},P_{\infty_-},P_0\}$, $(x,x_0)\in\mathbb{R}^2$. Then ϕ satisfies the Riccati-type equation*

$$\phi_x(P,x)-z^{-1}\phi(P,x)^2-2\phi(P,x)+4u(x)-u_{xx}(x)=0, \tag{3.19}$$

as well as

$$\phi(P,x)\phi(P^*,x)=-\frac{zH_n(z,x)}{F_n(z,x)}, \tag{3.20}$$

$$\phi(P,x)+\phi(P^*,x)=-2\frac{zG_n(z,x)}{F_n(z,x)}, \tag{3.21}$$

$$\phi(P,x)-\phi(P^*,x)=\frac{2y}{F_n(z,x)}, \tag{3.22}$$

while Ψ fulfills

$$\Psi_x(P,x,x_0)=U(z,x)\Psi(P,x,x_0), \tag{3.23}$$

$$-\,y\Psi(P,x,x_0)=zV_n(z,x)\Psi(P,x,x_0), \tag{3.24}$$

$$\psi_1(P,x,x_0)=\left(\frac{F_n(z,x)}{F_n(z,x_0)}\right)^{1/2}\exp\left(-(y/z)\int_{x_0}^{x}dx'F_n(z,x')^{-1}\right), \tag{3.25}$$

$$\psi_1(P,x,x_0)\psi_1(P^*,x,x_0)=\frac{F_n(z,x)}{F_n(z,x_0)}, \tag{3.26}$$

$$\psi_2(P,x,x_0)\psi_2(P^*,x,x_0)=-\frac{H_n(z,x)}{zF_n(z,x_0)}, \tag{3.27}$$

$$\psi_1(P,x,x_0)\psi_2(P^*,x,x_0)+\psi_1(P^*,x,x_0)\psi_2(P,x,x_0)=2\frac{G_n(z,x)}{F_n(z,x_0)}, \tag{3.28}$$

$$\psi_1(P,x,x_0)\psi_2(P^*,x,x_0)-\psi_1(P^*,x,x_0)\psi_2(P,x,x_0)=\frac{2y}{zF_n(z,x_0)}. \tag{3.29}$$

In addition, as long as the zeros of $F_n(\,\cdot\,,x)$ are all simple for $x\in\Omega$, $\Omega\subseteq\mathbb{R}$ an open interval, $\Psi(\,\cdot\,,x,x_0)$, $x,x_0\in\Omega$, is meromorphic on $\mathcal{K}_n\backslash\{P_0\}$.

Next, we recall the Dubrovin-type equations for μ_j. Since in the remainder of this section we will frequently assume $\mathcal{K}_n$ to be nonsingular, we list all restrictions on $\mathcal{K}_n$ in this case,

$$E_m \in \mathbb{C}\backslash\{0\},\ E_m \neq E_{m'} \text{ for } m \neq m',\ m, m' = 0, \dots, 2n+1,\ E_{2n+1} = 0. \tag{3.30}$$

Lemma 3.2. *Suppose* (3.1) *and the* nth *stationary CH equation* (2.22) *holds subject to the constraint* (3.30) *on an open interval* $\widetilde{\Omega}_\mu \subseteq \mathbb{R}$. *Moreover, suppose that the zeros* μ_j, $j = 1, \dots, n$, *of* $F_n(\,\cdot\,)$ *remain distinct and nonzero on* $\widetilde{\Omega}_\mu$. *Then* $\{\hat\mu_j\}_{j=1,\dots,n}$, *defined by* (3.7), *satisfies the following first-order system of differential equations*

$$\mu_{j,x}(x) = 2\frac{y(\hat\mu_j(x))}{\mu_j(x)} \prod_{\substack{\ell=1 \\ \ell\neq j}}^n (\mu_j(x) - \mu_\ell(x))^{-1}, \quad j = 1, \dots, n,\ x \in \widetilde{\Omega}_\mu. \tag{3.31}$$

Next, assume $\mathcal{K}_n$ *to be nonsingular and introduce the initial condition*

$$\{\hat\mu_j(x_0)\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{3.32}$$

for some $x_0 \in \mathbb{R}$, *where* $\mu_j(x_0) \neq 0$, $j = 1, \dots, n$, *are assumed to be distinct. Then there exists an open interval* $\Omega_\mu \subseteq \mathbb{R}$, *with* $x_0 \in \Omega_\mu$, *such that the initial value problem* (3.31), (3.32) *has a unique solution* $\{\hat\mu_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ *satisfying*

$$\hat\mu_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1, \dots, n, \tag{3.33}$$

and μ_j, $j = 1, \dots, n$, *remain distinct and nonzero on* Ω_μ.

Combining the polynomial approach in Section 2 with (3.5) yields trace formulas for the CH invariants. For simplicity we just record two simple cases.

Lemma 3.3. *Suppose* (3.1), *assume the* nth *stationary CH equation* (2.22) *holds, and let* $x \in \mathbb{R}$. *Then*

$$u(x) = \frac{1}{2}\sum_{j=1}^n \mu_j(x) - \frac{1}{4}\sum_{m=0}^{2n+1} E_m, \tag{3.34}$$

$$4u(x) - u_{xx}(x) = -\Bigg(\prod_{\substack{m=0 \\ E_m\neq 0}}^{2n+1} E_m\Bigg)\Bigg(\prod_{j=1}^n \mu_j(x)^{-2}\Bigg). \tag{3.35}$$

Next we turn to asymptotic properties of ϕ and ψ_j, $j = 1, 2$.

Lemma 3.4. *Suppose* (3.1), *assume the* nth *stationary CH equation* (2.22) *holds, and let* $P = (z, y) \in \mathcal{K}_n\backslash\{P_{\infty_+}, P_{\infty_-}, P_0\}$, $x \in \mathbb{R}$. *Then*

$$\phi(P,x) \underset{\zeta\to 0}{=} \begin{cases} -2\zeta^{-1} - 2u(x) + u_x(x) + O(\zeta), & P \to P_{\infty_+}, \\ 2u(x) + u_x(x) + O(\zeta), & P \to P_{\infty_-}, \end{cases} \quad \zeta = z^{-1}, \tag{3.36}$$

$$\phi(P,x) \underset{\zeta\to 0}{=} \Bigg(\prod_{\substack{m=0 \\ E_m\neq 0}}^{2n+1} E_m\Bigg)^{1/2} f_n(x)^{-1}\zeta + O(\zeta^2), \quad P \to P_0, \quad \zeta = z^{1/2}, \tag{3.37}$$

and

$$\psi_1(P,x,x_0) \underset{\zeta\to 0}{=} \exp(\pm(x - x_0))(1 + O(\zeta)), \quad P \to P_{\infty_\pm}, \quad \zeta = 1/z, \tag{3.38}$$

$$\psi_2(P,x,x_0) \underset{\zeta\to 0}{=} \exp(\pm(x-x_0)) \begin{cases} -2+O(\zeta), & P\to P_{\infty_+}, \\ (2u(x)+u_x(x))\zeta + O(\zeta^2), & P\to P_{\infty_-}, \end{cases} \tag{3.39}$$
$$\zeta = 1/z,$$

$$\psi_1(P,x,x_0) \underset{\zeta\to 0}{=} \exp\bigg(-\frac{1}{\zeta}\int_{x_0}^{x} dx' \bigg(\prod_{\substack{m=0\\ E_m\neq 0}}^{2n+1} E_m\bigg)^{1/2} f_n(x')^{-1} + O(1)\bigg), \tag{3.40}$$
$$P\to P_0,\ \zeta = z^{1/2},$$

$$\psi_2(P,x,x_0) \underset{\zeta\to 0}{=} O\big(\zeta^{-1}\big)\exp\bigg(-\frac{1}{\zeta}\int_{x_0}^{x} dx' \bigg(\prod_{\substack{m=0\\ E_m\neq 0}}^{2n+1} E_m\bigg)^{1/2} f_n(x')^{-1} + O(1)\bigg),$$
$$P\to P_0,\ \zeta = z^{1/2}. \tag{3.41}$$

Since the representations of ϕ and u in terms of the Riemann theta function associated with $\mathcal{K}_n$ (assuming $\mathcal{K}_n$ to be nonsingular) are not explicitly needed in this paper, we omit the corresponding details and refer to the detailed treatment in [13, Ch. 5] and [12] instead.

Finally, we will recall that solvability of the Dubrovin equations (3.31) on $\Omega_\mu \subseteq \mathbb{R}$ in fact implies equation (2.22) on Ω_μ.

Theorem 3.5. *Fix $n\in\mathbb{N}$, assume* (3.30), *and suppose that $\{\hat\mu_j\}_{j=1,\dots,n}$ satisfies the stationary Dubrovin equations* (3.31) *on an open interval $\Omega_\mu \subseteq \mathbb{R}$ such that μ_j, $j=1,\dots,n$, remain distinct and nonzero on Ω_μ. Then $u\in C^\infty(\Omega_\mu)$ defined by*

$$u(x) = \frac{1}{2}\sum_{j=1}^{n}\mu_j(x) - \frac{1}{4}\sum_{m=0}^{2n+1} E_m \tag{3.42}$$

satisfies the nth stationary CH equation (2.22), *that is,*

$$\text{s-CH}_n(u) = 0 \ \text{ on } \Omega_\mu. \tag{3.43}$$

4. Basic Facts on Self-adjoint Hamiltonian Systems

We now turn to the Weyl–Titchmarsh theory for singular Hamiltonian (canonical) systems and briefly recall the basic material needed in the following section. This material is standard and can be found, for instance, in [6], [16], [17], [18], [19], [20], and the references therein.

Hypothesis 4.1. (i) *Define the 2×2 matrix $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$, and suppose $a_{j,k}, b_{j,k} \in L^1_{\text{loc}}(\mathbb{R})$, $j,k=1,2$ and $A(x) = \big(a_{j,k}(x)\big)_{j,k=1,2} \geq 0$, $B(x) = \big(b_{j,k}(x)\big)_{j,k=1,2} = B(x)^*$ for a.e. $x\in\mathbb{R}$. We consider the Hamiltonian system*

$$J\Psi'(z,x) = (zA(x)+B(x))\Psi(z,x), \quad z\in\mathbb{C} \tag{4.1}$$

for a.e. $x\in\mathbb{R}$, where z plays the role of the spectral parameter, and where

$$\Psi(z,x) = (\psi_1(z,x)\ \psi_2(z,x))^\top, \quad \psi_j(z,\cdot\,)\in AC_{\text{loc}}(\mathbb{R}),\ j=1,2. \tag{4.2}$$

Here $AC_{\text{loc}}(\mathbb{R})$ denotes the set of locally absolutely continuous functions on $\mathbb{R}$ and the M^* and $M^\top$ denote the adjoint and transpose of a matrix M, respectively.

(ii) For all nontrivial solutions Ψ of (4.1) *we assume the definiteness hypothesis* (*cf.* [4, Sect. 9.1])

$$\int_c^d dx\, \Psi(z,x)^* A(x) \Psi(z,x) > 0\,, \tag{4.3}$$

on every interval $(c,d) \subset \mathbb{R}$, $c < d$.

A simple example of a Hamiltonian system satisfying (4.3) is obtained when $A(x) = \left(\begin{smallmatrix} w(x) & 0 \\ 0 & 0 \end{smallmatrix}\right)$, for some weight function $w \in L^1_{\text{loc}}(\mathbb{R})$, $w > 0$ a.e. on $\mathbb{R}$, and $b_{2,2}(x) > 0$ a.e. on $\mathbb{R}$ (cf. Section 5). Hypothesis 4.1 (ii) clearly holds in this case.

Next, we introduce the vector space $(-\infty \leq a < b \leq \infty)$

$$L^2_A((a,b)) = \left\{ \phi : (a,b) \to \mathbb{C}^2 \text{ measurable} \,\middle|\, \int_a^b dx\, (\phi(x), A(x)\phi(x))_{\mathbb{C}^2} < \infty \right\}, \tag{4.4}$$

where $(\phi,\psi)_{\mathbb{C}^2} = \sum_{j=1}^2 \overline{\phi_j}\psi_j$ denotes the standard scalar product in $\mathbb{C}^2$. Fix a point $x_0 \in \mathbb{R}$. Then the Hamiltonian system (4.1) is said to be in the *limit point case* at ∞ (resp., $-\infty$) if for some (and hence for all) $z \in \mathbb{C}\backslash\mathbb{R}$, precisely one solution of (4.1) lies in $L^2_A((x_0,\infty))$ (resp., $L^2_A((-\infty,x_0))$). (By the analog of Weyl's alternative, if (4.1) is not in the limit point case at $\pm\infty$, all solutions of (4.1) lie in $L^2_A((x_0,\pm\infty))$ for all $z \in \mathbb{C}$. In the latter case the Hamiltonian system (4.1) is said to be in the *limit circle case* at $\pm\infty$.)

To simplify matters for the remainder of this section, we will always suppose the limit point case at $\pm\infty$ from now on.

Hypothesis 4.2. *Assume Hypothesis* 4.1 *and suppose that the Hamiltonian system* (4.1) *is in the limit point case at $\pm\infty$.*

An elementary example of a Hamiltonian system satisfying Hypothesis 4.2 is given by the case where all entries of A and B are essentially bounded on $\mathbb{R}$ (cf. Section 5).

When considering the Hamiltonian system (4.1) on the half-line $[x_0,\infty)$ (resp., $(-\infty,x_0]$), a self-adjoint (separated) boundary condition at the point x_0 is of the type

$$\alpha\Psi(x_0) = 0, \tag{4.5}$$

where $\alpha = (\alpha_1\ \alpha_2) \in \mathbb{C}^{1\times 2}$ satisfies

$$\alpha\alpha^* = I, \quad \alpha J \alpha^* = 0 \ \ (\text{equivalently, } |\alpha_1|^2 + |\alpha_2|^2 = 1,\, \operatorname{Im}(\alpha_2\overline{\alpha_1}) = 0). \tag{4.6}$$

In particular, the boundary condition (4.5) (with α satisfying (4.6)) is equivalent to $\alpha_1\psi_1(x_0) + \alpha_2\psi_2(x_0) = 0$ with $\alpha_1/\alpha_2 \in \mathbb{R}$ if $\alpha_2 \neq 0$ and $\alpha_2/\alpha_1 \in \mathbb{R}$ if $\alpha_1 \neq 0$. The special case $\alpha_0 = (1\ 0)$ will be of particular relevance in Section 5. Due to our limit point assumption at $\pm\infty$ in Hypothesis 4.2, no additional boundary condition at $\pm\infty$ needs to be introduced when considering (4.1) on the half-lines $[x_0,\infty)$ and $(-\infty,x_0]$. The resulting full-line and half-line Hamiltonian systems are said to be self-adjoint on $\mathbb{R}$, $[x_0,\infty)$, and $(-\infty,x_0]$, respectively (assuming of course a boundary condition of the type (4.5) in the two half-line cases).

Next we digress a bit and briefly turn to Herglotz functions and their representations in terms of measures, the focal point of Weyl–Titchmarsh theory (and hence spectral theory) of self-adjoint Hamiltonian systems.

Definition 4.3. Any analytic map $m\colon \mathbb{C}_+ \to \mathbb{C}_+$ is called a *Herglotz* function (here $\mathbb{C}_+ = \{z \in \mathbb{C} \,|\, \operatorname{Im}(z) > 0\}$). Similarly, any analytic map $M\colon \mathbb{C}_+ \to \mathbb{C}^{k\times k}$, $k \in \mathbb{N}$, is called a $k \times k$ matrix-valued Herglotz function if $\operatorname{Im}(M(z)) \geq 0$ for all $z \in \mathbb{C}_+$.

Herglotz functions are characterized by a representation of the form

$$m(z) = a + bz + \int_{-\infty}^{\infty} d\omega(\lambda)\,\big((\lambda - z)^{-1} - \lambda(1+\lambda^2)^{-1}\big), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{4.7}$$

$$a \in \mathbb{R},\ b \geq 0, \quad \int_{-\infty}^{\infty} d\omega(\lambda)\,(1+\lambda^2)^{-1} < \infty, \tag{4.8}$$

$$\omega((\lambda,\mu]) = \lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda+\delta}^{\mu+\delta} d\nu\,\operatorname{Im}\left(m(\nu + i\epsilon)\right), \tag{4.9}$$

in the following sense: Every Herglotz function admits a representation of the type (4.7), (4.8) and conversely, any function of the type (4.7), (4.8) is a Herglotz function. Moreover, local singularities and zeros of m are necessarily located on the real axis and at most of first order in the sense that

$$\omega(\{\lambda\}) = \lim_{\varepsilon\downarrow 0}\left(\omega(\lambda+\varepsilon) - \omega(\lambda-\varepsilon)\right) = -\lim_{\varepsilon\downarrow 0} i\varepsilon\, m(\lambda + i\varepsilon) \geq 0, \quad \lambda \in \mathbb{R}, \tag{4.10}$$

$$\lim_{\varepsilon\downarrow 0} i\varepsilon\, m(\lambda+i\varepsilon)^{-1} \geq 0, \quad \lambda \in \mathbb{R}. \tag{4.11}$$

In particular, isolated poles of m are simple and located on the real axis, the corresponding residues being negative. Analogous results hold for matrix-valued Herglotz functions (see, e.g., [14] and the literature cited therein).

For subsequent purpose in Section 5 we also note that $-1/z$ is a Herglotz function and compositions of Herglotz functions remain Herglotz functions. In addition, diagonal elements of a matrix-valued Herglotz function are Herglotz functions.

Returning to Hamiltonian systems on the half-lines $[x_0, \pm\infty)$ satisfying Hypotheses 4.1 and 4.2, we now denote by $\Psi_\pm(z,x,x_0)$ the unique solution of (4.1) satisfying $\Psi_\pm(z,\cdot\,,x_0) \in L^2_A([x_0,\pm\infty))$, $z \in \mathbb{C}\backslash\mathbb{R}$, normalized by $\psi_{1,\pm}(z,x_0,x_0) = 1$. Then the half-line Weyl–Titchmarsh function $m_\pm(z,x)$, associated with the Hamiltonian system (4.1) on $[x,\pm\infty)$ and the fixed boundary condition $\alpha_0 = (1\ 0)$ at the point $x \in \mathbb{R}$, is defined by

$$m_\pm(z,x) = \psi_{2,\pm}(z,x,x_0)/\psi_{1,\pm}(z,x,x_0), \quad z \in \mathbb{C}\backslash\mathbb{R},\ \pm x \geq x_0. \tag{4.12}$$

The actual normalization of $\Psi_\pm(z,x,x_0)$ was chosen for convenience only and is clearly immaterial in the definition of $m_\pm(z,x)$ in (4.12).

One easily verifies that $m_\pm(z,x)$ satisfies the following Riccati-type differential equation,

$$\begin{aligned} &m'(z,x) + [b_{2,2}(x) + a_{2,2}(x)z]m(z,x)^2 \\ &\quad + [b_{1,2}(x) + b_{2,1}(x) + (a_{1,2}(x) + a_{2,1}(x))z]m(z,x) + b_{1,1}(x) + a_{1,1}(x)z = 0. \end{aligned} \tag{4.13}$$

Finally, the 2×2 Weyl–Titchmarsh matrix $M(z,x)$ associated with the Hamiltonian system (4.1) on $\mathbb{R}$ is then defined in terms of the half-line Weyl–Titchmarsh functions $m_\pm(z,x)$ by

$$\begin{aligned} M(z,x) &= \big(M_{j,j'}(z,x)\big)_{j,j'=1,2}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \\ M_{1,1}(z,x) &= [m_-(z,x) - m_+(z,x)]^{-1}, \\ M_{1,2}(z,x) &= M_{2,1}(z,x) \end{aligned} \tag{4.14}$$

$$= 2^{-1}[m_-(z,x) - m_+(z,x)]^{-1}[m_-(z,x) + m_+(z,x)], \tag{4.15}$$
$$M_{2,2}(z,x) = [m_-(z,x) - m_+(z,x)]^{-1}m_-(z,x)m_+(z,x).$$

One verifies that $M(z,x)$ is a 2×2 matrix-valued Herglotz function. We emphasize that for any fixed $x_0 \in \mathbb{R}$, $M(z,x_0)$ contains all the spectral information of the self-adjoint Hamiltonian system (4.1) on $\mathbb{R}$ (assuming Hypotheses 4.1 and 4.2).

5. Real-Valued Algebro-geometric CH Solutions and the Associated Isospectral Torus

In our final and principal section we study real-valued algebro-geometric solutions of the CH hierarchy associated with curves $\mathcal{K}_n$ whose affine part is nonsingular and determine the isospectral manifold of smooth bounded CH solutions. We focus on the stationary case as this is the primary concern in this context.

To study the direct spectral problem we first introduce the following assumptions.

Hypothesis 5.1. *Suppose*

$$E_0 < E_1 < \cdots < E_{2n} < E_{2n+1} = 0 \tag{5.1}$$

and let u be a real-valued solution of the nth stationary CH equation (2.22),

$$\text{s-CH}_n(u) = 0, \tag{5.2}$$

(*i.e., u is a particular algebro-geometric CH potential*), *satisfying*

$$u \in C^\infty(\mathbb{R}), \quad \partial_x^k u \in L^\infty(\mathbb{R}),\ k = 0,1,2, \tag{5.3}$$
$$4u - u_{xx} > 0. \tag{5.4}$$

We start by noticing that the basic stationary equation (3.23),

$$\Psi_x(z,x) = U(z,x)\Psi(z,x), \quad \Psi = (\psi_1,\psi_2)^\top,\ (z,x) \in \mathbb{C}\times\mathbb{R}, \tag{5.5}$$

is equivalent to the following Hamiltonian (canonical) system

$$J\widetilde{\Psi}_x(\tilde z,x) = [\tilde z A(x) + B(x)]\widetilde{\Psi}(\tilde z,x), \quad \widetilde{\Psi} = (\tilde\psi_1,\tilde\psi_2)^\top,\ (\tilde z,x) \in \mathbb{C}\times\mathbb{R}, \tag{5.6}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tilde\Psi(\tilde z,x) = \Psi(z,x), \quad \tilde z = -1/z, \tag{5.7}$$
$$A(x) = \begin{pmatrix} 4u(x) - u_{xx}(x) & 0 \\ 0 & 0 \end{pmatrix}, \quad B(x) = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \quad x \in \mathbb{R}. \tag{5.8}$$

In particular, due to assumptions (5.3) and (5.4), the Hamiltonian system (5.6) satisfies Hypotheses 4.1 and 4.2. Explicitly, the Hamiltonian system (5.6) boils down to

$$\tilde\psi_{1,x}(\tilde z,x) = \tilde\psi_2(\tilde z,x) - \tilde\psi_1(\tilde z,x), \tag{5.9}$$
$$\tilde\psi_{2,x}(\tilde z,x) = -\tilde z(4u(x) - u_{xx}(x))\tilde\psi_1(\tilde z,x) + \tilde\psi_2(\tilde z,x), \quad (z,x) \in \mathbb{C}\times\mathbb{R}, \tag{5.10}$$

and upon eliminating $\tilde\psi_2$ results in a particular case of the weighted Sturm–Liouville problem

$$\frac{1}{r}\left[-\frac{d}{dx}p\frac{d}{dx} + q\right] \tag{5.11}$$

of the type

$$-\tilde\psi_{1,xx}(\tilde z,x) + \tilde\psi_1(\tilde z,x) = \tilde z(4u(x) - u_{xx}(x))\tilde\psi_1(\tilde z,x), \quad (z,x) \in \mathbb{C}\times\mathbb{R}, \tag{5.12}$$

with "weight" $r = (4u - u_{xx})$ and constant coefficients $p = q = 1$.

Introducing

$$\Sigma = \bigcup_{\ell=0}^{n} [E_{2\ell}, E_{2\ell+1}], \tag{5.13}$$

we define

$$\begin{aligned} R_{2n+2}(\lambda)^{1/2} &= |R_{2n+2}(\lambda)^{1/2}| \\ &\times \begin{cases} -1 & \text{for } \lambda \in (E_{2n+1}, \infty), \\ (-1)^{n+j} & \text{for } \lambda \in (E_{2j-1}, E_{2j}),\ j = 1, \dots, n, \\ (-1)^n & \text{for } \lambda \in (-\infty, E_0), \\ i(-1)^{n+j+1} & \text{for } \lambda \in (E_{2j}, E_{2j+1}),\ j = 0, \dots, n, \end{cases} \qquad \lambda \in \mathbb{R}, \end{aligned} \tag{5.14}$$

and

$$R_{2n+2}(\lambda)^{1/2} = \lim_{\varepsilon \downarrow 0} R_{2n+2}(\lambda + i\varepsilon)^{1/2}, \quad \lambda \in \Sigma, \tag{5.15}$$

and analytically continue $R_{2n+2}(\,\cdot\,)^{1/2}$ to $\mathbb{C}\backslash\Sigma$. We also note the property

$$\overline{R_{2n+2}(\overline{z})^{1/2}} = R_{2n+2}(z)^{1/2}. \tag{5.16}$$

For notational convenience we will occasionally call (E_{2j-1}, E_{2j}), $j = 1, \dots, n$, spectral gaps and E_{2j-1}, E_{2j} the corresponding spectral gap endpoints.

Next, we introduce the cut plane

$$\Pi = \mathbb{C}\backslash\Sigma, \tag{5.17}$$

and the upper, respectively, lower sheets $\Pi_\pm$ of $\mathcal{K}_n$ by

$$\Pi_\pm = \{(z, \pm R_{2n+2}(z)^{1/2}) \in \mathcal{K}_n \mid z \in \Pi\} \tag{5.18}$$

with the associated charts

$$\zeta_\pm \colon \Pi_\pm \to \Pi, \quad P = (z, \pm R_{2n+2}(z)^{1/2}) \mapsto z. \tag{5.19}$$

The two branches $\Psi_\pm(z, x, x_0)$ of the Baker–Akhiezer vector $\Psi(P, x, x_0)$ in (3.16) are then given by

$$\Psi_\pm(z, x, x_0) = \Psi(P, x, x_0), \quad P = (z, y) \in \Pi_\pm, \quad \Psi_\pm = (\psi_{1,\pm}, \psi_{2,\pm})^\top, \tag{5.20}$$

and one infers from (3.38) that

$$\psi_{1,\pm}(z, \,\cdot\,, x_0) \in L^2((x_0, \mp\infty)) \text{ for } |z| \text{ sufficiently large.} \tag{5.21}$$

Thus, introducing

$$\widetilde{\Psi}_\pm(\tilde{z}, x, x_0) = \Psi_\mp(z, x, x_0), \quad \widetilde{\Psi}_\pm = (\tilde{\psi}_{1,\pm}, \tilde{\psi}_{2,\pm})^\top, \quad \tilde{z} = -1/z, \tag{5.22}$$

and the two branches $\phi_\pm(z, x)$ of $\phi(P, x)$ on $\Pi_\pm$ by

$$\phi_\pm(z, x) = \phi(P, x), \quad P = (z, y) \in \Pi_\pm, \tag{5.23}$$

one infers from (4.12) and (5.21) that the Weyl–Titchmarsh functions $\tilde{m}_\pm(\tilde{z}, x)$ associated with the self-adjoint Hamiltonian system (5.6) on the half-lines $[x, \pm\infty)$ and the Dirichlet boundary condition indexed by $\alpha_0 = (1\ 0)$ at the point $x \in \mathbb{R}$, are given by

$$\begin{aligned} \tilde{m}_\pm(\tilde{z}, x) &= \tilde{\psi}_{2,\pm}(\tilde{z}, x, x_0)/\tilde{\psi}_{1,\pm}(\tilde{z}, x, x_0) = \psi_{2,\mp}(z, x, x_0)/\psi_{1,\mp}(z, x, x_0) \\ &= (-1/z)\phi_\mp(z, x), \quad z \in \mathbb{C}\backslash\Sigma. \end{aligned} \tag{5.24}$$

More precisely, (5.21) yields (5.24) only for $|z|$ sufficiently large. However, since by general principles $\tilde m_\pm(\,\cdot\,,x)$ are analytic in $\mathbb{C}\backslash\mathbb{R}$, and by (3.12), $\phi_\pm(\,\cdot\,,x)$ are analytic in $\mathbb{C}\backslash\Sigma$, one infers (5.24) by analytic continuation. In particular, (5.21) extends to all $z\in\mathbb{C}\backslash\Sigma$, that is,

$$\psi_{1,\pm}(z,\,\cdot\,,x_0)\in L^2((x_0,\mp\infty)),\quad z\in\mathbb{C}\backslash\Sigma. \tag{5.25}$$

Next, we mention a useful fact concerning a special class of Herglotz functions closely related to the problem at hand. The result must be well-known to experts, but since we could not quickly locate a proof in the literature, we provide the simple contour integration argument below.

Lemma 5.2. *Let P_N be a monic polynomial of degree N. Then $P_N/R_{2n+2}^{1/2}$ is a Herglotz function if and only if one of the following alternatives applies:*
(i) $N=n$ and

$$P_n(z)=\prod_{j=1}^n (z-a_j),\quad a_j\in[E_{2j-1},E_{2j}],\ j=1,\dots,n. \tag{5.26}$$

If (5.26) is satisfied, then $P_n/R_{2n+2}^{1/2}$ admits the Herglotz representation

$$\frac{P_n(z)}{R_{2n+2}(z)^{1/2}}=\frac{1}{\pi}\int_\Sigma \frac{|P_n(\lambda)|\,d\lambda}{|R_{2n+2}(\lambda)^{1/2}|}\frac{1}{\lambda-z},\quad z\in\mathbb{C}\backslash\Sigma. \tag{5.27}$$

(ii) $N=n+1$ and

$$P_{n+1}(z)=\prod_{\ell=0}^n (z-b_\ell),\quad b_0\in(-\infty,E_0],\ b_j\in[E_{2j-1},E_{2j}],\ j=1,\dots,n. \tag{5.28}$$

If (5.28) is satisfied, then $P_{n+1}/R_{2n+2}^{1/2}$ admits the Herglotz representation

$$\frac{P_{n+1}(z)}{R_{2n+2}(z)^{1/2}}=\mathrm{Re}\bigg(\frac{P_{n+1}(i)}{R_{2n+2}(i)^{1/2}}\bigg)+\frac{1}{\pi}\int_\Sigma \frac{|P_{n+1}(\lambda)|\,d\lambda}{|R_{2n+2}(\lambda)^{1/2}|}\bigg(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\bigg),\quad z\in\mathbb{C}\backslash\Sigma. \tag{5.29}$$

Proof. Since Herglotz functions are $O(z)$ as $|z|\to\infty$ and cannot vanish faster than $O(1/z)$ as $|z|\to\infty$, we can confine ourselves to the range $N\in\{n,n+1,n+2\}$. We start with the case $N=n$ and employ the following contour integration approach. Consider a closed oriented contour $\Gamma_{R,\varepsilon}$ which consists of the clockwise oriented semicircle $C_\varepsilon=\{z\in\mathbb{C}\,|\,z=E_0-\varepsilon\exp(-i\alpha),\ -\pi/2\le\alpha\le\pi/2\}$ centered at E_0, the straight line $L_+=\{z\in\mathbb{C}_+\,|\,z=E_0+x+i\varepsilon,\ 0\le x\le R\}$ (oriented from left to right), the following part of the counterclockwise oriented circle of radius $(R^2+\varepsilon^2)^{1/2}$ centered at E_0, $C_R=\{z\in\mathbb{C}\,|\,z=E_0+(R^2+\varepsilon^2)^{1/2}\exp(i\beta),\ \arctan(\varepsilon/R)\le\beta\le 2\pi-\arctan(\varepsilon/R)\}$, and the straight line $L_-=\{z\in\mathbb{C}_-\,|\,z=E_0+x-i\varepsilon,\ 0\le x\le R\}$ (oriented from right to left). Then, for $\varepsilon>0$ small enough and $R>0$ sufficiently large, one infers

$$\begin{aligned}\frac{P_n(z)}{R_{2n+2}(z)^{1/2}}&=\frac{1}{2\pi i}\oint_{\Gamma_{R,\varepsilon}}\frac{1}{\zeta-z}\frac{P_n(\zeta)}{R_{2n+2}(\zeta)^{1/2}}d\zeta\\&\underset{\varepsilon\downarrow 0,R\uparrow\infty}{=}\frac{1}{\pi}\int_\Sigma\frac{1}{\lambda-z}\frac{P_n(\lambda)d\lambda}{iR_{2n+2}(\lambda)^{1/2}}.\end{aligned} \tag{5.30}$$

Here we used (5.14) to compute the contributions of the contour integral along $[E_0, R]$ in the limit $\varepsilon \downarrow 0$ and note that the integral over C_R tends to zero as $R \uparrow \infty$ since

$$\frac{P_n(\zeta)}{R_{2n+2}(\zeta)^{1/2}} \underset{\zeta\to\infty}{=} O\big(|\zeta|^{-1}\big). \tag{5.31}$$

Next, utilizing the fact that P_n is monic and using (5.14) again, one infers that $F_n(\lambda)d\lambda/[iR_{2n+2}(\lambda)^{1/2}]$ represents a positive measure supported on Σ if and only if P_n has precisely one zero in each of the intervals $[E_{2j-1}, E_{2j}]$, $j = 1, \dots, n$. In other words,

$$\frac{P_n(\lambda)}{iR_{2n+1}(\lambda)^{1/2}} = \frac{|P_n(\lambda)|}{|R_{2n+1}(\lambda)^{1/2}|} \geq 0 \text{ on } \Sigma \tag{5.32}$$

if and only if P_n has precisely one zero in each of the intervals $[E_{2j-1}, E_{2j}]$, $j = 1, \dots, n$. The Herglotz representation (4.7), (4.8) then finishes the proof of (5.27).

In the case where $N = n+1$, the proof of (5.28) follows along similar lines taking into account the additional residues at $\pm i$ inside $\Gamma_{R,\varepsilon}$ which are responsible for the real part on the right-hand side of (5.29).

Finally, in the case $N = n + 2$, assume that $P_{n+2}/R_{2n+2}^{1/2}$ is a Herglotz function. Then necessarily,

$$\frac{P_{n+2}(z)}{R_{2n+2}(z)^{1/2}} = a + bz + \int_{E_0}^{0} d\omega(\lambda)\,(\lambda - z)^{-1}, \quad z \in \mathbb{C}\backslash\Sigma \tag{5.33}$$

for some $a \in \mathbb{R}$, $b \geq 0$, and some finite (positive) measure ω supported on $[E_0, 0]$, since

$$\lim_{\varepsilon\downarrow 0} \operatorname{Im}(P_{n+2}(\lambda)R_{2n+2}(\lambda + i\varepsilon)^{-1/2}) = 0 \text{ for } \lambda > E_{2n+2} = 0 \text{ and } \lambda < E_0. \tag{5.34}$$

In particular, (5.33) implies

$$P_{n+2}(z)R_{2n+2}(z)^{-1/2} \underset{|z|\to\infty}{=} bz + O(1), \quad b \geq 0. \tag{5.35}$$

However, by (5.14), one immediately infers

$$P_{n+2}(\lambda)R_{2n+2}(\lambda)^{-1/2} \underset{\lambda\uparrow\infty}{=} -\lambda + O(1). \tag{5.36}$$

This contradiction dispenses with the case $N = n + 2$. ☐

Now we are in position to state the following result concerning the half-line and full-line Weyl–Titchmarsh functions associated with the self-adjoint Hamiltonian system (5.6). We denote by $\tilde{m}_\pm(\tilde{z}, x)$ the Weyl–Titchmarsh m-functions corresponding to (5.6) associated with the half-lines $(x, \pm\infty)$ and the Dirichlet boundary condition indexed by $\alpha_0 = (1\ 0)$ at the point $x \in \mathbb{R}$, and by $\widetilde{M}(\tilde{z}, x)$ the 2×2 Weyl–Titchmarsh matrix corresponding to (5.6) on $\mathbb{R}$ (cf. (4.12), (4.14), and (4.15)). Moreover, Σ^o denotes the open interior of Σ and the real part of a matrix M is defined as usual by $\operatorname{Re}(M) = (M + M^*)/2$.

Theorem 5.3. *Assume Hypothesis* 5.1 *and let* $(z, x) \in \mathbb{R} \times (\mathbb{C}\backslash\Sigma)$, $\tilde{z} = -1/z$. *Then*

$$\tilde{m}_\pm(\tilde{z}, x) = \frac{\pm R_{2n+2}(z)^{1/2} + zG_n(z, x)}{zF_n(z, x)} \tag{5.37}$$

$$= 1 \pm \operatorname{Re}\Big(\frac{R_{2n+2}(i)^{1/2}}{iF_n(i,x)}\Big) + \sum_{j=1}^n \frac{G_n(\mu_j(x),x)(1\mp\varepsilon_j(x))}{dF_n(\mu_j(x),x)/dz}\frac{1}{z-\mu_j(x)}$$
$$\pm \frac{1}{\pi}\int_\Sigma \frac{|R_{2n+2}(\lambda)^{1/2}|\,d\lambda}{|\lambda F_n(\lambda,x)|}\Big(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\Big), \tag{5.38}$$

where $\varepsilon_j(x)\in\{1,-1\}$, $j=1,\dots,n$, *is chosen such that*

$$\frac{G_n(\mu_j(x),x)\varepsilon_j(x)}{dF_n(\mu_j(x),x)/dz} \geq 0, \quad j=1,\dots,n. \tag{5.39}$$

Moreover,

$$\widetilde{M}(\tilde z,x) = \frac{-1}{2R_{2n+2}(z)^{1/2}}\begin{pmatrix} -H_n(z,x) & zG_n(z,x) \\ zG_n(z,x) & zF_n(z,x)\end{pmatrix} \tag{5.40}$$
$$= \operatorname{Re}(\widetilde{M}(i,x)) + \int_\Sigma d\Omega(\lambda,x)\left(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\right), \tag{5.41}$$

where

$$\Omega(\lambda,x) = \frac{1}{2\pi i R_{2n+2}(\lambda)^{1/2}}\begin{pmatrix} H_n(\lambda,x) & -\lambda G_n(\lambda,x) \\ -\lambda G_n(\lambda,x) & -\lambda F_n(\lambda,x)\end{pmatrix}, \quad \lambda\in\Sigma^o. \tag{5.42}$$

The essential spectrum of the half-line Hamiltonian systems (5.6) *on* $[x,\pm\infty)$ (*with any self-adjoint boundary condition at* x) *as well as the essential spectrum of the Hamiltonian system* (5.6) *on* $\mathbb{R}$ *is purely absolutely continuous and given by*

$$\bigcup_{\ell=0}^{n-1}\big[-E_{2l}^{-1}, -E_{2\ell+1}^{-1}\big]\cup\big[-E_{2n}^{-1},\infty\big). \tag{5.43}$$

The spectral multiplicities are simple in the half-line cases and of uniform multiplicity two in the full-line case.

Proof. Equation (5.37) follows from (3.12), (5.14), and (5.24). Equation (5.40) is then a consequence of (3.20)–(3.22), (4.14), (4.15), (5.24), and (5.37). Different self-adjoint boundary conditions at the point x lead to different half-line Hamiltonian systems whose Weyl–Titchmarsh functions are related by a linear fractional transformation (cf., e.g., [6]), which leads to the invariance of the essential spectrum with respect to the boundary condition at x. In order to prove the Herglotz representation (5.38) one can follow the corresponding computation for Schrödinger operators with algebro-geometric potentials in [21, Sect. 8.1]. For this purpose one first notes that by (5.29) also $R_{2n+2}(z)^{1/2}/[zF_n(z,x)]$ is a Herglotz function. A contour integration as in the proof of Lemma 5.2 then proves

$$\frac{R_{2n+2}(z)^{1/2}}{zF_n(z,x)} = \operatorname{Re}\Big(\frac{R_{2n+2}(i)^{1/2}}{iF_n(i,x)}\Big) + \sum_{j=1}^n \frac{|R_{2n+2}(\mu_j(x))^{1/2}|}{\mu_j(x)|dF_n(\mu_j(x),x)/dz|}\frac{1}{z-\mu_j(x)}$$
$$+ \frac{1}{\pi}\int_\Sigma \frac{|R_{2n+2}(\lambda)^{1/2}|\,d\lambda}{|\lambda F_n(\lambda,x)|}\Big(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\Big). \tag{5.44}$$
$$= \operatorname{Re}\Big(\frac{R_{2n+2}(i)^{1/2}}{iF_n(i,x)}\Big) - \sum_{j=1}^n \frac{G_n(\mu_j(x),x)\varepsilon_j(x)}{dF_n(\mu_j(x),x)/dz}\frac{1}{z-\mu_j(x)}$$
$$+ \frac{1}{\pi}\int_\Sigma \frac{|R_{2n+2}(\lambda)^{1/2}|\,d\lambda}{|\lambda F_n(\lambda,x)|}\Big(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\Big). \tag{5.45}$$

The only difference compared to the corresponding argument in the proof of Lemma 5.2 concerns additional (approximate) semicircles of radius ε centered at each $\mu_j(x)$, $j = 1, \dots, n$, in the upper and lower complex half-planes. Whenever, $\mu_j(x) \in (E_{2j-1}, E_{2j})$, the limit $\varepsilon \downarrow 0$ picks up a residue contribution displayed in the sum over j in (5.44). This contribution vanishes, however, if $\mu_j(x) \in \{E_{2j-1}, E_{2j}\}$. In this case $dF_n(\mu_j(x), x)/dz \neq 0$ by (4.10) and $R_{2n+2}(\mu_j(x)) = 0$ by (2.17). Equation (5.45) then follows from (3.7) and the sign of $\varepsilon_j(x)$ must be chosen according to (5.39) in order to guarantee nonpositive residues in (5.45) (cf. (4.10)).

Next, we apply the Lagrange interpolation formula. If Q_{n-1} is a polynomial of degree $n-1$, then

$$Q_{n-1}(z) = F_n(z) \sum_{j=1}^{n} \frac{Q_{n-1}(\mu_j)}{dF_n(\mu_j)/dz} \frac{1}{z - \mu_j}, \quad z \in \mathbb{C}. \tag{5.46}$$

Since F_n and G_n are monic polynomials of degree n, we can apply (5.46) to $Q_{n-1} = G_n - F_n$ and obtain

$$\frac{G_n(z,x)}{F_n(z,x)} = 1 + \sum_{j=1}^{n} \frac{G_n(\mu_j(x), x)}{dF_n(\mu_j(x), x)/dz} \frac{1}{z - \mu_j(x)}. \tag{5.47}$$

Insertion of (5.47) into (5.37) then yields

$$\tilde{m}_\pm(\tilde{z}, x) = \frac{\pm R_{2n+2}(z)^{1/2}}{z F_n(z,x)} + \sum_{j=1}^{n} \frac{G_n(\mu_j(x), x)[\varepsilon_j(x) + (1 - \varepsilon_j(x))]}{dF_n(\mu_j(x), x)/dz} \frac{1}{z - \mu_j(x)} + 1, \tag{5.48}$$

and hence (5.38) follows by inserting (5.45) into (5.48). Equations (5.41) and (5.42) are clear from the matrix analog of (4.9).

The statement (5.43) for the essential half-line spectra then follows from the fact that the measure in the Herglotz representation (5.38) of $\tilde{m}_\pm$ (as a function of z) is supported on the set Σ in (5.13), with a strictly positive density on the open interior Σ^o of Σ. The transformation $z \to -1/z$ then yields (5.43) and since half-line spectra with a regular endpoint x have always simple spectra this completes the proof of our half-line spectral assertions. The full-line case follows in exactly the same manner since the corresponding 2×2 matrix-valued measure Ω in the Herglotz representation (5.41) of $\widetilde{M}$ (as a function of z) also has support Σ and rank equal to two on Σ^o. ☐

Returning to direct spectral theory, we note that the two spectral problems (5.6) on $\mathbb{R}$ associated with the vanishing of the first and second component of $\widetilde{\Psi}$ at x, respectively, are clearly self-adjoint since they correspond to the choices $\alpha = (1\ 0)$ and $\alpha = (0\ 1)$ in (4.5). Hence a comparison with (3.5), (3.26), and (3.27) necessarily yields $\{\mu_j(x)\}_{j=1,\dots,n}, \{\nu_j(x)\}_{j=1,\dots,n} \subset \mathbb{R}$. Thus we will assume the convenient eigenvalue orderings

$$\mu_j(x) < \mu_{j+1}(x), \quad \nu_j(x) < \nu_{j+1}(x) \text{ for } j = 1, \dots, n-1, \ x \in \mathbb{R}. \tag{5.49}$$

The zeros of $\tilde{\psi}_1(\,\cdot\,, x)$ belong to the Dirichlet spectral problem associated with the Hamiltonian system (5.6) (resp., the weighted Sturm–Liouville problem (5.12)) on $\mathbb{R}$. A comparison with (3.26) then relates the zeros $\mu_j(x_1)$, $j = 1, \dots, n$, of $F_n(\,\cdot\,, x_1)$ in (3.5) to the Dirichlet spectrum of (5.6) (resp., (5.12)) on $\mathbb{R}$. The correspondence between each μ_j and the related spectral point of the Dirichlet problem (5.6) (resp., (5.12)) on $\mathbb{R}$ is of course effected by the transformation $z \to -1/z$. In contrast to this, the zeros of $\tilde{\psi}_2(\,\cdot\,, x_1)$ do not belong to the Neumann spectrum associated with the Hamiltonian system (5.6) (resp., the weighted Sturm–Liouville problem (5.12)) on $\mathbb{R}$. In fact, by (5.9), zeros of $\tilde{\psi}_2(\,\cdot\,, x_1)$ correspond to a mixed boundary condition of the type $\tilde{\psi}_{1,x}(x_1) + \tilde{\psi}_1(x_1) = 0$. By (3.27), this relates the zeros $\nu_j(x_1)$, $j = 1, \dots, n$, of $H_n(\,\cdot\,, x_1)$ in (3.5) to the spectrum of (5.6) (resp., (5.12)) on $\mathbb{R}$ corresponding to the self-adjoint boundary condition $\tilde{\psi}_{1,x}(x_1) + \tilde{\psi}_1(x_1) = 0$.

Combining Lemma 5.2 with the Herglotz property of the 2×2 Weyl–Titchmarsh matrix $\widetilde{M}(\,\cdot\,, x)$ then yields the following refinement of Theorem 3.2.

Theorem 5.4. *Assume Hypothesis 5.1. Then* $\{\hat{\mu}_j\}_{j=1,\dots,n}$*, with the projections* $\mu_j(x)$*,* $j = 1, \dots, n$*, the zeros of* $F_n(\,\cdot\,, x)$ *in* (3.5)*, satisfies the first-order system of differential equations* (3.31) *on* $\Omega_\mu = \mathbb{R}$ *and*

$$\hat{\mu}_j \in C^\infty(\mathbb{R}, \mathcal{K}_n), \quad j = 1, \dots, n. \tag{5.50}$$

Moreover,

$$\mu_j(x) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots, n, \ x \in \mathbb{R}. \tag{5.51}$$

In particular, $\hat{\mu}_j(x)$ *changes sheets whenever it hits* E_{2j-1} *or* E_{2j} *and its projection* $\mu_j(x)$ *remains trapped in* $[E_{2j-1}, E_{2j}]$ *for all* $j = 1, \dots, n$ *and* $x \in \mathbb{R}$*. The analogous statements apply to* $\hat{\nu}_j(x)$ *and one infers*

$$\nu_j(x) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots, n, \ x \in \mathbb{R}. \tag{5.52}$$

Proof. Since $\widetilde{M}(\,\cdot\,, x)$ is a 2×2 Herglotz matrix, its diagonal elements are Herglotz functions. Thus,

$$\widetilde{M}_{1,1}(\tilde{z}, x) = \frac{H_n(z, x)}{2R_{2n+2}(z)^{1/2}}, \quad \widetilde{M}_{2,2}(\tilde{z}, x) = \frac{-zF_n(z, x)}{2R_{2n+2}(z)^{1/2}}, \quad \tilde{z} = -1/z \tag{5.53}$$

are Herglotz functions (the left-hand sides with respect to $\tilde{z}$, the right-hand sides with respect to z) and the interlacing properties (5.51), (5.52) then follow from (5.28) and (5.26). □

Remark 5.5. *Combining the interlacing property* (5.51) *with* (2.18)*,* (2.19)*, and* (2.20) *yields* (*cf. also* (3.35))

$$4u(x) - u_{xx}(x) = -\bigg(\prod_{m=0}^{2n} E_m\bigg)\bigg(\prod_{j=1}^{n} \mu_j(x)^{-2}\bigg) > 0, \quad x \in \mathbb{R}, \tag{5.54}$$

in accordance with (5.4)*. Moreover, since by* (5.52) *the* $\nu_j(x)$ *also remain trapped in the intervals* $[E_{2j-1}, E_{2j}]$ *for all* $x \in \mathbb{R}$*, none of the* $\hat{\nu}_j$ *can reach* P_{∞_-} *and hence* $h_0 = 4u + 2u_x \neq 0$ *on* $\mathbb{R}$ (*cf. the discussion surrounding* (3.15))*. Actually,*

$$h_0(x) > 0, \quad x \in \mathbb{R}, \tag{5.55}$$

since $H_n(\,\cdot\,, x)/R_{2n+2}^{1/2}$ *is a Herglotz function* (*cf.* (5.27)).

Remark 5.6. *The zeros* $\mu_j(x) \in (E_{2j-1}, E_{2j})$*,* $j = 1, \dots, n$ *of* $F_n(\,\cdot\,, x)$ *which are related to eigenvalues of the Hamiltonian system* (5.6) *on* $\mathbb{R}$ *associated with the boundary condition* $\tilde{\psi}_1(x) = 0$*, in fact, are related to left and right half-line eigenvalues of the corresponding Hamiltonian system restricted to the half-lines* $(-\infty, x]$ *and* $[x, \infty)$*, respectively. Indeed, by* (5.22) *and* (5.25)*, depending on whether* $\hat{\mu}_j(x) \in \Pi_+$ *or* $\hat{\mu}_j(x) \in \Pi_-$*,* $\mu_j(x)$ *is related to a left or right half-line eigenvalue associated with* *the Dirichlet boundary condition $\tilde{\psi}_1(x) = 0$. A careful investigation of the sign of the right-hand sides of the Dubrovin equations* (3.30) (*combining* (5.1), (5.14), *and* (5.18)), *then proves that the $\mu_j(x)$ related to right* (*resp., left*) *half-line eigenvalues of the Hamiltonian system* (5.6) *associated with the boundary condition $\tilde{\psi}_1(x) = 0$, are strictly monotone increasing* (*resp., decreasing*) *with respect to x, as long as the μ_j stay away from the right* (*resp., left*) *endpoints of the corresponding spectral gaps (E_{2j-1}, E_{2j}). Here we purposely avoided the limiting case where some of the $\mu_k(x)$ hit the boundary of the spectral gaps, $\mu_k(x) \in \{E_{2k-1}, E_{2k}\}$, since the half-line eigenvalue interpretation is lost as there is no $L^2((x, \pm\infty))^2$ eigenfunction $\widetilde{\Psi}(x)$ satisfying $\tilde{\psi}_1(x) = 0$ in this case. In fact, whenever an eigenvalue $\mu_k(x)$ hits a spectral gap endpoint, the associated point $\hat{\mu}_j(x)$ on $\mathcal{K}_n$ crosses over from one sheet to the other* (*equivalently, the corresponding left half-line eigenvalue becomes a right half-line eigenvalue and vice versa*) *and accordingly, strictly increasing half-line eigenvalues become strictly decreasing half-line eigenvalues and vice versa. In particular, using the appropriate local coordinate $(z-E_{2k})^{1/2}$* (*resp., $(z-E_{2k-1})^{1/2}$*) *near E_{2k}* (*resp., E_{2k-1}*), *one verifies that $\mu_k(x)$ does not pause at the endpoints E_{2k} and E_{2k-1}.*

Next, we turn to the inverse spectral problem and determine the isospectral manifold of real-valued, smooth, and bounded CH solutions.

Our basic assumptions then will be the following:

Hypothesis 5.7. *Suppose*

$$E_0 < E_1 < \cdots < E_{2n} < E_{2n+1} = 0, \tag{5.56}$$

fix $x_0 \in \mathbb{R}$, and assume that the initial data

$$\{\hat{\mu}_j(x_0) = (\mu_j(x_0), -\mu_j(x_0)G_n(\mu_j(x_0), x_0))\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{5.57}$$

for the Dubrovin equations (3.31) *are constrained by*

$$\mu_j(x_0) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots, n. \tag{5.58}$$

Theorem 5.8. *Assume Hypothesis* 5.7. *Then the Dubrovin initial value problem* (3.31), (5.57), (5.58) *has a unique solution $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ satisfying*

$$\hat{\mu}_j \in C^\infty(\mathbb{R}, \mathcal{K}_n), \quad j = 1, \dots, n, \tag{5.59}$$

and the projections μ_j remain trapped in the intervals $[E_{2j-1}, E_{2j}]$, $j = 1, \dots, n$, for all $x \in \mathbb{R}$,

$$\mu_j(x) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots, n, \ x \in \mathbb{R}. \tag{5.60}$$

Moreover, u defined by the trace formula (3.34), *that is,*

$$u(x) = \frac{1}{2}\sum_{j=1}^{n} \mu_j(x) - \frac{1}{4}\sum_{m=0}^{2n+1} E_m, \quad x \in \mathbb{R}, \tag{5.61}$$

satisfies Hypothesis 5.1, *that is,*

$$u \in C^\infty(\mathbb{R}), \quad u \text{ is real-valued}, \tag{5.62}$$

$$\partial_x^k u \in L^\infty(\mathbb{R}), \ k \in \mathbb{N}_0, \tag{5.63}$$

$$4u - u_{xx} > 0, \quad x \in \mathbb{R}, \tag{5.64}$$

and the n*th stationary CH equation*

$$\text{s-CH}_n(u) = 0 \text{ on } \mathbb{R}, \tag{5.65}$$

with integration constants c_ℓ *in* (5.65) *given by* $c_\ell = c_\ell(\underline{E})$, $\ell = 1, \dots, n$, *according to* (2.26), (2.27).

Proof. Given initial data constrained by $\mu_j(x_0) \in (E_{2j-1}, E_{2j})$, $j = 1, \dots, n$, one concludes from the Dubrovin equations (3.31) and the sign properties of $R_{2n+2}^{1/2}$ on the intervals $[E_{2k-1}, E_{2k}]$, $k = 1, \dots, n$, described in (5.14), that the solution $\mu_j(x)$ remains in the interval $[E_{2j-1}, E_{2j}]$ as long as $\hat{\mu}_j(x)$ stays away from the branch points $(E_{2j-1}, 0), (E_{2j}, 0)$. In case $\hat{\mu}_j$ hits such a branch point, one can use the local chart around $(E_m, 0)$, with local coordinate $\zeta = \sigma(z - E_m)^{1/2}$, $\sigma \in \{1, -1\}$, $m \in \{2j-1, 2j\}$, to verify (5.59) and (5.60). Relations (5.61)–(5.63) are then evident from (5.59), (5.60), and

$$|\partial_x^k \mu_j(x)| \leq C_k, \quad k \in \mathbb{N}_0,\ j = 1, \dots, n,\ x \in \mathbb{R}. \tag{5.66}$$

In the course of the proof of Theorem 3.5 presented in [12] (cf. also [13, Sect. 5.3]), one constructs the polynomial formalism (F_n, G_n, H_n, R_{2n+2}, etc.) and then obtains identity (3.35) as an elementary consequence. The latter immediately proves (5.64). Finally, (5.65) follows from Theorem 3.5 (with $\Omega_\mu = \mathbb{R}$). □

Corollary 5.9. *Fix* $\{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{R}$ *and assume the ordering* (5.56). *Then the isospectral manifold of smooth bounded real-valued solutions* $u \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *of* $\text{s-CH}_n(u) = 0$ *is given by a real* n*-dimensional torus* $\mathbb{T}^n$.

Proof. The discussion in Remark 5.6 and Theorem 5.8, shows that the motion of each $\hat{\mu}_j(x)$ on $\mathcal{K}_n$ proceeds topologically on a circle and is uniquely determined by the initial data $\hat{\mu}_k(x_0)$, $k = 1, \dots, n$. More precisely, the initial data

$$\begin{aligned} &\hat{\mu}_j(x_0) = (\mu_j(x_0), y(\hat{\mu}_j(x_0))) = (\mu_j(x_0), -\mu_j(x_0) G_n(\mu_j(x_0), x_0)), \\ &\mu_j(x_0) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots, n \end{aligned} \tag{5.67}$$

are topologically equivalent to data of the type

$$(\mu_j(x_0), \sigma_j(x_0)) \in [E_{2j-1}, E_{2j}] \times \{+, -\}, \quad j = 1, \dots, n, \tag{5.68}$$

the sign of $\sigma_j(x_0)$ depending on $\hat{\mu}_j(x_0) \in \Pi_\pm$. If some of the $\mu_k(x_0) \in \{E_{2k-1}, E_{2k}\}$, then the determination of the sheet $\Pi_\pm$ and hence the sign $\sigma_k(x_0)$ in (5.68) becomes superfluous and is eliminated from (5.68). Indeed, since by (2.16),

$$\mu_j(x_0)^2 G_n(\mu_j(x_0), x_0) = R_{2n+2}(\mu_j(x_0)), \tag{5.69}$$

$G_n(\mu_j(x_0), x_0)$ is determined up to a sign unless $\mu_j(x_0)$ hits a spectral gap endpoint E_{2j-1}, E_{2j} in which case $G_n(\mu_j(x_0), x_0) = R_{2n+2}(\mu_j(x_0)) = 0$ and the sign ambiguity disappears. The n data in (5.68) (properly interpreted if $\mu_j(x_0) \in \{E_{2j-1}, E_{2j}\}$) can be identified with circles. Since the latter are independent of each other, the isospectral manifold of real-valued, smooth, and bounded solutions of $\text{s-CH}_n(u) = 0$ is given by a real n-dimensional torus $\mathbb{T}^n$. □

Remark 5.10.
(*i*) *For simplicity we only focused on the case* $4u - u_{xx} > 0$. *The opposite case* $4u - u_{xx} < 0$ *is completely analogous and results in a reflection of* E_m, $m = 0, \dots, 2n+1$, *and* $\mu_j(x), \nu_j(x)$, $j = 1, \dots, n$, *about* $z = 0$, *etc.*
(*ii*) *The time-dependent case also offers nothing new. Higher-order* CH_r *flows drive*

each $\hat{\mu}_j(x,t_r)$ around the same circles as in the stationary case in complete analogy to the familiar KdV case.

In summary, one observes that the reality problem for smooth bounded solutions of the CH hierarchy, assuming the ordering (5.56) (resp., the one obtained upon reflection with respect to $z=0$), parallels that of the KdV hierarchy with the basic self-adjoint Lax operator (the one-dimensional Schrödinger operator) replaced by the self-adjoint Hamiltonian system (5.6).

The following result was found in response to a query of Igor Krichever, who inquired about the significance of the eigenvalue ordering (5.56) (or the one obtained upon reflection at $z=0$). As it turns out, such an ordering is crucial if one is interested in smooth algebro-geometric solutions on $\mathbb{R}$.

Theorem 5.11. *Suppose*

$$\begin{aligned}&E_m < E_{m+1}, \quad m = 0, \dots 2n,\\ &\text{and } E_{2j_0-1} = 0 \text{ (resp., } E_{2j_0} = 0) \text{ for some } j_0 \in \{1,\dots,n\}.\end{aligned} \tag{5.70}$$

Fix $x_0 \in \mathbb{R}$, and assume that the initial data

$$\{\hat{\mu}_j(x_0) = (\mu_j(x_0), -\mu_j(x_0)G_n(\mu_j(x_0),x_0))\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{5.71}$$

for the Dubrovin equations (3.31) *are constrained by*

$$\begin{aligned}&\mu_j(x_0) \in [E_{2j-1}, E_{2j}], \quad j \in \{1,\dots,n\}\backslash\{j_0\},\\ &\text{and } \mu_{j_0}(x_0) \in (E_{2j_0-1}, E_{2j_0}] \text{ (resp., } \mu_{j_0}(x_0) \in [E_{2j_0-1}, E_{2j_0})).\end{aligned} \tag{5.72}$$

Then there exists a set $\Omega_\mu \subset \mathbb{R}$ of the type

$$\Omega_\mu = \mathbb{R}\backslash\{\xi_k\}_{k\in\mathbb{Z}}, \quad \xi_k < \xi_{k+1},\ k \in \mathbb{Z}, \quad \lim_{k\downarrow-\infty} \xi_k = -\infty,\ \lim_{k\uparrow\infty} \xi_k = \infty, \tag{5.73}$$

such that the Dubrovin initial value problem (3.31), (5.57), (5.58) *has a unique solution $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1,\dots,n, \tag{5.74}$$

and the projections μ_j remain trapped in the intervals $[E_{2j-1}, E_{2j}]$, $j = 1,\dots,n$, for all $x \in \Omega_\mu$,

$$\mu_j(x) \in [E_{2j-1}, E_{2j}], \quad j = 1,\dots,n,\ x \in \Omega_\mu. \tag{5.75}$$

Moreover, u defined by the trace formula (3.34)*, that is,*

$$u(x) = \frac{1}{2}\sum_{j=1}^n \mu_j(x) - \frac{1}{4}\sum_{m=0}^{2n+1} E_m, \quad x \in \Omega_\mu, \tag{5.76}$$

satisfies

$$u \in C^\infty(\Omega_\mu), \quad u \text{ is real-valued,} \tag{5.77}$$

$$4u - u_{xx} > 0, \quad x \in \Omega_\mu, \tag{5.78}$$

and the nth stationary CH equation

$$\text{s-CH}_n(u) = 0 \text{ on } \Omega_\mu, \tag{5.79}$$

with integration constants c_ℓ in (5.65) *given by $c_\ell = c_\ell(\underline{E})$, $\ell = 1,\dots,n$, according to* (2.26), (2.27)*. At each ξ_k, u_x exhibits a singularity of the type*

$$u_x(x) \underset{x\to\xi_k}{=} C_k(x-\xi_k)^{-1/3} + o\big((x-\xi_k)^{-1/3}\big), \quad C_k \neq 0,\ k \in \mathbb{Z}. \tag{5.80}$$

In particular, $u \notin C^1(\mathbb{R})$, $u_x \notin L^\infty(\mathbb{R})$. *The isospectral manifold corresponding to* (5.70)–(5.72) *is then given by* $\mathbb{T}^{n-1} \times \mathbb{R}$.

Proof. One can follow the proof of Theorem 5.8 and Corollary 5.9 with one important twist, though, since the right-hand side of the Dubrovin equation of μ_{j_0} blows up as $\mu_{j_0} \to E_{2j_0-1}$ (resp., E_{2j_0}). For notational convenience and without loss of generality we may assume $E_1 = 0$ (and hence $j_0 = 1$) in the following. Recalling the Dubrovin equations (3.31), one verifies that its solutions are smooth with respect to x as long as μ_1 stays away from $E_1 = 0$. Varying $x \in \mathbb{R}$, the sign restrictions on $\mu_{1,x}$ in terms of the right-hand side of the corresponding equation in (3.31) eventually accelerate μ_1 into $E_1 = 0$ as x tends to some $\xi_k \in \mathbb{R}$, and we now analyze what happens to all μ_j for x in a neighborhood of ξ_k. Recalling the local coordinate $\sigma z^{1/2}$, $\sigma = \pm 1$, near $E_1 = 0$ and hence introducing

$$\zeta_1(x) = \sigma(-\mu_1(x))^{1/2} \text{ for } \mu_1(x) \text{ sufficiently close to } E_1 = 0 \text{ as } x \to \xi_{k'}, \tag{5.81}$$

and the corresponding point $\hat{\mu}_1 = \widehat{-\zeta_1^2} \in \mathcal{K}_n$, the Dubrovin equation for μ_j becomes for x near ξ_k,

$$\begin{aligned} \zeta_{1,x}(x) &= \frac{y(\widehat{-\zeta_1(x)^2})}{\zeta_1(x)^3} \prod_{\ell=2}^n (-\zeta_1(x)^2 - \mu_\ell(x))^{-1}, \\ &\underset{x \to \xi_{k'}}{=} C_1 \zeta_1(x)^{-2}(1 + o(1)), \end{aligned} \tag{5.82}$$

$$\mu_{j,x}(x) = 2\frac{y(\hat{\mu}_j(x))}{\mu_j(x)} \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\mu_j(x) - \mu_\ell(x))^{-1}, \quad j = 2, \dots, n. \tag{5.83}$$

for some constant $C_1 \neq 0$. (Here we implicitly assume that no other μ_j, $j = 2, \dots, n$ simultaneously hits E_{2j-1} or E_{2j} as $x \to \xi_k$. Otherwise one simply resorts to the proper local coordinate for such a μ_j. We omit the details.) To treat the singularity of $\zeta_{1,x}$ as $\zeta_1(x) \to 0$ for $x \to \xi_k$, we now resort to a well-known trick described, for instance, in [15, Theorem 3.2.2] in the context of scalar first-order differential equations. Instead of looking for solutions ζ_1, μ_j as functions of x, we now look for $x = x(\zeta_1)$, $\tilde{\mu}_j = \tilde{\mu}_j(\zeta_1)$ as functions of ζ_1, where we denote $\tilde{\mu}_j(\zeta_1) = \mu_j(x)$, $j = 2, \dots, n$. Then (5.82), (5.83) turn into

$$x'(\zeta_1) = \frac{\zeta_1^3}{y(\widehat{-\zeta_1^2})} \prod_{\ell=2}^n (-\zeta_1^2 - \tilde{\mu}_\ell(\zeta_1)), \tag{5.84}$$

$$\tilde{\mu}_{j,\zeta_1}(x) = 2\frac{y(\hat{\tilde{\mu}}_j(\zeta_1))}{y(\widehat{-\zeta_1^2})} \frac{\zeta_1^3}{\tilde{\mu}_j(\zeta_1)} \prod_{\ell=2}^n (-\zeta_1^2 - \tilde{\mu}_\ell(\zeta_1)) \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\tilde{\mu}_j(\zeta_1) - \tilde{\mu}_\ell(\zeta_1))^{-1}, \quad j = 2, \dots, n. \tag{5.85}$$

Since the right-hand sides in (5.84), (5.85) are holomorphic with respect to the n variables $\zeta_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n$ for ζ_1 near zero and $\tilde{\mu}_j$ near $[E_{2j-1}, E_{2j}]$, $j = 2, \dots, n$, equations (5.84) and (5.85) yield solutions $x, \tilde{\mu}_2, \dots, \tilde{\mu}_n$ holomorphic with respect to ζ_1 near $\zeta_1 = 0$. In particular, since

$$x'(\zeta_1) \underset{\zeta_1 \to 0}{=} \zeta_1^2/C_1 + O(\zeta_1^3), \quad \mu_j'(\zeta_1) \underset{\zeta_1 \to 0}{=} C_j\zeta_1^2 + O(\zeta_1^3), \ j = 2, \dots, n, \tag{5.86}$$

for some constants $C_j \neq 0$, $j = 1, \dots, n$, one obtains

$$x(\zeta_1) \underset{\zeta_1 \to 0}{=} \xi_k + \zeta_1^3/(3C_1) + O(\zeta_1^4), \tag{5.87}$$

$$\tilde{\mu}_j(\zeta_1) \underset{\zeta_1 \to 0}{=} \tilde{\mu}_j(0) + C_j \zeta_1^3/3 + O(\zeta_1^4) = \mu_j(\xi_k) + C_j \zeta_1^3/3 + O(\zeta_1^4). \tag{5.88}$$

Thus, inverting $x(\zeta_1)$, one observes

$$\zeta_1(x) \underset{x \to \xi_k}{=} [3C_1(x - \xi_k)]^{1/3} + O\big((x - \xi_k)^{2/3}\big), \tag{5.89}$$

$$\mu_j(x) \underset{x \to \xi_k}{=} \mu_j(\xi_k) + C_1 C_j (x - \xi_k) + O\big((x - \xi_k)^{4/3}\big), \quad j = 2, \dots, n, \tag{5.90}$$

and hence,

$$\begin{aligned} \mu_{1,x}(x) &= -2\zeta_1(x)\zeta_{1,x}(x) \\ &\underset{x \to \xi_k}{=} -(2/3)(3C_1)^{2/3}(x - \xi_k)^{-1/3} + o\big((x - \xi_k)^{-1/3}\big) \end{aligned} \tag{5.91}$$

and

$$\begin{aligned} u_x(x) &= (1/2) \sum_{j=1}^n \mu_{j,x}(x) \\ &\underset{x \to \xi_k}{=} -(1/3)(3C_1)^{2/3}(x - \xi_k)^{-1/3} + o\big((x - \xi_k)^{-1/3}\big). \end{aligned} \tag{5.92}$$

The singular behavior (5.91), (5.92) repeats itself after each revolution of μ_1 around its circle and occurs whenever μ_1 passes again through $E_1 = 0$, giving rise to the exceptional set $\{\xi_k\}_{k \in \mathbb{Z}}$ in (5.73). Hence, $\mu_j \in C^1(\mathbb{R})$, $j = 2, \dots, n$, while $\mu_{1,x} = -\zeta_1(x)^2$ blows up whenever x approaches an element of $\{\xi_k\}_{k \in \mathbb{Z}}$. The rest of the discussion follows as in Theorem 5.8 and Corollary 5.9. Since $\mu_1(x_0) = E_1 = 0$ is not an admissible initial condition in (5.72), one point must be removed from the circle associated to μ_1 which topologically results in $\mathbb{R}$ instead of S^1 and hence in the noncompact isospectral manifold $\mathbb{T}^{n-1} \times \mathbb{R}$. □

Thus, smooth algebro-geometric CH solutions require $E_0 = 0$ or $E_{2n+1} = 0$.

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