EXPLICIT FINITE VOLUME CRITERIA FOR LOCALIZATION IN CONTINUOUS RANDOM MEDIA AND APPLICATIONS

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Abstract. We give finite volume criteria for localization of quantum or classical waves in continuous random media. We provide explicit conditions, depending on the parameters of the model, for starting the bootstrap multiscale analysis. A simple application yields localization for Anderson Hamiltonians on the continuum at the bottom of the spectrum in an interval of size $\mathcal{O}(\lambda)$ for large λ, where λ stands for the disorder parameter. A more sophisticated application proves localization for two-dimensional random Schrödinger operators in a constant magnetic field (random Landau Hamiltonians) up to a distance $\mathcal{O}(\frac{\log B}{B})$ from the Landau levels, where B is the strength of the magnetic field.

1. Introduction

We give finite volume criteria for localization of waves in random media. The emphasis is on providing explicit conditions, depending on the various parameters of the model, for starting the bootstrap multiscale analysis [GK1]. These criteria thus yield Anderson localization, strong dynamical localization, SULE, etc. (See also [GK3] for a discussion of the consequences of the bootstrap multiscale analysis.)

In the lattice finite volume criteria for localization were provided by Aizenman et al [ASFH]. Although we were motivated by continuum models, our criteria are also valid on the lattice; in particular, they can be used to satisfy the criteria in [ASFH].

The explicit finite volume criteria given in this article yield localization in situations where the crucial quantities of the model that enter the multiscale analysis (the constant in Wegner's estimate and the constant in the Simon-Lieb inequality) depend on the parameters of the model (e.g., the disorder parameter, the energy where localization is to be proven, the strength of the magnetic field).

To illustrate the need for such explicit criteria, let us consider the simplest Anderson Hamiltonian in the continuum,

$$H_{\lambda,\omega} = -\Delta + \lambda V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x), \tag{1.1}$$

where $\lambda > 0$ is the disorder parameter and the random potential V_ω is of the form

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^d} \omega_i u(x-i)\,, \tag{1.2}$$

with $u(x) \geq 0$ a bounded measurable function with compact support such that $0 < U_- \leq \sum_{i\in\mathbb{Z}^d} u(x-i)$, and $\omega = \{\omega_i;\ i \in \mathbb{Z}^d\}$ a family of independent, identically distributed random variables taking values in the interval $[0,1]$, whose common probability distribution μ has a bounded density $g > 0$ a.e. in $[0,1]$. This model

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was studied by Combes and Hislop [CH1] and Kirsch [Ki], who proved that for any fixed energy $E_1 > 0$ we have Anderson localization in the interval $[0, E_1]$ for sufficiently large disorder. (Note that 0 is the bottom of the spectrum.)

But suppose we ask a different question: If $[0, E_\lambda]$ is the largest interval at the bottom of the spectrum in which $H_{\lambda,\omega}$ is localized, how does E_λ grow with λ? To provide a lower bound on E_λ we need explicit criteria, because both the constant Q_E in Wegner's estimate and the constant γ_E in the Simon-Lieb inequality increase as the energy E increases, so as we increase E the initial length scale for the multiscale analysis also increases. Thus if E is increasing with λ we cannot fix a length scale and satisfy the initial condition for the multiscale analysis at the λ dependent energy E by taking the disorder large enough; the initial length scale also grows with λ and hence is a moving target. Nevertheless, by using our explicit criterion we will show that there is a constant C such that we have $E_\lambda \geq C\lambda$ for large λ. In addition, we show that eigenfunctions with eigenvalues in the interval $[0, E_\lambda]$ decay exponentially with a rate $\geq C'\sqrt{\lambda}$ for some constant C'. To our knowledge these are the first results of this kind in the large disorder regime..

In [GK4] we extend this result by having the random variables ω_i take values in the interval $[-1, 1]$, with a bounded density $\rho > 0$ a.e. in $[-1, 1]$. Now the bottom of the spectrum of H_λ is at about $-C\lambda$ for some constant C, and the γ_E grow like $\sqrt{\lambda}$ for fixed E. It follows that even to prove localization in an interval of fixed length at the bottom of the spectrum we need explicit criteria, since the initial length scale now grows with λ. Our explicit criteria can be used to prove Anderson localization in an interval of size $\mathcal{O}(\lambda)$ for large λ at the bottom of the spectrum. The result is still true if $u \in L^p$ with $p > \frac{d}{2}$ instead of bounded.

For a more sophisticated application of our explicit criteria we revisit the two-dimensional random Schrödinger operator in a constant magnetic field (random Landau Hamiltonian). This model was studied by Combes and Hislop [CH2] and Wang [W1], who obtained Wegner estimates and probability estimates on the decay of local resolvents using percolation arguments. (See [DMP1, DMP2] for a related model.) However the effect of the dependency of the initial length scale for the multiscale analysis on the strength B of the magnetic field and on the distance to the Landau level was overlooked. It is fair to say that the multiscale analyses available at the time did not provide the tools to handle this dependency. Combining the Wegner estimate and the probability estimate on the decay of local resolvents obtained in [CH2] with our explicit criterion, we give a complete proof of Anderson localization up to a distance $\mathcal{O}(\frac{\log B}{B})$ from the Landau levels for large B. It tuns out that the initial length scale for this problem is a "Goldilock" scale: it cannot be too big or too small, it must be just right for the strength of the magnetic field.

We also obtain estimates on the rate of exponential decay of eigenfunctions of random Landau Hamiltonians: for eigenvalues at a distance $\mathcal{O}(\frac{\log B}{B})$ from a Landau level the rate of decay is $\geq \mathcal{O}\left(\left(\frac{\log B}{B}\right)^\nu\right)$, where the exponent $\nu > 0$ comes from the probability distribution of the random variables and two-dimensional percolation, typically $\nu > 1$; at a fixed distance of a Landau level, i.e., $\mathcal{O}(1)$, the rate of decay is $\geq \mathcal{O}\left(\sqrt{B}\right)$.

The issue of the dependency of the initial length scale for the multiscale analysis in terms of the parameters of the model also appears when investigating localization at the bottom of the spectrum for the Anderson model at low disorder, as in the work of Wang [W2] and Klopp [Klo2], since the constant in the Wegner estimate blows up as the disorder goes to zero. Our explicit criteria may be used to handle this dependency and perform a multiscale analysis.

Localization for continuous random operators has been usually proven by a multiscale analysis. (But note that the Aizenman-Molchanov method has just been extended to the continuum [AENSS].) In this context the multiscale analysis is a technique, initially developed by Fröhlich and Spencer [FS] and Fröhlich, Martinelli, Spencer and Scoppolla [FMSS], and simplified by von Dreifus [vD] and von Dreifus and Klein [vDK], for the purpose of proving Anderson localization. i.e., pure point spectrum and exponential decay of eigenfunctions. (See also [HM, CKM, Sp, KLS, vDK2, K, CH1, Klo1, CH2, FK1, FK2, W1, BCH, KSS1, KSS2, CHT, FLM, W2, St, KK2, U].) It was later shown to also yield dynamical localization (non spreading of the wave packets) [GDB, DS, GK1].

Recently, the authors developed a more involved procedure, built out of four different multiscale analyses, called a bootstrap multiscale analysis [GK1]. It yields Anderson localization, SULE-like estimates, sub-exponential decay of the kernel of the evolution operator, and strong Hilbert-Schmidt dynamical localization. A new feature of this bootstrap multiscale analysis is that, unlike the formerly standard multiscale analysis (e.g. [FS, FMSS, vD, vDK, CH1, St]), the strength of the conclusions is not affected by the rate of decay in the starting condition.

We give three explicit finite volume criteria for starting the bootstrap multiscale analysis. Theorem 2.4 gives a criterion for starting the bootstrap multiscale analysis at a "a priori" specified initial length scale; we may thus play with the parameters of the model to satisfy the starting condition at the specified length scale. Theorem 2.5 provides an explicit expression for the minimum initial length scale required to start the bootstrap multiscale analysis with a "a priori" specified decay; it typically requires large scales, but not *large enough* scales: how large the initial length scale has to be is made precise by the criterion. Theorem 2.6 is an analog of Theorem 2.5 but for the second multiscale analysis of the bootstrap scheme described in [GK1]; it also gives an estimate of the exponential rate of decay of the eigenfunctions in terms of the constants of the problems. These criteria reduce the requirements of the multiscale analysis to explicit and transparent conditions. (The criterion of Theorem 2.4 is specially simple.)

This article is organized as follows: Our explicit criteria for localization are stated in Section 2; their proofs are given in Section 5 (Theorems 2.4 and 2.5), Section 6 (Theorem 2.7, a general criterion that gives Theorems 2.4 and 2.5 as special cases), and Section 7 (Theorem 2.6). Section 3 is dedicated to localization at the bottom of the spectrum of the Anderson model (in the continuum) at large disorder; the main result is Theorem 3.1. Section 4 contains our discussion of localization for random Landau Hamiltonians, the detailed description of our results is given in Theorem 4.1.

2. Explicit criteria for localization

Our random medium is modeled by a $\mathbb{Z}^d$-ergodic random self-adjoint operator H_ω on $\mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x; \mathbb{C}^n)$, where ω belongs to a probability set Ω with a probability measure $\mathbb{P}$ and expectation $\mathbb{E}$. Our results apply to random Schrödinger operators (e.g., [HM, CH1, Klo1, KSS1, GK3, AENSS]), random magnetic Schrödinger operators (e.g., [CH2, W1]), and classical wave operators (e.g., [FK1, FK2, CHT, KK1, KK2]).

We recall that it follows from the ergodicity that there exists a nonrandom set Σ such that $\sigma(H_\omega) = \Sigma$ with probability one, and that the decomposition of $\sigma(H_\omega)$ into pure point spectrum, absolutely continuous spectrum, and singular continuous spectrum is also independent of the choice of ω with probability one.

As in [GK1], we take as assumptions the properties of the random operator H_ω that are required for the multiscale analysis. These properties are routinely verified for the operators of interest, except for the Wegner estimate which usually requires more hypotheses on the parameters of the random operator.

Throughout this paper we use the sup norm in $\mathbb{R}^d$:

$$|x| = |x|_\infty = \max\{|x_i|,\ i = 1, \dots, d\}.$$

By $\Lambda_L(x)$ we denote the open box (or cube) of side $L > 0$:

$$\Lambda_L(x) = \left\{y \in \mathbb{R}^d;\ \ |y - x| < \tfrac{L}{2}\right\}, \tag{2.1}$$

and by $\overline{\Lambda}_L(x)$ the closed box. The characteristic function of a set $\Lambda \subset \mathbb{R}^d$ is denoted by χ_Λ; we set

$$\chi_{x,L} = \chi_{\Lambda_L(x)}\,, \quad \text{with} \ \ \chi_x = \chi_{x,1}\,. \tag{2.2}$$

In this article we will take boxes centered at sites $x \in \mathbb{Z}^d$ with side $L \in 2\mathbb{N}$. For such a box we set

$$\Upsilon_L(x) = \left\{y \in \mathbb{Z}^d;\ |y - x| = \tfrac{L}{2} - 1\right\}, \tag{2.3}$$

and define its boundary belt by

$$\tilde{\Upsilon}_L(x) = \overline{\Lambda}_{L-1}(x) \backslash \Lambda_{L-3}(x) = \bigcup_{y \in \Upsilon_L(x)} \overline{\Lambda}_1(y)\ ; \tag{2.4}$$

it has the characteristic function

$$\Gamma_{x,L} = \chi_{\tilde{\Upsilon}_L(x)} = \sum_{y \in \Upsilon_L(x)} \chi_y \ \ a.e. \tag{2.5}$$

Note that $|\Upsilon_L(x)| \le d(L-1)^{d-1}$. We will also write

$$[K]_{6\mathbb{N}} = \max\{L \in 6\mathbb{N};\ L \le K\}. \tag{2.6}$$

We need a notion of *finite volume operator*, i.e. a "restriction" $H_{\omega,x,L}$ of H_ω where the "randomness based outside the box $\Lambda_L(x)$" is not taken into account. Usually $H_{\omega,x,L}$ is defined as the restriction of H_ω, either to the open box $\Lambda_L(x)$ with Dirichlet boundary condition, or to the closed box $\overline{\Lambda}_L(x)$ with periodic boundary condition. The operator $H_{\omega,x,L}$ then acts on $\mathrm{L}^2(\Lambda_L(x), \mathrm{d}x; \mathbb{C}^n)$. But $H_{\omega,x,L}$ may also be defined as acting on the whole space, by throwing away the random coefficients "based outside the box $\Lambda_L(x)$"; this is usually used for random magnetic Schrödinger operators [CH2, W1]. In all cases the finite volume operators have either compact resolvent or are perturbations of the free finite volume operators by relatively compact operators.

We assume that an appropriate choice of finite volume operators $H_{\omega,x,L}$ has been made. (The multiscale analysis is not sensitive to such a choice as long as the required properties can be proven.) We write $R_{\omega,x,L} = (H_{\omega,x,L} - z)^{-1}$ for the resolvent of $H_{\omega,x,L}$.

We assume that the random operator H_ω satisfy the requirements for the multiscale analysis as in [GK1], i.e., it satisfies Assumptions SLI (Simon-Lieb inequality), EDI (eigenfunction decay inequality), IAD (independence at a distance), NE (average number of eigenvalues), W (Wegner estimate), and SGEE (strong generalized eigenfunction expansion) in a given open interval $\mathcal{I}$.

We will now state Assumptions SLI, IAD, NE and W, since the constants in these assumptions will be used explicitly in this article. We refer to [GK1] for Assumptions EDI and SGEE and for a discussion of all these assumptions.

In what follows $\mathcal{I}$ will be a fixed open interval.

Assumption SLI. *There exists a constant $\gamma_{\mathcal{I}} \geq 1$, such that given $L, \ell', \ell'' \in 2\mathbb{N}$, $x, y, y' \in \mathbb{Z}^d$ with $\Lambda_{\ell''}(y) \subset \Lambda_{\ell'-3}(y')$ and $\Lambda_{\ell'}(y') \subset \Lambda_{L-3}(x)$, then for a.e ω, if $E \in \mathcal{I}$ with $E \notin \sigma(H_{\omega,x,L}) \cup \sigma(H_{\omega,y',\ell'})$, we have*

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{y,\ell''}\| \leq \gamma_{\mathcal{I}} \|\Gamma_{y',\ell'} R_{\omega,y',\ell'}(E) \chi_{y,\ell''}\| \, \|\Gamma_{x,L} R_{\omega,x,L}(E) \Gamma_{y',\ell'}\| \, . \tag{2.7}$$

We say that an event is based on the box $\Lambda_L(x)$ if it is determined by conditions on the finite volume operator $H_{\omega,x,L}$. Given $\varrho \geq 0$, we say that two boxes $\Lambda_L(x)$ and $\Lambda_{L'}(x')$ are *ϱ-nonoverlapping* if $|x - x'| > \frac{L+L'}{2} + \varrho$ (i.e., if $\operatorname{dist}(\Lambda_L(x), \Lambda_{L'}(x')) > \varrho$).

Assumption IAD. *There exists $\varrho \geq 0$ such that events based on ϱ-nonoverlapping boxes are independent.*

Assumption NE. *There exists a constant $C_{\mathcal{I}}$ such that*

$$\mathbb{E}\left(\operatorname{tr} E_{H_{\omega,x,L}}(\mathcal{I})\right) \leq C_{\mathcal{I}} L^d \tag{2.8}$$

for all $x \in \mathbb{Z}^d$ and $L \in 2\mathbb{N}$.

Assumption W. *There exist constants $b \geq 1$, $0 < \eta_{\mathcal{I}} \leq 1$, and $Q_{\mathcal{I}}$, such that*

$$\mathbb{P}\{\operatorname{dist}(\sigma(H_{\omega,x,L}), E) \leq \eta\} \leq Q_{\mathcal{I}} \eta L^{bd} \ , \tag{2.9}$$

for all $E \in \mathcal{I}$, $0 < \eta \leq \eta_{\mathcal{I}}$, $x \in \mathbb{Z}^d$, and $L \in 2\mathbb{N}$.

In practice we have either $b = 1$ or $b = 2$ in the Wegner estimate (2.9). Recently the correct volume dependency (i.e., $b = 1$) was obtained for certain operators [CHN, CHKN, HK], at the price of losing a bit in the η dependency. In this paper, we shall use (2.9) as stated, the modifications in our methods required for the other forms of (2.9) being obvious. (For a discussion of possible modifications see [GK1, Remark 2.4].) Note also that usually $\eta_{\mathcal{I}} = 1$, but not always, as in the case of random Landau Hamiltonians (see Section 4).

We will look for localization by studying the decay of the finite volume resolvent from the center of a box $\Lambda_L(x)$ to its boundary as measured by

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,L/3}\| \, . \tag{2.10}$$

We use the convention that $\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,L/3}\| = \infty$ if $E \in \sigma(H_{\omega,x,L})$.

We start with two deterministic (i.e., for a given ω, which is omitted from the notation) definitions.

Definition 2.1. *Given $\theta > 0$, $E \in \mathbb{R}$, $x \in \mathbb{Z}^d$, and $L \in 6\mathbb{N}$, we say that the box $\Lambda_L(x)$ is (θ, E)-suitable if $E \notin \sigma(H_{x,L})$ and*

$$\|\Gamma_{x,L} R_{x,L}(E) \chi_{x,L/3}\| \leq \frac{1}{L^\theta} \ . \tag{2.11}$$

Definition 2.2. *Given $m > 0$, $E \in \mathbb{R}$, $x \in \mathbb{Z}^d$, and $L \in 6\mathbb{N}$, we say that the box $\Lambda_L(x)$ is (m,E)-regular if $E \notin \sigma(H_{x,L})$ and*

$$\|\Gamma_{x,L} R_{x,L}(E) \chi_{x,L/3}\| \leq \mathrm{e}^{-m\frac{L}{2}}. \tag{2.12}$$

We define the multiscale analysis region $\Sigma_{\mathrm{MSA}} \subset \Sigma$ by requiring the conclusions of the bootstrap multiscale analysis [GK1, Theorem 3.4].

Definition 2.3. *The multiscale analysis region Σ_{MSA} for the random operator H_ω is the set of $E \in \Sigma$ for which there exists some open interval $I \ni E$, such that Assumptions SLI, EDI, IAD, NE, W, and SGEE hold in I, and given any ζ, $0 < \zeta < 1$, and α, $1 < \alpha < \zeta^{-1}$, there is a length scale L_0 and a mass $m_\zeta > 0$, so if we set $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$, $k = 0, 1, \ldots$, we have*

$$\mathbb{P}\left\{R\left(m_\zeta, L_k, I, x, y\right)\right\} \geq 1 - \mathrm{e}^{-L_k^\zeta} \tag{2.13}$$

for all $k = 0, 1, \ldots$, and $x, y \in \mathbb{Z}^d$ with $|x-y| > L_k + \varrho$, where

$$\begin{aligned} &R(m, L, I, x, y) = \\ &\qquad \{\text{for every } E' \in I, \ \text{either } \Lambda_L(x) \text{ or } \Lambda_L(y) \text{ is } (m,E')\text{-regular}\}\,. \end{aligned} \tag{2.14}$$

On Σ_{MSA} we have all desired properties of localization: Anderson localization, SULE-like estimates, strong Hilbert-Schmidt dynamical localization, and more [GK1]. We will give explicit criteria for $E \in \Sigma_{\mathrm{MSA}}$.

In [GK3] we defined the strong insulator region Σ_{SI} for H_ω by

$$\Sigma_{\mathrm{SI}} = \{E \in \Sigma\,;\ H_\omega \text{ exhibits strong HS-dynamical localization at } E\}\,. \tag{2.15}$$

The operator H_ω is said to exhibit strong Hilbert-Schmidt dynamical localization at E if it exhibits strong Hilbert-Schmidt dynamical localization in some open interval around E. (We refer to [GK3] for precise definitions.) Note that Σ_{SI} is an open set, and H_ω has pure point spectrum in Σ_{SI}. We have $\Sigma_{\mathrm{MSA}} \subset \Sigma_{\mathrm{SI}}$ from [GK1, Proof of Theorem 3.8 and Corollary 3.10]. (Note that Σ_{MSA} in [GK3] is the same set as in this article, although the two definitions are somewhat different.) If in addition we have the decay estimates of [GK2, Theorem 2], uniformly for a.e. ω (true under very natural hypotheses), it is shown in [GK3] that if Assumptions SLI, EDI, IAD, NE, W, and SGEE hold in an open interval $\mathcal{I}$, then $\Sigma_{\mathrm{MSA}}^{\mathcal{I}} = \Sigma_{\mathrm{SI}}^{\mathcal{I}}$, where $B^{\mathcal{I}} = B \cap \mathcal{I}$ (see also [GK3, Theorem 4.2]), and we must have nontrivial transport in $\Sigma^{\mathcal{I}} \backslash \Sigma_{\mathrm{MSA}}^{\mathcal{I}}$.

In this article we provide three explicit finite volume criteria for localization: Theorems 2.4 and 2.5 which correspond to the first step (i.e., the first multiscale analysis [GK1, Theorem 5.1]) in the Bootstrap Multiscale Analysis, and Theorem 2.6 which corresponds to the second step ([GK1, Theorem 5.2]) in the Bootstrap Multiscale Analysis. Theorems 2.4 and 2.5 are special cases of a general explicit criterion, Theorem 2.7.

The first criterion works for a *prescribed* value of the initial length scale L_0. In Section 3 we illustrate its use by proving localization for Anderson Hamiltonians on the continuum at the bottom of the spectrum in an interval of size $\mathcal{O}(\lambda)$ for large λ, where λ stands for the disorder parameter.

Theorem 2.4. *Let H_ω be a random operator such that Assumptions SLI, EDI, IAD, NE, W, and SGEE hold in an open interval $\mathcal{I}$. Fix a length scale $L_0 \in 6\mathbb{N}$,*

$$L_0 \geq \max\left\{6, 3\varrho, \eta_{\mathcal{I}}^{-\left[\left(\frac{5}{3}+b\right)d\right]^{-1}}\right\}. \tag{2.16}$$

Let $E_0 \in \Sigma \cap \mathcal{I}$ and suppose

$$\mathbb{P}\left\{ D_{\mathcal{I}} L_0^{\left(\frac{5}{3}+b\right)d} \|\Gamma_{0,L_0} R_{\omega,0,L_0}(E_0)\chi_{0,L_0/3}\| < 1 \right\} \geq 1 - \tfrac{2}{368^d}\,, \tag{2.17}$$

with

$$D_{\mathcal{I}} = 39^{(3+b)d} \max\left\{16 \cdot 60^d Q_{\mathcal{I}}, 1\right\} \gamma_{\mathcal{I}}^2 \,. \tag{2.18}$$

Then $E_0 \in \Sigma_{MSA}$.

The following criterion is for *large* initial length scale L_0 and *weak* initial decay: any rate that is faster than the volume (if $b = 1$ in Wegner) or the volume squared (if $b = 2$) is allowed. It provides a precise estimate on how large L_0 has to be, depending on the parameters $Q_{\mathcal{I}}$ and $\gamma_{\mathcal{I}}$ of the model, and on the prescribed rate of decay of the resolvent.

Theorem 2.5. *Let H_ω be a random operator such that Assumptions SLI, EDI, IAD, NE, W, and SGEE hold in an open interval $\mathcal{I}$. Given $s > bd$, set*

$$\mathcal{L} = \max\left\{ 3\varrho, 42, 3\left(\tfrac{107^d}{2}\right)^{\frac{2}{s-bd}}, \tfrac{1}{37}\left(16 \cdot 60^d Q_{\mathcal{I}}\right)^{\frac{2}{s-bd}}, \eta_{\mathcal{I}}^{-\frac{1}{s}} \right\}. \tag{2.19}$$

Suppose that for some $L_0 \geq \mathcal{L}$, $L_0 \in 6\mathbb{N}$, and $E_0 \in \Sigma \cap \mathcal{I}$,

$$\mathbb{P}\left\{ 90^d \gamma_{\mathcal{I}}^2 (37L_0)^s \|\Gamma_{0,L_0} R_{\omega,0,L_0}(E_0)\chi_{0,L_0/3}\| < 1 \right\} \geq 1 - \tfrac{2}{344^d}\,. \tag{2.20}$$

Then $E_0 \in \Sigma_{MSA}$.

The next criterion is an analog of Theorem 2.5 but for the second multiscale analysis of the bootstrap scheme described in [GK1], i.e., for the multiscale analysis with exponential decay of the resolvent and polynomial decay of the probabilities as in von Dreifus and Klein [vDK], modified as in Figotin and Klein [FK1, Theorem 32] to allow the mass in the starting hypotheses to decrease as the initial length scale increases. Theorem 2.6 keeps track of the dependency of how large the initial length scale has to be in terms of the constants coming from Assumptions SLI, NE, and W. It will be used in Section 4 to prove localization for random Schrödinger operators in a constant magnetic field (random Landau Hamiltonians) up to a distance $\mathcal{O}(\frac{\log B}{B})$ from the Landau levels, where B is the strength of the magnetic field.

While Theorem 2.6 has stronger hypotheses than Theorem 2.5, it delivers more: an estimate of the exponential rate of decay of the eigenfunctions in terms of the constants of the problems. This is of great interest in the large disorder regime for the Anderson model as discussed in Section 3, for the random Landau Hamiltonian as in Section 4, and also in the regime of small disorder for the Anderson model as studied in [W2, Klo2].

If the probability estimate in the initial length scale is sufficiently good, one may use Theorem 2.6 to start the bootstrap multiscale analysis, instead of applying first Theorem 2.4 or Theorem 2.5 and then bootstrapping to Theorem 2.6 to obtain the exponential rate of decay of eigenfunctions. This is the case for the random Landau Hamiltonians as in Section 4 and in the weak disorder regime for the Anderson model as in [Klo2].

Theorem 2.6. *Let H_ω be a random operator such that Assumptions SLI, EDI, IAD, NE, W and SGEE hold in an open interval $\mathcal{I}$. Given $E_0 \in \Sigma \cap \mathcal{I}$, $p > 0$,*

$\alpha = 1 + \frac{p/2}{p+2d}$, $s > 2p + (b+2)d$, and $\theta > 4\alpha s$, set

$$\mathcal{L} = \max\left\{3\varrho, 417^{\frac{1}{\alpha-1}}, \eta_{\mathcal{I}}^{-\frac{1}{s}}, \operatorname{dist}(E_0, \mathbb{R}\backslash\mathcal{I})^{-\frac{1}{s}}, Q_{\mathcal{I}}^{\frac{1}{s-p-bd}},\right. \tag{2.21}$$
$$\left.\left(16 \cdot 36^d 2^{s-2p} C_{\mathcal{I}} Q_{\mathcal{I}}\right)^{\frac{1}{\alpha(s-2p-(b+2)d)}}, \left(69^d \gamma_{\mathcal{I}}^2\right)^{\frac{4}{\theta-4\alpha s}}, \left(3^d \gamma_{\mathcal{I}}\right)^{\frac{16\alpha}{\theta(\alpha-1)}}\right\}.$$

If for some $L_0 \geq \mathcal{L}$, $L_0 \in 6\mathbb{N}$, we have

$$\mathbb{P}\{\Lambda_{L_0}(0) \text{ is } (\theta, E_0)\text{-suitable}\} > 1 - \frac{1}{L_0^p}, \tag{2.22}$$

then there exists an open interval $I = I(\theta, s, L_0)$, with $E_0 \in I \subset \mathcal{I}$, such that if we set $m_0 = 2\theta \frac{\log L_0}{L_0}$, and $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$, $k = 0, 1, \dots,$ we have

$$\mathbb{P}\{\Lambda_{L_k}(0) \text{ is } (\tfrac{m_0}{4}, E)\text{-regular}\} \geq 1 - \frac{2}{L_k^p} \quad \text{for all } E \in I, \tag{2.23}$$

and

$$\mathbb{P}\left[R\left(\tfrac{m_0}{4}, L_k, I, x, y\right)\right] \geq 1 - \frac{4}{L_k^{2p}} \text{ for } x, y \in \mathbb{Z}^d, |x-y| > L_k + \varrho, \tag{2.24}$$

for all $k = 0, 1, \dots,$ where

$$R(m, L, I, x, y) = \tag{2.25}$$
$$\{\text{for every } E \in I, \text{ either } \Lambda_L(x) \text{ or } \Lambda_L(y) \text{ is } (m, E)\text{-regular}\}.$$

Moreover,

- **(i):** *$I \cap \Sigma \subset \Sigma_{MSA}$.*
- **(ii):** *For almost every ω, an eigenfunction $\varphi_{\omega,E}$ of H_ω with eigenvalue $E \in I$ decays exponentially (in the L^2-sense) with a rate $\geq \frac{\theta}{2}\frac{\log L_0}{L_0}$, i.e.,*

$$\liminf_{|x|\to\infty} -\frac{\log \|\chi_x \varphi_{\omega,E}\|}{|x|} \geq \frac{\theta \log L_0}{2L_0}. \tag{2.26}$$

Theorems 2.5 and 2.4 are corollaries of a general criterion we will now state. We set

$$\begin{aligned} \beta(Y,S) &= \frac{(3Y-4)^d((3Y-4)^d - 4^d)\cdots((3Y-4)^d - (S-1)4^d)}{S!} \\ &< \frac{(3Y-4)^{Sd}}{S!}. \end{aligned} \tag{2.27}$$

Theorem 2.7. *Let H_ω be a random operator such that Assumptions SLI, EDI, IAD, NE, W, and SGEE hold in an open interval $\mathcal{I}$. Pick $p > 0$, $S \geq 2$, and $S \in \mathbb{N}$. Take $s > p + bd$, $Y \geq 8S+5$, Y odd, and $L_0 \in 6\mathbb{N}$, $L_0 \geq \max\left\{6, 3\varrho, \eta_{\mathcal{I}}^{-\frac{1}{s}}\right\}$, such that*

$$Y - \frac{2}{\log L_0}\log Y \geq 8S + 3, \tag{2.28}$$

and

$$(YL_0)^{s-p-bd} \geq 2^{d+2} Q_{\mathcal{I}} \frac{S(7S+2)^{bd}}{(3Y)^{(b-1)d}}. \tag{2.29}$$

In addition assume that

$$\text{either} \qquad L_0 \geq (2\beta(Y, S+1)Y^p)^{\frac{1}{(S+1)p}}\,, \tag{2.30}$$

$$\text{or} \qquad p > d,\ S \geq \frac{p+d}{p-d}. \tag{2.31}$$

Define $\theta_0 > s$ *by*

$$L_0^{\theta_0} = \gamma_{\mathcal{I}}^2 3^d (7S+2)^d (YL_0)^s. \tag{2.32}$$

Then, if for some $E_0 \in \Sigma \cap \mathcal{I}$ *and* $\theta > \theta_0$ *,*

$$\mathbb{P}\{\Lambda_{L_0}(0) \text{ is } (\theta, E_0)\text{-suitable}\} > 1 - (2\beta(Y, S+1))^{-\frac{1}{S}}\,, \tag{2.33}$$

where $\beta(Y, S+1)$ *is defined in* (2.27)*, then setting* $L_{k+1} = YL_k$*,* $k = 0, 1, 2, \ldots$*, we have*

$$\mathbb{P}\{\Lambda_{L_k}(0) \text{ is } (\theta, E_0)\text{-suitable}\} \geq 1 - \frac{1}{L_k^p}\,, \tag{2.34}$$

for all $k \geq \mathcal{K}$*, where* $\mathcal{K} = \mathcal{K}(d, p, Y, S) < \infty$*. In particular* $E_0 \in \Sigma_{MSA}$*.*

Theorems 2.4 and 2.5 are derived from Theorem 2.7 in Section 5. Theorem 2.7 is proven in Section 6, and Theorem 2.6 is proven in Section 7.

Remark 2.8. *The initial probabilities* $p_0 = \frac{2}{368^d}$ *in Theorem 2.4 and* $p_0 = \frac{2}{344^d}$ *in Theorem 2.5 are mostly indicative. We are applying Theorem 2.7 with* $S = 4, Y = 39$ *and* $S = 4, Y = 37$*, respectively, which are not optimal in terms of probabilities. Indeed, from Theorem 2.7 one gets that* $p_0 = (2\beta(Y, S+1))^{-1/S}$ *(with* $\beta(Y, S)$ *defined in* (2.27)*), with* $Y = 8S + 5$ *as the best choice. By computing* p_0 *for different values of* S *(*$S = 2, 3, \cdots$*), one can check (by numerical computation) that the optimal probabilities are:*

- $d = 1$*;* $p_0 = (24\mathrm{e})^{-1} \approx 0.015$ *for* $S = +\infty$*,* $p_0 \approx 0.013$ *for* $S = 13$*.*
- $d = 2$*;* $p_0 \approx 2.9 \cdot 10^{-5}$ *for* $S = 7$*,*
- $d = 3$*;* $p_0 \approx 7.9 \cdot 10^{-8}$ *for* $S = 6$*.*

These probabilities are far from being optimal, but one can compare them to $p_0 = \frac{1}{841^d}$ *given in* [GK1, Theorem 3.4].

3. Localization at large disorder

In this section we focus our attention on the simplest continuous Anderson model,

$$H_{\lambda,\omega} = -\Delta + \lambda V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x), \tag{3.1}$$

where $\lambda > 0$ is the disorder parameter and the random potential V_ω is of the form

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} \omega_i u(x - i)\,, \tag{3.2}$$

with $u(x) \geq 0$ a bounded measurable function with compact support, say $\operatorname{supp} u \subset \Lambda_\varrho(0)$, such that $0 < U_- \leq \sum_{i \in \mathbb{Z}^d} u(x-i)$, and $\omega = \{\omega_i;\ i \in \mathbb{Z}^d\}$ a family of independent, identically distributed random variables taking values in the interval $[0, 1]$, whose common probability distribution μ has a bounded density g with $g > 0$ a.e. in $[0, a)$ for some $a > 0$. Note we have $\Sigma_\lambda = [0, \infty)$, where Σ_λ denotes the almost-sure spectrum of $H_{\lambda,\omega}$ [KM, Theorem 3].

The finite volume operators $H_{\lambda,\omega,x,L}$ are taken as the restriction of $H_{\lambda,\omega}$, either to the open box $\Lambda_L(x)$ with Dirichlet boundary condition, or to the closed box $\overline{\Lambda}_L(x)$ with periodic boundary condition.

For each $\lambda > 0$, $H_{\lambda,\omega}$ is known to be a random operator such that Assumptions SLI, EDI, IAD, NE, W, and SGEE hold in any open interval (e.g., [CH1, Ki, St, GK3]). In particular, [GK3, Theorem A.1] proves Asumption SLI with the constant $\gamma_{\mathcal{I}}$ in (2.7) given by

$$\gamma_{\lambda,\mathcal{I}} = \sup_{E\in\mathcal{I}} \gamma_{\lambda,E} \ , \quad \text{with} \quad \gamma_{\lambda,E} = 6\sqrt{2d}\,\sqrt{\max\{E,0\}+50d}\ . \tag{3.3}$$

The Wegner estimate (2.9) can be derived as in [Ki, Proposition 1] or [FK1, Theorem 2.3] with $b=2$, $\eta_{\mathcal{I}}=1$, and

$$Q_{\lambda,\mathcal{I}} = \sup_{E\in\mathcal{I}} Q_{\lambda,E} \ , \quad \text{with} \quad Q_{\lambda,E} = \tfrac{C_{d,U_-}}{\lambda}\|g\|_\infty \left(\max\{E+1,0\}\right)^{\frac{d}{2}} , \tag{3.4}$$

where C_{d,U_-} is a constant depending only on d and U_-.

Assumption (NE) is satisfied with a constant

$$C_{\lambda,\mathcal{I}} = \sup_{E\in\mathcal{I}} C_{\lambda,E} \ , \quad \text{with} \quad C_{\lambda,E} = C_{d,U_-}\left(\max\{E+1,0\}\right)^{\frac{d}{2}} , \tag{3.5}$$

Note that this random operator satisfies the conditions of [GK3], so we have $\Sigma_{\mathrm{MSA}} = \Sigma_{\mathrm{SI}}$.

Our result is

Theorem 3.1. *Let $H_{\lambda,\omega}$ be the Anderson Hamiltonian as in* (3.1), *and set*

$$L_0 = \min\{L \in 6\mathbb{N};\ L \geq \max\{3\varrho, 3(2+\sqrt{d})\}\}\,. \tag{3.6}$$

Then there exists λ^*, depending only on d, U_-, $\|g\|_\infty$, and ϱ , such that for any $\lambda \geq \lambda^*$ we have

$$[0, c\lambda] \subset \Sigma_{MSA} \ \ \textit{with}\ \ c = \tfrac{U_-}{(368L_0)^d\|g\|_\infty}\ . \tag{3.7}$$

Moreover there exists $c' > 0$, depending on d, U_-, $\|g\|_\infty$, and ϱ, but not on $\lambda \geq \lambda^*$, such that if $\lambda \geq \lambda^*$, then for a.e. ω, if $\varphi_{\lambda,\omega,E}$ is an eigenfunction of $H_{\lambda,\omega}$ with eigenvalue $E \in [0,c\lambda]$, then

$$\liminf_{|x|\to\infty} -\frac{\log\|\chi_x\varphi_{\omega,E}\|}{|x|} \geq c'\sqrt{\lambda}. \tag{3.8}$$

Proof. For any $\delta > 0$ and $L_0 \in 2\mathbb{N}$ we have

$$\begin{aligned}\mathbb{P}\{\omega_i > \delta \text{ for all } i \in \Lambda_{L_0}(0)\} &= 1 - \mathbb{P}\{\omega_i \in [0,\delta] \text{ for some } i \in \Lambda_{L_0}(0)\}\\ &\geq 1 - \delta\|g\|_\infty L_0^d. \end{aligned}\tag{3.9}$$

We fix δ and L_0, and set

$$E_\lambda = \tfrac{1}{2}\delta U_-\lambda\,. \tag{3.10}$$

It follows that

$$\mathbb{P}\{A_\lambda\} \geq 1 - \delta\|g\|_\infty L_0^d\,, \tag{3.11}$$

where A_λ denotes the event

$$A_{\lambda,L} = \{\inf\sigma(H_{\lambda,\omega,L_0}) \geq 2E_\lambda\}\,. \tag{3.12}$$

We now use Theorem 2.4. To maximize δ, and hence E_λ, we need to minimize L_0; hence we pick L_0 as in (3.6) and choose δ by matching the right hand sides of (3.11) and (2.17), i.e.,

$$\delta\|g\|_\infty L_0^d = \tfrac{2}{368^d}\,. \tag{3.13}$$

If $\omega \in A_\lambda$, it follows that

$$\operatorname{dist}(E, \sigma(H_{\lambda,\omega,L})) \geq E_\lambda \quad \text{fo any} \quad E \in [0, E_\lambda]\,, \tag{3.14}$$

so we can use the Combes-Thomas estimate to get the decay of the resolvent for any $E \in [0, E_\gamma]$. Note that the exact dependency of the exponential decay rate in the Combes-Thomas estimate in terms of the energy parameter and the distance to the spectrum is crucial in our argument, since we deal with large energies and large distances from the spectrum. Such a precise estimate is provided in [GK2, Eq. (19) in Theorem 1]. Although these estimates are for operators on $\mathbb{R}^d$, the results in [GK2] can adapted for finite volume. In a box $\Lambda_{L_0}(0)$, with either Dirichlet or periodic boundary condition, the same estimates as in [GK2, Eq. (19) in Theorem 1] hold for $x, y \in \bar{\Lambda}_{L_0-2}(0)$ (i.e., at a distance ≥ 1 from the boundary) but the exponential rates of decay get divided by $1 + 2\sqrt{d}L_0$. This can be seen as follows: in the proof of [GK2, Theorem 1] we replace $\mathrm{e}^{\pm \alpha\cdot x}$ by $\mathrm{e}^{\pm(\alpha\cdot x)\varphi(x)}$, where $\varphi(x)$ is a smooth function such that $0 \leq \varphi(x) \leq 1$, $\varphi(x) = 1$ for $x \in \bar{\Lambda}_{L_0-2}(0)$, $\operatorname{supp}\varphi \subset \Lambda_{L_0}(0)$, and $|\nabla\varphi(x)| \leq 3\sqrt{d}$. It follows that $|\nabla((\alpha\cdot x)\varphi(x))| \leq |\alpha|(1 + 3\sqrt{d}\frac{L}{2}) \leq |\alpha|(1 + 2\sqrt{d}L)$ for all $x \in \Lambda_{L_0}(0)$. The proof then proceeds as in [GK2, Proof of Theorem 1] using this estimate. (We can do better if we specify periodic boundary condition and use the distance on the torus, see[FK1, Lemma 18] and [KK1, Theorem 3.6].)

Thus, using the adaptation of [GK2, Eq. (19) in Theorem 1] to finite boxes with $\gamma = \frac{\sqrt{2}}{2}$, we conclude that, if $\omega \in A_\lambda$ and $E \in [0, E_\lambda]$, we have that for any $x, y \in \mathbb{Z}^d \cap \Lambda_{L_0}(0) = \mathbb{Z}^d \cap \bar{\Lambda}_{L_0-2}(0)$,

$$\begin{aligned} \|\chi_x R_{\lambda,\omega,L_0}(E)\chi_y\| &\leq \tfrac{2}{2E_\lambda - E}\, \mathrm{e}^{-\sqrt{2(2E_\lambda-E)}\left(1+2\sqrt{d}L_0\right)^{-1}\left(|x-y|-\sqrt{d}\right)} \\ &\leq \tfrac{2}{E_\lambda}\, \mathrm{e}^{-\sqrt{2E_\lambda}\left(1+2\sqrt{d}L_0\right)^{-1}\left(|x-y|-\sqrt{d}\right)}\,. \end{aligned} \tag{3.15}$$

Summing over the support of Γ_{L_0} and of $\chi_{L_0/3}$, noting that if $x \in \Lambda_{L_0/3}(0)$, $y \in \Upsilon_{L_0}(0)$ we have $|x - y| \geq \frac{L_0}{3} - 1$, and using (3.6) yields

$$\|\Gamma_{L_0} R_{\lambda,\omega,L_0}(E)\chi_{L_0/3}\| \leq \tfrac{2d}{E_\lambda} L_0^{2d-1} \mathrm{e}^{-\sqrt{2E_\lambda}\left(1+2\sqrt{d}L_0\right)^{-1}}\,. \tag{3.16}$$

We will show that (2.17) of Theorem 2.4 is satisfied for all $E \in [0, E_\lambda]$ for large disorder. Let $\mathcal{I}_\lambda = (-\infty, 2E_\lambda - 1)$, we have, using (2.18), (3.3), (3.4), (3.10), and (3.16),

$$D_{\mathcal{I}_\lambda} L_0^{\left(\frac{5}{3}+b\right)d} \|\Gamma_{L_0} R_{\lambda,\omega,L_0}(E)\chi_{L_0/3}\| \tag{3.17}$$

$$= \; 39^{5d} \max\left\{16 \cdot 60^d Q_{\mathcal{I}_\lambda}, 1\right\} \gamma_{\mathcal{I}_\lambda}^2 L_0^{\frac{11}{3}d} \|\Gamma_{L_0} R_{\lambda,\omega,,L_0}(E)\chi_{L_0/3}\|$$

$$\leq \; c_1 \lambda^{\max\left\{\frac{d}{2}-1,0\right\}} \mathrm{e}^{-c_2\sqrt{\lambda}} \tag{3.18}$$

$$< \; 1\,, \tag{3.19}$$

where (3.18) holds for $\omega \in A_\lambda$ with c_1 and c_2 constants depending only on d, U_-, $\|g\|_\infty$, and ϱ (note that we *fixed* L_0 and δ in (3.6) and (3.13)). We conclude that there exists λ^*, depending only on d, U_-, $\|g\|_\infty$, and ϱ, such that we have (3.19) for all $\lambda > \lambda^*$ and $\omega \in A_\lambda$.

Thus condition (2.17) holds for $\lambda > \lambda^*$ and $E \in [0, E_\lambda]$ by (3.11), hence Theorem 2.4 implies that $[0, \frac{1}{2}\delta U_-\lambda] \subset \Sigma_{\mathrm{MSA}}$.

Let us turn to the rate of the exponential decay of the eigenfunctions with energies in $[0, \frac{1}{2}\delta U_- \lambda]$. Let θ_λ be defined by

$$L_0^{-\theta_\lambda} = \tfrac{2d}{E_\lambda} L_0^{2d-1} \mathrm{e}^{-\sqrt{2E_\lambda}\left(1+2\sqrt{d}L_0\right)^{-1}} . \tag{3.20}$$

Then (3.16) says $\Lambda_{L_0}(0)$ is (θ_λ, E)-suitable for any $E \in [0, E_\lambda]$. Note that from (3.17) and (2.18) (see (5.4)) one checks that $\theta_\lambda > \theta_{0,\lambda}$, where $\theta_{0,\lambda}$ is as in (2.32). Moreover, it follows from (3.20) that

$$\theta_\lambda \geq c_0 \sqrt{\lambda} \quad \text{for all} \quad \lambda \geq \lambda^* , \tag{3.21}$$

where c_0 is some constant and λ^* is taken large enough (both depending only on d, U_-, $\|g\|_\infty$, and ϱ).

As in [GK1, Proof of Theorem 3.4], we bootstrap from Theorem 2.7 (of which Theorem 2.4 is a special case) to Theorem 2.6. Note that we have the conclusions of Theorem 2.7 with $p = \frac{5}{3}$, $S = 4$, and $Y = 39$, i.e., if we set $L_k = 39^k L_0$, $k = 0, 1, 2, \ldots$, where L_0 is as in (3.6), then for all $\lambda > \lambda^*$ we have

$$\mathbb{P}\{\Lambda_{L_k}(0) \text{ is } (\theta_\lambda, E)\text{-suitable}\} \geq 1 - \frac{1}{L_k^{\frac{5}{3}}} , \tag{3.22}$$

for all $E \in [0, E_\lambda]$ and $k \geq \mathcal{K}$, where $\mathcal{K} = \mathcal{K}(d, \varrho) < \infty$ is a constant depending only on d and ϱ. A key fact is that $\mathcal{K}$ does not depend on λ.

We now want to feed (3.22) into the hypotheses of Theorem 2.6. We have already fixed $p = \frac{5}{3}$, and hence α is fixed. On the other hand for each λ we have $\theta = \theta_\lambda$ as in (3.20), and $Q_{\mathcal{I}_\lambda}$, $C_{\mathcal{I}_\lambda}$, and $\gamma_{\mathcal{I}_\lambda}$ grow polynomially with λ. To control $\mathcal{L}_\lambda$ as given in (2.21) with all this dependence in λ, we pick the remaining parameter, s, to also depend on λ by $s_\lambda = \log \lambda$. By taking λ^* sufficiently large, as before depending only on d, U_-, $\|g\|_\infty$, and ϱ, and using (3.21), we can also guarantee $\theta_\lambda > 4\alpha s_\lambda$ and $s_\lambda > 2p + (b+2)d$ for all $\lambda > \lambda^*$. It follows from the explicit form of (2.21) that

$$\mathcal{L}_\infty = \sup_{\lambda > \lambda^*} \sup_{E \in [0, E_\lambda]} \mathcal{L}_\lambda(E) < \infty , \tag{3.23}$$

where $\mathcal{L}_\lambda(E)$ is given by (2.21) with $E_0 = E$, $\mathcal{I} = \mathcal{I}_\lambda$, $p = \frac{5}{3}$, $\theta = \theta_\lambda$, and $s = s_\lambda$. Note that $\mathcal{L}_\infty$ depends only on d, U_-, $\|g\|_\infty$, and ϱ.

We now fix κ to be that smallest $k \geq \mathcal{K}$ such that $L_k = 39^k L_0 \geq \mathcal{L}_\infty$. It follows from Theorem 2.6 that if $\lambda > \lambda^*$, then for almost every ω, the eigenfunctions $\varphi_{\lambda,\omega,E}$ with energy $E \in [0, E_\lambda]$ decay exponentially (in the L^2-sense) with

$$\liminf_{|x| \to \infty} -\frac{\log \|\chi_x \varphi_{\lambda,\omega,E}\|}{|x|} \geq \frac{\theta_\lambda \log L_\kappa}{2 L_\kappa} . \tag{3.24}$$

so (3.8) follows from (3.21). ☐

4. Application to Landau Hamiltonians with random potentials

In this section we study a two-dimensional random Schrödinger operator in the presence of a constant transverse magnetic field, the Landau Hamiltonian, with a random potential:

$$H_{B,\omega} = H_B + V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x), \tag{4.1}$$

with

$$H_B = (i\nabla + A)^2 \quad , \quad A = \frac{B}{2}(x_2, -x_1) \tag{4.2}$$

where A is the vector potential, $B > 0$ is the strength of the magnetic field, and the random potential V_ω is of the form

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^2} \omega_i \, u(x - i)\,, \tag{4.3}$$

where, following Combes and Hislop [CH2], we require conditions (R1) and (R2):

(R1): $0 \leq u(x) \in C^2$, with $u(0) > 0$ and $\operatorname{supp} u \subset B\,(0, r_u)$, $0 < r_u < \frac{1}{\sqrt{2}}$. (Note $B\,(0,r) = \left\{x \in \mathbb{R}^2;\ |x|_2 = (x_1^2 + x_2^2)^{\frac{1}{2}} < r\right\}$.)

(R2): $\omega = \{\omega_i;\ i \in \mathbb{Z}^d\}$ a family of independent, identically distributed random variables, whose common probability distribution μ has a bounded density g, even (i.e., $g(t) = g(-t)$), with $g > 0$ a.e. in $[-M, M]$ and $g = 0$ on $\mathbb{R}\backslash[-M, M]$ for some finite $M > 0$.

We set $\|u\|_\infty = 1$ without loss of generality. (We absorb it in M and g.) Note that V_ω is uniformly bounded:

$$|V_\omega(x)| \leq 2M \ \text{ for a.e. } \omega \text{ and all } x \in \mathbb{R}^2\,. \tag{4.4}$$

In addition, we make a further requirement in Theorem 4.1:

(R3): For some $\zeta > 0$ and some finite constant $C_1 > 0$ we have

$$\mu([0,t]) = \int_0^t g(s)\,\mathrm{d}s \geq C_1 \, \min\{t, M\}^\zeta \quad \text{for all} \quad t > 0. \tag{4.5}$$

The hypothesis (R3) is new; it allows the use of nontrivial results from percolation theory, as in (4.24) and (4.27). It is equivalent to $\liminf_{t\to 0} \frac{\mu([0,t])}{t^\zeta} > 0$. We set $\nu = \nu_0\zeta > 0$, $C_3 = C_2 C_1^{\nu_0}$, where $\nu_0 \geq 1$ and C_2 are constants given by two-dimensional percolation theory (see (4.24)). The exponent ν plays an important role in our analysis as, e.g., in (4.9). Note that if the density g is continuous at 0 with $g(0) > 0$, then we may take $\zeta = 1$ and hence $\nu = \nu_0 \geq 1$.

$H_{B,\omega}$ is a $\mathbb{Z}^d$-ergodic random self-adjoint operator on $\mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x)$ [CH2]; Σ_B will denote its almost sure spectrum. Recall that the spectrum of the free Landau Hamiltonian H_B consists of a sequence of infinitely degenerate eigenvalues, the Landau levels:

$$B_n = (2n+1)B\,, \quad n = 0, 1, 2, \ldots\,. \tag{4.6}$$

It follows that

$$\Sigma_B \subset \bigcup_{n=0}^{\infty} \mathcal{B}_n\,, \quad \text{with} \quad \mathcal{B}_n = [B_n - 2M, B_n + 2M]\,. \tag{4.7}$$

We will assume $B > 2M$ so the bands $\mathcal{B}_n$ are disjoint. We recall that the size of a possible spectral gap in $\Sigma_B \cap \mathcal{B}_n$ is at most $\mathcal{O}(B^{-\frac{1}{2}})$ for large B (depending on n) [CH2, Theorem 7.1]. (Note that if $u \in C^\infty$, then the size of possible spectral gaps is at most $\mathcal{O}(B^{-\infty})$, see [W1, p. 3].) This is enough to ensure that the result of Theorem 4.1 below is not empty.

Theorem 4.1. *Let $H_{B,\omega}$ be the random Landau Hamiltonian (as in (4.1)), satisfying (R1), (R2), and (R3). For any $n = 0, 1, 2, \ldots$ there exists a finite constant $\beta_n > 0$, depending only on n, M, r_u, $\|\nabla u\|_\infty$, and $\|\Delta u\|_\infty$, such that, given $0 < \widetilde{M} < M$, there exists a finite positive constant $\mathbf{B}(n)$, depending only on n, M, r_u, $\|\nabla u\|_\infty$, $\|\Delta u\|_\infty$, $\|g\|_\infty$, C_3, ν, and $\widetilde{M}$, such that for all $B \geq \mathbf{B}(n)$ we have*

$$\Sigma_{B,n} = \Sigma_B \cap \left\{ E \in \mathcal{B}_n,\ |E - B_n| \geq 2K_n \tfrac{\log B}{B} \right\} \subset \Sigma_{\mathrm{MSA}}\,, \tag{4.8}$$

with $K_n = \frac{(43+20\nu)}{\beta_n}$, and for a.e. ω, if $\varphi_{B,\omega,E}$ is an eigenfunction of $H_{B,\omega}$ with eigenvalue $E \in \Sigma_{B,n}$, then

$$\begin{aligned} &\liminf_{x\to\infty} -\frac{\log \|\chi_x \varphi_{B,\omega,E}\|}{|x|} \\ &\geq \begin{cases} \frac{\tau_n |E-B_n|^{1+\nu} B}{|\log|E-B_n||} \geq \frac{\tau_n' (\log B)^\nu}{B^\nu} & \text{if } 2K_n \frac{\log B}{B} \leq |E - B_n| \leq \frac{2}{\sqrt{B}} \\ \frac{\tau_n |E-B_n|^{\nu} \sqrt{B}}{|\log|E-B_n||} \geq \frac{\tau_n' B^{\frac{1-\nu}{2}}}{\log B} & \text{if } \frac{2}{\sqrt{B}} \leq |E - B_n| \leq 2\widetilde{M} \\ \tau_n \sqrt{B} & \text{if } 2\widetilde{M} \leq |E - B_n| \leq 2M \end{cases}, \end{aligned} \tag{4.9}$$

where $\tau_n, \tau_n' > 0$ are constants depending only on M, $\widetilde{M}$, β_n, C_3, and ν.

Theorem 4.1 is proven by using the results of [CH2] to satisfy the hypotheses of Theorem 2.6. Note that we could also work with the hypotheses used by Wang [W1], in which case we obtain the results of Theorem 4.1 for energies at a fixed distance from a Landau level, i.e., $2\widetilde{M} \leq |E - B_n| \leq 2M$ for a given $0 < \widetilde{M} < M$, including a rate of exponential decay for the eigenfunctions $\propto \sqrt{B}$.

Following [CH2], we introduce the lattice

$$\Gamma = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) + \mathrm{e}^{i\frac{\pi}{4}} \sqrt{2}\, \mathbb{Z}^2\,, \tag{4.10}$$

which is just the lattice $\mathbb{Z}^2$ with the origin shifted to $\left(\frac{1}{2}, \frac{1}{2}\right)$, rotated by 45°, and rescaled by $\sqrt{2}$. We will denote sites in Γ by $\breve{x}, \breve{y}, \ldots$. Each site in $\mathbb{Z}^2$ is the middle point of a bond in Γ. We will work with open boxes in the lattice Γ:

$$\breve{\Lambda}_L(\breve{x}) = \left\{ y \in \mathbb{R}^d;\ \ |y - \breve{x}|_1 < L \right\}\,, \tag{4.11}$$

where $\breve{x} \in \Gamma$, $L \in 4\mathbb{N}$, and $|x|_1 = |x_1| + |x_2|$, so the the boundary of $\breve{\Lambda}_L(\breve{x})$ is made out of bonds in Γ. Note that $\frac{1}{2}\left|x - \left(\frac{1}{2}, \frac{1}{2}\right)\right|_1$ is just the sup norm in the coordinates given by axes parallel to the bonds of Γ, centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$, and rescaled so of bonds in Γ have length one. (That's why we have $|y - \breve{x}|_1 < L$ and not $< \frac{L}{2}$ in (4.11).) We also adapt (2.2)-(2.5) to the lattice Γ:

$$\chi_{\breve{x},L} = \chi_{\breve{\Lambda}_L(\breve{x})}\,, \ \text{ with } \ \chi_{\breve{x}} = \chi_{\breve{x},1}\,, \tag{4.12}$$

$$\Upsilon_L(\breve{x}) = \{\breve{y} \in \Gamma; |\breve{y} - \breve{x}|_1 = L - 2\}\,, \tag{4.13}$$

$$\tilde{\Upsilon}_L(\breve{x}) = \overline{\breve{\Lambda}}_{L-1}(\breve{x}) \backslash \breve{\Lambda}_{L-2}(\breve{x}) = \bigcup_{\breve{y} \in \Upsilon_L(\breve{x})} \overline{\breve{\Lambda}}_1(\breve{y})\,, \tag{4.14}$$

$$\Gamma_{\breve{x},L} = \chi_{\tilde{\Upsilon}_L(\breve{x})} = \sum_{\breve{y} \in \Upsilon_L(\breve{x})} \chi_{\breve{y}} \ \ a.e. \tag{4.15}$$

Note that all of our results are valid with this choice of boxes with no change in the constants in our theorems. (The boxes and the corresponding boundary belts are just the usual ones (e.g., (2.1)-(2.5)) in the rescaled lattice Γ.)

We define (as in [CH2]) the finite volume operators by

$$H_{B,\omega,\breve{x},L} = H_B + V_{\omega,\breve{x},L}\,, \tag{4.16}$$

with $\breve{x} \in \Gamma$, $L \in 4\mathbb{N}$, and

$$V_{\omega,\breve{x},L} = \sum_{i\in\mathbb{Z}^2\cap\breve{\Lambda}_L(\breve{x})} \omega_i\, u(x-i) = \sum_{i\in\mathbb{Z}^2,\ \operatorname{supp} u(x-i)\subset\breve{\Lambda}_L(\breve{x})} \omega_i\, u(x-i)\,. \tag{4.17}$$

Note that $V_{\omega,\breve{x},L}$ has support in $\breve{\Lambda}_L(\breve{x})$ and is relatively compact with respect to H_B, so the essential spectrum of the finite volume operators consists of the Landau levels given in (4.6). Note that (4.7) also holds for these finite volume operators.

By the definition of $V_{\omega,\breve{x},L}$ we clearly have Assumption IAD with $\varrho = 0$. Assumption SLI inequality can be proven as in [GK3, Theorem A.1] for any open interval $\mathcal{I}$, with the constant $\gamma_{\mathcal{I}}$ in (2.7) given by

$$\gamma_{\mathcal{I}} = \sup_{E\in\mathcal{I}} \gamma_E\,, \quad \text{with} \quad \gamma_E = \tilde{\gamma}\sqrt{\max\{E,0\}+1}\,, \tag{4.18}$$

where the constant $\tilde{\gamma}$ depends only on M; it does not depend on B. (Although, as noted in [CH2], we have

$$\chi_{\breve{y},\ell} V_{\omega,\breve{x},L} - V_{\omega,\breve{y},\ell} \neq 0 \quad if \quad \breve{\Lambda}_\ell(\breve{y}) \subset \breve{\Lambda}_L(\breve{x})\,, \tag{4.19}$$

we also have

$$\chi_{\breve{y},\ell-1}(V_{\omega,\breve{x},L} - V_{\omega,\breve{y},\ell}) = 0 \quad if \quad \breve{\Lambda}_\ell(\breve{y}) \subset \breve{\Lambda}_L(\breve{x})\,, \tag{4.20}$$

so the proof of Assumption SLI is not affected.) Assumption EDI also follows in the same way.

Assumption SGEE follows from [Si, Theorem B.13.2] and properties of Schrödinger operators, as in [CH2, Section 5].

Combes and Hislop [CH2, Theorem 3.1] proved the following form of Assumptions NE and W (note that $b = 1$):

Theorem 4.2 ([CH2]). *Let $H_{B,\omega}$ be the random Landau Hamiltonian (as in (4.1)), satisfying (R1) and (R2). For any $n = 0,1,2,\ldots$, there exists a constant $Q_n = \tilde{Q}_n\|g\|_\infty$, with the constant $\tilde{Q}_n$ depending only on r_u and M, such that for any closed interval $I \subset \mathcal{B}_n\backslash\{B_n\}$ we have*

$$\mathbb{E}\left\{E_{H_{B,\omega,\breve{x},L}}(I)\right\} \leq \tfrac{1}{2} Q_n \tfrac{B}{\{\operatorname{dist}(I,B_n)\}^2}\, |I|\, L^2\,, \tag{4.21}$$

for all $B > 2M$, $\breve{x} \in \Gamma$, and $L \in 2\mathbb{N}$.

In particular, for any $E \in \mathcal{B}_n\backslash\{B_n\}$ we have

$$\mathbb{P}\{\operatorname{dist}(\sigma(H_{B,\omega,\breve{x},L}),E) \leq \eta\} \leq Q_n \frac{B}{(|E-B_n|-\eta)^2}\,\eta\, L^2\,, \tag{4.22}$$

for all $0 < \eta < |E-B_n|$, $B > 2M$, $\breve{x} \in \Gamma$, and $L \in 2\mathbb{N}$.

In addition, Combes and Hislop [CH2, Proposition 5.1] obtained length scale estimates. Their argument relies on independent bond percolation on the lattice Γ.

We will need some notions of independent bond percolation in $\mathbb{Z}^2$ (see [Gr]). We denote by p the probability of a bond being occupied. The connectivity function

$\tau_p(x)$ is given by the probability that 0 is connected to $x \in \mathbb{Z}^2$ by occupied bonds. We have [Gr, Proposition 5.47)]

$$\tau_p(x) \leq \mathrm{e}^{-\varphi(p)|x|}\,, \quad \text{with} \quad \varphi(p) = -\tfrac{1}{n}\log\tau_p((n,0)) \geq 0\,. \tag{4.23}$$

The function $\varphi(p)$ is continuous and nonincreasing on $[0,1]$, with $\varphi(\frac{1}{2}) = 0$ and $\varphi(0) = \infty$ [Gr, Theorem 5.14], where we used that in two dimensions the critical probability $p_c = \frac{1}{2}$ [Gr, Theorem 9.11]. Moreover, it follows from [Gr, Eqs. (5.51), (9.95), (9.97)] that there are finite constants $C_2 > 0$ and $\nu_0 \geq 1$ such that

$$\varphi(p) \geq C_2\left(\tfrac{1}{2} - p\right)^{\nu_0} \quad if \quad p < \tfrac{1}{2}\,. \tag{4.24}$$

To make the connection to random Landau Hamiltonians satisfying (R1) and (R2), given $a \geq 0$ we let

$$p(a) = 1 - \int_{-\infty}^{a} g(t)\,\mathrm{d}t = \tfrac{1}{2} - \textstyle\int_0^a g(t)\,\mathrm{d}t\,, \tag{4.25}$$

and set

$$m(a) = \varphi(p(a))\,. \tag{4.26}$$

(Note that $p(a) = 0$ if $a \geq M$.) If in addition we have hypothesis (R3), it follows from (4.5) and (4.24) that

$$m(a) \geq C_3\,\min\{a, M\}^{\nu} \text{ for } a \geq 0\,,\ with\ \nu = \nu_0\zeta > 0\,,\ C_3 = C_2 C_1^{\nu_0}. \tag{4.27}$$

(In fact $m(a) = \infty$ if $a \geq M$.) This lower bound on $m(a)$ will play a crucial role in the proof of Theorem 4.1 as we let $a = \frac{|E-B_n|}{2} \to 0$ as $B \to \infty$.

We may now restate [CH2, Proposition 5.1] as follows:

Theorem 4.3 ([CH2]). *Let $H_{B,\omega}$ be the random Landau Hamiltonian (as in (4.1)), satisfying (R1) and (R2). Let $E = B_n \pm 2a$ for some $n = 0, 1, 2, \ldots$, with $0 < a < \frac{B}{2}$. There exist a geometric constant $C_4 > 0$, and constants $Y_n > 0$ and $\beta_n > 0$, depending only on n, M, r_u,$\|\nabla u\|_\infty$, and $\|\Delta u\|_\infty$, such that for any $0 < \delta \leq a$, $\breve{x} \in \Gamma$, and $L \in 6\mathbb{N}$,*

$$\mathbb{P}\left\{\|\Gamma_{\breve{x},L} R_{B,\omega,\breve{x},L}(E)\chi_{\breve{x},L/3}\| \leq Y_n \frac{B}{a\delta^2}\,\mathrm{e}^{-\beta_n \min\left\{aB, \sqrt{B}\right\}}\right\} \tag{4.28}$$
$$\geq 1 - C_4 L \mathrm{e}^{-m(a)L} - Q_n \frac{\delta B}{a^2} L^2\,,$$

where the constant Q_n is as in Theorem 4.2.

We are now ready to prove Theorem 4.1. As in [CH2], Theorem 4.3 will be used to obtain the initial length estimate for the multiscale analysis. The decay given in (4.28) is independent of the scale, so the rate of exponential decay given at the length scale L_0 is $\approx \frac{2\beta_n \min\left\{aB, \sqrt{B}\right\}}{L_0}$. Thus L_0 cannot be too big, depending on the strength of B through the constants of the model and on the distance a to the Landau level. On the other hand the multiscale analysis requires the initial length scale L_0 to be large enough, depending also on the strength of B through the constants of the model and on the distance a to the Landau level. For these reasons Theorem 4.1 cannot be proven by a naive application of the multiscale analysis as stated in [FS, vDK, CH1, GK1, St]. Theorem 2.6 will take care of these conflicting demands.

Proof of Theorem 4.1. Let us fix $n = 0, 1, 2, \ldots$. For each $E \in \mathcal{B}_n \backslash \{B_n\}$ we set

$$a = a(E) = \frac{|E - B_n|}{2} \quad (\text{note } \ 0 < a \leq M)\,, \tag{4.29}$$

and define open intervals

$$\mathcal{I}_E = (E - a, E + a), \tag{4.30}$$

where we have Assumptions SLI, NE, and W with constants

$$\begin{aligned} \gamma_{\mathcal{I}_E} &= \gamma_n \sqrt{B}\,, \ \text{ with } \ \gamma_n = \tilde{\gamma}\sqrt{2(n+1)}\,; & (4.31)\\ C_{\mathcal{I}_E} &= Q_n \frac{B}{a}\,; & (4.32)\\ Q_{\mathcal{I}_E} &= 4Q_n \frac{B}{a^2}\,, \ \ \eta_{\mathcal{I}_E} = \tfrac{a}{2}, \ \ b = 1\,; & (4.33) \end{aligned}$$

where (4.31) follows from (4.18), (4.32) from (4.21), and (4.33) from (4.22).

To apply Theorem 2.6 at a given energy $E \in \mathcal{B}_n \backslash \{B_n\}$, we must pick appropriate $p > 0$, $\alpha = 1 + \frac{p/2}{p+4}$, $s > 2p + 6$, and $\theta > 4\alpha s$, and use the estimates of Theorem 4.3 to satisfy (2.22) at some length scale $L_0 \geq \mathcal{L}$, with $\mathcal{L} = \mathcal{L}(E, B)$ given by (2.21) with $\mathcal{I} = \mathcal{I}_E$.

We start by fixing $p > 0$, to be specified later, and setting $\alpha = 1 + \frac{p/2}{p+4}$; note that $1 < \alpha < \frac{3}{2}$. We will choose s and θ later depending on a, B, and p.

Given $E \in \mathcal{B}_n \backslash \{B_n\}$, to satisfy (2.22) from (4.28), it suffices to verify

$$\begin{aligned} Y_n \frac{B}{a\delta^2}\, \mathrm{e}^{-\beta_n \min\{aB, \sqrt{B}\}} &\leq \frac{1}{L_0^{\theta}}\,, & (4.34)\\ C_4 L_0\, \mathrm{e}^{-m(a) L_0} &< \frac{1}{2L_0^{p}}\,, & (4.35)\\ Q_n \frac{\delta B}{a^2} L_0^2 &\leq \frac{1}{2L_0^{p}}\,, & (4.36) \end{aligned}$$

for some $0 < \delta < a$ and an appropriate choice of the initial length scale L_0.

We start with condition (4.35). Using (4.27), it suffices to establish

$$C_4 L_0\, \mathrm{e}^{-C_3\, a^{\nu} L_0} \ < \ \frac{1}{2L_0^{p}}\,. \tag{4.37}$$

To do so, we pick $0 < \widetilde{M} < M$ and choose (keeping in mind (2.21))

$$L_0 = L_0(a) = \left[\max\left\{ \frac{C_5 \log \frac{M}{a}}{a^{\nu}}, \frac{C_5 \log \frac{M}{\widetilde{M}}}{\widetilde{M}^{\nu}} \right\} + 6 \right]_{6\mathbb{N}} \geq \frac{C_5 \log \frac{M}{a}}{a^{\nu}}\,, \tag{4.38}$$

where the constant C_5 is chosen large enough so that (4.37) holds for all $0 < a \leq \widetilde{M}$, and hence for all $0 < a \leq M$, and we also have $C_5 > \max\left\{ \frac{(p+1)(\nu+1)}{C_3}, \frac{417^{2+\frac{8}{p}} \widetilde{M}^{\nu}}{\log \frac{M}{\widetilde{M}}} \right\}$. The constant C_5 depends only on C_3, M, $\widetilde{M}$, ν, and p (C_4 is a geometric constant). By construction

$$L_0(a) \geq \frac{C_5 \log \frac{M}{\widetilde{M}}}{\widetilde{M}^{\nu}} \geq 417^{2+\frac{8}{p}} \tag{4.39}$$

for all $0 < a \leq M$.

We now fix $K_n > 0$, to be specified later, and require

$$K_n \frac{\log B}{B} \leq a \leq M\,. \tag{4.40}$$

For B large enough (so $K_n \log B \leq \sqrt{B}$), it follows from (4.40) that

$$\min\{aB, \sqrt{B}\} \geq K_n \log B\,. \tag{4.41}$$

We have

$$L_0(a) \leq \begin{cases} L_0\left(K_n \frac{\log B}{B}\right) \leq \frac{C_5}{K_n^{\nu}} (\log B)^{1-\nu} B^{\nu} & \text{if } K_n \frac{\log B}{B} \leq a \leq \frac{1}{\sqrt{B}} \\ L_0\left(\frac{1}{\sqrt{B}}\right) \leq C_5 (\log B) B^{\frac{\nu}{2}} & \text{if } \frac{1}{\sqrt{B}} \leq a \leq \widetilde{M} \\ L_0(\widetilde{M}) \leq \frac{C_5 \log \frac{M}{\widetilde{M}}}{\widetilde{M}^{\nu}} + 6 & \text{if } \widetilde{M} \leq a \leq M \end{cases} \tag{4.42}$$

for $B \geq \mathbf{B}_1(n)$, with $\mathbf{B}_1(n)$ depending only only on C_3, M, $\widetilde{M}$, ν, p, and K_n. We take $\mathbf{B}_1(n)$ sufficiently large so we also have

$$\log L_0(a) \leq 2\nu \log B \quad \text{for all } a \text{ as in (4.40)}\,. \tag{4.43}$$

To satisfy (4.36), we set

$$\delta = \delta(a,B) = \frac{a^2}{2Q_n B L_0(a)^{p+2}} < a\,, \tag{4.44}$$

the last inequality holding for all a as in (4.40) and $B \geq \mathbf{B}_2(n)$ in view of (4.39), where the constant $\mathbf{B}_2(n)$ depends only on M, $\widetilde{M}$, ν, C_3, Q_n, p, and K_n.

It remains to verify condition (4.34) for this choice of δ, which follows if

$$4Y_n Q_n^2 B^3 a^{-5}\, \mathrm{e}^{-\beta_n \min\{aB, \sqrt{B}\}} \leq \frac{1}{L_0(a)^{\theta + 2(p+2)}}\,, \tag{4.45}$$

i.e.,

$$\theta \leq \frac{\beta_n \min\{aB, \sqrt{B}\} - \log\left(4Y_n Q_n^2 B^3 a^{-5}\right)}{\log L_0(a)} - 2(p+2)\,. \tag{4.46}$$

We take

$$\theta = \theta(a,B) = \frac{\beta_n' \min\{aB, \sqrt{B}\}}{\log L_0(a)}\,, \tag{4.47}$$

where we require

$$\beta_n' = \beta_n - \frac{8 + 4\nu(p+2)}{K_n} > 0\,, \tag{4.48}$$

and, using (4.43) and (4.41), check that $\theta(a,B)$ satisfies (4.46) for all a as in (4.40), as long as and $B \geq \mathbf{B}_3(n)$, where the constant $\mathbf{B}_3(n)$ depends only on M, $\widetilde{M}$, ν, C_3, Y_n, Q_n, p, and K_n.

We now choose $s = s(a,B) > 2p+6$, satisfying $\theta(a,B) > 4\alpha s(a,B)$, and show that for sufficiently large B we have, for all a as in (4.40), that $L_0(a) \geq \mathcal{L}$, with $\mathcal{L} = \mathcal{L}(E,B)$ given by (2.21) with $\mathcal{I} = \mathcal{I}_E$.

To satisfy

$$Q_{\mathcal{I}_E}^{\frac{1}{s-p-2}} \leq L_0(a) \quad\text{and}\quad \left(16 \cdot 36^2 2^{s-2p} C_{\mathcal{I}_E} Q_{\mathcal{I}_E}\right)^{\frac{1}{\alpha(s-2p-6)}} \leq L_0(a) \tag{4.49}$$

for large B and all a as in (4.40), it suffices to require $(s-p-2)\log L_0(a) \geq 3\log B$ and $(s-2p-6)\log L_0(a) \geq 5\log B$, respectively. Thus we set

$$s = s(a,B) = \frac{5\log B}{\log L_0(a)} + 2p + 6\,, \tag{4.50}$$

and satisfy (4.49) for all a as in (4.40) if $B \geq \mathbf{B}_4(n)$, with $\mathbf{B}_4(n)$ depending only only on M, $\widetilde{M}$, C_3, ν, p, Q_n, and K_n. Moreover, since it follows from (4.43) that $s(a,B) \geq \frac{5}{2\nu} > \frac{1}{\nu}$ for $B \geq \mathbf{B}_1(n)$, we may take $\mathbf{B}_4(n)$ large enough so we also have $\eta_{\mathcal{I}_E}^{-\frac{1}{s}}, \operatorname{dist}(E, \mathbb{R}\backslash\mathcal{I}_E)^{-\frac{1}{s}} < L_0(a)$ for $B \geq \mathbf{B}_4(n)$.

In view of (4.31), $\alpha < \frac{3}{2}$, (4.47), (4.50), and (4.43), if we require

$$\beta_n' > \frac{34 + 4\nu(p+3)}{K_n} > 0\,, \tag{4.51}$$

we have

$$\left(69^2 \gamma_{\mathcal{I}_E}^2\right)^{\frac{4}{\theta - 4\alpha s}} \leq L_0(a) \tag{4.52}$$

if $B \geq \mathbf{B}_5(n)$, with $\mathbf{B}_5(n)$ depending only only on M, $\widetilde{M}$, C_3, ν, p, and K_n.

Recalling (4.39), it only remains to satisfy

$$\left(3^2 \gamma_{\mathcal{I}_E}\right)^{\frac{16\alpha}{\theta(\alpha-1)}} \leq L_0(a)\,, \tag{4.53}$$

which, again using $\alpha < \frac{3}{2}$, follows from

$$\beta_n' > \frac{24(p+4)}{pK_n}\,, \tag{4.54}$$

and $B \geq \mathbf{B}_6(n)$, with $\mathbf{B}_6(n)$ depending only only on M, $\widetilde{M}$, ν, C_3, p, and K_n.

In pursuit of simplification, we now choose $p = \frac{8}{\nu}$ and set

$$K_n = \frac{43 + 20\nu}{\beta_n}\,, \tag{4.55}$$

so conditions (4.48), (4.51), and (4.54) are satisfied. We also set $\mathbf{B}(n) = \max\{\mathbf{B}_i(n),\ i = 1, 2, \ldots, 6\}$, which depends only on n, M, $\widetilde{M}$, r_u ,$\|\nabla u\|_\infty$, $\|\Delta u\|_\infty$, $\|g\|_\infty$, C_3, and ν, and take $B \geq \mathbf{B}(n)$. For any $E \in \Sigma_{B,n}$ we can now verify the hypotheses of Theorem 2.6 with L_0 and θ as in (4.38) and (4.47), and conclude that (4.8) holds, i.e., $\Sigma_{B,n} \subset \Sigma_{\mathrm{MSA}}$. Moreover, for a.e. ω, if $\varphi_{B,\omega,E}$ is an eigenfunction of $H_{B,\omega}$ with eigenvalue $E \in \Sigma_{B,n}$, then

$$\liminf_{\breve{x}\to\infty} -\frac{\log\|\chi_{\breve{x}}\varphi_{B,\omega,E}\|}{|\breve{x}|_1} \geq \frac{\beta_n' \min\{aB, \sqrt{B}\}}{2L_0(a)} \tag{4.56}$$

$$\geq \begin{cases} \dfrac{\beta_n' a^{1+\nu} B}{2\left(C_5 \log \frac{M}{a} + 6a^\nu\right)} \geq \dfrac{\beta_n' K_n^{1+\nu} (\log B)^\nu}{2C_5 B^\nu} & \text{if } K_n \frac{\log B}{B} \leq a \leq \frac{1}{\sqrt{B}} \\ \dfrac{\beta_n' a^\nu \sqrt{B}}{2\left(C_5 \log \frac{M}{a} + 6a^\nu\right)} \geq \dfrac{\beta_n' B^{\frac{1-\nu}{2}}}{2C_5 \log B} & \text{if } \frac{1}{\sqrt{B}} \leq a \leq \widetilde{M} \\ \dfrac{\beta_n' \widetilde{M}^\nu \sqrt{B}}{2\left(C_5 \log \frac{M}{\widetilde{M}} + 6\widetilde{M}^\nu\right)} & \text{if } \widetilde{M} \leq a \leq M \end{cases} \tag{4.57}$$

where we used (4.42). The estimate (4.9) follows. ∎

5. Proof of Theorems 2.4 and 2.5

The following lemma lists explicit scenarios under which condition (2.28) of Theorem 2.7 holds.

Lemma 5.1. *Let $S \geq 2$. If one of the following conditions holds:*
(i) $Y \geq 8S+5$, $L_0 \geq 8S+5$;
(ii) $Y \geq 9S+3$, $L_0 \geq (9S+3)^{\frac{2}{S}}$;
then condition (2.28) *of Theorem 2.7 is satisfied, i.e.*

$$Y - \frac{2}{\log L_0} \log Y \geq 8S+3.$$

Proof. The lemma follows from the fact that the function $t \to t - t_0 \log t$ is increasing for $t \geq t_0$.

If $Y \geq 8S+5$ and $\log(8S+5)/\log L_0 \leq 1$, condition (2.28) follows. This proves (i).

To prove (ii), take $Y \geq 9S+3$ and note that (2.28) will hold if

$$S - \frac{2}{\log L_0} \log(9S+3) \geq 0. \tag{5.1}$$

□

Proof of Theorem 2.4. We take $p = \frac{5}{3}d$, $S = 4$, and $Y = 9S+3 = 39$ in Theorem 2.7, so we satisfy (2.31), and obtain (2.28) from Lemma 5.1 (ii) if $L_0 \geq 6$.

Given $L_0 \geq \max\left\{6, 3\varrho, \eta_{\mathcal{I}}^{-[(\frac{5}{3}+b)d]^{-1}}\right\}$, $L_0 \in 6\mathbb{N}$, we define s_0 by

$$\begin{aligned} (39L_0)^{s_0 - \left(\frac{5}{3}+b\right)d} &= \max\left\{16 \cdot 60^d Q_{\mathcal{I}}, 1\right\} \qquad (5.2)\\ &\geq \max\left\{16 \cdot 234^d \left(\tfrac{30}{117}\right)^{bd} Q_{\mathcal{I}}, 1\right\}, \qquad (5.3)\end{aligned}$$

so $s_0 \geq \left(\frac{5}{3}+b\right)d$ and (2.29) is satisfied for $s \geq s_0$. We also define $\theta_0 > s_0$ by (2.32) with $s = s_0$; note that $D_{\mathcal{I}}$ is defined in (2.18) so

$$L_0^{\theta_0} \leq D_{\mathcal{I}} L_0^{\left(\frac{5}{3}+b\right)d}. \tag{5.4}$$

Thus condition (2.17) in Theorem 2.4 then implies

$$\begin{aligned} \mathbb{P}\left\{\|\Gamma_{0,L} R_{0,L}(E_0)\chi_{0,L/3}\|_{0,L} < \frac{1}{L^{\theta_0}}\right\} &\geq 1 - \tfrac{2}{368^d} \qquad (5.5)\\ &> 1 - 60^{\frac{1}{4}}(113)^{-\frac{5}{4}d} > 1 - (2\beta(39,5))^{-\frac{1}{4}}, \end{aligned}$$

and hence condition (2.33) in Theorem 2.7 holds for some $\theta > \theta_0$, so Theorem 2.4 follows from Theorem 2.7. □

Proof of Theorem 2.5. We take $S = 4$ and $Y = 8S+5 = 37$ in Theorem 2.7; condition (2.27) holds for $L_0 \geq 37$ by Lemma 5.1(i). We will use (see (2.27))

$$\beta(37,5) \leq \frac{107^{5d}}{120}. \tag{5.6}$$

Given $s > bd$, we set $p = \frac{1}{2}(s - bd)$ and

$$\mathcal{L}' = \max\left\{3\varrho, 37, 37^{\frac{1}{5}}\left(\frac{107^d}{60^{\frac{1}{5}}}\right)^{\frac{2}{s-bd}}, \frac{1}{37}\left(16 \cdot 222^d Q_{\mathcal{I}}\right)^{\frac{2}{s-bd}}, \eta_{\mathcal{I}}^{-\frac{1}{s}}\right\} \leq \mathcal{L}, \tag{5.7}$$

where $\mathcal{L}$ is as in (2.19). . Conditions (2.29) and (2.30) of Theorem 2.7 are satisfied for $L_0 \geq \mathcal{L}'$ by its definition.

Now set $\theta_0 > s$ (recall $\gamma_{\mathcal{I}} \geq 1$) by $L_0^{\theta_0} = 90^d \gamma_{\mathcal{I}}^2 (37 L_0)^s$, i.e., by (2.32). Then condition (2.20) is that same as

$$\mathbb{P}\left\{ \|\Gamma_{0,L} R_{0,L}(E_0)\chi_{0,L/3}\|_{0,L} < \frac{1}{L^{\theta_0}} \right\} \geq 1 - \tfrac{2}{344^d} \tag{5.8}$$

$$> 1 - 60^{\frac{1}{4}}(107)^{-\frac{5}{4}d} > 1 - (2\beta(37,5))^{-\frac{1}{4}},$$

and hence condition (2.33) in Theorem 2.7 holds for some $\theta > \theta_0$, so Theorem 2.5 follows from Theorem 2.7. □

6. Proof of Theorem 2.7

Theorem 2.7 is a refinement of [GK1, Theorem 5.1], the main difference are the explicit conditions.

Proof of Theorem 2.7. We proceed as in the proof of [GK1, Theorem 5.1]. We start by picking $E_0 \in \mathcal{I}$, $p > 0$, $S \geq 2$ with $S \in \mathbb{N}$, $s > p + bd$, $\theta > \theta_0$, where θ_0 is defined by condition (2.32) (note $\theta > s$), $Y \geq 8S + 5$, Y odd, and scales $L \geq \max\left\{6, 3\varrho, \eta_{\mathcal{I}}^{-\frac{1}{s}}\right\}$ with $L \in 6\mathbb{N}$. We set

$$p_L = \mathbb{P}\{\Lambda_L(0) \text{ is not } (\theta, E_0)\text{-suitable}\}. \tag{6.1}$$

The proof proceeds by induction. For the induction step, let $\ell \in 6\mathbb{N}$, $\ell \geq \max\left\{6, 3\varrho, \eta_{\mathcal{I}}^{-\frac{1}{s}}\right\}$, and $L = Y\ell$. We estimate p_L from p_ℓ as in [GK1, (Eq. (5.22)], but being slightly more careful in our use of the Wegner estimate (2.9), obtaining

$$\begin{aligned}
p_L &\leq \beta(Y, S+1) p_\ell^{S+1} + Q_{\mathcal{I}} \left[S \left(\frac{7S+2}{3Y} \right)^{bd} (6Y)^d + 1 \right] L^{bd} L^{-s} \\
&\leq \beta(Y, S+1) p_\ell^{S+1} + 2^{d+1} Q_{\mathcal{I}} S \frac{(7S+2)^{bd}}{(3Y)^{(b-1)d}} L^{-s+bd+p} \frac{1}{L^p} &\quad (6.2)\\
&\leq \beta(Y, S+1) p_\ell^{S+1} + \frac{1}{2}\frac{1}{L^p}, &\quad (6.3)
\end{aligned}$$

where $\beta(Y, S)$ is as in (2.27). (Note that (2.27) is a slightly better estimate than [GK1, (Eq. (5.20)].) To obtain (6.3) we assume

$$(Y L_0)^{s-bd-p} \geq 2^{d+2} Q_{\mathcal{I}} S \frac{(7S+2)^{bd}}{(3Y)^{(b-1)d}} \tag{6.4}$$

which is exactly condition (2.29) in Theorem 2.7.

As in the proof of [GK1, Theorem 5.1], we next must satisfy [GK1, (Eq. (5.24)]. i.e., we must show that

$$[\gamma_{\mathcal{I}} (7S+2)^d L^s][\gamma_{\mathcal{I}} 3^d \ell^{-\theta}] < 1, \tag{6.5}$$

which follows if

$$L_0^{\theta - s} > \gamma_{\mathcal{I}}^2 3^d (7S+2)^d Y^s, \tag{6.6}$$

which is true if $\theta > \theta_0$, where θ_0 is defined by condition (2.32).

Next, we define $N(Y)$ as in [GK1, (Eq. (5.26)], note that since we specified $Y \geq 8S+5$, we have

$$N(Y) = Y - 8S - 1 \geq 4\,. \tag{6.7}$$

We now need to satisfy [GK1, (Eq. (5.27)]:

$$Y^d \left[\frac{\gamma_{\mathcal{I}} 3^d}{\ell^\theta}\right]^{N(Y)} L^s < \frac{1}{L^\theta} \tag{6.8}$$

i.e.,

$$(3^d \gamma_{I_0})^{N(Y)} Y^{s+\theta+d} < \ell^{\theta(N(Y)-1)-s}\,. \tag{6.9}$$

But (6.9) is automatically fulfilled for any $\ell \geq L_0 \geq 6$ in view of condition (2.28). First note that $\theta > s$ and $N(Y) \geq 4$ imply that $\theta(N(Y)-1) - s > 0$. As a consequence (6.9) certainly holds if

$$\left(\frac{L_0^{N(Y)-1}}{Y}\right)^\theta \geq (3^d \gamma_{\mathcal{I}})^{N(Y)} Y^d (Y L_0)^s. \tag{6.10}$$

Second, we always have

$$\frac{L_0^{N(Y)-1}}{Y} \geq L_0^{N(Y)/2}, \tag{6.11}$$

provided $N(Y) \geq (2\log Y/\log L_0) + 2$, which is exactly (2.28). Thus plugging (6.6) (which is the same as (2.32)) into (6.10) leads to

$$\begin{aligned} \left(\frac{L_0^{N(Y)-1}}{Y}\right)^\theta &\geq \left(L_0^\theta\right)^{\frac{N(Y)}{2}} \geq \left(\gamma_{\mathcal{I}}^2 3^d (7S+2)^d (Y L_0)^s\right)^{\frac{N(Y)}{2}} \\ &\geq \left(3^d \gamma_{\mathcal{I}}\right)^{N(Y)} Y^d (Y L_0)^s, \end{aligned}$$

where we used $N(Y) \geq 2$ and $s > d$. Thus we have (6.10), and hence (6.8).

We now set $L_{k+1} = Y L_k$, $k = 0, 1, 2, \cdots$, $p_k = p_{L_k}$. It follows from (6.3) that

$$p_{k+1} \leq \beta(Y, S+1) p_k^{S+1} + \frac{1}{2} L_{k+1}^{-p} \quad \text{for} \quad k = 0, 1, 2, \ldots\,. \tag{6.12}$$

If condition (2.30) holds, we may finish the proof as in [GK1]. If $p_k < L_k^{-p}$, then

$$\begin{aligned} p_{k+1} &\leq \beta(Y, S+1) \frac{1}{L_k^{(S+1)p}} + \frac{1}{2L_{k+1}^p} \\ &\leq \frac{2\beta(Y, S+1) Y^p}{L_0^{(S+1)p}} \frac{1}{2L_{k+1}^p} + \frac{1}{2L_{k+1}^p} \\ &\leq \frac{1}{L_{k+1}^p}, \end{aligned} \tag{6.13}$$

where we used (2.30). If not, we must have $p_{k+1} \geq L_{k+1}^{-p}$ for $k = 0, 1, 2, \ldots, n$, so (6.12) implies

$$\frac{1}{2} L_{k+1}^{-p} \leq \beta(Y, S+1) p_k^{S+1} \quad \text{for} \quad k = 0, 1, 2, \ldots, n\,, \tag{6.14}$$

which we plug back into (6.12) to get

$$p_{k+1} \leq 2\beta(Y, S+1) p_k^{S+1} \quad \text{for} \quad k = 0, 1, 2, \ldots, n\,. \tag{6.15}$$

Hence

$$\frac{1}{Y^{(n+1)p}L_0^p} \leq p_{n+1} \leq (2\beta(Y,S+1))^{-\frac{1}{S}} \left((2\beta(Y,S+1))^{\frac{1}{S}} p_0 \right)^{(S+1)^{n+1}}. \tag{6.16}$$

If condition (2.33) holds, i.e.,

$$p_0 < (2\beta(Y,S+1))^{-\frac{1}{S}}, \tag{6.17}$$

we get a contradiction for n large, uniformly in $L_0 \geq 6$, which finishes the proof.

If If condition (2.30) does not hold, we work with the scale $L_1 = YL_0$, instead of L_0. That provides an extra term in Y that will supply the needed control uniformly in $L_0 \geq 6$. The price we pay is expressed in condition (2.31): $p > d$ and $S \geq (p+d)/(p-d)$.

We now suppose condition (2.31) holds. If $p_k < L_k^{-p}$ for some $k \geq 1$, we have

$$\begin{aligned} p_{k+1} &\leq \beta(Y,S+1)\frac{1}{L_k^{(S+1)p}} + \frac{1}{2L_{k+1}^p} \\ &\leq 2\beta(Y,S+1)\frac{Y^p}{L_1^{Sp}}\frac{1}{2L_{k+1}^p} + \frac{1}{2L_{k+1}^p} \qquad (6.18)\\ &\leq \frac{1}{L_{k+1}^p}, \qquad (6.19) \end{aligned}$$

since $p > d$ and $S \geq \frac{p+d}{p-d}$. Indeed, in this case we have

$$\begin{aligned} 2\beta(Y,S+1)\frac{Y^p}{L_1^{Sp}} &\leq 2\frac{(3Y)^{(S+1)d}}{S!}\frac{Y^p}{(YL_0)^{Sp}} \qquad (6.20)\\ \leq \frac{2\,3^{(S+1)d}}{6^{Sp}S!}\frac{Y^{(S+1)d+p}}{Y^{Sp}} &\leq \frac{3^d}{2^{Sd}}Y^{(p+d)-S(p-d)} \leq \left(\frac{3}{4}\right)^d < 1\,, \end{aligned}$$

where we also used $L_0 \geq 6$ and $S \geq 2$. Otherwise, we must have $p_{k+1} \geq L_{k+1}^{-p}$ for $k = 1,2,\ldots,n$, so it follows from (6.12) that $\beta(Y,S+1)p_k^{S+1} \geq \frac{1}{2}L_{k+1}^{-p}$ for $k = 1,2,\ldots,n$, so we again get (6.14) and (6.15), but we start from L_1 instead of L_0, so instead of (6.16) we get

$$\frac{1}{Y^{(n+1)p}L_0^p} \leq p_{n+1} \leq (2\beta(Y,S+1))^{-\frac{1}{S}} \left((2\beta(Y,S+1))^{\frac{1}{S}} p_1 \right)^{(S+1)^n}. \tag{6.21}$$

We will get a contradiction for large n, uniformly in $L_0 \geq 6$, if we have

$$(2\beta(Y,S+1))^{\frac{1}{S}} p_1 < 1\,. \tag{6.22}$$

In other terms, we need condition (6.17) but with p_1 instead of p_0. Thus we need to estimate p_1 in terms of p_0. Note that (6.12) holds for $k = 0$ as well, so

$$p_1 \leq \beta(Y,S+1)p_0^{S+1} + \frac{1}{2(YL_0)^p}\,. \tag{6.23}$$

Hence (6.22) is certainly true if

$$2^{\frac{1}{S}}\beta(Y,S+1)^{\frac{S+1}{S}}p_0^{S+1} + \frac{(2\beta(Y,S+1))^{\frac{1}{S}}}{2(YL_0)^p} < 1 \tag{6.24}$$

for all $L_0 \geq 6$. Since the first term on the right hand side of (6.24) is $\leq \frac{1}{2}$ by (6.17), i.e., condition (2.33), it suffices to show that the second term is also $\leq \frac{1}{2}$, which can be seen as follows:

$$\frac{(2\beta(Y,S+1))^{\frac{1}{S}}}{(YL_0)^p} = \left(\frac{2\beta(Y,S+1)}{(YL_0)^{Sp}}\right)^{\frac{1}{S}} < 1\ , \tag{6.25}$$

using (6.20). Hence (6.22) holds, so we get in (6.21) a contradiction for large n.

The contradictions to either (6.16) or (6.21) happen at some large n, how large depending only on d, p, Y, S and L_0. Thus there is $\mathcal{K} = \mathcal{K}(d, p, Y, S, L_0)$ such that $p_k < L_k^{-p}$ for all $k \geq \mathcal{K}$.

That $E_0 \in \Sigma_{MSA}$ now follows from [GK1]. $\square$

7. Proof of Theorem 2.6

Theorem 2.6 is a refinement of [GK1, Theorem 5.2], again the the main difference being the explicit conditions.

Proof of Theorem 2.6. We proceed as in the proof of [GK1, Theorem 5.2] (see [GK1, Section 5.4]), where we fix $S = 3$ and take $\alpha = 1 + \frac{p/2}{p+2d} \in (1, 1 + \frac{p}{p+2d})$. We prove (2.24), note that (2.23) is easier and may be proven in a similar way.

We start by deriving from (2.22) the initial step of the inductive process, i.e.,

$$\mathbb{P}\left[R\left(\tfrac{m_0}{2}, L_0, I, x, y\right)\right] \geq 1 - \frac{4}{L_0^{2p}} \tag{7.1}$$

for all $x, y \in \mathbb{Z}^d$ with $|x - y| > L_0 + \varrho$, which is just (2.24) with $k = 0$ and $\frac{m_0}{2}$ substituted for $\frac{m_0}{4}$. To do that, we recall that

$$\Lambda_L(x) \text{ is } (\theta, E)\text{-suitable} \iff \Lambda_L(x) \text{ is } (2\theta \tfrac{\log L}{L}, E)\text{-regular}\,. \tag{7.2}$$

As in [GK1, p. 440], we set

$$\delta_2 = \delta_2(m_0, L_0, s) = \frac{1}{2L_0^{2s}}\left[\mathrm{e}^{-\frac{m_0}{2}\frac{L_0}{2}} - \mathrm{e}^{-m_0\frac{L_0}{2}}\right] \tag{7.3}$$

and $I(\delta_2) = (E_0 - \delta_2, E_0 + \delta_2) \cap \mathcal{I}$. Using (2.22), Assumption W with $\eta = L_0^{-s}$, and requiring

$$L_0 \geq Q_{\mathcal{I}}^{\frac{1}{s-p-bd}}\ , \tag{7.4}$$

obtain the equivalent to [GK1, Eq (5.37)]:

$$\begin{aligned}&\mathbb{P}\{\text{for every } E \in I(\delta_2),\ \Lambda_{L_0}(0) \text{ is } (\tfrac{m_0}{2}, E)\text{-regular}\} \\ &\qquad \geq 1 - \frac{1}{L_0^p} - Q_{\mathcal{I}} L_0^{bd-s} \geq 1 - \frac{2}{L_0^p}\,.\end{aligned} \tag{7.5}$$

Using Assumption IAD, (7.1) follows from (7.5).

For reasons that will become clear later we will work with a smaller open interval I, with $E_0 \in I \subset \mathcal{I}$, defined as

$$I = I(\delta_2) \cap \{E \in \mathcal{I};\ \operatorname{dist}(E, \mathbb{R}\backslash\mathcal{I}) > L_0^{-s}\}\,. \tag{7.6}$$

The induction now proceeds as in [GK1]. The induction step goes from scale $\ell \geq L_0 \ \geq 3\varrho$ to scale $L = [\ell^\alpha]_{6\mathbb{N}}$, using (2.9) with $\eta = L^{-s}$. We need to satisfy the equivalent to [GK1, Eq. (5.42)] (recall $S = 3$),

$$\left[\gamma_{\mathcal{I}}(21+2)^d L^s\right]\left[\gamma_{\mathcal{I}} 3^d \mathrm{e}^{-m\frac{\ell}{2}}\right] < 1, \tag{7.7}$$

for all $\ell \geq L_0$ and $\frac{m_0}{4} \leq m \leq \frac{m_0}{2}$, $m_0 = 2\theta\frac{\log L_0}{L_0}$. To do so, it suffices to require

$$69^d \gamma_{\mathcal{I}}^2 \ell^{\alpha s} \mathrm{e}^{-m_0\frac{\ell}{8}} < 1 \ \text{ for all } \ \ell \geq L_0\,, \tag{7.8}$$

i.e.,

$$69^d \gamma_{\mathcal{I}}^2 t^{\alpha s} L_0^{-\frac{\theta t - 4\alpha s}{4}} < 1 \ \text{ for all } \ t \geq 1\,. \tag{7.9}$$

To satisfy the inequality at $t = 1$ we require

$$\theta > 4\alpha s \ \text{ and } \ L_0 \geq \left(69^d \gamma_{\mathcal{I}}^2\right)^{\frac{4}{\theta - 4\alpha s}}\,. \tag{7.10}$$

Since $\log L_0 \geq 1$ as $L_0 \geq 6$, it follows that (7.9) then holds for all $t \geq 1$. Thus (7.7) is satisfied if (7.10) holds.

Now, the equivalent of [GK1, Eq. (5.47)] is

$$N_\ell \ \geq \ \frac{L}{\ell} - 8\cdot 3 - 1 = \frac{L}{\ell} - 25 \geq \frac{L}{\ell}\left(1 - 50\ell^{-(\alpha-1)}\right), \tag{7.11}$$

where we used, and hence require,

$$\ell \geq L_0 \geq 12 \tag{7.12}$$

Thus [GK1, Eq. (5.48)] follows with

$$\begin{aligned} m' \ &\geq \ m\left(1 - \frac{50}{\ell^{\alpha-1}}\right) - 2\left[\frac{(\alpha-1)d\log\ell}{\ell^\alpha - 6} + \frac{\log(3^d\gamma_{\mathcal{I}})}{\ell} + \frac{\alpha s\log\ell}{(\ell^\alpha - 6)}\right] \\ &\geq \ m\left[1 - \frac{50}{\ell^{\alpha-1}} - \frac{4\log(3^d\gamma_{\mathcal{I}})}{\theta\log\ell} - \frac{4(\alpha-1)d + 4\alpha s}{\theta\ell^{\alpha-1}}\right] \qquad (7.13) \\ &\geq \ m\left[1 - \frac{52}{\ell^{\alpha-1}} - \frac{4\log(3^d\gamma_{\mathcal{I}})}{\theta\log\ell}\right]. \qquad (7.14) \end{aligned}$$

where we used the fact that $\frac{\log L}{L}$ is decreasing in L for $L \geq 3$, $1 < \alpha < 2$, d, s, and $\theta > 4\alpha s$.

Continuing as in [GK1], we construct the sequence of length scales $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$, $k = 0, 1, \ldots$. Applying the inductive step from scale L_k to scale L_{k+1}, we obtain a decreasing sequence of masses m_k', with $m_0' = \frac{m_0}{2}$, satisfying [GK1, Eq. (5.48)] and (7.14) at scale L_k. We thus need to verify [GK1, (5.51)], i.e.,

$$0 \leq \sum_{k=0}^{\infty}(m_k' - m_{k+1}') < \frac{m_0}{4}\,. \tag{7.15}$$

To do so, note that it follows from (7.14) that it suffices to show that for suitable L_0 we have

$$\sum_{k=0}^{\infty}\frac{52}{L_k^{\alpha-1}} \ < \ \frac{1}{4}\,, \tag{7.16}$$

$$\sum_{k=0}^{\infty}\frac{4\log(3^d\gamma_{\mathcal{I}})}{\theta\log L_k} \ < \ \frac{1}{4}\,. \tag{7.17}$$

Recall $L_k = L_0^{\alpha^k}$. For (7.16), we use $\alpha^k \geq k\alpha$, so

$$\sum_{k=0}^{\infty} \frac{1}{L_0^{\alpha^k(\alpha-1)}} \leq \frac{1}{L_0^{\alpha-1}} + \sum_{k=1}^{\infty} \frac{1}{L_0^{\alpha(\alpha-1)k}} = \frac{1}{L_0^{\alpha-1}} + \frac{1}{L_0^{\alpha-1}\left(1 - L_0^{-(\alpha-1)}\right)} \leq \frac{2}{L_0^{\alpha-1} - 1}, \tag{7.18}$$

and hence (7.16) holds if we require

$$L_0 \geq 417^{\frac{1}{\alpha-1}}. \tag{7.19}$$

Since $\sum_{k=0}^{\infty} \alpha^{-k} = \frac{\alpha}{\alpha-1}$, (7.17) holds if we require

$$L_0 \geq \left(3^d \gamma_{\mathcal{I}}\right)^{\frac{16\alpha}{\theta(\alpha-1)}}. \tag{7.20}$$

Note that $417^{\frac{1}{\alpha-1}} \geq 417^{\frac{1}{2}} \geq 20 > 12$ since $\alpha \leq \frac{3}{2}$, and hence (7.12) follows from (7.19).

We continue as in [GK1, Eqs. (5.54)-(5.55)], with $\tilde{I}_0 = \mathcal{I}$ and $I_0 = I$. We need the equivalent of [GK1, Eq. (5.55)], i.e.,

$$2C_{\mathcal{I}}Q_{\mathcal{I}}(3+1)^2\left(\frac{6L}{\ell}\right)^{2d} L^{(b+1)d}L^{-s} \leq \frac{2}{L^{2p}}, \tag{7.21}$$

so it suffices to establish the first inequality in

$$16 \cdot 36^d C_{\mathcal{I}}Q_{\mathcal{I}}\ell^{2d(\alpha-1)+\alpha(b+1)d} \leq \left(\frac{\ell^\alpha}{2}\right)^{s-2p} \leq (\ell^\alpha - 6)^{s-2p} \leq L^{s-2p}. \tag{7.22}$$

Thus we must show that for all $\ell \geq L_0$ we have

$$16 \cdot 36^d 2^{s-2p} C_{\mathcal{I}}Q_{\mathcal{I}} \leq \ell^{\alpha(s-2p-(b+3)d)+2d}, \tag{7.23}$$

which is true if

$$s > 2p + (b+2)d > 2p + (b+3)d - \frac{2d}{\alpha} \tag{7.24}$$

and

$$L_0 \geq \left(16 \cdot 36^d 2^{s-2p} C_{\mathcal{I}}Q_{\mathcal{I}}\right)^{\frac{1}{\alpha(s-2p-(b+2)d)}}. \tag{7.25}$$

Using the definition of the interval I, given in (7.6), the equivalent of [GK1, Eq. (5.56)] follows from (7.21). To ensure the equivalent of [GK1, Eqs. (5.57)-(5.58)], we need

$$\left[\left(\frac{3L}{\ell}\right)^{2d}\frac{4}{\ell^{2p}}\right]^{\frac{3+1}{2}} \leq \frac{1}{L^{2p}}, \tag{7.26}$$

so it suffices to ensure

$$\left[9^d \ell^{2(\alpha-1)d}\frac{4}{\ell^{2p}}\right]^2 \leq \frac{1}{\ell^{2\alpha p}} \quad \text{for all} \quad \ell \geq L_0, \tag{7.27}$$

i.e.,

$$4 \cdot 9^d \leq \ell^{2p-\alpha p-2(\alpha-1)d} \quad \text{for all} \quad \ell \geq L_0. \tag{7.28}$$

Since $2p - \alpha p - 2(\alpha - 1)d = \frac{7}{2}p$ by the definition of α, (7.28) follows from (7.19) since

$$(4 \cdot 9^d)^{\frac{2}{7p}} = (4 \cdot 9^d)^{\frac{1}{7(\alpha-1)(p+2d)}} \leq (4 \cdot 9^d)^{\frac{1}{d(\alpha-1)}} \leq 417^{\frac{1}{\alpha-1}} . \tag{7.29}$$

The desired conclusion (2.24) now follows as in [GK1]. The fact that $I \subset \Sigma_{MSA}$ now follows from (2.24) and [GK1, Theorem 3.4].

The exponential decay of the eigenfunctions given in (2.26) follows from (2.24) as in [vDK, Theorem 2.3]. □

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