BOOTSTRAP OF REGULARITY FOR INTEGRABLE SOLUTIONS OF COHOMOLOGY EQUATIONS

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ABSTRACT. We present an elementary argument to bootstrap the regularity of integrable solutions of cohomology equations. if $f : M \to M$ is a smooth Anosov (partially hyperbolic with uniform accessibility) we show that if $\varphi \in L^p$, $\eta \in C^\beta$, $\beta > 0$ and

$$\varphi = \eta\varphi \circ f,$$

p high enough, then $\varphi \in \alpha'$, where β' depends on the hyperbolicity properties of f. ($\beta' = \beta$ if f is Anosov.)

1. Introduction

The goal of this note is to report a very simple trick to bootstrap the regularity of measurable solutions of cohomology equations over systems with some hyperbolic behavior which preserve an absolutely continuous invariant measure.

More precisely, given M a d dimensional compact manifold, G a Lie group – which we will assume to be a matrix group or, more generally a subgroup of a Banach algebra $\mathcal{B}$, possibly infinite dimensional —, $f : M \to M$ a diffeomorphism preserving a measure μ equivalent to Lebesgue. Given $\eta : M \to G$ we consider the equation for $\varphi : M \to G$

$$\varphi(x) = \eta(x)\varphi\big(f(x)\big) \ . \tag{1}$$

We will always assume that η is rather regular, at least C^β for some $\beta > 0$. Similarly we will always assume that f is C^r, $r > 1+\beta$. We will also assume that the density of μ is rather smooth. At least continuous. We note that, by the results of [dlLMM86], when f is Anosov, the measure μ has a density which is $C^{r-1-\delta}$.

The main goal is to see whether, just assuming that $\varphi \in L^p$ for some suitable $1 \le p \le \infty$, high enough. allows us to conclude that φ is actually rather regular. Of course, (1) is supposed to hold almost everywhere μ.

There is an extensive literature for these problems. In particular, we note [NP99], which considers the case $p = \infty$. We also note [dlL01], which uses similar techniques than the techniques developed here. In particular, we will make use of Sobolev-like spaces (the potential spaces) as intermediate steps. (See Section 2.3 for some notions on these spaces.) We note that both [NP99] and [dlL01] – as well as this paper – use some remarkable cancellations that were discovered in [NT98].

As soon as we establish some regularity, there are other bootstrap arguments that can be employed – at least in the case that the diffeomorphism f is Anosov. Notably, when when we know that φ is continuous, it is possible to use the results of [NT98]. When $\varphi \in W^{1,p}$, it is possible to use the results in [dlL01] to obtain further regularity when the regularity of η is expressed in Sobolev classes. Hence, in this paper we will only detail the bootstrap till we obtain that η is continuous or $W^{1,p}$.

We also point out that the bootstrap regularities is extremely easy when we assume that η is small or that the group G has certain growth properties. We will collect these results in Section 5.4 but in the main part of the paper, we will not make any assumptions either on the fact that η is small or in the structure of the group.

We also make assumptions of hyperbolicity on f weaker than being Anosov. See Theorem 3.1 for a precise statement of the results.

The argument presented here is extremely simple. Hence, we decided to present it in its simplest form and present some of the further developments in a somewhat more informal way in Section 5. If all the improvements were taken at the same time, the paper – even if conceptually simple – would have been hard to read.

2. Preliminaries and notation

2.1. **Some conventions that will be used throughout the paper.**

We will assume, that G is a linear group. (i.e. a subgroup of $GL(n,\mathbb{R})$ or, more generally of a Banach algebra $\mathcal{B}$ of operators, possibly infinite dimensional.

This will be used in two ways in the paper. First, it will allow to write to use the product notation for the group operation and, at the same time to use $\varphi_1 - \varphi_2$ and the like. Of course, differences should be interpreted as elements of the Lie algebra. This is only a typographical convenience and can be eliminating just by complicating the notation.

A more subtle point is that we will be able to write the derivative of a product of factors as a sum of terms. This, implicitly uses that the group operation is linear.

Similarly, given a vector field δ on the manifold we will write $x + \delta(x)$ to denote $\exp_x \delta(x)$. This is only for typographical convenience. Of course, one has to be careful because this addition is not associative.

We emphasize that for functions taking values in groups we understand that $\varphi \in L^p(M,G)$ means that $\varphi, \varphi^{-1} \in L^p\big(M,\mathcal{B}\big)$ when the later is understood in the usual sense of Banach valued L^p spaces.

We note that the fact that $\eta \in C^\beta$ and that the manifold is compact implies that η remains at a finite distance of non-invertible elements in the algebra. Hence, η^{-1} is C^β.

2.2. **Some notions of hyperbolicity.**

We recall a very weak notion of hyperbolicity.

Definition 2.1. *We say that a diffeomorphism f of a manifold M preserves a dominated splitting when there is a decomposition of the tangent bundle $T_xM = E^< \oplus E^>$ and numbers $C > 0$, $\lambda_< < \lambda_>$ in such a way that*

$$
\begin{aligned}
v \in E_x^< &\iff |Df^n(x)v| \le C(\lambda_<)^n|v| \quad n \ge 0\\
v \in E_x^> &\iff |Df^n(x)v| \le C(\lambda_>)^n|v| \quad n \le 0
\end{aligned}
\tag{2}
$$

As it is well known, it is always possible to introduce an analytic metric in M in such a way that $C = 1$ in (2) at the only price of redefining slightly $\lambda_<, \lambda_>$. We will henceforth assume that this redefinition has taken place.

In case that $\lambda_< < 1$, $\lambda_> < 1$ the system is Anosov.

In the case that $\lambda_< < 1$, we say that the system has a dominated splitting with a stable component. When $\lambda_> > 1$, we say that it has a dominated splitting with an unstable component. When the component is stable, we will use the notation E^s, λ_s to denote $E^<, \lambda_<$. Analogously for the unstable case.

A very important property of systems with a dominated splitting with a stable component is the fact that the foliation W^s associated to the stable component of the splitting is absolutely continuous. Analogously, for the unstable foliation. See [Ano69] Theorem 9.

Lemma 2.2. *If the mapping f is $C^{1+\alpha}$, $\alpha > 0$, and preserves a stable distribution around every point, it is possible to find a neighborhood U_x and a mapping $\Lambda_{U_x} : U_x \to \mathbb{R}^d$ which is a local homeomorphism and which transforms the stable foliation into a product foliation in $\mathbb{R}^d$.*

Moreover, it can be arranged that the mapping Λ_{U_x} is continuous and absolutely continuous. The Jacobian of Λ_{U_x} is C^α along the leaves of the stable foliation, continuous over U_x and bounded away from zero.

The Lemma 2.2 can also be formulated by saying that the holonomy maps of the stable and unstable foliations are absolutely continuous. See [PS72].

It is possible to show that if the map f is C^r, the Jacobian of Λ_{U_x} can be made to C^{r-1} along the leaves of the stable foliation and Hölder with its derivatives across the manifold. See [dlLMM86].

Definition 2.3. *Let W^s be the stable foliation of a $C^{1+\alpha}$, $\alpha > 0$ dynamical system, (in particular absolutely continuous).*

We say that a continuous vector field δ is adapted to the foliation W^s when:

i) $x + \delta(x) \in W^s_x$
ii) *Denoting by $h_\delta(x) = x + \delta(x)$, h_δ is absolutely continuous.*

A system is called partially hyperbolic when it has two dominated splittings, one with an stable component, another one with an unstable component.

The following definition is somewhat standard.

Definition 2.4. *We say that a partially hyperbolic system is uniformly accessible when $\exists\ \ \varepsilon_0 > 0$, $N \in \mathbb{N}$, $K \in [1, \infty)$, $C > 0$ such that:*

Given $x, y \in M$, $d(x,y) \le \varepsilon$, we can find

$$u_1 \cdots u_N, v_1 \cdots v_N, \in M$$

in such a way that

a) $u_1 = x$, $v_N = y$
b) $v_i \in W^s_{u_i}$, $u_{i+1} \in W^u_{v_i}$
c) $d(u_i, v_i) \le C\varepsilon^{1/K}$, $d(v_i, u_{i+1}) \le C\varepsilon^{1/K}$.

The notion of uniformly accessible systems means that all sufficiently close points can be connected by a sequence of steps along stable and unstable manifolds. The intuition is that, if we want to connect two points, at the beginning we may have to go rather far away, but then, we come back. The number of steps is uniformly bounded and the length of each step is bounded is a root of the length of distance among the original points.

The importance of conditions similar to this for cohomology equations was emphasized in [KK96]. The amount to the stable and unstable foliations being quite non-integrable and they have played a role in many studies of partially hyperbolic systems, starting with [BP73].

We will need the following property which is a version of Definition 2.4 including measurability assumptions.

Definition 2.5. *We say that a partially hyperbolic system is measurably uniformly accessible when for some $\varepsilon > 0$, $n \in \mathbb{N}$, $K \in [1, \infty)$, $L \in \mathbb{R}^+$ we have the following property:*

Given a C^1 measure preserving vector field with $\|h - \mathrm{Id}\|_{C^1} \le \varepsilon_0$, we can find vector fields $u_1, \ldots u_N$, $v_1, \ldots, v_N$.

a) *$u_1, \ldots u_N$ adapted to the stable foliation, $v_1, \ldots, v_N$ adapted to the unstable foliation.*
b) *$h = h_{v_N} \circ h_{u_N} \circ \ldots h_{v_1} \circ h_{u_1}$.*
c) *$\|v_i\|_{L^\infty}, \|u_i\|_{L^\infty} \le L\|h - \mathrm{Id}\|^{1/K}_{L^\infty}$.*
d) *$\|J[h_{u_i}]\|_{L^\infty}, \|J[h_{v_i}]\|_{L^\infty} \le L$. Where $J[h]$ denotes the Jacobian of the function h.*

Note that Definition 2.5 is a version of Definition 2.4 in which we consider the motions connecting x to $h(x)$ but we require that the decompositions are not only small – which more or less follows from Definition 2.4– but also we have condition d) which requires that the maps h_u are absolutely continuous.

The absolute continuity of the maps and the fact that they have bounded Jacobians comes into play when we consider their action on L^p functions under composition. Note that $\int |\varphi \circ h(x)|^p \, dx = \int |\varphi(y)|^p J_h(y) \, dy$ where J_h is the Jacobian of h^{-1}.

Therefore:

$$\|\varphi \circ h\|_{L^p} \le \|\varphi\|_{L^p} \|J_h\|^{1/p}_{L^\infty} \tag{3}$$

We do not know how to characterize systems satisfying Definition 2.5.

We will just give two examples of systems that satisfy Definition 2.5.

Proposition 2.6. *An Anosov system satisfies Definition 2.5.*
The algebraic examples in [BPSW01] *p. 330 satisfy Definition 2.5.*

Proof. Given a manifold W^s_x, we consider the manifold $h(W^s_x)$. Both of them are C^1 manifolds. Restricted to W^s_x, the map h is absolutely continuous and the Jacobian is bounded uniformly when h ranges in a C^1 neighborhood of the identity.

Moreover, W^s_x, $h(W^s_x)$ are C^1 close. We can, therefore define a holonomy map H_x along the unstable foliation from $h(W^s_x)$ to W^s_x. Because of Lemma 2.2, the holonomy map will be absolutely continuous when the manifolds are endowed with the Riemannian measures and the Jacobian of the holonomy map is bounded. We note that the bound is uniform over all the points in the manifold and over all the maps h which are in a C^1 neighborhood of the identity.

We denote by $\tilde{h} = H_x \circ h$. The map $\tilde{h}$ is adapted to the stable foliations and absolutely continuous when restricted to each leaf. Moreover, we have bounds on the Jacobian which are uniform over the point in the manifold and over the maps h in a C^1 neighborhood.

Given the absolute continuity of the stable foliation, it is clear that a map which preserves the leaves and is uniformly absolutely continuous over all of them, is absolutely continuous and we can bound the Jacobian.

The fact that the algebraic examples satisfy te definition is clear because the stable and unstable foliations are smooth. $\square$

2.3. **The potential spaces.** In this subsection, we briefly recall the definition and a few facts about the so-called potential spaces and adapt them to an slightly more geometric meaning.

The results we will present here are quite well known and can all be found in [Ste70] p. 134 – 165. [Tay97], [Nik61]. More in depth treatments of Sobolev spaces can be found in [Ada75], [Nik75].

In this paper we will only be concerned with those spaces for which there exist characterizations in terms of modulus of continuity. The reason is that the modulus of continuity of solutions of cohomology equations can be studied using dynamical arguments.

Definition 2.7. *For $1 \le p \le \infty$, given a $\psi \in L^p(\mathbb{R}^d)$, we say that $\psi \in \mathfrak{L}^p_\alpha(\mathbb{R}^d)$ when $\|\psi(\cdot) - \psi(\cdot + t)\|_{L^p} \le C|t|^\alpha$ for all t, $|t| \le \gamma_0$ for some γ_0.*

These spaces can be endowed the norm

$$\max(\|\psi\|_{L^p}, \sup_{0<|t|\le\gamma_0} \|\psi(\cdot) - \psi(\cdot + t)\|_{L^p}|t|^{-\alpha}$$

which makes them Banach spaces.

We recall that the potential spaces, defined in [Ste70] p. 134 are the image of L^p under $(-\Delta + \mathrm{Id})^{-\alpha/2}$. (i.e. $W^{\alpha,p} = \{f|(-\Delta + \mathrm{Id})^{\alpha/2} \in L^p\}$ endowed with the norm $\|f\|_{W^{\alpha,p}} = \|(-\Delta + \mathrm{Id})^{\alpha/2} f\|_{L^p}$.

Hence, it is natural to consider them as Sobolev spaces for fractional regularities. Hence, we use the notation $W^{\alpha,p}$.

Remark 2.8. In [Ste70], the spaces that we call W^p_α are called $\mathcal{L}^p_\alpha$ and, in p. 135, it is shown that, for $\alpha \in \mathbb{N}$, this definition agrees with a more customary definition in terms of L^p norms of derivatives. We will follow the more customary notation $W^{\alpha,p}$ to avoid having to resort to different fonts for L as in [Ste70].

The notation $\mathfrak{L}^p_\alpha$ does not occur in [Ste70]. $\square$

Some relations between these spaces are summarized in Lemma 2.9 below.

Lemma 2.9. *With the above definitions:*

a) *When* $\alpha = 1, 1 < p < \infty$ *or when* $0 < \alpha < 1$ *and* $p = 2$, *then*

$$\mathfrak{L}^p_\alpha = W^{\alpha,p}$$

b) *When* $0 \le \alpha < 1$ *we have*

$$\mathfrak{L}^p_\alpha \subset W^p_{\alpha'}$$

for every $\alpha' < \alpha < 1$.

Part a) of Lemma 2.9 can be found in [Ste70] §3. 3 p. 138. Part b) in and in §3.5.2 p. 141. (It suffices to work out the integrals in the sufficient conditions stated there.)

Two important results for $\mathcal{L}^p_\alpha$ spaces that we will need are:

Lemma 2.10. *We have:*

a) $\mathfrak{L}^p_\alpha \subset L^q$ *where* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$.

b) *If* $p > d/\alpha$, *then* $W^p_\alpha \subset C^0$.

In both cases, the embeddings are continuous.

Both results are versions of the Sobolev embedding theorem which can be found in [Ste70] p. 124 – for integer α. The standard result for $0 < \alpha < 1$ can be found e.g. in [Tay97] p. 22 ff.

It is also true that in case b) of Lemma 2.10 one can conclude that the function is Hölder. For us, this is not so important, since, once we know that the functions we are interested in are continuous, we can use dynamic arguments to obtain results which are sharper than those obtained by applying embedding theorems.

The definition of spaces above generalize straightforwardly to manifolds. The results mentioned above remain valid. All of them are local (notice that we need only to consider only displacements that are small enough. Hence we can use partition of unity arguments).

To apply the results to dynamical systems, we will find it useful to introduce also analogues of the $\mathfrak{L}^p_\alpha$ associated to the foliations that appear in dynamical systems.

Definition 2.11. *Let* $0 < \alpha \le 1$, $1 \le p \le \infty$. *Given a function* $\psi \in L^p$, *we say that* $\psi \in \mathfrak{L}^{p,(s)}_\alpha$ *when*

$$\|\psi - \psi \circ h_\delta\|_{L^p} \le C \|\delta\|^\alpha_{L^\infty}$$

for all the vector fields δ that are adapted to the foliation W^s and $\|\delta\|_{L^\infty} \leq \gamma_0$ for some $\gamma_0 > 0$.

Analogous definitions, of course, are true for the unstable foliations.

The following proposition is a very simple argument.

Proposition 2.12. *Assume that the system satisfies the Definition 2.5 Then, for $1 \leq p \leq \infty$, $0 < \alpha \leq 1$,*

$$\mathfrak{L}_\alpha^{p,(s)} \cap \mathfrak{L}_\alpha^{p,(u)} \subset \mathfrak{L}_{\alpha/K}^p \tag{4}$$

The Proposition 2.12 is very elementary. The only thing that is used is adding and subtracting increments along the stable and unstable directions.

$$\psi \circ h - \psi = \left(\psi \circ h_{\delta_N^u} - \psi\right) \circ h_{\delta_N^s} \circ \ldots \circ h_{\delta_1^u} \circ h_{\delta_1^s} + \cdots + \psi \circ h_{\delta_1^s} - \psi$$

For other higher regularity Sobolev spaces, when $\alpha \in \mathbb{N}$, there are results similar to (4) for Anosov systems. We refer for more details to [dlLMM86] and [dlL01]. The proos use methods from elliptic regularity theory.

3. Statement of results

We recall that our standing assumptions are that M is a compact manifold, f is a $C^{1+\alpha}$ diffeomorphism and η is C^β $0 < \beta \leq \mathrm{Lip}$.

The main technical result of this note is:

Theorem 3.1. *Assume that f admits a dominated splitting with a stable component E^s.*

Then, if $\varphi \in L^p(M, G)$, $1 \leq p < \infty$, then,

$$\varphi \in \mathfrak{L}_\beta^{p/3,(s)}$$

Of course, we have similar results for φ^{-1} in place of φ and for the unstable foliation.

Once we have Theorem 3.1, using the known properties of the potential spaces summarized in Section 2.3, we obtain in the conditions of Theorem 3.1

Corollary 3.2. *Assume that f is uniformly accessible as in Definition 2.4.*

Assume that

$$p \geq 2\frac{dK}{\alpha} \tag{5}$$

where d is the dimension of the manifold.

Then φ is continuous. (Therefore we have $\varphi \in C^{\beta/K}$.)

Moreover, if f is Anosov and $\beta = \mathrm{Lip}$, then $\varphi \in W^{1,p}$.

The proof of the Corollary is quite simple given the material mentioned in Section 2.3.

We note that, by Theorem 3.1 if $\varphi, \varphi^{-1} \in \mathfrak{L}_\beta^{p/3,s} \cap \mathfrak{L}_\beta^{p/3,u} \subset \mathfrak{L}_{\beta/K}^p$.

By Sobolev embedding theorem, we conclude $\varphi, \varphi^{-1} \in L^q$ where $\frac{1}{q} = \frac{3}{p} - \frac{dK}{\beta} - \varepsilon$.

We note that, if p satisfies (5), we have that $q > p$. We can keep repeating the process till we obtain that $\varphi, \varphi^{-1} \in \mathfrak{L}^q_{\beta/K}$ with $q \geq \frac{dK}{\beta}$. Then, we can conclude that φ, φ^{-1} are continuous by Sobolev embedding theorem.

Once that they are continuous, there are arguments in [NT98] to conclude that φ, φ^{-1} are Hölder when restricted to the stable and unstable manifolds. Then, we obtain that φ, φ^{-1} are Hölder. Since we will use variants of the method of [NT98], we we postpone till Remark 4.1 the discussion of the bootstrap from continuous to $C^{\beta/K}$.

4. Proof of Theorem 3.1

We will use a method which is similar to the method used in [dlL01] in the commutative case – iterate the equation (1) and derived formulas for the increments. for the non-commutative case, we use several cancellations that were discovered in [NT98].

4.1. **Some representation formulas.** We note the equation (1) implies that for every, almost every x and for every $n \in \mathbb{N}$ we have

$$\varphi(x) = \eta(x) \cdots \eta \circ f^n(x) \varphi \circ f^{n+1}(x) \tag{6}$$

for μ-almost all x.

Let $\delta(x)$ be a vector field on the manifold M. For a function $\Psi : M \to G$ we define $h_\delta(x) = x + \delta(x)$ and

$$\big[\Delta\varphi\big](x) = \varphi\big(x + \delta(x)\big) - \varphi(x) \ . \tag{7}$$

Adding and subtracting extra terms in (7) we have:

$$\big[\Delta(\Psi\varphi)\big](x) = \Psi\big(x + \delta(x)\big)\big[\Delta\varphi\big](x) + \big[\Delta\Psi\big](x)\varphi(x) \ . \tag{8}$$

Note that this is one of the places where we use the fact that we are considering a group which is a Banach algebra. For a general group, the derivative of the composition operation would appear.

Substituting (8) into (6), we have

$$\begin{aligned} \Delta\varphi(x) = \sum_{i=0}^{n} \eta \circ h_\delta(x) \cdots \eta \circ f^{i-1} \circ h_\delta(x) \\ \cdot \big[\Delta(\eta \circ f^i)\big](x) \eta \circ f^{i+1}(x) \cdots \eta \circ f^n(x) \varphi \circ f^{n+1}(x) \\ + \eta \circ h_\delta(x) \cdots \eta \circ f^n \circ h_\delta(x) \big[\Delta(\varphi \circ f^{n+1})\big](x) \end{aligned} \tag{9}$$

Using again (6) we have:

$$\begin{aligned} \Delta\varphi(x) = \sum_{i=0}^{n} \varphi \circ h_\delta(x) \varphi^{-1} \circ f^i \circ h_\delta(x) \big[\Delta(\eta \circ f^i)\big](x) \varphi \circ f^{i+1}(x) \\ + \varphi \circ h_\delta(x) \varphi^{-1} \circ f^{n+1} \circ h_\delta(x) \big[\Delta(\varphi \circ f^{n+1})\big](x) \ . \end{aligned} \tag{10}$$

Even if formula (6) is quite general, we restrict our attention to the case that δ is adapted to the stable foliation.

In such a case, we estimate the general term of the sum in (10) as:

$$\begin{aligned} &\|\varphi \circ h_\delta \varphi^{-1} \circ f^i \circ h_\delta \Delta(\eta \circ f^i) \varphi \circ f^{i+1}\|_{L^{p/3}} \\ &\qquad \le C\|\varphi\|_{L^p}^2 \|\varphi^{-1}\|_{L^p} \|\Delta(\eta \circ f^i)\|_{L^\infty} \\ &\qquad \le C\|\varphi\|_{L^p}^2 \|\varphi^{-1}\|_{L^p} \|\eta\|_{C^\beta} \|\delta\|_{L^\infty}^\beta \lambda_s^{i\beta} \end{aligned} \tag{11}$$

Hence we see that the $L^{p/3}$ limit as $n \to \infty$ in the sum in (10) exists.

Now we turn to study the last term in (10). We note that

$$\|\varphi \circ h_\delta(x) \varphi^{-1} \circ f^{n+1} \circ h_\delta\|_{L^{p/2}}$$

is bounded uniformly in n. We also note that, when φ is continuous,

$$\|\Delta(\varphi \circ f^{n+1})\|_{L^\infty} \to 0.$$

Since the space of continuous functions is L^p-dense on L^p, an $\epsilon/3$ argument shows that

$$\|\varphi \circ h_\delta(x) \varphi^{-1} \circ f^{n+1} \circ h_\delta(x) \left[\Delta(\varphi \circ f^{n+1})\right]\|_{L^{p/3}} \to 0 \tag{12}$$

Hence, if we take limits in (10), we obtain that

$$\Delta\varphi(x) = \sum_{i=0}^{\infty} \varphi \circ h_\delta(x) \varphi^{-1} \circ f^i \circ h_\delta(x) \left[\Delta(\eta \circ f^i)\right](x) \varphi \circ f^{i+1}(x) \tag{13}$$

From (13) and summing the estimates in (11), we obtain

$$\|\Delta\varphi\|_{L^{p/3}} \le C\|\delta\|_{L^\infty}^\beta$$

where C may depend on φ, φ^{-1} and, of course, of η, f but is independent on δ.

That is, we conclude that $\varphi \in \mathfrak{L}^{p/3}_{\beta,(s)}$.

A similar argument – or just use this result for replacing f, η, φ by f^{-1}, $(\eta^{-1})^t$ $(\varphi^{-1})^t$ – shows that $\varphi \in \mathfrak{L}^{p/3}_{\beta,(u)}$.

An appeal to Lemma 2.12 ends the proof of Theorem 3.1. □

Remark 4.1. We also note that the formula (13) also shows that if φ, φ^{-1} are continuous and η is C^β, then, we can bound $||\Delta\varphi||_{L^\infty} \le C||\delta||_{L^\infty}$. This means that φ is C^β along the stable leafs. A similar argument shows that φ is C^β along the unstable leaves. It is then, elementary to show that $\varphi \in C^{\beta/K}$ provided that the system satisfies Definition 2.4. This is the argument promised in the proof of Corollary3.2 □

5. Some generalizations

We note that the present method can be generalized in several respects. All of them are reasonably straightforward and they can all be carried out simultaneously. The formulation, however would obscure the simple idea.

5.1. **Different integrability conditions for φ, φ^{-1}.** The main result, Theorem 3.1 can be modified so that we do not assume the same integrability for φ and φ^{-1}.

A more precise version of the argument, proved by the same method is

Theorem 5.1. *Assume that f admits a dominated splitting with an stable component E^s.*

Then, if $\varphi \in L^p(M, \mathcal{B})$, $1 \le p < \infty$, $\varphi^{-1} \in L^q(M, \mathcal{B})$, $1 \le q < \infty$, then,

$$\varphi \in \mathfrak{L}_\beta^{p',(s)} \qquad \varphi^{-1} \in \mathfrak{L}_\beta^{q',(s)}$$

where

$$\frac{1}{p'} = \frac{2}{p} + \frac{1}{q}; \qquad \frac{1}{q'} = \frac{1}{p} + \frac{2}{q}.$$

If the system satisfies Definition 2.5, from the assumptions of integrability in Theorem 5.1 we can conclude that

$$\varphi \in \mathfrak{L}_{\beta/K}^{p'} \subset L^{p''-\varepsilon}$$
$$\varphi^{-1} \in \mathfrak{L}_{\beta/K}^{q'} \subset L^{q''-\varepsilon}$$

where

$$\frac{1}{p''} = \frac{2}{p} + \frac{1}{q} - \frac{\beta}{dK}; \qquad \frac{1}{q''} = \frac{1}{p} + \frac{2}{q} - \frac{\beta}{dK}$$

Of course, when the RHS of the above equation is negative, it means that the corresponding function is continuous.

We can see that when $\frac{1}{p} + \frac{1}{q} < \frac{\beta}{dK}$ we can apply repeatedly the bootstrap argument and get that the function is continuous.

5.2. **$\mathfrak{L}_\beta^q$ regularity of η.** The following result is true:

Theorem 5.2. *Assume that f admits a dominated splitting with an stable component E^s.*

Then, if $\varphi \in L^p(M, \mathcal{B})$, $\eta, \eta^{-1} \in \mathfrak{L}_\beta^q$, $q, p \in [1, \infty]$ $0 \le \beta \le 1$. then,

$$\varphi \in \mathfrak{L}_\beta^{p',(s)}$$

where $1/p' = 3/p + 1/q$.

The proof of the result is the same as in the previous cases. We just bound the $L^{p'}$ norm of the general term in (10) using the Hölder inequality. The last term goes to zero.

If $q > p$, one can do better. Using the embedding theorems, we obtain that in the conditions of Theorem 5.2, we have $\varphi \in L^{p''}$ where $1/p'' = 3/p + 1/q - \beta/(Kd) - \varepsilon$. Under the condition $2/p + 1/q \le -\beta/(Kd)$, $p'' > p$ we can repeat the process till we conclude that $\varphi \in \mathfrak{L}_{\beta/K}^{q-\varepsilon}$.

5.3. **Weakening the assumption that f preserves a smooth measure.** We point out that the fact that f preserves a smooth measure is not used much. The only place is in the estimates of the iterated maps. Similar arguments to those outlined here appear in [dlL01].

Hence, instead of (11), we could bound:

$$
\begin{aligned}
&\|\varphi\circ h_\delta\varphi^{-1}\circ f^i\circ h_\delta\Delta(\eta\circ f^i)\varphi\circ f^{i+1}\|_{L^{p/3}}\\
&\qquad\le C\|\varphi\|^2_{L^p}\|\varphi^{-1}\|_{L^p}\|\eta\|_{C^\beta}\|\delta\|^\beta_{L^\infty}\lambda_s^{i\beta}\|J_f\|^{i/p}_{L^\infty}
\end{aligned}
\tag{14}
$$

We see that, provided that

$$\lambda_s^\beta\|J_f\|^{1/p}_{L^\infty}<1 \tag{15}$$

the series in (10) converges.

The analogue of (12) will require more analysis and more assumptions. We start by studying the effect on highly iterated compositions on the differences.

Note that we have: $\varphi\circ f^n\circ h_\delta=\varphi\circ h_{\tilde\delta_n}\circ f^n$ where $h_{\tilde\delta_n}$ is defined so that the equation is satisfied. By the contractivity of the fibers we have

$$\|\tilde\delta_n\|_{L^\infty}\le\|\delta\|_{L^\infty}\lambda_s^n$$

We also have

$$\|J_{\tilde h_n}\|_{L^\infty}\le C(\|J_f\|_{L^\infty}\|J_{f^{-1}}\|_{L^\infty})^n$$

If we assume that $\varphi\in\mathfrak{L}^p_\nu$ for some $\nu>0$ we see that

$$\|\Delta(\varphi\circ f^{n+1}\|_{L^p}\le C\lambda_s^{\nu(n+1)}\|\delta\|^\nu_{L^\infty}(\|J_f\|_{L^\infty}\|J_{f^{-1}}\|_{L^\infty})^n$$

Since

$$\|\varphi\circ h_\delta\varphi^{-1}\circ f^{n+1}\circ h_\delta\|_{L^{p/2}}\le\|J_f\|^{n/p}_{L^\infty}$$

We see that we recover (12) provided that

$$\|J_f\|^{1+1/p}_{L^\infty}\|J_{f^{-1}}\|_{L^\infty}\lambda_s^\nu<1. \tag{16}$$

The rest of the proof goes through. Hence, we can see that the assumption that f preserves a smooth measure can be somewhat relaxed to assuming that $\varphi\in\mathfrak{L}^p_\gamma$ where the parameters satisfy (15), (16).

5.4. **Exploiting the structure of the group and smallness assumptions on η.** We assume that we have bounds

$$|\eta(x)\cdots\eta\circ f^i(x)|\le C\rho^n \tag{17}$$

with $C\ge 0$, $1\le\rho$ sufficiently small. (of course, if $\rho<1$, it is trivial to show that there are only trivial solutions) Such bounds happen for example when η takes values in unipotent groups or when η is uniformly small.

We also assume that $\varphi\in\mathcal{L}^p_\nu$ for some $\nu>0$.

Then, we can proceed to we could proceed to bound the general term in (9) rather than bounding the general term in (10). We see that the general term in the sum in (9) can be bounded

(18)
$$\|\eta\circ h_\delta \cdots \eta\circ f^{i-1}\circ h_\delta\big[\Delta(\eta\circ f^i)\big]\eta\circ f^{i+1}\cdots\eta\circ f^n(x)\varphi\circ f^{n+1}\|_{L^p} \le C\rho^n\lambda_<^{i\beta}\|\delta\|_{L^\infty}^\beta$$

The last term in (9) can be bounded by $C\rho^n\lambda_<^{n\nu}\|\delta\|_{L^\infty}^\nu$

Summing (18) from $i=0$ to n and adding the bound for the last term we obtain that for every $n\in\mathbb{N}$, we have:

$$\|\Delta\varphi\|_{L^p} \le C\rho^n(\|\delta\|_{L^\infty}^\beta + \lambda_<^{n\nu}\|\delta\|_{L^\infty}^\nu) \tag{19}$$

Choosing $n=\log\delta^{\beta-\nu}/\log(\lambda_<^\nu)$, we obtain that (19) implies

$$\|\Delta\varphi\|_{L^p} \le \|\delta\|_{L^\infty}^\gamma$$

with

$$\gamma = (\beta-\nu)\frac{\log\rho}{\log(\lambda_<^\nu)} + \beta$$

In the nilpotent case, when we can take ρ as close to 1 as desired we see that we can make γ as close to β as desired.

This tells that, in the nilpotent case, from $\varphi\in L^p$ we can conclude $\varphi\in L^{p,(s)}_{\beta-\varepsilon}$.

This is very similar to the result Theorem 3.1 except because we integrability exponent is not divided by 3. This makes much easier to bootstrap.

Using the Sobolev embedding Theorem, we conclude that $\varphi\in L^{p'}$ where

$$\frac{1}{p'} = \frac{1}{p} - \frac{\beta}{dK}.$$

Since we can repeat the steps, we end up concluding that φ is continuous.

Hence, we have established that in the case that η is nilpotent, in the same conditions of Corollary 3.2, from the fact that $\varphi\in\mathfrak{L}^p_\nu$ with $\nu\ge 0$, $1\le p\le\infty$ we conclude that φ is continuous.

5.5. **Results for flows.** There are very similar results for flows. We note that Theorem 3.1 applied to the time one map, gives a result for the strong stable direction of the flow. We also note that, for the cohomology equations for flows, regularity along the flow is automatic.

We also note that Proposition 2.12 is elementary and, hence, holds whenever we have a number of transversal foliations satisfying an analogue of Definition 2.5.

Another rather straightforward case is cohomology over Cartan actions (we refer to [Hur94] for the definitions) provided that we have an analogous of Definition 2.5.

5.6. **Some possible future extensions and open problems.** The main idea of this paper that we sometimes we can obtain information on Sobolev regularity by examining modulus of continuity in some integral norm.

This is useful in the study of cohomology equations because information on modulus of continuity can be obtained by iterating the equations and deriving asymptotic formulas for increments when we displace along stable or unstable manifolds.

We point out that our study is very preliminary and we have only started with the easiest cases where uniform moduli of continuity are used. There are many more sophisticated spaces which could be useful in these studies. (In particular the spaces that [Ste70] calls $\Lambda_\alpha^{p,q}$.) It seems to us that the main obstacle in using these spaces is the somewhat technical problem that many of the proofs use the Euclidean structure.

One important result that is still missing is an analogue of (2.12) for higher differentiability properties. Since the methods of proof the regularity results Lemma 2.5 in [dlLMM86] are based in Schauder estimates, which generalize well to Sobolev spaces, it seems that this program of study has good chances of success. Of course, the regularity results for systems with Definition 2.4 remain wide open.

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