Convergence results for a coarsening model using global linearization

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September 13, 2002

Abstract

We study a coarsening model describing the dynamics of interfaces in the one-dimensional Allen-Cahn equation. Given a partition of the real line into intervals of length greater than one, the model consists in constantly eliminating the shortest interval of the partition by merging it with its two neighbors. We show that the mean-field equation for the time-dependent distribution of interval lengths can be explicitly solved using a global linearization transformation. This allows us to derive rigorous results on the long-time asymptotics of the solutions. If the average length of the intervals is finite, we prove that all distributions approach a uniquely determined self-similar solution. We also obtain global stability results for the family of self-similar profiles which correspond to distributions with infinite expectation.

1 Introduction

Consider a domain $D \subset \mathbb{R}^n$ which is divided into a large number of subdomains (or cells) of different sizes, separated by domain walls, and assume that the system evolves in such a way that the larger subdomains grow with time while the smaller ones shrink and eventually disappear. In particular, the average size of the cells increases, so that the subdivision of D becomes rougher and rougher. Such a *coarsening* dynamics is observed in many physical situations, especially near a phase transition when a system is quenched from a homogeneous state into a state of coexisting phases. Typical examples are the formation of microstructure in alloy solidification [LiS61] and the phase separation in lattice spin systems [De97, KBN97]. Closely related to coarsening is the coagulation (or aggregation) process which describes the dynamics of growing and coalescing droplets [DGY91, PeR92, Vo85]. In this case, the system consists of a large number of particles of different masses which interact by forming clusters. Again, the total mass is preserved, so that the average mass per cluster increases with time.

Given a coarsening or a coagulation model, the main task is to predict the long-time evolution of the size distribution of the cells, or the mass distribution of the clusters. In many cases, experiments and numerical calculations show that this behavior is asymptotically self-similar: the system can be described by a single length scale $\mathcal{L}(t)$, and the distribution approaches the scaling form $\mathcal{L}(t)^{-1}\Phi(x/\mathcal{L}(t))$ as $t \to \infty$. The profile Φ and the asymptotics of $\mathcal{L}(t)$ can sometimes be determined exactly [NaK86, BDG94]. However, even in simple situations, it is very difficult to prove that the size distribution actually converges to a self-similar profile.

In this work, we consider a simple coarsening model related to the one-dimensional Allen-Cahn equation $\partial_t u = \partial_x^2 u + \frac{1}{2}(u - u^3)$, where $x \in \mathbb{R}$. The equilibria of this system are the homogeneous steady states $u = \pm 1$, together with the kinks $u(x) = \pm \tanh(x/2)$ which represent domain walls separating regions of different "phases". If u is any bounded solution of this equation, then for $t > 0$ sufficiently large the graph of $u(t, \cdot)$ will typically look like a (countable) family of kinks separated by large intervals on which $u \approx \pm 1$. If we denote by $x_j(t)$ the position of the j^{th} kink and if we assume that $x_{j+1}(t) - x_j(t) \gg 1$ for all $j \in \mathbb{Z}$, a rigorous asymptotic analysis shows that $\dot{x}_j \approx F(x_{j+1} - x_j) - F(x_j - x_{j-1})$, where $F(y) = 24\mathrm{e}^{-y}$ [CaP89]. In other words, the positions of the domain walls behave like a system of point particles with short range attractive pair interactions. Thus, on an appropriate time scale, only the closest pairs of kinks will really move; in such pairs, kinks will attract each other until they eventually annihilate.

This kink dynamics suggests the following coarsening model [NaK86, DGY91, CaP92, BDG94, RuB94, BrD95, CaP00]. Consider a partition of the real line $\mathbb{R}$ into a countable union of disjoint intervals I_j, with $\ell(I_j) \geq 1$ for all $j \in \mathbb{Z}$. In the previous picture, the intervals I_j correspond to regions where u is close to ± 1. A dynamics on this configuration space is defined by iterating the following coarsening step: choose the "smallest" interval in the partition, and merge it with its two nearest neighbors. This model clearly mimics the dynamics of the domain walls in the one-dimensional Allen-Cahn equation. However, proving that the formal procedure described above actually defines a well-posed evolution (e.g. for almost all initial configurations) and investigating its statistical properties after many coarsening iterations is a non-trivial task, which has not been accomplished so far. Instead, the coarsening model has been studied in the *mean field* approximation, which consists in merging the minimal interval not with its true neighbors, but with two intervals chosen at random in the configuration $\{I_j\}_{j\in\mathbb{Z}}$. This approximation is valid provided the lengths of consecutive intervals stay uncorrelated during the coarsening process, an assumption that seems reasonable [BDG94].

Under this hypothesis, it is possible to write a closed evolution equation for the distribution $f(t,x)$ (per unit length) of intervals of length $x \geq 1$ at time t [CaP92]. Denoting by $N(t) = \int_0^\infty f(t,x)\,\mathrm{d}x$ the total number of intervals per unit length, and by $\mathcal{L}(t)$ the length of the smallest interval, the equation reads

$$\partial_t f(t,x) = \frac{\dot{\mathcal{L}}(t) f(t,\mathcal{L}(t))}{N(t)^2} \left(\int_0^{x-\mathcal{L}(t)} f(t,y) f(t,x-y-\mathcal{L}(t))\,\mathrm{d}y - 2f(t,x)N(t) \right), \tag{1.1}$$

for $x \geq \mathcal{L}(t)$, whereas $f(t,x) = 0$ for $x < \mathcal{L}(t)$ by the definition of $\mathcal{L}(t)$. By construction, $N(t)$ decreases with time, while the total length of the intervals $\int_0^\infty x f(t,x)\,\mathrm{d}x$ is conserved.

We prefer to work with the distribution density $\rho(t,x) = f(t,x)/N(t)$, which satisfies $\rho(t,x) = 0$ for $x < \mathcal{L}(t)$ and the normalization $\int_0^\infty \rho(t,x)\,\mathrm{d}x = 1$ for all t. The evolution equation for ρ reads

$$\partial_t\rho(t,x) = \dot{\mathcal{L}}(t)\rho(t,\mathcal{L}(t))\int_0^{x-\mathcal{L}(t)} \rho(t,y)\rho(t,x-y-\mathcal{L}(t))\,\mathrm{d}y \quad \text{for } x \ge \mathcal{L}(t). \tag{1.2}$$

Of course, systems (1.1) and (1.2) are equivalent. In particular, once the density $\rho(t,x)$ is known, the total number $N(t)$ can be recovered by solving the ordinary differential equation $\dot{N}(t) = -2\dot{\mathcal{L}}(t)\rho(t,\mathcal{L}(t))N(t)$, and the distribution $f(t,x)$ is then given by $N(t)\rho(t,x)$.

It is important to note that equations (1.1), (1.2) are invariant under reparametrizations of time. As a consequence, the minimal length $\mathcal{L}(t)$ is not determined by the initial data, but can be prescribed to be an arbitrary (increasing) function of time. In [CaP92], the authors define an "intrinsic time" by imposing the relation $f(t,\mathcal{L}(t))\dot{\mathcal{L}}(t) = 1$, which means that the number of merging events per unit time is constant. We find it more convenient to use the "coarsening time" defined by the simple relation $\mathcal{L}(t) = t$. In other words, we choose to parameterize the coarsening process by the length of the smallest remaining interval, forgetting about how much physical time elapses between or during the merging events. With our choice, equation (1.2) becomes

$$\partial_t\rho(t,x) = \rho(t,t)\int_0^{x-t} \rho(t,y)\rho(t,x-y-t)\,\mathrm{d}y \quad \text{for } x \ge t. \tag{1.3}$$

Since we do not allow for intervals of length smaller than 1, we impose our initial condition at time $t = 1$: $\rho(1,x) = \rho_1(x)$.

The aim of this paper is to show that the dynamics of (1.3) can be completely understood using a global linearization transformation. As a consequence, we are able to prove that solutions of (1.3) satisfying $\int_0^\infty x\rho(t,x)\,\mathrm{d}x < \infty$ approach a non-trivial self-similar profile as $t \to \infty$. To achieve this goal, we first rewrite (1.3) in similarity coordinates by setting

$$\rho(t,x) = \frac{1}{t}\eta(\log t, x/t), \quad \text{or} \quad \eta(\tau,y) = \mathrm{e}^\tau\rho(\mathrm{e}^\tau, \mathrm{e}^\tau y),$$

where $\tau = \log t \ge 0$ and $y = x/t \in [1,\infty)$. Then the rescaled density $\eta(\tau,\cdot)$ lies in the time-independent space

$$\mathbb{P} = \Big\{\eta \in L^1((1,\infty),\mathbb{R}_+) \;\Big|\; \int_1^\infty \eta(y)\,\mathrm{d}y = 1\Big\}, \tag{1.4}$$

which is a closed convex subset of $L^1((1,\infty))$. Moreover, (1.3) is transformed into the autonomous evolution equation

$$\partial_\tau\eta(\tau,y) = \partial_y\big(y\,\eta(\tau,y)\big) + \eta(\tau,1)\int_1^{y-2} \eta(\tau,z)\eta(\tau,y-z-1)\,\mathrm{d}z \quad \text{for } y \ge 1. \tag{1.5}$$

In Section 3 we show that, for all initial data $\eta_0 \in \mathbb{P}$, (1.5) has a unique global solution $\eta \in C^0([0,\infty),\mathbb{P})$ with $\eta(0) = \eta_0$.

We now define a nonlinear map $\mathcal{N} : \mathbb{P} \to L^1_{\mathrm{loc}}([1,\infty),\mathbb{R}_+)$ by

$$\mathcal{N} = \mathcal{F}^{-1} \circ \phi \circ \mathcal{F},$$

where $\mathcal{F}$ is the Fourier transform and $\phi(z) = \frac{1}{2}\log\frac{1+z}{1-z}$. If $\eta(\tau,\cdot)$ is a solution of (1.5) in $\mathbb{P}$, a direct calculation reveals that $w(\tau,\cdot) = \mathcal{N}(\eta(\tau,\cdot))$ satisfies the linear equation $\partial_\tau w(\tau,y) = \partial_y(yw(\tau,y))$. As a consequence,

$$w(\tau,y) = (S_\tau w_0)(y) = \begin{cases} \mathrm{e}^\tau w_0(\mathrm{e}^\tau y) & \text{if } y \geq 1, \\ 0 & \text{if } y < 1, \end{cases}$$

where $w_0 = \mathcal{N}(\eta_0)$. It follows that any solution $\eta \in C^0([0,\infty),\mathbb{P})$ of (1.5) satisfies $\mathcal{N}(\eta(\tau)) = S_\tau \mathcal{N}(\eta_0)$ for all $\tau \geq 0$. In other words, the nonlinear evolution defined by (1.5) is *conjugated* (via the map $\mathcal{N}$) to the linear semigroup (S_τ). Thus, the difficulty of solving (1.5) is carried over to the study of the mapping $\mathcal{N}$ and of its inverse $\mathcal{N}^{-1} = \mathcal{F}^{-1}\circ\phi^{-1}\circ\mathcal{F}$. Although the properties of these maps are not fully understood, it possible to obtain some information on them using the analyticity properties of the Fourier-Laplace transform.

In Section 4 we investigate the steady states of (1.5), which form a one-parameter family $\{\eta_\theta^*\}_{\theta\in\mathbb{R}}$. Here $\eta_\theta^* = \mathcal{N}^{-1}(\frac{\theta}{2}w^*)$, where $w^*(y) = y^{-1}\mathbf{1}_{\{y\geq 1\}}$. More explicitly, we have

$$\widehat{\eta_\theta^*}(\xi) = (\mathcal{F}\eta_\theta^*)(\xi) = \tanh\Big(\frac{\theta}{2}\,\mathrm{E}_1(\mathrm{i}\xi)\Big) \quad \text{for } \xi \in \mathbb{R}, \tag{1.6}$$

where E_1 is the exponential integral [AS72]. We prove that $\eta_\theta^* \in \mathbb{P}$ if and only if $\theta \in (0,1]$. Moreover, $\eta_1^*(y)$ decays exponentially as $y \to \infty$, while $\eta_\theta^*(y) \sim y^{-(1+\theta)}$ if $0 < \theta < 1$. In particular, η_1^* is the only steady state for which the average length $\int_1^\infty y\eta_1^*(y)\,\mathrm{d}y$ is finite.

Finally, Section 5 is devoted to the convergence results. If the initial data $\eta_0 \in \mathbb{P}$ satisfy $y^\gamma\eta_0 \in L^2((1,\infty))$ for some $\gamma > 3/2$ (so that $\int_1^\infty y\eta_0(y)\,\mathrm{d}y < \infty$), we prove that the corresponding solution of (1.5) converges exponentially to the steady state η_1^*:

$$\|y^{\gamma-1}(\eta(\tau) - \eta_1^*)\|_{L^2((1,\infty))} = \mathcal{O}(\mathrm{e}^{-(\gamma-3/2)\tau}) \quad \text{for } \tau \to \infty.$$

In terms of the original variables, this shows that the density $\rho(t,x)$ asymptotically approaches the self-similar solution $t^{-1}\eta_1^*(x/t)$ of (1.3). Moreover, the remainder is $\mathcal{O}(t^{-(\gamma-3/2)})$, so that the convergence is very fast if γ is large, i.e., the initial data decay rapidly at infinity. Similarly, if $0 < \theta < 1$ and if $\eta_0 \in \mathbb{P}$ satisfies $y^\gamma(\eta_0 - \nu\eta_\theta^*) \in L^2((1,\infty))$ for some $\gamma > \theta + 1/2$ and some $\nu > 0$, we prove that the solution of (1.5) with initial data η_0 converges to the steady state η_θ^*.

To conclude this section, we briefly comment on previous results and possible generalizations. The mean field equations (1.1) and especially the self-similar solutions (1.6) can be found in many physics papers [NaK86, DGY91, BDG94, RuB94, BrD95]. The first mathematical work is [CaP92], where the authors prove the existence of global solutions to (1.1). They also show that the profile η_1^* is a positive function (a crucial property that is tacitly assumed in the physics literature!) and study its asymptotic behavior as $y \to \infty$. Our main contribution is the introduction of the linearization transformation $\mathcal{N}$ which allows to prove the convergence results. We also extend the analysis of [CaP92] to the equilibria η_θ^* with $0 < \theta < 1$.

The "two-sided" coarsening model discussed in this introduction is clearly not the most general system to which our analysis applies. For instance, we can consider the "one-sided" variant in which the minimal interval is only merged with *one* of its neighbors [CaP00]. More generally, we can assume that, for $j = 1,\ldots,N$, the minimal interval has a probability p_j of being merged with j of its neighbors, where $p_1 + \cdots + p_N = 1$. In the mean field approximation, this leads to an evolution equation similar to (1.3), where the quadratic convolution in the right-hand side is replaced by a more general convolution polynomial. Except for a modified definition of the mapping $\mathcal{N}$, this extension does not affect our analysis in any essential way. Therefore, in the rest of this paper, all results will be stated and proved in this general situation.

Acknowledgments. The authors are grateful for financial support through the French-German grant: PROCOPE 00307 TK, *Attractors for Extended Systems.*

2 The coarsening equation and its solution

As is explained in the introduction, we shall study a general coarsening model for which the number of intervals involved in each merging event is not necessarily fixed. Instead, we allow for some randomness by choosing nonnegative real numbers $p_1, \ldots, p_N$ satisfying $p_1 + \cdots + p_N = 1$, where p_j is interpreted as the probability for an interval of minimal length to merge with j other intervals. We define the polynomial

$$Q(z) = \sum_{j=1}^{N} p_j z^j,$$

which satisfies $Q(1) = 1$. The original coarsening model related to the Allen-Cahn equation corresponds to the particular case where $Q(z) = z^2$.

If $\rho \in L^1(\mathbb{R})$, we set

$$\mathbb{Q}[\rho] = \sum_{j=1}^{N} p_j \rho^{*j}, \tag{2.1}$$

where $\rho^{*j} = \rho * \rho * \cdots * \rho$ (j factors) and $*$ denotes the convolution product in $L^1(\mathbb{R})$. In particular, we have $\int_0^\infty \mathbb{Q}[\rho](x)\,\mathrm{d}x = Q(\int_0^\infty \rho(x)\,\mathrm{d}x)$. In what follows, we shall mainly use the space $\mathbb{P}$ of probability densities defined by (1.4). Any $\rho \in \mathbb{P}$ can be extended to the whole real line by setting $\rho(x) = 0$ for $x < 1$. This natural extension, still denoted by ρ, will be used in the sequel without further mention. As an example of this abuse of notation, if $\rho \in \mathbb{P}$, we have $\mathbb{Q}[\rho] \in \mathbb{P}$ and $\mathrm{support}(\mathbb{Q}[\rho]) \subset [2, \infty)$.

The problem we are interested in can now be stated as follows. Given $\rho_1 \in \mathbb{P}$, find a density $\rho : [1,\infty)^2 \to \mathbb{R}_+$ satisfying $\rho(1,x) = \rho_1(x)$ for $x \geq 1$, $\rho(t,x) = 0$ for $1 \leq x < t$, and

$$\partial_t \rho(t,x) = \rho(t,t)\mathbb{Q}[\rho(t,\cdot)](x-t) \quad \text{for } x \geq t \geq 1. \tag{2.2}$$

If $Q(z) = z^2$, the evolution equation (2.2) reduces to (1.3).

By assumption, the density $\rho(t,x)$ is nonzero only in the sector $\{(t,x) \in \mathbb{R}^2 | 1 \leq t \leq x\}$, where it satisfies (2.2). An important role will be played by the values of ρ on the boundaries of this domain, namely the initial density ρ_1 and the trace of ρ on the diagonal $x = t$, which we denote by α:

$$\alpha(t) = \rho(t,t) \quad \text{for } t \geq 1.$$

Any sufficiently smooth solution of (2.2) satisfies $\rho(t,\cdot) \in \mathbb{P}$ for all $t \geq 1$ provided $\rho_1 \in \mathbb{P}$. Indeed, it is obvious from (2.2) that ρ stays nonnegative. Moreover, if $m(t) = \int_t^\infty \rho(t,x)\,\mathrm{d}x$, a direct calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) = \alpha(t)\big(Q(m(t)) - 1\big) \quad \text{for } t \geq 1. \tag{2.3}$$

Therefore, if $m(1) = 1$, then $m(t) = 1$ for all $t \geq 1$.

A very remarkable property of equation (2.2) is that it can be explicitly solved using Fourier (or Laplace) transform. If $\rho \in \mathbb{P}$, we define

$$\widehat{\rho}(\xi) = (\mathcal{F}\rho)(\xi) = \int_1^\infty \mathrm{e}^{-\mathrm{i}\xi x}\rho(x)\,\mathrm{d}x \quad \text{for } \xi \in \mathbb{R}.$$

Then $\widehat{\rho} \in C^0(\mathbb{R},\mathbb{C})$ satisfies $\widehat{\rho}(0) = 1$, $|\widehat{\rho}(\xi)| < 1$ for all $\xi \neq 0$, and $\widehat{\rho}(\xi) \to 0$ as $\xi \to \pm\infty$. Moreover, $\widehat{\rho}$ is a positive definite function (in the sense of Bochner). Since $\mathrm{support}(\rho) \subset [1,\infty)$, the Fourier transform $\widehat{\rho}$ can be continuously extended to the lower complex half plane

$$\mathbb{L}^- = \{\xi \in \mathbb{C} \mid \operatorname{Im}\xi \leq 0\}.$$

This extension (still denoted by $\widehat{\rho}$) is analytic in the interior of $\mathbb{L}^-$ and satisfies the bound $|\widehat{\rho}(\xi)| \leq \mathrm{e}^{\operatorname{Im}\xi}$ for all $\xi \in \mathbb{L}^-$.

Remark. The closely related Laplace transform is defined by

$$\widetilde{\rho}(p) = (\mathcal{L}\rho)(p) = \int_1^\infty \mathrm{e}^{-px}\rho(x)\,\mathrm{d}x \quad \text{for } \operatorname{Re}p \geq 0,$$

so that $\widetilde{\rho}(p) = \widehat{\rho}(-\mathrm{i}p)$. In the sequel, we prefer using Fourier transform instead of Laplace because the inversion formula is more natural.

Applying Fourier transform to (2.2) and using the fact that convolutions are turned into multiplications, we find the equation

$$\partial_t\widehat{\rho}(t,\xi) = \alpha(t)\,\mathrm{e}^{-\mathrm{i}\xi t}\Big(Q(\widehat{\rho}(t,\xi)) - 1\Big) \quad \text{for } t \geq 1, \tag{2.4}$$

where $\alpha(t) = \rho(t,t)$. To solve (2.4), we introduce the nonlinear complex transformation ϕ defined by

$$\phi'(z) = \frac{1}{1 - Q(z)}\,, \quad \phi(0) = 0. \tag{2.5}$$

Remark that $\phi'(z) = \sum_{k=0}^\infty [Q(z)]^k$, so that ϕ has a power series expansion with non-negative coefficients whose radius of convergence is equal to 1. In particular, the map $\phi : [0,1) \to [0,\infty)$ is one-to-one and onto. Let $\psi = \phi^{-1}$ be the inverse map, which satisfies

$$\psi'(w) = 1 - Q(\psi(w)), \quad \psi(0) = 0. \tag{2.6}$$

By construction, ψ is analytic in a neighborhood of the real positive axis. In the particular case where $Q(z) = z^2$, one finds

$$\phi(z) = \frac{1}{2}\log\frac{1+z}{1-z} \quad \text{and} \quad \psi(w) = \tanh(w).$$

Applying the nonlinear transformation ϕ simplifies equation (2.4) a lot. The function $\widehat{w}(t,\xi) = \phi(\widehat{\rho}(t,\xi))$, which is defined at least for $\operatorname{Im}\xi < 0$, satisfies the differential equation

$$\partial_t \widehat{w}(t,\xi) = -\alpha(t)\mathrm{e}^{-\mathrm{i}\xi t} \quad \text{for } t \geq 1,$$

which has the explicit solution

$$\widehat{w}(t,\xi) = \widehat{w}(1,\xi) - \int_1^t \alpha(s)\mathrm{e}^{-\mathrm{i}\xi s}\,\mathrm{d}s \quad \text{for } t \geq 1 \text{ and } \operatorname{Im}\xi < 0. \tag{2.7}$$

Remark that $|\widehat{\rho}(t,\xi)| \leq \mathrm{e}^{t\operatorname{Im}\xi}$ for all $t \geq 1$ and all $\xi \in \mathbb{L}^-$, because $\rho(t,\cdot) \in \mathbb{P}$ and $\operatorname{support}(\rho(t,\cdot)) \subset [t,\infty)$. Since $\phi(z) = z + \mathcal{O}(|z|^2)$ as $z \to 0$, it follows that $|\widehat{w}(t,\xi)| = |\phi(\widehat{\rho}(t,\xi))| \to 0$ as $t \to \infty$ if $\operatorname{Im}\xi < 0$. Thus, taking the limit $t \to \infty$ in (2.7), we find $\widehat{w}(1,\xi) = \int_1^\infty \alpha(t)\,\mathrm{e}^{-\mathrm{i}\xi t}\,\mathrm{d}t$, which in turn implies

$$\widehat{w}(t,\xi) = \int_t^\infty \alpha(s)\mathrm{e}^{-\mathrm{i}\xi s}\,\mathrm{d}s \quad \text{for } t \geq 1 \text{ and } \operatorname{Im}\xi < 0. \tag{2.8}$$

This formula has a very nice interpretation. Let $\mathcal{N}$ be the nonlinear transformation defined (at least formally) by

$$\mathcal{N} = \mathcal{F}^{-1} \circ \phi \circ \mathcal{F} \quad \text{or} \quad \mathcal{N}^{-1} = \mathcal{F}^{-1} \circ \psi \circ \mathcal{F}. \tag{2.9}$$

Setting $t = 1$ in (2.8), we obtain $\phi(\widehat{\rho}_1) = \widehat{\alpha}$, that is $\alpha = \mathcal{N}(\rho_1)$. In other words, the trace $\alpha(t) = \rho(t,t)$ is obtained from the initial density $\rho_1(x) = \rho(1,x)$ by applying the nonlinear map $\mathcal{N}$. Moreover, if $U(t)$ is the linear operator defined for $t \geq 1$ by

$$(U(t)w)(s) = \mathbf{1}_{\{s\geq t\}} w(s) = \begin{cases} 0 & \text{if } s < t, \\ w(s) & \text{if } s \geq t, \end{cases} \tag{2.10}$$

then (2.8) reads $\widehat{w}(t,\cdot) = \phi(\widehat{\rho}(t,\cdot)) = \mathcal{F}(U(t)\alpha)$, which means $\mathcal{N}(\rho(t,\cdot)) = U(t)\alpha$. Therefore, the solution of (2.2) satisfies

$$\mathcal{N}(\rho(t,\cdot)) = U(t)\mathcal{N}(\rho_1) \quad \text{for } t \geq 1. \tag{2.11}$$

This shows that the dynamics of the nonlinear system (2.2) is conjugated via the nonlinear mapping $\mathcal{N}$ to the linear evolution U. Since $\mathcal{N}(\rho_1)$ is the trace function defined by $\alpha(t) = \rho(t,t)$, it is very natural that the evolution of α is obtained just by cutting off the history in $[1,t)$.

It is not difficult to show that the map $\mathcal{N}$ is well-defined on the space $\mathbb{P}$, cf. (1.4):

Proposition 2.1 *If $\rho \in \mathbb{P}$, then $\mathcal{N}(\rho) \in L^1_{\mathrm{loc}}([1,\infty),\mathbb{R}_+)$, and the mapping $\rho \mapsto \mathcal{N}(\rho)$ is one-to-one.*

Proof. For $\rho \in \mathbb{P}$ we construct $w = \mathcal{N}(\rho)$ as follows. Define $\widehat{w} : \mathbb{L}^-_* \to \mathbb{C}$ by $\widehat{w}(\xi) = \phi(\widehat{\rho}(\xi))$, where $\mathbb{L}^-_* = \mathbb{L}^- \setminus \{0\}$. We recall that $\widehat{\rho}$ is continuous on $\mathbb{L}^-$, analytic in the interior of $\mathbb{L}^-$, and that $|\widehat{\rho}(\xi)| < 1$ for $\xi \neq 0$. Since ϕ is analytic in the unit disk of $\mathbb{C}$, it follows that $\widehat{w}$ is continuous on $\mathbb{L}^-_*$ and analytic in the interior of $\mathbb{L}^-$. Moreover, $|\widehat{w}(\xi)| \leq \phi(|\widehat{\rho}(\xi)|) \leq \phi(\mathrm{e}^{\operatorname{Im}\xi})$, hence $|\widehat{w}(\xi)| = \mathcal{O}(\mathrm{e}^{\operatorname{Im}\xi})$ as $\operatorname{Im}\xi \to -\infty$. These properties imply (see [Sch66], Ch. VIII) that $\widehat{w}$ is the Fourier transform of a uniquely determined distribution $w \in \mathcal{D}'(\mathbb{R})$ with support in $[1,\infty)$.

The injectivity of $\mathcal{N}$ follows from the facts that the mapping $\phi : \{z \mid |z| < 1\} \to \mathbb{C}$ is locally injective (as $\phi'(z) = 1/(1-Q(z)) \neq 0$) and that $\phi : [0,1) \to \mathbb{R}$ is globally injective (as $\phi'(s) \geq 1$ for $s \in [0,1)$). If $\mathcal{N}(\rho_1) = \mathcal{N}(\rho_2)$, then, by the above, we have $\phi(\widehat{\rho}_1(\xi)) = \phi(\widehat{\rho}_2(\xi))$ for $\xi \in \mathbb{L}_*^-$. This proves $\widehat{\rho}_1(-\mathrm{i}p) = \widehat{\rho}_2(-\mathrm{i}p)$ for $p > 0$, as $\widehat{\rho}_j(-\mathrm{i}p) \in [0,1)$. By continuity of $\widehat{\rho}_j$ and local invertibility we obtain $\widehat{\rho}_1 = \widehat{\rho}_2$ on $\mathbb{L}_*^-$, and hence $\rho_1 = \rho_2$.

To prove $w = \mathcal{N}(\rho) \in L^1_{\mathrm{loc}}([1,\infty))$, choose any $\varepsilon > 0$ and consider the distribution $w_\varepsilon : x \mapsto \mathrm{e}^{-\varepsilon x} w(x)$. It belongs to $\mathcal{S}'(\mathbb{R})$ (the space of tempered distributions) and its Fourier transform satisfies

$$\widehat{w}_\varepsilon(\xi) = \widehat{w}(\xi - \mathrm{i}\varepsilon) = \phi(\widehat{\rho}(\xi - \mathrm{i}\varepsilon)) \quad \text{for } \operatorname{Im}\xi \leq 0.$$

Now we observe that $\widehat{\rho}(\xi - \mathrm{i}\varepsilon) = \widehat{\rho}_\varepsilon(\xi)$, where $\rho_\varepsilon(x) = \mathrm{e}^{-\varepsilon x}\rho(x)$. Since $\|\rho_\varepsilon\|_{L^1} \leq \mathrm{e}^{-\varepsilon} < 1$, the series $\sum_{k=1}^\infty \frac{\phi^{(k)}(0)}{k!}\rho_\varepsilon^{*k}$ converges in $L^1(\mathbb{R})$ to some function $W_\varepsilon \in L^1([1,\infty),\mathbb{R}_+)$. (Here we use the crucial fact that $\phi^{(k)}(0) \geq 0$ for all $k \in \mathbb{N}$.) By construction,

$$\widehat{W}_\varepsilon(\xi) = \sum_{k=1}^\infty \frac{\phi^{(k)}(0)}{k!}\big(\widehat{\rho}_\varepsilon(\xi)\big)^k = \phi(\widehat{\rho}(\xi - \mathrm{i}\varepsilon)) = \widehat{w}_\varepsilon(\xi) \quad \text{for } \operatorname{Im}\xi \leq 0,$$

giving $w_\varepsilon = W_\varepsilon \in L^1((1,\infty),\mathbb{R}_+)$, and hence $w : x \mapsto \mathrm{e}^{\varepsilon x} w_\varepsilon(x)$ lies in $L^1_{\mathrm{loc}}([1,\infty),\mathbb{R}_+)$. ■

Remarks.

1. Under the assumptions of Proposition 2.1, one has that $w = \mathcal{N}(\rho) \in \mathcal{S}'(\mathbb{R})$, i.e., w is a tempered distribution. In fact, there exists a constant $C > 0$ such that $|\widehat{w}(\xi)| = |\phi(\widehat{\rho}(\xi))| \leq C\max\{1, -\log|\xi|\}$ for $\xi \neq 0$, see the proof of Proposition 5.1 below. This means that the singularity of $\widehat{w}(\xi)$ at $\xi = 0$ is (not worse than) logarithmic.

2. More information on $\mathcal{N}$ can be extracted from the proof of Proposition 2.1. For instance, if $\rho \in \mathbb{P}$, then $\mathcal{N}(\rho)(x) = \rho(x)$ for a.a. $x \in (1, n+1)$, where

$$n = \min\big\{j \in \{1,\dots,N\} \,\big|\, p_j > 0\big\} \geq 1 \tag{2.12}$$

is the largest integer such that $|Q(z)| = \mathcal{O}(|z|^n)$ as $z \to 0$. Indeed, in view of (2.5), one has $\phi(z) = z + \mathcal{O}(|z|^{n+1})$ as $z \to 0$. It follows that

$$W_\varepsilon = \rho_\varepsilon + \sum_{k=n+1}^\infty \frac{\phi^{(k)}(0)}{k!}\rho_\varepsilon^{*k},$$

where the second term in the right-hand side is supported in the interval $[n+1,\infty)$. Thus $W_\varepsilon = \rho_\varepsilon$ a.e. in $[1, n+1]$, which proves the claim. Similarly, using the observation that $\mathrm{support}(\rho_\varepsilon^{*k}) \subset [k,\infty)$, it is easy to show that, if $\rho : [1,\infty) \to \mathbb{R}_+$ is continuous, so is $\mathcal{N}(\rho)$.

The formula (2.11) is very nice, but does not provide an effective method for solving the Cauchy problem associated with (2.2). Indeed, Proposition 2.1 does not give a sufficient characterization of the set $\mathcal{N}(\mathbb{P})$, which is also the domain of $\mathcal{N}^{-1}$. It is not even clear a priori that this set in left invariant by the linear evolution $U(t)$. For this reason, we shall use standard PDE techniques to prove existence of solutions to (2.2) in the next section. But the representation (2.11) will be very useful to find self-similar solutions of (2.2) in Section 4, and to study their stability in Section 5.

3 The Cauchy problem for the rescaled system

The evolution equation (2.2) is not autonomous, and it is defined on the time-dependent domain $\{x \in \mathbb{R}_+ \,|\, x \geq t\}$. These drawbacks are eliminated if we rescale the density $\rho(t,x)$ by setting

$$\rho(t,x) = \frac{1}{t}\eta(\log t, x/t) \quad \text{for } x \geq t \geq 1, \tag{3.1}$$

or equivalently

$$\eta(\tau,y) = \mathrm{e}^{\tau}\rho(\mathrm{e}^{\tau}, \mathrm{e}^{\tau}y) \quad \text{for } \tau \geq 0,\ y \geq 1. \tag{3.2}$$

In what follows, we denote by $\tau = \log t$ and $y = x/t$ the new time and space coordinates. The rescaled density $\eta(\tau,\cdot)$ now belongs to the fixed space $\mathbb{P}$ defined in (1.4). Moreover, it satisfies the autonomous evolution equation

$$\partial_\tau \eta(\tau,y) = \partial_y(y\,\eta(\tau,y)) + \beta(\tau)\mathbb{Q}[\eta(\tau,\cdot)](y{-}1) \quad \text{for } y \geq 1, \tag{3.3}$$

where $\beta(\tau) = \eta(\tau,1)$ is the new trace which relates to $\alpha(t)$ via $\beta(\tau) = \mathrm{e}^{\tau}\alpha(\mathrm{e}^{\tau})$. The initial condition for (3.3) is $\eta(0,y) = \eta_0(y)$, where $\eta_0 = \rho_1 \in \mathbb{P}$.

The nonlinearity in (3.3) has the form $\beta(\tau)T_1\mathbb{Q}[\eta(\tau)]$, where $T_1 : \mathbb{P} \to \mathbb{P}$ is the shift operator defined by

$$(T_1\eta)(y) = \begin{cases} \eta(y{-}1) & \text{if } y \geq 2; \\ 0 & \text{if } y < 2. \end{cases} \tag{3.4}$$

In particular, for all $\eta \in \mathbb{P}$, the support of $T_1\mathbb{Q}[\eta]$ is contained in $[2,\infty)$, or even in $[n{+}1,\infty)$, where $n \geq 1$ is defined in (2.12). Thus, any solution of (3.3) satisfies the linear equation $\partial_\tau\eta = \partial_y(y\eta)$ in the strip $\{(\tau,y) \,|\, \tau \geq 0,\ 1 \leq y \leq 2\}$. It follows that $\eta(\tau,y) = \mathrm{e}^{\tau-\tau_0}\eta(\tau_0, \mathrm{e}^{\tau-\tau_0}y)$ for all $\tau \geq \tau_0 \geq 0$ and all $y \geq 1$ such that $\mathrm{e}^{\tau-\tau_0}y \leq 2$. Setting $y = 1$, we obtain the important relation

$$\beta(\tau) = \mathrm{e}^{\tau-\tau_0}\,\eta(\tau_0, \mathrm{e}^{\tau-\tau_0}) \quad \text{for } 0 \leq \tau{-}\tau_0 \leq \log 2, \tag{3.5}$$

which means that the trace $\beta(\tau)$ for $\tau \in [\tau_0, \tau_0{+}\log 2]$ can be determined from the solution $\eta(\tau_0,\cdot)$. This formula will be useful to define the trace β properly when the solution $\eta(\tau,\cdot)$ of (3.3) is not continuous. For instance, if $\eta(\tau,\cdot) \in \mathbb{P}$ for all $\tau \geq 0$ and if β satisfies (3.5), then $\beta \in L^1_{\mathrm{loc}}([0,\infty),\mathbb{R}_+)$.

The main purpose of this section is to show that (3.3) defines a well-posed evolution in the space $\mathbb{P}$. To do this, we consider the associated integral equation

$$\eta(\tau) = S_\tau\eta_0 + \int_0^\tau \beta(s)S_{\tau-s}T_1\mathbb{Q}[\eta(s)]\,\mathrm{d}s \quad \text{for } \tau \geq 0, \tag{3.6}$$

where $(S_\tau)_{\tau\geq 0}$ is the linear semigroup on $\mathbb{P}$ defined by

$$(S_\tau\eta)(y) = \begin{cases} \mathrm{e}^{\tau}\eta(\mathrm{e}^{\tau}y) & \text{if } y \geq 1; \\ 0 & \text{if } y < 1. \end{cases} \tag{3.7}$$

To formulate our convergence results in Section 5, we shall need some weighted L^p spaces which we now introduce. For $p \in [1,\infty)$ and $\gamma \geq 0$, we denote by L^p_γ the function space

$$L^p_\gamma = \{w \in L^1_{\mathrm{loc}}([1,\infty),\mathbb{R}) \,|\, \|w\|_{p,\gamma} < \infty\}, \tag{3.8}$$

where

$$\|w\|_{p,\gamma} = \|y^\gamma w\|_{L^p} = \left(\int_1^\infty (y^\gamma |w(y)|)^p \, \mathrm{d}y\right)^{1/p}.$$

When $\gamma = 0$, we simply write L^p instead of L^p_0 and $\|w\|_p$ instead of $\|w\|_{p,0}$. Remark that $L^p_\gamma \hookrightarrow L^1$ if and only if $\gamma > 1 - 1/p$ (when $p > 1$) or $\gamma \geq 0$ (when $p = 1$). In what follows, we shall often restrict ourselves to such values of p, γ.

We first give a few basic estimates on the semigroup (S_τ) and the nonlinearity $\mathbb{Q}$ acting on L^p_γ.

Lemma 3.1 *Let $p \in [1, \infty)$ and $\gamma \geq 0$. Then* (3.7) *defines a strongly continuous semigroup $(S_\tau)_{\tau \geq 0}$ in L^p_γ, and*

$$\|S_\tau \eta\|_{p,\gamma} \leq \mathrm{e}^{-\tau(\gamma - 1 + 1/p)} \|\eta\|_{p,\gamma}, \tag{3.9}$$

for all $\eta \in L^p_\gamma$ and all $\tau \geq 0$. Moreover, equality holds in (3.9) *if and only if $\eta(y) = 0$ for almost all $y \in [1, \mathrm{e}^\tau]$.*

Lemma 3.2 *Let $\mathbb{Q}$ be the nonlinear map defined by* (2.1).
a) *If $\eta \in L^1$, then $\mathbb{Q}[\eta] \in L^1$ and $\|\mathbb{Q}[\eta]\|_1 \leq Q(\|\eta\|_1)$. If $\eta, \tilde{\eta} \in L^1$, then*

$$\|\mathbb{Q}[\eta] - \mathbb{Q}[\tilde{\eta}]\|_1 \leq Q'(r) \|\eta - \tilde{\eta}\|_1,$$

where $r = \max\{\|\eta\|_1, \|\tilde{\eta}\|_1\}$. Finally, if $\eta \in \mathbb{P}$, then $\mathbb{Q}[\eta] \in \mathbb{P}$.
b) *Let $p \in [1, \infty)$ and $\gamma > 1 - 1/p$. If $\eta \in L^p_\gamma$, then $\mathbb{Q}[\eta] \in L^p_\gamma$, and there exists $C > 0$ (independent of η) such that*

$$\|T_1 \mathbb{Q}[\eta]\|_{p,\gamma} \leq C Q'(\|\eta\|_1) \|\eta\|_{p,\gamma}. \tag{3.10}$$

If $\eta, \tilde{\eta} \in L^p_\gamma$ and $R = \max\{\|\eta\|_{p,\gamma}, \|\tilde{\eta}\|_{p,\gamma}\}$, then

$$\|T_1 \mathbb{Q}[\eta] - T_1 \mathbb{Q}[\tilde{\eta}]\|_{p,\gamma} \leq C Q'(R) \|\eta - \tilde{\eta}\|_{p,\gamma}.$$

Proof. Estimate (3.9) is a straightforward calculation, and the proof of Lemma 3.2 will be outlined in Appendix C. ∎

We are now ready to state the main result of this section:

Theorem 3.3 *For any $\eta_0 \in L^1((1, \infty), \mathbb{R})$ with $\|\eta_0\|_1 \leq 1$, equations* (3.6), (3.5) *have a unique global solution $\eta \in C^0([0, \infty), L^1)$, which satisfies $\|\eta(\tau)\|_1 \leq 1$ for all $\tau \geq 0$. In addition,*
1) if $\eta_0 \in \mathbb{P}$, then $\eta(\tau) \in \mathbb{P}$ for all $\tau \geq 0$;
2) if $\eta_0 \in L^p_\gamma$ for some $p \geq 1$ and some $\gamma > 1 - 1/p$, then $\eta \in C^0([0, \infty), L^p_\gamma)$.

Proof. Fix $\eta_0 \in B_1$, where $B_1 = \{\eta \in L^1 \mid \|\eta\|_1 \leq 1\}$. Setting $\tau_0 = 0$ in (3.5), we obtain

$$\beta(\tau) = \mathrm{e}^\tau \eta_0(\mathrm{e}^\tau) \quad \text{for } 0 \leq \tau \leq \log 2. \tag{3.11}$$

The first step is to show that (3.6),(3.11) have a unique solution $\eta \in C^0([0, \log 2], L^1)$.

Let $q = Q'(1) \geq 1$, and let $T = (\log 2)/m$, where $m \in \mathbb{N}^*$ is sufficiently large so that, for all $k = 1, \ldots, m$,

$$\int_{(k-1)T}^{kT} \mathrm{e}^s |\eta_0(\mathrm{e}^s)| \,\mathrm{d}s < \frac{1}{q}. \tag{3.12}$$

We introduce the Banach space $X = C^0([0,T], L^1)$ equipped with the norm

$$\|\eta\|_X = \sup_{0 \leq \tau \leq T} \|\eta(\tau)\|_1.$$

Let $B = \{\eta \in X \mid \|\eta\|_X \leq 1\}$, and let $F : X \mapsto X$ be the nonlinear map defined by

$$(F[\eta])(\tau) = S_\tau \eta_0 + \int_0^\tau \beta(s) S_{\tau-s} T_1 \mathbb{Q}[\eta(s)] \,\mathrm{d}s \quad \text{for } 0 \leq \tau \leq T,$$

where $\beta(s)$ is given by (3.11). We claim that $F(B) \subset B$ and that F is a strict contraction in B. Indeed:

a) Assume that $\eta \in B$. Using Lemmas 3.1 and 3.2, we find, for all $\tau \in [0,T]$,

$$\begin{aligned}
\|(F[\eta])(\tau)\|_1 &\leq \|S_\tau \eta_0\|_1 + \int_0^\tau |\beta(s)| \|S_{\tau-s} T_1 \mathbb{Q}[\eta(s)]\|_1 \,\mathrm{d}s \\
&= \int_1^\infty \mathrm{e}^\tau |\eta_0(\mathrm{e}^\tau y)| \,\mathrm{d}y + \int_0^\tau |\beta(s)| \|T_1 \mathbb{Q}[\eta(s)]\|_1 \,\mathrm{d}s \\
&= \int_{\mathrm{e}^\tau}^\infty |\eta_0(y)| \,\mathrm{d}y + \int_0^\tau \mathrm{e}^s |\eta_0(\mathrm{e}^s)| \|\mathbb{Q}[\eta(s)]\|_1 \,\mathrm{d}s \\
&\leq \int_{\mathrm{e}^\tau}^\infty |\eta_0(y)| \,\mathrm{d}y + Q(\|\eta\|_X) \int_1^{\mathrm{e}^\tau} |\eta_0(y)| \,\mathrm{d}y \;\leq\; 1,
\end{aligned} \tag{3.13}$$

since $Q(\|\eta\|_X) \leq Q(1) = 1$ and $\|\eta_0\|_1 \leq 1$. This shows that $F(B) \subset B$.

b) If $\eta, \tilde\eta \in B$, then for all $\tau \in [0,T]$,

$$\begin{aligned}
\|(F[\eta])(\tau) - (F[\tilde\eta])(\tau)\|_1 &\leq \int_0^\tau |\beta(s)| \|S_{\tau-s}(T_1 \mathbb{Q}[\eta(s)] - T_1 \mathbb{Q}[\tilde\eta(s)])\|_1 \,\mathrm{d}s \\
&= \int_0^\tau \mathrm{e}^s |\eta_0(\mathrm{e}^s)| \|\mathbb{Q}[\eta(s)] - \mathbb{Q}[\tilde\eta(s)])\|_1 \,\mathrm{d}s \\
&\leq \int_0^\tau \mathrm{e}^s |\eta_0(\mathrm{e}^s)| Q'(1) \|\eta(s) - \tilde\eta(s)\|_1 \,\mathrm{d}s \\
&\leq q \Big(\int_0^T \mathrm{e}^s |\eta_0(\mathrm{e}^s)| \,\mathrm{d}s \Big) \|\eta - \tilde\eta\|_X.
\end{aligned}$$

In view of (3.12), this shows that F is a strict contraction in B.

Let $\eta \in X$ be the unique fixed point of F in the ball B. Then η satisfies (3.6), and using Gronwall's lemma it is readily verified that η is in fact the unique solution of (3.6) in the whole space $X = C^0([0,T], L^1)$. Repeating the same argument m times (where m is such that (3.12) holds), we conclude that equations (3.6), (3.11) have a unique solution $\eta \in C^0([0, \log 2], L^1)$, which satisfies $\|\eta(\tau)\|_1 \leq 1$ for all $\tau \in [0, \log 2]$. Moreover, it is clear that (3.5) holds for all $\tau_0 \in [0, \log 2]$ and almost all $\tau \in [\tau_0, \log 2]$.

For $\tau \in [0, \log 2]$, let $\Xi_\tau : B_1 \to B_1$ be the nonlinear map defined by $\Xi_\tau \eta_0 = \eta(\tau)$, where $\eta(\tau)$ is the solution of (3.6) we have just constructed. Then it is easy to verify that $\Xi_{\tau_1+\tau_2} = \Xi_{\tau_1} \circ \Xi_{\tau_2}$ for $0 \le \tau_1 + \tau_2 \le \log 2$. It follows that the family (Ξ_τ) can be extended to a continuous semiflow $(\Xi_\tau)_{\tau \ge 0}$. By construction, if $\eta_0 \in B_1$ and if we set $\eta(\tau) = \Xi_\tau \eta_0$ for all $\tau \ge 0$, then $\eta \in C^0([0,\infty), L^1)$ is the unique solution of (3.6), (3.5), and $\eta(\tau) \in B_1$ for all $\tau \ge 0$. This proves the first part of Theorem 3.3.

Assume now that $\eta_0 \in \mathbb{P}$. Keeping the same notations as above, we define

$$\tilde{B} = \{\eta \in X \mid \eta(\tau) \in \mathbb{P} \text{ for all } \tau \in [0,T]\}.$$

In particular, $\tilde{B}$ is a closed subset of B, as $\mathbb{P}$ is closed in $B_1 \subset L^1$. If $\eta \in \tilde{B}$, it is clear that $(F[\eta])(\tau) \in L^1((1,\infty), \mathbb{R}_+)$ for all $\tau \in [0,T]$, and that all inequalities in (3.13) can be replaced by equalities. Thus $F(\tilde{B}) \subset \tilde{B}$, hence the solution $\eta \in C^0([0,\infty), L^1)$ of (3.6) satisfies $\eta(\tau) \in \mathbb{P}$ for all $\tau \in [0,T]$. Proceeding as above, we then show that $\eta(\tau) \in \mathbb{P}$ for all $\tau \in [0, \log 2]$, hence for all $\tau \ge 0$. This proves assertion 1) in Theorem 3.3.

Finally, assume that $\eta_0 \in L^p_\gamma$ for some $p \ge 1$ and some $\gamma > 1 - 1/p$, and that $\|\eta_0\|_1 \le 1$. Using Lemmas 3.1, 3.2 and a fixed point argument as before, it is straightforward to show that the solution $\eta \in C^0([0,\infty), L^1)$ of (3.6) satisfies $\eta \in C^0([0,T], L^p_\gamma)$ for some $T > 0$ (depending on η_0). Let

$$T^* = \sup\bigl\{T > 0 \bigm| \eta \in C^0([0,T], L^p_\gamma)\bigr\} \in (0,\infty].$$

We claim that $T^* = \infty$. Indeed, assume on the contrary that $0 < T^* < \infty$. Since $\|\eta(\tau)\|_1 \le 1$ for all $\tau \ge 0$, it follows from (3.6), (3.9), (3.10) that

$$\|\eta(\tau)\|_{p,\gamma} \le \|\eta_0\|_{p,\gamma} + Cq \int_0^\tau |\beta(s)| \|\eta(s)\|_{p,\gamma} \mathrm{d}s \quad \text{for } 0 \le \tau < T^*.$$

Using Gronwall's lemma and the fact that $\beta \in L^1_{\mathrm{loc}}([0,\infty))$, we deduce that $\|\eta(\tau)\|_{p,\gamma} \le C'$ for all $\tau \in [0, T^*)$. In view of (3.6), (3.10), this in turn implies that $\eta(\tau)$ has a limit in L^p_γ as $\tau \nearrow T^*$, giving $\eta \in C^0([0,T^*], L^p_\gamma)$. Since we have a local existence result in L^p_γ, we conclude that $\eta \in C^0([0,T], L^p_\gamma)$ for some $T > T^*$, which contradicts the definition of T^*. This proves assertion 2) in Theorem 3.3. ∎

The nonlinear map $\mathcal{N}$ introduced in the previous section can also be used to linearize (3.3). Indeed, the Fourier transforms of ρ and η are related via $\widehat{\rho}(t,\xi) = \widehat{\eta}(\log t, t\xi)$, so that (3.1) is just a rescaling of the Fourier variable ξ. As is clear from (2.9), this transformation commutes with the action of $\mathcal{N}$. Thus, if ρ is a solution of (2.2) with initial data ρ_1 and if η is the corresponding solution of (3.3) given by (3.2), it follows from (2.11) that

$$\frac{1}{t}\mathcal{N}\bigl(\eta(\log t, \cdot)\bigr)(x/t) = \mathcal{N}(\eta_0)(x) \quad \text{for } x \ge t \ge 1, \tag{3.14}$$

where $\eta_0 = \rho_1$. Setting $\tau = \log t$ and $y = x/t$, we obtain the representation formula

$$\mathcal{N}(\eta(\tau)) = S_\tau \mathcal{N}(\eta_0) \quad \text{for } \tau \ge 0, \tag{3.15}$$

where (S_τ) is the linear semigroup (3.7). The last result of this section shows that this formula is indeed correct:

Proposition 3.4 *Let $\eta_0 \in \mathbb{P}$, and let $\eta \in C^0([0,\infty),\mathbb{P})$ be the solution of* (3.6) *given by Theorem 3.3. Then $\mathcal{N}(\eta(\tau)) = S_\tau \mathcal{N}(\eta_0)$ for all $\tau \geq 0$.*

Proof. We establish the formula by returning to the unscaled variables (t,x) and by showing that the formal steps of Section 2 can be made rigorous for the solutions of (3.3). Define $\rho : [1,\infty)^2 \to \mathbb{R}_+$ by $\rho(t,x) = \frac{1}{t}\eta(\log t, x/t)$ if $x \geq t \geq 1$ and $\rho(t,x) = 0$ if $1 \leq x < t$. Then $\rho \in C^0([1,\infty),\mathbb{P})$, and rescaling (3.6) we find

$$\rho(t) = U(t)\left(\rho_1 + \int_1^t \alpha(s) T_s \mathbb{Q}[\rho(s)]\,\mathrm{d}s\right) \quad \text{for } t \geq 1, \tag{3.16}$$

where $\rho_1 = \eta_0 \in \mathbb{P}$, $\alpha(t) = \frac{1}{t}\beta(\log t)$, $U(t)$ is the linear operator (2.10), and T_s is the shift operator defined as in (3.4). To simplify the notation, we set $f(s,x) = (T_s\mathbb{Q}[\rho(s)])(x)$. Then $f \in C^0([1,\infty),L^1)$, so that $(s,x) \mapsto \alpha(s)f(s,x) \in L^1_{\mathrm{loc}}([1,\infty),L^1)$. By construction, the trace α satisfies the identity

$$\alpha(t) = \rho_1(t) + \int_1^t \alpha(s) f(s,t)\,\mathrm{d}s \quad \text{for a.a. } t \geq 1.$$

We now apply the Fourier transform to (3.16). For any $\xi \in \mathbb{L}^-$ and any $t \geq 1$, we find

$$\widehat{\rho}(t,\xi) = \int_t^\infty \rho_1(x)\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x + \int_t^\infty \left\{\int_1^t \alpha(s)f(s,x)\,\mathrm{d}s\right\}\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x.$$

Since $\rho_1 \in \mathbb{P}$, the first term in the right-hand side is absolutely continuous with respect to t, and

$$\partial_t \int_t^\infty \rho_1(x)\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x = -\rho_1(t)\mathrm{e}^{-\mathrm{i}\xi t} \quad \text{for a.a. } t \geq 1.$$

The second term can be decomposed as $h_1(t,\xi) - h_2(t,\xi)$, where

$$\begin{aligned} h_1(t,\xi) &= \int_1^\infty \left\{\int_1^t \alpha(s)f(s,x)\,\mathrm{d}s\right\}\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x = \int_1^t \alpha(s)\widehat{f}(s,\xi)\,\mathrm{d}s, \\ h_2(t,\xi) &= \int_1^t \left\{\int_1^t \alpha(s)f(s,x)\,\mathrm{d}s\right\}\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x. \end{aligned}$$

Clearly, $h_1(t,\xi)$ is absolutely continuous with respect to t, and

$$\partial_t h_1(t,\xi) = \alpha(t)\widehat{f}(t,\xi) = \alpha(t)\mathrm{e}^{-\mathrm{i}\xi t}Q(\widehat{\rho}(t,\xi)) \quad \text{for a.a. } t \geq 1.$$

Next, since $f(s,x) = 0$ for $x < s$, we have $\int_1^t \alpha(s)f(s,x)\,\mathrm{d}s = \int_1^x \alpha(s)f(s,x)\,\mathrm{d}s$, and this expression is a locally integrable function of x. It follows that $h_2(t,\xi)$ is absolutely continuous with respect to t, and

$$\partial_t h_2(t,\xi) = \mathrm{e}^{-\mathrm{i}\xi t}\int_1^t \alpha(s)f(s,t)\,\mathrm{d}s \quad \text{for a.a. } t \geq 1.$$

Summarizing, we have shown that, for any $\xi \in \mathbb{L}^-$, the Fourier transform $\widehat{\rho}(t,\xi)$ is absolutely continuous with respect to t and satisfies

$$\begin{aligned} \partial_t\widehat{\rho}(t,\xi) &= -\mathrm{e}^{-\mathrm{i}\xi t}\left(\rho_1(t) + \int_1^t \alpha(s)f(s,t)\,\mathrm{d}s\right) + \alpha(t)\mathrm{e}^{-\mathrm{i}\xi t}Q(\widehat{\rho}(t,\xi)) \\ &= \alpha(t)\,\mathrm{e}^{-\mathrm{i}\xi t}\Big(Q(\widehat{\rho}(t,\xi)) - 1\Big) \qquad \text{for a.a. } t \geq 1. \end{aligned}$$

This gives (2.4). Now, proceeding exactly as in Section 2, we deduce that (2.8) holds for all $t \geq 1$ if $\operatorname{Im} \xi < 0$, and this in turn is equivalent to (2.11). Finally, using the transformation (3.14) we obtain (3.15). ■

4 Properties of the steady states

This section is devoted to the time-independent solutions of (3.3) in the space $\mathbb{P}$ defined by (1.4).

Definition. We say that $\eta_0 \in \mathbb{P}$ is a *steady state* of (3.3) if the solution $\eta \in C^0([0,\infty), \mathbb{P})$ of (3.6) given by Theorem 3.3 satisfies $\eta(\tau) = \eta_0$ for all $\tau \geq 0$.

The steady states of (3.3) will also be called "equilibria" or "stationary solutions".

Lemma 4.1 *If $\eta_0 \in \mathbb{P}$ is a steady state of* (3.3)*, there exists $\beta \geq 0$ such that $\eta_0(y) = \beta/y$ for almost all $y \in [1,2]$.*

Proof. If $\eta(\tau) \equiv \eta_0$, (3.6) implies that $\eta_0(y) = \mathrm{e}^\tau \eta_0(\mathrm{e}^\tau y)$ for all $\tau \in [0, \log 2]$ and a.a. $y \in [1, 2\,\mathrm{e}^{-\tau}]$, because the nonlinearity in (3.6) vanishes identically for such values of τ, y. We define $F : x \mapsto \int_1^{\mathrm{e}^x} \eta_0(y)\,\mathrm{d}y \geq 0$ and obtain

$$F(x+y) = F(x) + F(y) \quad \text{for } x, y \geq 0 \text{ and } x+y \leq \log 2.$$

Since F is continuous, we conclude that $F(x) = \beta x$ for some $\beta \geq 0$. Differentiating implies $\beta = \mathrm{e}^x \eta_0(\mathrm{e}^x)$ for a.a. $x \in [0, \log 2]$ which gives the desired result. ■

Let $\eta_0 \in \mathbb{P}$ be a steady state. Since η_0 coincides almost everywhere in $[1,2]$ with a continuous function, the constant β in Lemma 4.1 can be identified with $\eta_0(1)$. Clearly, the trace function defined by (3.5) satisfies $\beta(\tau) = \beta$ for all $\tau \geq 0$. In particular, the integral equation (3.6) reduces to

$$\eta_0 = S_\tau \eta_0 + \beta \int_0^\tau S_s T_1 \mathbb{Q}[\eta_0]\,\mathrm{d}s \quad \text{for } \tau \geq 0. \tag{4.1}$$

From $\eta_0 \in \mathbb{P}$ we now conclude that $\beta > 0$.

On the other hand, if $\eta_0 \in \mathbb{P}$ and $w = \mathcal{N}(\eta_0)$, it follows from Propositions 2.1 and 3.4 that η_0 is a steady state if and only if $S_\tau w = w$ for all $\tau \geq 0$. In view of (3.7), this is the case if and only if there exists $\beta' \in \mathbb{R}$ such that $w = \beta' w^*$, where

$$w^*(y) = \begin{cases} 1/y & \text{if } y \geq 1, \\ 0 & \text{if } y < 1. \end{cases} \tag{4.2}$$

But since $w(y) = \eta_0(y)$ for a.a. $y \in [1,2]$ (see Remark 2 after Proposition 2.1), we necessarily have $\beta' = \beta = \eta_0(1)$.

Finally, since equilibria are time-independent solutions of (3.3), we certainly expect them to solve the ordinary differential equation

$$(y\eta)'(y) + \beta(T_1 \mathbb{Q}[\eta])(y) = 0 \quad \text{for } y \geq 1, \quad \eta(1) = \beta. \tag{4.3}$$

Remark that the initial value β also appears as a parameter in front of the nonlinear term. It is not difficult to show that (4.3) has global solutions:

Lemma 4.2 *For any $\beta \in \mathbb{R}$, equation* (4.3) *has a unique global solution $\eta : [1,\infty) \to \mathbb{R}$.*

Proof. For any $k \in \mathbb{N}$, let $I_k = [kn+1, (k+1)n+1]$, where $n \in \mathbb{N}_*$ is defined in (2.12). For any $\eta \in L^1_{\rm loc}([1,\infty),\mathbb{R})$, the nonlinear term $(T_1\mathbb{Q}[\eta])(y)$ only depends on the values of $\eta(z)$ for $z \le y-n$. In particular, $(T_1\mathbb{Q}[\eta])(y) = 0$ for $y \le n+1$, so that any solution of (4.3) satisfies $\eta(y) = \beta/y$ for $y \in I_0 = [1, n+1]$. Using this information, one can compute $(T_1\mathbb{Q}[\eta])(y)$ explicitly for $y \in I_1 = [n+1, 2n+1]$, and then solve (4.3) on this interval to determine $\eta(y)$ for $y \in I_1$. By construction, η is smooth on both I_0 and I_1, but η has a discontinuity of order n at $y = n+1$, in the sense that the derivatives $\eta^{(k)}(y)$ are continuous for $k = 0, \dots, n-1$, whereas $\eta^{(n)}(y)$ has different limits to the left and to the right at $y = n+1$ (if $\beta \neq 0$). Iterating this procedure, we find that (4.3) has a unique global solution $\eta \in C^{n-1,1}([1,\infty),\mathbb{R})$, which satisfies $\eta \in C^\infty(I_k)$ for all $k \in \mathbb{N}$. ∎

The following result shows that equilibria of (3.3) indeed correspond to solutions of the differential equation (4.3).

Proposition 4.3 *If $\eta_0 \in \mathbb{P}$ and $\beta > 0$, the following assertions are equivalent:*
a) *η_0 is a steady state of* (3.3) *with $\eta_0(1) = \beta$.*
b) *η_0 coincides almost everywhere with the solution of* (4.3).
c) *$\mathcal{N}(\eta_0) = \beta w^*$.*

Proof. We already proved that **a**) ⇔ **c**). If $\eta_0 \in \mathbb{P}$ is a steady state with $\eta_0(1) = \beta$, it follows from (4.1) that

$$\frac{S_\tau\eta_0 - \eta_0}{\tau} + \frac{\beta}{\tau}\int_0^\tau S_s T_1 \mathbb{Q}[\eta_0]\,\mathrm{d}s \;=\; 0,$$

for all $\tau > 0$. Using (3.7), it is not difficult to verify that the first term converges to $(y\eta_0)'$ in $\mathcal{D}'((1,\infty))$ as $\tau \to 0$, while the second one tends to $\beta T_1\mathbb{Q}[\eta_0]$ in $L^1((1,\infty))$. This shows that (after modification on a set of measure zero) η_0 is absolutely continuous on $(1,\infty)$ and satisfies the differential equation (4.3) for almost all $y > 1$. It follows easily that η_0 is the solution of (4.3) in the sense of Lemma 4.2. Thus **a**) ⇒ **b**).

Conversely, assume that $\eta_0 \in \mathbb{P}$ satisfies (4.3). Applying the semi-group S_τ to (4.3) and integrating over τ, we immediately obtain (4.1), which implies that η_0 is a steady state. This proves that **b**) ⇒ **a**). ∎

The main goal of this section is to determine for which values of $\beta > 0$ the solution η of (4.3) belongs to $\mathbb{P}$. Our strategy is to use the characterization **c**) in Proposition 4.3. Therefore, we are led to study the image of βw^* under the map $\mathcal{N}^{-1}$, and this requires very precise information on the complex transformations (2.5) and (2.6). The following quantities, related to the polynomial $Q(z)$, will play an important role in the sequel:

$$q = Q'(1) \ge 1 \quad \text{and} \quad \kappa = \exp\left(\int_0^1 \Bigl(\frac{1}{1-z} - \frac{q}{1-Q(z)}\Bigr)\mathrm{d}z\right) \le 1. \tag{4.4}$$

Lemma 4.4 *Let*

$$\Phi(z) = 1 - \mathrm{e}^{-q\phi(z)} \quad \textit{for } |z| < 1,$$

where ϕ is defined in (2.5)*. Then Φ can be extended analytically to a neighborhood of the real positive axis $\mathbb{R}_+$. This extension satisfies $\Phi(z) \geq 0$ and $\Phi'(z) > 0$ for all $z \geq 0$. Moreover, $\Phi(0) = 0$, $\Phi'(0) = q$, $\Phi(1) = 1$, $\Phi'(1) = \kappa$, and $\Phi(z) \to R$ as $z \to \infty$, where*

$$R = 1 + \exp\left(\int_0^2 \Big(\frac{1}{1-z} - \frac{q}{1-Q(z)}\Big)\,\mathrm{d}z - \int_2^\infty \frac{q}{1-Q(z)}\,\mathrm{d}z\right). \tag{4.5}$$

Note that $R = \infty$ if $Q(z) = z$ and $1 < R < \infty$ otherwise.

Proof. Since the polynomial $1 - Q(z)$ has the unique real positive root $z = 1$, which is a simple root because $Q'(1) = q \neq 0$, it is clear that the function

$$\chi(z) = \exp\left(\int_0^z \Big(\frac{1}{1-t} - \frac{q}{1-Q(t)}\Big)\,\mathrm{d}t\right) = \frac{\mathrm{e}^{-q\phi(z)}}{1-z} \quad \text{for } |z| < 1,$$

can be extended to an analytic map in a neighborhood of the real positive axis $\mathbb{R}_+$. Moreover, $\chi(0) = 1$, $\chi(1) = \kappa$, and using $z{-}1 = \exp(-\int_2^z \frac{\mathrm{d}t}{1-t})$ shows that $(z{-}1)\chi(z) \to R{-}1$ for $z \to \infty$, where R is defined in (4.5). Since $\Phi(z) = 1 - (1{-}z)\chi(z)$, we conclude that the function Φ has the desired properties. In particular,

$$\Phi'(z) = q\chi(z)\frac{1-z}{1-Q(z)},$$

so that $\Phi'(z) > 0$ for all $z \geq 0$. ∎

It follows from Lemma 4.4 that the map $\Phi : [0,\infty) \to [0,R)$ is one-to-one and onto. Let $\Psi = \Phi^{-1} : [0,R) \to [0,\infty)$ be the inverse map. Then $\Psi(0) = 0$, $\Psi'(0) = 1/q$, $\Psi(1) = 1$, $\Psi'(1) = 1/\kappa$, and $\Psi'(u) > 0$ for all $u \in [0,R)$. By construction,

$$\Psi(u) = \psi\Big(-\frac{1}{q}\log(1{-}u)\Big) \quad \text{for } 0 \leq u < 1. \tag{4.6}$$

Lemma 4.5 *The function $\Psi : [0,R) \to [0,\infty)$ is absolutely monotone, i.e. $\Psi^{(k)}(u) \geq 0$ for all $k \in \mathbb{N}$ and all $u \in [0,R)$. In particular, Ψ can be extended to an analytic function on the disc $|u| < R$, and there exist nonnegative coefficients $(\Psi_k)_{k\in\mathbb{N}_*}$ such that*

$$\Psi(u) = \sum_{k=1}^\infty \Psi_k u^k \quad \textit{for } |u| < R.$$

Proof. Since $\Psi = \Phi^{-1}$, we already know that Ψ is analytic in a neighborhood of $[0,R)$. We first show by induction that, for all $n \in \mathbb{N}_*$, there exists a polynomial P_n such that

$$\Psi^{(n)}(u) = \frac{P_n(\Psi(u))}{q^n\,(1{-}u)^n} \quad \text{for } 0 < u < 1. \tag{4.7}$$

Indeed, differentiating (4.6) and using (2.6), we obtain

$$\Psi'(u) = \frac{1 - Q(\Psi(u))}{q\,(1{-}u)} \quad \text{for } 0 < u < 1. \tag{4.8}$$

Thus (4.7) holds for $n = 1$ with $P_1(z) = 1 - Q(z)$. On the other hand, differentiating (4.7) and using (4.8), we find, for $0 < u < 1$,

$$\Psi^{(n+1)}(u) = \frac{P_{n+1}(\Psi(u))}{q^{n+1}\,(1-u)^{n+1}} \quad \text{with } P_{n+1}(z) = P_n'(z)(1-Q(z)) + nqP_n(z). \tag{4.9}$$

Therefore, (4.7) is established.

We next show that, for all $n \in \mathbb{N}_*$, there exists a polynomial $R_n(z)$ with *nonnegative coefficients* such that

$$P_n(z) = (1-Q(z))(1-z)^{n-1}R_n(z). \tag{4.10}$$

Obviously, (4.10) holds for $n = 1$ with $R_1(z) = 1$. Combining (4.9) and (4.10), we obtain the recursion relation

$$R_{n+1}(z) = A_1(z)R_n'(z) + A_2(z)R_n(z) + (n-1)A_3(z)R_n(z),$$

where the coefficient functions A_j are given by

$$\begin{aligned}
A_1(z) &= \frac{1 - Q(z)}{1 - z} = \sum_{j=1}^{N} p_j \frac{1 - z^j}{1 - z}, \\
A_2(z) &= \frac{q - Q'(z)}{1 - z} = \sum_{j=2}^{N} j p_j \frac{1 - z^{j-1}}{1 - z}, \\
A_3(z) &= \frac{q}{1 - z} - \frac{1 - Q(z)}{(1 - z)^2} = \sum_{j=2}^{N} p_j \sum_{k=1}^{j-1} \frac{1 - z^k}{1 - z}.
\end{aligned}$$

Because of $p_j \geq 0$ all A_1, A_2, A_3 are polynomials (in z) with nonnegative coefficients. Thus, the same property holds for R_n by induction over n.

Since $0 < \Psi(u) < 1$ and $0 < Q(\Psi(u)) < 1$ for all $u \in (0, 1)$, it follows from (4.7) and (4.10) that $\Psi^{(n)}(u) \geq 0$ for all $n \in \mathbb{N}$ and all $u \in (0, 1)$, hence also for $u \in [0, 1]$. By a classical result of Bernstein (see [Fe71], Section VII.2), the power series

$$\sum_{k=1}^{\infty} \Psi_k u^k, \quad \text{where } \Psi_k = \frac{1}{k!}\Psi^{(k)}(0) \geq 0, \tag{4.11}$$

converges absolutely and uniformly for $|u| \leq 1$, and defines an analytic continuation of Ψ to the unit disk. Moreover, if $R_1 \geq 1$ denotes the radius of convergence of the series (4.11), it is well-known (see for instance [Ru87], exercise 16.1) that the analytic function defined by (4.11) has a singularity at $u = R_1$. Since $\Psi(u) \to \infty$ as $u \nearrow R$, it follows that $R = R_1$. This concludes the proof. ∎

Example. To conclude this study of the mappings Φ and Ψ, we give an explicit example of a nonlinearity Q for which these functions can be calculated explicitly. Let $Q(z) = (1-a)z + az^2$, where $a \in [0, 1]$. The value $a = 1$ corresponds to the coarsening equation (1.3), while $a = 0$ is a particular case of a model studied in [CaP00]. Then $q = 1+a = 1/\kappa$, $R = 1 + 1/a$, and

$$\phi(z) = \frac{1}{1 + a} \log \frac{1 + az}{1 - z}, \qquad \psi(w) = \frac{1 - \mathrm{e}^{-qw}}{1 + a\mathrm{e}^{-qw}}.$$

The auxiliary functions Φ, Ψ are:

$$\Phi(z) = \frac{(1{+}a)z}{1+az}, \qquad \Psi(u) = \frac{u}{1{+}a-au}.$$

We are now ready to state and prove the main result of this section.

Theorem 4.6 (Steady states of (3.3))
Fix $\theta > 0$ and let $\eta^*_\theta : [1,\infty) \to \mathbb{R}$ be the solution of (4.3) with $\beta = \theta/q$. Then
a) *$\eta^*_\theta \in \mathbb{P}$ if and only if $0 < \theta \le 1$.*
b) *If $\theta \in (0,1]$, $\eta^*_\theta \in \mathbb{P}$ is positive and strictly decreasing, so that $y\eta^*_\theta(y) \to 0$ as $y \to \infty$.*
c) *If $0 < \theta < 1$, then*

$$\lim_{y\to\infty} y^{1+\theta}\eta^*_\theta(y) = \frac{\theta\,\mathrm{e}^{\theta\gamma_\mathrm{E}}}{\kappa\,\Gamma(1{-}\theta)}, \tag{4.12}$$

where Γ is the Gamma function and $\gamma_\mathrm{E} = -\Gamma'(1) \approx 0.577216$ is Euler's constant.
d) *If $\theta = 1$, then*

$$\int_1^\infty y\eta^*_1(y)\,\mathrm{d}y = \frac{\mathrm{e}^{\gamma_\mathrm{E}}}{\kappa}. \tag{4.13}$$

Moreover, if $\deg Q > 1$, there exists $\lambda > 0$ such that

$$\lim_{y\to\infty} \frac{\log\eta^*_1(y)}{y} = -\lambda. \tag{4.14}$$

For $Q(z) = z$ we have

$$\lim_{y\to\infty} \frac{\log\eta^*_1(y)}{y\log y} = -1. \tag{4.15}$$

Remark. It follows from Theorem 4.6 and Proposition 4.3 that (3.3) has a *unique* steady state $\eta^*_1 \in \mathbb{P}$ such that $\int_1^\infty y\eta^*_1(y)\,\mathrm{d}y < \infty$.

Proof. We first show that $\eta^*_\theta \in \mathbb{P}$ if $0 < \theta \le 1$. According to Proposition 4.3, it is sufficient to prove that there exists an element of $\mathbb{P}$ (still denoted by η^*_θ) such that $\mathcal{N}(\eta^*_\theta) = (\theta/q)w^*$. Since $\mathcal{N}^{-1} = \mathcal{F}^{-1}\circ\psi\circ\mathcal{F}$ and $\psi(w) = \Psi(1{-}\mathrm{e}^{-qw})$ by (4.6), this relation is equivalent to

$$\widehat{\eta}^*_\theta = \Psi\big(1{-}\mathrm{e}^{-\theta\widehat{w}^*}\big), \tag{4.16}$$

where $\widehat{\eta}^*_\theta = \mathcal{F}\eta^*_\theta$ and $\widehat{w}^* = \mathcal{F}w^*$. In view of (4.2),

$$\widehat{w}^*(\xi) = \int_1^\infty \frac{\mathrm{e}^{-\mathrm{i}\xi y}}{y}\,\mathrm{d}y = \mathrm{E}_1(\mathrm{i}\xi), \tag{4.17}$$

where E_1 is the exponential integral, see [AS72]. It is well-known that

$$\mathrm{E}_1(z) = -\log z - \gamma_\mathrm{E} + \chi(z) \quad \text{for } |\arg z| < \pi, \tag{4.18}$$

where $\chi : \mathbb{C} \to \mathbb{C}$ is an entire function with $\chi(0) = 0$ and $\chi'(0) = 1$. Thus, $\widehat{w}^*$ is analytic in the interior of $\mathbb{L}^-$, where $\mathbb{L}^- = \{\xi \in \mathbb{C} \mid \operatorname{Im}\xi \le 0\}$. Moreover, $\operatorname{Re}(\widehat{w}^*(\xi)) \to \infty$ as $\xi \to 0$ within $\mathbb{L}^-$.

In Appendix A, we prove that $|1-\mathrm{e}^{-\theta\widehat{w}^*(\xi)}|<1$ for all $\xi\in\mathbb{L}^-\setminus\{0\}$ and all $\theta\in(0,1]$, see also Figure A.1. From Lemma 4.5, we also know that Ψ is analytic in the disk of radius $R>1$ centered at the origin. Therefore, the map $\widehat{\eta}^*_\theta$ defined by (4.16) is continuous over $\mathbb{L}^-$ (with $\widehat{\eta}^*_\theta(0)=1$) and analytic in the interior of $\mathbb{L}^-$. In addition, since $|\Psi(u)|\le|u|$ whenever $|u|\le 1$, we have the bound

$$|\widehat{\eta}^*_\theta(\xi)|\le|1-\mathrm{e}^{-\theta\widehat{w}^*(\xi)}|\le 2\theta|\widehat{w}^*(\xi)|\quad\text{for }\xi\in\mathbb{L}^-\setminus\{0\}.$$

In particular, $|\widehat{\eta}^*_\theta(\xi)|=\mathcal{O}(\mathrm{e}^{\mathrm{Im}\,\xi})$ as $\mathrm{Im}\,\xi\to-\infty$. By the Paley-Wiener Theorem (see for instance [Ru87]), we conclude that $\eta^*_\theta=\mathcal{F}^{-1}\widehat{\eta}^*_\theta\in L^2((1,\infty))$.

To prove that η^*_θ is nonnegative, we argue as in [CaP92]. Consider the Laplace transform $\widetilde{\eta}^*_\theta=\mathcal{L}\eta^*_\theta$, which satisfies $\widetilde{\eta}^*_\theta(p)=\widehat{\eta}^*_\theta(-\mathrm{i}p)$. As is well-known (see [Fe71], Section XIII.4), positivity of η^*_θ is equivalent to *complete monotonicity* of $\widetilde{\eta}^*_\theta$, namely $(-1)^k\widetilde{\eta}^{*(k)}_\theta(p)\ge 0$ for all $k\in\mathbb{N}$ and $p>0$. Recall that

$$\widetilde{\eta}^*_\theta(p)=\Psi(1-\mathrm{e}^{-\theta\widetilde{w}^*(p)})=\Psi(1-\mathrm{e}^{-\theta\mathrm{E}_1(p)})\quad\text{for }p>0. \tag{4.19}$$

We apply Lemma 4.7 below with

$$f_1:\begin{cases}(0,1)&\to&\mathbb{R},\\ u&\mapsto&\Psi(1-u);\end{cases}\qquad\text{and}\qquad g_1:\begin{cases}(0,\infty)&\to&(0,1),\\ p&\mapsto&\mathrm{e}^{-\theta\mathrm{E}_1(p)}.\end{cases}$$

By Lemma 4.5, f_1 is completely monotone, thus it remains to show that g_1' is completely monotone. Observe that $g_1'=f_2\circ g_2$, where $f_2:\mathbb{R}\to\mathbb{R}$ is defined by $f_2(w)=\theta\mathrm{e}^{-w}$ and $g_2:(0,\infty)\to\mathbb{R}$ by

$$g_2(p)=\theta\mathrm{E}_1(p)-\log(-\mathrm{E}_1'(p))=\theta\mathrm{E}_1(p)+p+\log p.$$

Clearly, f_2 is completely monotone, thus (again by Lemma 4.7) it remains to prove that g_2' is completely monotone. This follows from the representation

$$g_2'(p)=-\theta\frac{\mathrm{e}^{-p}}{p}+1+\frac{1}{p}=1+(1-\theta)\frac{1}{p}+\theta\int_0^1\mathrm{e}^{-sp}\,\mathrm{d}s.$$

Thus, we have shown that $\eta^*_\theta\in L^2((1,\infty),\mathbb{R}_+)$. Since $\widetilde{\eta}(p)\to 1$ as $p\searrow 0$, we conclude that $\eta^*_\theta\in L^1$ and $\int_1^\infty\eta^*_\theta(y)\,\mathrm{d}y=1$, i.e., $\eta^*_\theta\in\mathbb{P}$.

Now, fix $\theta>1$ and assume that $\eta^*_\theta\in\mathbb{P}$, where η^*_θ is the solution of (4.3) with $\beta=\theta/q$. According to Proposition 4.3, $\mathcal{N}(\eta^*_\theta)=(\theta/q)w^*$, so that (4.19) holds. Thus, in view of (4.18), the Laplace transform of η^*_θ satisfies

$$\widetilde{\eta}^*_\theta(p)=\Psi\Big(1-p^\theta\,\mathrm{e}^{\theta(\gamma_{\mathrm{E}}-\chi(p))}\Big)=1-\kappa^{-1}p^\theta\,\mathrm{e}^{\theta\gamma_{\mathrm{E}}}+\mathcal{O}(p^{1+\theta})\quad\text{for }p\searrow 0. \tag{4.20}$$

Since $\theta>1$, it follows that $\int_1^\infty y\eta^*_\theta(y)\,\mathrm{d}y=-(\widetilde{\eta}^*_\theta)'(0)=0$, which clearly contradicts the hypothesis $\eta^*_\theta\in\mathbb{P}$. This proves **a)**.

Next, fix $\theta\in(0,1]$. To prove that η^*_θ is strictly decreasing, it is sufficient to show that $\eta^*_\theta(y)>0$ for all $y\ge 1$, since $y(\eta^*_\theta)'(y)+\eta^*_\theta(y)\le 0$ by (4.3). Assume on the contrary that there exists $y_0>1$ such that $\eta^*_\theta(y_0)=0$ and $\eta^*_\theta(y)>0$ for $1\le y<y_0$. It is clear that

$y_0 > n+1$, where n is defined in (2.12). Thus, $(T_1\mathbb{Q}[\eta])(y_0) > 0$, hence $\eta_\theta^{*\prime}(y_0) < 0$ by (4.3), which contradicts the fact that $\eta_\theta^* \in \mathbb{P}$. This proves **b)**.

Assume now that $0 < \theta < 1$. In Appendix B, we prove that the limit in the left-hand side of (4.12) exists. Let $L(\theta)$ denote this limit, and let

$$H_\theta(y) = \int_y^\infty \eta_\theta^*(x)\,\mathrm{d}x \quad \text{for } y \geq 1.$$

Clearly, $y^\theta H_\theta(y) \to L(\theta)/\theta$ as $y \to \infty$. Thus, the Laplace transform of H_θ satisfies

$$p^{1-\theta}\widetilde{H}_\theta(p) = \int_p^\infty \mathrm{e}^{-t}t^{-\theta}\Big(\frac{t}{p}\Big)^\theta H_\theta\Big(\frac{t}{p}\Big)\,\mathrm{d}t \to \Gamma(1-\theta)\frac{L(\theta)}{\theta} \quad \text{as } p \searrow 0.$$

Since $\widetilde{\eta}_\theta^*(p) = \mathrm{e}^{-p} - p\widetilde{H}_\theta(p) = 1 - p^\theta\Gamma(1-\theta)L(\theta)/\theta + o(p^\theta)$ as $p \searrow 0$, it follows from (4.20) that $\Gamma(1-\theta)L(\theta)/\theta = \mathrm{e}^{\theta\gamma_{\mathrm{E}}}/\kappa$. This proves (4.12).

Finally, let $\theta = 1$. Then (4.19), (4.20) show that the Laplace transform $\widehat{\eta}_1^*$ is analytic in the half-plane $\{p \in \mathbb{C} \mid \operatorname{Re} p > -\lambda\}$, where $\lambda > 0$ is the unique real root of the equation $1 - \mathrm{e}^{-\mathrm{E}_1(-\lambda)} = R$ (if $Q(z) = z$, then $R = \infty$, hence also $\lambda = \infty$.) In particular, $\eta_1^*(y)$ decays exponentially as $y \to \infty$, and

$$-\widetilde{\eta}_1^{*\prime}(0) = \int_1^\infty y\eta_1^*(y)\,\mathrm{d}y = \frac{\mathrm{e}^{\gamma_{\mathrm{E}}}}{\kappa}.$$

If $\deg Q > 1$, then $\lambda < \infty$, and the arguments given in [CaP92] (in the particular case $Q(z) = z^2$) show that (4.14) holds. If $Q(z) = z$, then $\lambda = \infty$ and $\eta_1^*(y) = \rho(y-1)/y$, where $\rho : [0,\infty) \to \mathbb{R}_+$ is the Dickmann function studied in [CaP00]. From the asymptotics of ρ given there, we deduce that (4.15) holds. This concludes the proof. ∎

The following lemma was used in the proof of Theorem 4.6. For its proof see [Fe71], Section XIII.4.

Definition. Let $I \subset \mathbb{R}$ be an open interval, and let $f \in C^\infty(I,\mathbb{R})$. The function f is called *completely monotone* if $(-1)^k f^{(k)}(x) \geq 0$ for all $x \in I$ and all $k \in \mathbb{N}$.

Lemma 4.7 (Composition lemma)
Let $I, J \subset \mathbb{R}$ be open intervals. If $f : J \to \mathbb{R}$ is completely monotone and $g : I \to J$ has a derivative g' which is completely monotone, then $f \circ g : I \to \mathbb{R}$ is completely monotone.

Remarks. The (generalized) steady states η_θ^* with $\theta > 1$ will not be studied in this paper, because they do not lie in our function space $\mathbb{P}$. We just mention here a few properties that can established using the techniques developed in the proof of Theorem 4.6. There exists a critical value $\theta_* \in (1,\infty]$ such that

1) If $1 < \theta < \theta_*$, then $\eta_\theta^* \in L^1((1,\infty),\mathbb{R})$ and $\int_1^\infty \eta_\theta^*(y)\,\mathrm{d}y = 1$. However, η_θ^* is not a positive function. In particular, $\|\eta_\theta^*\|_1 > 1$, so that η_θ^* does not belong to the unit ball of L^1 where existence of global solutions is known from Theorem 3.3.

2) If $\theta > \theta_*$, then $\eta_\theta^* \notin L^1((1,\infty),\mathbb{R})$.

Moreover, $\theta_* = \infty$ if $Q(z) = z$, whereas $\theta_* < \infty$ if $\deg Q > 1$. In the particular case where $Q(z) = z^2$, one has $\theta_* \approx 3.24826$. These statements are illustrated in Figure 4.1.

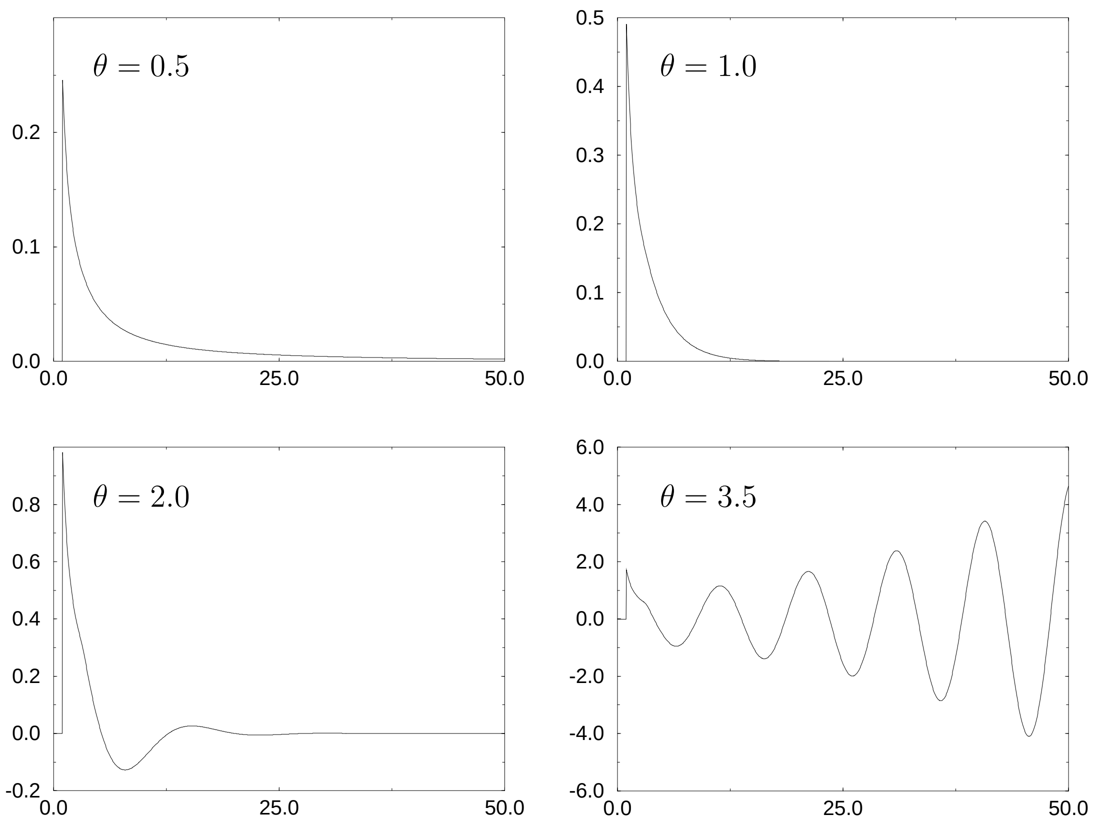

Figure 4.1: The steady state η^*_θ of the coarsening equation (3.3) with $Q(z) = z^2$ is represented for four values of the parameter θ. The first two graphs ($\theta = 0.5$ and $\theta = 1.0$) illustrate the conclusions of Theorem 4.6, and the other two ($\theta = 2.0$ and $\theta = 3.5$) the remarks after Lemma 4.7. The pictures were produced using the explicit formula (1.6) and a FFT routine to compute the Fourier transforms.

5 Global convergence results

In this final section, we use the explicit representation formula (3.15) to study the long-time behavior of the solutions of (3.3). In particular, we obtain global stability results for the steady states η^*_θ with $0 < \theta \le 1$.

Since the nonlinear map $\mathcal{N}$, which allows us to linearize (3.3), has a simple expression in Fourier variables, it is convenient to use L^2–based function spaces instead of the L^1–based function spaces which are more natural for the existence theory. Our basic space will be

$$P_\gamma = \mathbb{P} \cap L^2_\gamma \quad \text{for } \gamma \ge 0,$$

where $\mathbb{P}$ is defined in (1.4) and L^2_γ in (3.8). Remark that P_γ is a closed subspace of L^2_γ if $\gamma > 1/2$, since $L^2_\gamma \hookrightarrow L^1$.

The image of P_γ under the Fourier transform $\mathcal{F}$ can be characterized completely. Let $\mathbb{H}^\gamma_1$ be the space of all functions $z : \mathbb{L}^- \to \mathbb{C}$ satisfying the following three conditions:

(i) z is analytic in the interior of $\mathbb{L}^-$,

(ii) for each $\xi_2 \le 0$, the map $\xi_1 \mapsto z(\xi_1 + \mathrm{i}\xi_2)$ lies in the Sobolev space $H^\gamma(\mathbb{R})$,

(iii)

$$\|z\|_{\mathbb{H}_1^\gamma} = \sup_{\xi_2 \le 0} \mathrm{e}^{-\xi_2} \|z(\cdot{+}\mathrm{i}\xi_2)\|_{H^\gamma(\mathbb{R})} < \infty. \tag{5.1}$$

(In (5.1), the supremum over $\xi_2 \le 0$ is always attained at $\xi_2 = 0$.)

Then $\eta \in L^2_\gamma$ if and only if $\widehat{\eta} = \mathcal{F}\eta \in \mathbb{H}_1^\gamma$ (when $\gamma = 0$, this is just the Paley-Wiener theorem, see [Ru87]; the general case follows using the Fourier characterization of the Sobolev space $H^\gamma(\mathbb{R})$.) Moreover, the map $\eta \mapsto \|\widehat{\eta}\|_{\mathbb{H}_1^\gamma} = \|\widehat{\eta}\|_{H^\gamma(\mathbb{R})}$ is a norm on L^2_γ which is equivalent to $\|\eta\|_{2,\gamma}$. If in addition $\eta \in P_\gamma$, then $\widehat{\eta}(0) = 1$ and $\xi_1 \mapsto \widehat{\eta}(\xi_1)$ is a positive definite function on $\mathbb{R}$ (in the sense of Bochner). Furthermore, $|\widehat{\eta}(\xi)| < 1$ for all $\xi \in \mathbb{L}^- \setminus \{0\}$.

Assume now that $\eta \in P_\gamma$ for some $\gamma > 1/2$, and let $w = \mathcal{N}(\eta)$, namely $\widehat{w}(\xi) = \phi(\widehat{\eta}(\xi))$. Since ϕ is analytic in the unit disk, it is clear that $\widehat{w}$ is analytic in the interior of $\mathbb{L}^-$. Moreover, the fact that $\phi(z) = z + \mathcal{O}(|z|^2)$ as $z \to 0$ guarantees that $\widehat{w}(\xi)$ has the same decay properties as $\widehat{\eta}(\xi)$ as $|\xi| \to \infty$. However, since $\widehat{\eta}(0) = 1$ and since $\phi(z)$ has a singularity at $z = 1$, we see that $\widehat{w}(\xi)$ necessarily has a singularity at $\xi = 0$. This is the reason why the nonlinear transformation $\mathcal{N}$ *does not* map P_γ into itself. To handle this difficulty, our strategy is to subtract from $\widehat{w}(\xi)$ a suitable function with the same singularity at $\xi = 0$ and whose inverse Fourier transform is explicitly known.

If $\gamma > 3/2$, a natural candidate for this counter-term is $\frac{1}{q}\widehat{w}^*(\xi) = \phi(\widehat{\eta}_1^*(\xi))$, where η_1^* is the unique steady state of (3.3) that belongs to L^2_γ, see Theorem 4.6. We recall that w^* is defined in (4.2).

Proposition 5.1 *Let $\gamma > 3/2$ and $\eta \in P_\gamma$. Then $\mathcal{N}(\eta) = \frac{1}{q}w^* + d$ with $d \in L^2_{\gamma-1}$.*

Proof. We first show that $\mathcal{N}(\eta) \in L^2 = L^2((1,\infty))$. As explained above, it is sufficient to prove that $\widehat{w} \equiv \phi(\widehat{\eta})$ satisfies (5.1) with $\gamma = 0$. Choose $a > 0$ sufficiently small so that $|\phi(z)| \le 2|z|$ for all $z \in \mathbb{C}$ with $|z| \le a$. Since $\eta \in L^2_\gamma$, there exists $b < 0$ such that $|\widehat{\eta}(\xi)| \le a$ whenever $\operatorname{Im}\xi \le b$. Thus

$$\sup_{\xi_2 \le b} \mathrm{e}^{-\xi_2} \|\widehat{w}(\cdot{+}\mathrm{i}\xi_2)\|_{L^2} \le 2 \sup_{\xi_2 \le b} \mathrm{e}^{-\xi_2} \|\widehat{\eta}(\cdot{+}\mathrm{i}\xi_2)\|_{L^2} < \infty.$$

On the other hand, by a variant of the Riemann-Lebesgue lemma, there exists $c > 0$ such that $|\widehat{\eta}(\xi)| \le a$ for all $\xi \in \mathbb{L}^-$ with $|\operatorname{Re}\xi| \ge c$. Arguing as before, we thus get

$$\sup_{b \le \xi_2 \le 0} \int_{|\xi_1| \ge c} |\widehat{w}(\xi_1{+}\mathrm{i}\xi_2)|^2 \,\mathrm{d}\xi_1 < \infty\ .$$

It remains to verify that

$$\int_{|\xi_1| \le c} |\widehat{w}(\xi_1{+}\mathrm{i}\xi_2)|^2 \,\mathrm{d}\xi_1 \le C \quad \text{uniformly in } \xi_2 \in [b, 0]. \tag{5.2}$$

Since $\widehat{w} : \mathbb{L}^- \to \mathbb{C}$ is continuous except at the origin, it is sufficient to establish (5.2) for b, c sufficiently small. Now, as $\xi \to 0$ in $\mathbb{L}^-$, we have the expansion

$$\widehat{\eta}(\xi) = 1 - \mathrm{i}\mu\xi + r_1(\xi), \quad \text{with } r_1(\xi) = \begin{cases} \mathcal{O}(|\xi|^2) & \text{if} \quad \gamma > 5/2, \\ \mathcal{O}(|\xi|^{\gamma-1/2}) & \text{if} \quad 3/2 < \gamma < 5/2, \end{cases}$$

where $\mu = \int_1^\infty y\eta(y)\,\mathrm{d}y > 1$. Using the representation $\phi(z) = -(1/q)\log(1-\Phi(z))$ together with the properties of Φ listed in Lemma 4.4, we thus obtain

$$\widehat{w}(\xi) = -\frac{1}{q}\log(1-\Phi(\widehat{\eta}(\xi))) = -\frac{1}{q}\log(\mathrm{i}\kappa\mu\xi + r_2(\xi)),$$

where $\kappa = \Phi'(1)$ and $r_2(\xi)$ satisfies the same bounds as $r_1(\xi)$. This expansion immediately implies (5.2) if b, c are sufficiently small. Thus, we have shown that $\mathcal{N}(\eta) \in L^2$. Since obviously $\frac{1}{q}w^* \in L^2$, we deduce that $d = \mathcal{N}(\eta) - \frac{1}{q}w^* \in L^2$, too.

To prove that $d \in L^2_{\gamma-1}$, it remains to verify that $\widehat{d} \in H^{\gamma-1}(\mathbb{R})$. Again, by a localization argument, it is sufficient to show that $\widehat{d} \in H^{\gamma-1}((-c,c))$ for some $c > 0$ sufficiently small. If $\xi \in \mathbb{R}$, $|\xi| < c$, we use the representation

$$\widehat{d}(\xi) = \phi(\widehat{\eta}(\xi)) - \frac{1}{q}\widehat{w}^*(\xi) = -\frac{1}{q}\log\Bigl(\frac{1-\Phi(\widehat{\eta}(\xi))}{\mathrm{e}^{-\widehat{w}^*(\xi)}}\Bigr). \tag{5.3}$$

From (4.18), we know that $\mathrm{e}^{-\widehat{w}^*(\xi)} = \mathrm{i}\xi\mathrm{e}^{\gamma_{\mathrm{E}}}\mathrm{e}^{-\chi(\mathrm{i}\xi)}$, where χ is an entire function vanishing at the origin. It follows that $\widehat{d}(\xi) = -(1/q)\log(D(\xi)/\xi)$, where $D \in H^\gamma((-c,c))$ satisfies $D(0) = 0$ and $D'(0) = \kappa\mu\mathrm{e}^{-\gamma_{\mathrm{E}}}$. The claim is now a direct consequence of Lemma 5.2 below. This concludes the proof of Proposition 5.1. ∎

Remark. It follows immediately from the proof of Proposition 5.1 that

$$\widehat{d}(0) = \int_1^\infty d(y)\,\mathrm{d}y = \frac{1}{q}(\gamma_{\mathrm{E}} - \log(\kappa\mu)), \quad \text{where } \mu = \int_1^\infty y\eta(y)\,\mathrm{d}y. \tag{5.4}$$

Lemma 5.2 *Let $\gamma \geq 1$, and let $I \subset \mathbb{R}$ be an open interval containing 0. There exists a constant $C(I,\gamma) > 0$ such that, for each $f \in H^\gamma(I)$ with $f(0) = 0$, there exists $g \in H^{\gamma-1}(I)$ such that $f(x) = xg(x)$ for all $x \in I$ and*

$$\|g\|_{H^{\gamma-1}(I)} \leq C(I,\gamma)\|f\|_{H^\gamma(I)}.$$

Proof. It is sufficient to prove the claim for $I = \mathbb{R}$ (the general case can be reduced to this one using a bounded extension operator). If $f \in H^\gamma(\mathbb{R})$ and $f(0) = 0$, the Fourier transform $\widehat{f}$ has zero mean and satisfies $\lambda^\gamma\widehat{f} \in L^2(\mathbb{R})$, where $\lambda(\xi) = (1+\xi^2)^{1/2}$. Define $g \in L^2(\mathbb{R})$ by its Fourier transform

$$\mathrm{i}\widehat{g}(\xi) = \int_{-\infty}^{\xi}\widehat{f}(s)\,\mathrm{d}s = -\int_{\xi}^{\infty}\widehat{f}(s)\,\mathrm{d}s \quad \text{for } \xi \in \mathbb{R}.$$

Then $xg(x) = f(x)$ for (almost) all $x \in \mathbb{R}$. Moreover, since

$$\lambda(\xi)^{\gamma-1}|\widehat{g}(\xi)| \leq \begin{cases} \int_\xi^\infty \lambda(s)^{\gamma-1}|\widehat{f}(s)|\,\mathrm{d}s & \text{if } \xi \geq 0, \\ \int_{-\infty}^\xi \lambda(s)^{\gamma-1}|\widehat{f}(s)|\,\mathrm{d}s & \text{if } \xi \leq 0, \end{cases}$$

it follows from Theorem 328 in [HLP59] that $\|\lambda^{\gamma-1}\widehat{g}\|_{L^2} \leq 2\|\lambda^\gamma\widehat{f}\|_{L^2}$, which is the desired bound. ∎

We next show that the inverse map $\mathcal{N}^{-1}$ is well-defined in a neighborhood of $\frac{1}{q}w^*$ in L^2_γ.

Proposition 5.3 *Let $\gamma > 1/2$. There exists $\varepsilon > 0$ such that, for all $d \in L^2_\gamma$ with $\|d\|_{2,\gamma} \leq \varepsilon$, the function $\mathcal{N}^{-1}(\frac{1}{q}w^*+d)$ is well-defined and lies in L^2_γ. Moreover, there exists $C > 0$ such that*

$$\|\mathcal{N}^{-1}(\tfrac{1}{q}w^*+d) - \eta_1^*\|_{2,\gamma} \leq C\|d\|_{2,\gamma},$$

where $\eta_1^* = \mathcal{N}^{-1}(\frac{1}{q}w^*)$.

Proof. Throughout the proof, we denote by $\|\cdot\|_\gamma$ instead of $\|\cdot\|_{\mathbb{H}_1^\gamma}$ the norm on $\mathbb{H}_1^\gamma$ defined by (5.1). We first remark that the space $\mathbb{H}_1^\gamma$ is an algebra if $\gamma > 1/2$: there exists $C_1 > 0$ such that $\|rs\|_\gamma \leq C_1\|r\|_\gamma\|s\|_\gamma$ for all $r, s \in \mathbb{H}_1^\gamma$. Moreover, as is easy to verify, there exists $C_2 > 0$ such that, for all integer $k \geq 1$,

$$\|r^k\|_\gamma \leq C_2 k^{\ell(k,\gamma)}\|r\|_\infty^{k-\ell(k,\gamma)}\|r\|_\gamma^{\ell(k,\gamma)}, \tag{5.5}$$

where $\|r\|_\infty = \sup\{|r(\xi)| \mid \xi \in \mathbb{L}^-\} \leq C\|r\|_\gamma$ and $\ell(k,\gamma) = \min\{n \in \mathbb{N} \mid n \geq k \text{ or } n \geq \gamma\}$.

Assume that $d \in L^2_\gamma$ for some $\gamma > 1/2$, so that $\widehat{d} \in \mathbb{H}_1^\gamma$. For all $\xi \in \mathbb{L}^-$, we define

$$r(\xi) = 1 - \mathrm{e}^{-\widehat{w}^*(\xi)} \text{ and } s(\xi) = \mathrm{e}^{-\widehat{w}^*(\xi)}(1-\mathrm{e}^{-q\widehat{d}(\xi)}). \tag{5.6}$$

From (4.17), (4.18), it is easy to see that $r \in \mathbb{H}_1^\gamma$, and we prove in Appendix A that $\|r\|_\infty \leq 1$. On the other hand, since $\mathbb{H}_1^\gamma$ is an algebra, it is clear that $\sigma = 1-\mathrm{e}^{-q\widehat{d}} \in \mathbb{H}_1^\gamma$, hence $s = (1-r)\sigma \in \mathbb{H}_1^\gamma$. In addition, if $\|d\|_{2,\gamma} \leq \varepsilon$ for some $\varepsilon \leq 1$, there exists $C_3 > 0$ such that $\|s\|_\gamma \leq C_3\|d\|_{2,\gamma} \leq C_3\varepsilon$. In particular, $\|s\|_\infty \leq C\varepsilon$.

We now fix $R_1 \in (1, R)$, where $R > 1$ is defined in (4.5), and we assume that $\varepsilon \leq 1$ is sufficiently small so that $\|s\|_\infty \leq R_1 - 1$. We then define $\widehat{\eta} \in \mathbb{H}_1^\gamma$ by

$$\widehat{\eta} = \psi(\tfrac{1}{q}\widehat{w}^*+\widehat{d}) = \Psi(1-\mathrm{e}^{-q(\frac{1}{q}\widehat{w}^*+\widehat{d})}) = \Psi(r+s),$$

where Ψ is given by (4.6). From Lemma 4.5, we know that Ψ is analytic in the disk $\{u \in \mathbb{C} \mid |u| < R\}$, with the expansion $\Psi(u) = \sum_{k\geq 1}\Psi_k u^k$. Since $r + s \in \mathbb{H}_1^\gamma$ and $\|r+s\|_\infty \leq R_1 < R$, it follows from (5.5) that the series $\Psi(r+s)$ converges in $\mathbb{H}_1^\gamma$, so that $\widehat{\eta} \in \mathbb{H}_1^\gamma$. By construction, $\widehat{\eta} = \mathcal{F}\eta$ for some $\eta \in L^2_\gamma$ with $\mathcal{N}(\eta) = \frac{1}{q}w^* + d$.

It remains to show that $\|\widehat{\eta} - \widehat{\eta}_1^*\|_\gamma \leq C\|d\|_{2,\gamma}$, where $\widehat{\eta}_1^* = \Psi(r)$, see (4.19). Using

$$\|(r+s)^k - r^k\|_\gamma \leq C_1 \sup\bigl\{\|k(r+\theta s)^{k-1}\|_\gamma \bigm| \theta \in [0,1]\bigr\}\, \|s\|_\gamma$$

and (5.5), it is straightforward to verify that there exists $C_4 > 0$ such that, for all $k \geq 1$,

$$\|(r+s)^k - r^k\|_\gamma \leq C_4 k^{2+\gamma}R_1^{k-1}\|s\|_\gamma. \tag{5.7}$$

Since

$$\widehat{\eta} - \widehat{\eta}_1^* = \Psi(r+s) - \Psi(r) = \sum_{k=1}^\infty \Psi_k((r+s)^k - r^k),$$

it follows that

$$\|\widehat{\eta} - \widehat{\eta}_1^*\|_\gamma \leq C_4\Bigl(\sum_{k=1}^\infty k^{2+\gamma}\Psi_k R_1^{k-1}\Bigr)\|s\|_\gamma \leq C_5\|d\|_{2,\gamma}.$$

This concludes the proof. ∎

Remark. Unlike $\mathcal{N}$, the inverse mapping $\mathcal{N}^{-1}$ is not positivity preserving. However, if in Proposition 5.3 we *assume* in addition that $\eta = \mathcal{N}^{-1}(\frac{1}{q}w^* + d)$ is a positive function, then $y \mapsto y\eta(y) \in L^1((1,\infty))$ and

$$\int_1^\infty y\eta(y)\,\mathrm{d}y = \frac{1}{\kappa}\,\mathrm{e}^{\gamma_{\mathrm{E}} - q d_0}, \quad \text{where } d_0 = \int_1^\infty d(y)\,\mathrm{d}y. \tag{5.8}$$

Indeed, on the one hand the Laplace transform $\widetilde{\eta}(p) = \widehat{\eta}(-\mathrm{i}p)$ satisfies

$$\frac{1-\widetilde{\eta}(p)}{p} = \frac{1}{p}\Big(1 - \Psi(1 - p\,\mathrm{e}^{\gamma_{\mathrm{E}} - \chi(p) - q\widetilde{d}(p)})\Big) \longrightarrow \frac{1}{\kappa}\,\mathrm{e}^{\gamma_{\mathrm{E}} - q d_0} \quad \text{for } p \searrow 0,$$

and on the other hand, using $\eta(y) \ge 0$, we find

$$\frac{1-\widetilde{\eta}(p)}{p} = \int_1^\infty y\eta(y)\frac{1-\mathrm{e}^{-py}}{py}\,\mathrm{d}y \longrightarrow \int_1^\infty y\eta(y)\,\mathrm{d}y = \|\eta\|_{1,1} \quad \text{for } p \searrow 0,$$

by Lebesgue's monotone convergence theorem.

In addition to Lemma 3.2, the following bounds on the nonlinearity $\mathbb{Q}[\eta]$ will be used to prove our convergence results:

Lemma 5.4 *Fix $\gamma > 3/2$. For any $M > 0$, there exists $C > 0$ such that the following estimates hold:*
a) *For all $\eta \in P_\gamma$ with $\|\eta\|_{2,\gamma-1} \le M$ and $\|\eta\|_{1,1} \le M$,*

$$\|T_1\mathbb{Q}[\eta]\|_{2,\gamma} \le q\|\eta\|_{2,\gamma} + C. \tag{5.9}$$

b) *If $\gamma \ge 2$, then for all $\eta, \tilde{\eta} \in P_\gamma$ with $\|\eta\|_{2,\gamma} \le M$ and $\|\tilde{\eta}\|_{2,\gamma} \le M$,*

$$\|T_1\mathbb{Q}[\eta] - T_1\mathbb{Q}[\tilde{\eta}]\|_{2,\gamma} \le q\|\eta - \tilde{\eta}\|_{2,\gamma} + C\|\eta - \tilde{\eta}\|_{2,\gamma-1}. \tag{5.10}$$

Proof. See Appendix C. ∎

We are now ready to state the main result of this section, which shows that all solutions of (3.3) in P_γ with $\gamma > 3/2$ converge towards the limiting profile η_1^*.

Theorem 5.5 *Assume that $\eta_0 \in P_\gamma$ for some $\gamma > 3/2$, and let $\eta \in C^0([0,\infty), P_\gamma)$ be the solution of (3.6) given by Theorem 3.3. Then η is bounded in L^2_γ and there exists $C > 0$ such that*

$$\|\eta(\tau) - \eta_1^*\|_{2,\gamma-1} \le C\,\mathrm{e}^{-(\gamma-3/2)\tau} \quad \textit{for } \tau \ge 0. \tag{5.11}$$

Moreover, if $\gamma \ge 2$, then

$$\|\eta(\tau) - \eta_1^*\|_{2,\gamma} \le C(1+\tau)\,\mathrm{e}^{-(\gamma-3/2)\tau} \quad \textit{for } \tau \ge 0.$$

Remarkably, the faster the initial data decay at infinity, the faster the solution converges to the steady state. For compactly supported data, it should be possible to obtain faster decay than exponential.

Proof. We first prove (5.11) using the representation formula (3.15). By Proposition (5.1), $\mathcal{N}(\eta_0) = \frac{1}{q}w^* + d$ for some $d \in L^2_{\gamma-1}$. Since the semigroup S_τ is linear and leaves $\frac{1}{q}w^*$ invariant, we have $S_\tau\mathcal{N}(\eta_0) = \frac{1}{q}w^* + S_\tau d$. By Lemma (3.1), $\|S_\tau d\|_{2,\gamma-1} \le \mathrm{e}^{-(\gamma-3/2)\tau}\|d\|_{2,\gamma-1}$, so that $S_\tau d \to 0$ in $L^2_{\gamma-1}$ as $\tau \to \infty$. Thus, when τ is sufficiently large, we can apply Proposition 5.3 which gives

$$\|\eta(\tau) - \eta_1^*\|_{2,\gamma-1} = \|\mathcal{N}^{-1}(\tfrac{1}{q}w^*{+}S_\tau d) - \eta_1^*\|_{2,\gamma-1} \le C_1\,\mathrm{e}^{-(\gamma-3/2)\tau},$$

for some $C_1 > 0$. This estimate holds in fact for all $\tau \ge 0$ with a possibly larger constant C_1, which proves (5.11). Remark that, since $\eta(\tau)$ is nonnegative and $\|S_\tau d\|_1 \to 0$, it follows from (5.8) that $\|\eta(\tau)\|_{1,1} \to \frac{1}{\kappa}\,\mathrm{e}^{\gamma_{\mathrm{E}}}$ as $\tau \to \infty$.

We next show that $\|\eta(\tau)\|_{2,\gamma}$ is uniformly bounded for all $\tau \ge 0$. We already know that $\|\eta(\tau)\|_{2,\gamma-1}$ and $\|\eta(\tau)\|_{1,1}$ remain bounded. Thus, using the integral equation (3.6) together with the bounds (3.9) and (5.9), we obtain

$$\|\eta(\tau)\|_{2,\gamma} \le \mathrm{e}^{-(\gamma-1/2)\tau}\|\eta_0\|_{2,\gamma} + \int_0^\tau \beta(s)\,\mathrm{e}^{-(\gamma-1/2)(\tau-s)}(q\|\eta(s)\|_{2,\gamma} + C)\,\mathrm{d}s,$$

for some $C > 0$. Remark that $\beta(\tau) = \eta(\tau,1) = \mathcal{N}(\eta(\tau))|_{y=1} = \frac{1}{q}{+}\mathrm{e}^\tau d(\mathrm{e}^\tau)$. Since $d \in L^2_{\gamma-1}$, it follows that $\beta(\tau) = \frac{1}{q} + \varepsilon(\tau)$ with $\varepsilon \in L^1(\mathbb{R}_+)$. Setting $H(\tau) = \mathrm{e}^{(\gamma-1/2)\tau}\|\eta(\tau)\|_{2,\gamma}$, we thus find

$$H(\tau) \le H(0) + \int_0^\tau (1{+}q|\varepsilon(s)|)H(s)\,\mathrm{d}s + C\,\mathrm{e}^{(\gamma-1/2)\tau} \quad \text{for } \tau \ge 0,$$

for some $C > 0$. Since $\gamma - 1/2 > 1$ and $\varepsilon \in L^1(\mathbb{R}_+)$, it follows from Gronwall's lemma that $H(\tau) \le C_2\,\mathrm{e}^{(\gamma-1/2)\tau}$ for some $C_2 > 0$, hence $\|\eta(\tau)\|_{2,\gamma} \le C_2$ for all $\tau \ge 0$.

Finally, if $\gamma \ge 2$, we show that $\eta(\tau)$ converges to η_1^* in L^2_γ. To do this, we consider the integral equation satisfied by $r(\tau) = \eta(\tau) - \eta_1^*$, namely

$$r(\tau) = S_\tau r(0) + \int_0^\tau S_{\tau-s}\Big\{\varepsilon(s)T_1\mathbb{Q}[\eta(s)] + \frac{1}{q}\big(T_1\mathbb{Q}[\eta_1^*{+}r(s)] - T_1\mathbb{Q}[\eta_1^*]\big)\Big\}\,\mathrm{d}s.$$

In view of (5.11) and Lemma 5.4, there exists $C_3 > 0$ such that

$$\|T_1\mathbb{Q}[\eta(s)]\|_{2,\gamma} \le C_3, \quad \|T_1\mathbb{Q}[\eta_1^*{+}r(s)] - T_1\mathbb{Q}[\eta_1^*]\|_{2,\gamma} \le q\|r(s)\|_{2,\gamma} + C_3\,\mathrm{e}^{-(\gamma-3/2)s}.$$

Using Lemma 3.1 again, we find that $R(\tau) = \|r(\tau)\|_{2,\gamma}$ satisfies the integral inequality

$$R(\tau) \le \mathrm{e}^{-(\gamma-1/2)\tau}R(0) + \int_0^\tau \mathrm{e}^{-(\gamma-1/2)(\tau-s)}\Big\{C_3|\varepsilon(s)| + R(s) + C_3\mathrm{e}^{-(\gamma-3/2)s}\Big\}\,\mathrm{d}s.$$

Since $\varepsilon(\tau) = \mathrm{e}^\tau d(\mathrm{e}^\tau)$ with $d \in L^2_{\gamma-1}$, we have

$$\int_0^\tau \mathrm{e}^{(\gamma-1/2)s}|\varepsilon(s)|\,\mathrm{d}s = \int_1^{\mathrm{e}^\tau} y^{\gamma-1/2}|d(y)|\,\mathrm{d}y \le \Big(\int_0^{\mathrm{e}^\tau} y\,\mathrm{d}y\Big)^{1/2}\|d\|_{2,\gamma-1} \le \mathrm{e}^\tau\|d\|_{2,\gamma-1},$$

hence there exists $C_4 > 0$ such that

$$R(\tau) \le C_4\,\mathrm{e}^{-(\gamma-3/2)\tau} + \int_0^\tau \mathrm{e}^{-(\gamma-1/2)(\tau-s)} R(s)\,\mathrm{d}s.$$

Using Gronwall's lemma, we conclude that $R(\tau) \le C_5(1+\tau)\,\mathrm{e}^{-(\gamma-3/2)\tau}$ for some $C_5 > 0$, which is the desired result. ∎

We now argue that the convergence towards the steady state η_1^* cannot be faster than $\mathrm{e}^{-(\gamma-3/2)\tau}$ in the norm of L^2_γ, so that the result of Theorem 5.5 is optimal. To see this, we study the linearization of (3.3) around η_1^*. Setting $\eta(\tau) = \eta_1^* + b(\tau)$, we obtain the linearized equation $\partial_\tau b = Ab$, where

$$(Ab)(y) = (yb)'(y) + \Big(\frac{1}{q}T_1(\mathbb{Q}'[\eta_1^*] * b) + b(1)T_1\mathbb{Q}[\eta_1^*]\Big)(y).$$

Since we are interested in solutions $\eta(\tau) \in P_\gamma$, we study this operator in the space

$$X_\gamma = \Big\{ b \in L^2_\gamma \;\Big|\; \int_1^\infty b(y)\,\mathrm{d}y = 0 \Big\}.$$

Proposition 5.6 *If $\gamma > 3/2$, the operator A on X_γ has $\sigma = -(\gamma-3/2)$ in its spectrum.*

Proof. For $\delta > \gamma+1/2$ we define a Lipschitz function $b_\delta \in X_\gamma$ by

$$b_\delta(y) = \begin{cases} -1 & \text{for } y \in [1, Y_\delta], \\ -1 + (1+(Y_\delta+1)^{-\delta})(y - Y_\delta) & \text{for } y \in (Y_\delta, Y_\delta + 1), \\ y^{-\delta} & \text{for } y \ge Y_\delta + 1, \end{cases}$$

where Y_δ is chosen such that b_δ has mean 0. Note that Y_δ has a finite limit as $\delta \searrow \gamma+1/2$.

Our aim is to show that $Ab_\delta + (\gamma-3/2)b_\delta$ stays bounded in X_γ as $\delta \searrow \gamma+1/2$, while b_δ is unbounded. For this purpose, we compute the asymptotic behavior of Ab_δ as $y \to \infty$. Since η_1^* decays faster than $\mathrm{e}^{-\lambda y}$ for some $\lambda > 0$ and since $\int_1^\infty \eta_1^*(y)\,\mathrm{d}y = 1$, we obtain $(\mathbb{Q}'[\eta_1^*] * b_\delta)(y) = qb_\delta(y) + \mathcal{O}(y^{-\delta-1})$ as $y \to \infty$, where $q = Q'(1)$. It follows that

$$\begin{aligned} (Ab_\delta)(y) &= (-\delta+1)y^{-\delta} + (y-1)^{-\delta} + \mathcal{O}(y^{-\delta-1}) + \mathcal{O}(\mathrm{e}^{-\lambda y}) \\ &= (-\delta+2)y^{-\delta} + \mathcal{O}(y^{-\delta-1}) \quad \text{for } y \to \infty, \end{aligned}$$

where the remainder term is uniform in δ for $\delta \approx \gamma+1/2$. This implies the estimate

$$\|Ab_\delta + (\gamma-3/2)b_\delta\|_{2,\gamma} \le (\delta-\gamma-1/2)\|b_\delta\|_{2,\gamma} + C \le 2C$$

as $\delta \searrow \gamma+1/2$, since $\|b_\delta\|_{2,\gamma} \approx 1/\sqrt{\delta-\gamma-1/2}$. This proves the claim. ∎

To conclude this section, we also give a global stability result for the steady states η_θ^* with $0 < \theta < 1$.

Theorem 5.7 *Let $0 < \theta < 1$ and $\theta+1/2 < \gamma < \min\{3/2, 2\theta+1/2\}$. Assume that the initial value $\eta_0 \in \mathbb{P}$ satisfies $\eta_0 - \nu\eta_\theta^* \in L^2_\gamma$ for some $\nu > 0$, and let $\eta \in C^0([0,\infty), \mathbb{P})$ be the solution of* (3.6) *given by Theorem 3.3. Then there exists $C > 0$ such that*

$$\|\eta(\tau) - \eta_\theta^*\|_{2,\gamma-\theta} \le C\mathrm{e}^{-(\gamma-\theta-1/2)\tau} \quad \textit{for } \tau \ge 0. \tag{5.12}$$

Remarks.

1. From (4.12), we know that $\eta^*_\theta(y) \sim y^{-1-\theta}$ as $y \to \infty$, so that $\eta^*_\theta \in L^2_{\gamma'}$ if and only if $\gamma' < \theta+1/2$. Thus, the assumption $\gamma > \theta+1/2$ guarantees that the difference $\eta_0 - \nu\eta^*_\theta$ decays faster than η^*_θ at infinity (otherwise, we could just choose $\eta_0 = \eta^*_{\theta'}$ for some $\theta' < \theta$, in which case $\eta(\tau) = \eta^*_{\theta'}$ for all $\tau \geq 0$ so that (5.12) certainly fails.) For instance, if $\eta_0 \in \mathbb{P} \cap L^2$ is such that

$$\eta_0(y) = \frac{C}{y^{1+\theta}} + \mathcal{O}\Big(\frac{1}{y^{1+\theta+\varepsilon}}\Big) \quad \text{for } y \to \infty,$$

where $C > 0$ and $\varepsilon > 0$, then the assumptions of Theorem 5.7 are satisfied for some ν and γ. On the other hand, the hypothesis $\gamma < 2\theta+1/2$ ensures that η^*_θ and hence η_0 lie in $L^2_{\gamma-\theta}$, so that $\eta(\tau) \in L^2_{\gamma-\theta}$ for all $\tau \geq 0$.

2. Setting formally $\theta = 1$ in (5.12), we recover (5.11). However, the main difference between the two results is the upper bound $\gamma < 3/2$ in Theorem 5.7 which limits the decay rate in time of the perturbations. Even for compactly supported perturbations, the convergence in (5.12) is not faster than $\mathcal{O}(\mathrm{e}^{-\delta\tau})$, where $\delta = \min\{\theta, 1-\theta\}$.

Proof. The proof is quite similar to that of (5.11), so we just indicate the main differences here. Proceeding as in the proof of Proposition 5.1, we first show that $\mathcal{N}(\eta_0) = \frac{\theta}{q}w^* + d$ for some $d \in L^2_{\gamma-\theta}$. In analogy with (5.3), we find

$$\widehat{d}(\xi) = -\frac{1}{q}\log\Big(\frac{1-\Phi(\widehat{\eta}_0(\xi))}{\mathrm{e}^{-\theta\widehat{w}^*(\xi)}}\Big).$$

The crucial point is the behavior of $\widehat{d}(\xi)$ as $\xi \to 0$, which we now analyze. By assumption, $\eta_0 = \nu\eta^*_\theta + \zeta$ for some $\zeta \in L^2_\gamma$ with $\int_1^\infty \zeta(y)\,\mathrm{d}y = 1-\nu$. From (4.16), we have

$$\widehat{\eta}^*_\theta(\xi) = \Psi\big(1-\mathrm{e}^{-\theta\widehat{w}^*(\xi)}\big) = 1 - \mathrm{e}^{-\theta\widehat{w}^*(\xi)} H\big(\mathrm{e}^{-\theta\widehat{w}^*(\xi)}\big),$$

where $H : z \mapsto (1-\Psi(1-z))/z$ is analytic in a neighborhood of zero, with $H(0) = \Psi'(1) = 1/\kappa$. We recall that $\mathrm{e}^{-\theta\widehat{w}^*(\xi)} = (\mathrm{i}\xi)^\theta \mathrm{e}^{\theta\gamma_{\mathrm{E}}}\mathrm{e}^{-\theta\chi(\mathrm{i}\xi)}$ where χ is entire, see (4.18). Since $\gamma < 2\theta + 1/2$, we deduce that $r_1 : \xi \mapsto H(\mathrm{e}^{-\theta\widehat{w}^*(\xi)})$ belongs to $H^{\gamma-\theta}((-c,c))$ for some $c > 0$, and that $r_1(0) = 1/\kappa$. Next, we observe that $\widehat{\zeta}(\xi) = 1-\nu - r_2(\xi)$, where $r_2 \in H^\gamma((-c,c))$ and $r_2(0) = 0$. Using the fractional analog of Lemma 5.2, we conclude that $r_2(\xi) = \mathrm{e}^{-\theta\widehat{w}^*(\xi)} r_3(\xi)$, where $r_3 \in H^{\gamma-\theta}((-c,c))$ and $r_3(0) = 0$. (Here, we need $\gamma < 3/2$: if $\gamma > 3/2$, the claim would be false unless $r_2'(0) = 0$.) Summarizing, we have shown

$$\widehat{\eta}_0(\xi) = \nu\widehat{\eta}^*_\theta(\xi) + \widehat{\zeta}(\xi) = 1 - \mathrm{e}^{-\theta\widehat{w}^*(\xi)}\Big(\frac{\nu}{\kappa} + r_4(\xi)\Big),$$

where $r_4 \in H^{\gamma-\theta}((-c,c))$ and $r_4(0) = 0$. We now apply the inverse map $\Phi = \Psi^{-1}$ which is analytic in a neighborhood of 1 with $\Phi'(1) = \kappa$. Using the fact that $H^{\gamma-\theta}$ is an algebra, we obtain

$$\Phi(\widehat{\eta}_0(\xi)) = 1 - \mathrm{e}^{-\theta\widehat{w}^*(\xi)}(\nu + r_5(\xi)),$$

where r_5 has the same properties as r_4. Since $\widehat{d}(\xi) = -\frac{1}{q}\log(\nu + r_5(\xi))$, we conclude that $\widehat{d} \in H^{\gamma-\theta}((-c,c))$ with $\widehat{d}(0) = -\frac{1}{q}\log\nu$.

Now, from (3.15) we have $\mathcal{N}(\eta(\tau)) = S_\tau(\frac{\theta}{q}w^*+d) = \frac{\theta}{q}w^* + S_\tau d$, and $\|S_\tau d\|_{2,\gamma-\theta} \le \mathrm{e}^{-(\gamma-\theta-1/2)\tau}\|d\|_{2,\gamma-\theta}$ for all $\tau \ge 0$. Moreover, it is easy to check that Proposition 5.3 and its proof remain valid if we replace everywhere w^* with θw^*, η_1^* with η_θ^*, and γ with $\gamma' = \gamma-\theta < \theta+1/2$ (as is explained above, this inequality ensures that $\eta_\theta^* \in L^2_{\gamma'}$.) Thus, we conclude that

$$\|\eta(\tau)-\eta_\theta^*\|_{2,\gamma-\theta} = \|\mathcal{N}^{-1}(\tfrac{\theta}{q}w^*+S_\tau d) - \eta_\theta^*\|_{2,\gamma-\theta} \le C\|S_\tau d\|_{2,\gamma-\theta} = \mathcal{O}(\mathrm{e}^{-(\gamma-\theta-1/2)\tau}),$$

as $\tau \to \infty$, which is the desired result. ■

A Bounds on the exponential integral

Let $\widehat{w}^*(\xi) = \mathrm{E}_1(\mathrm{i}\xi)$, where $\mathrm{E}_1(z) = \int_1^\infty y^{-1}\mathrm{e}^{-zy}\,\mathrm{d}y$ is the exponential integral. The goal of this section is to prove that

$$|1-\mathrm{e}^{-\theta\widehat{w}^*(\xi)}| < 1 \quad \text{for } \theta \in (0,1] \text{ and } \xi \in \mathbb{L}^- \setminus \{0\}. \tag{A.1}$$

For $\theta = 1$ and $\xi \in \mathbb{R}$, this property is illustrated in Figure A.1.

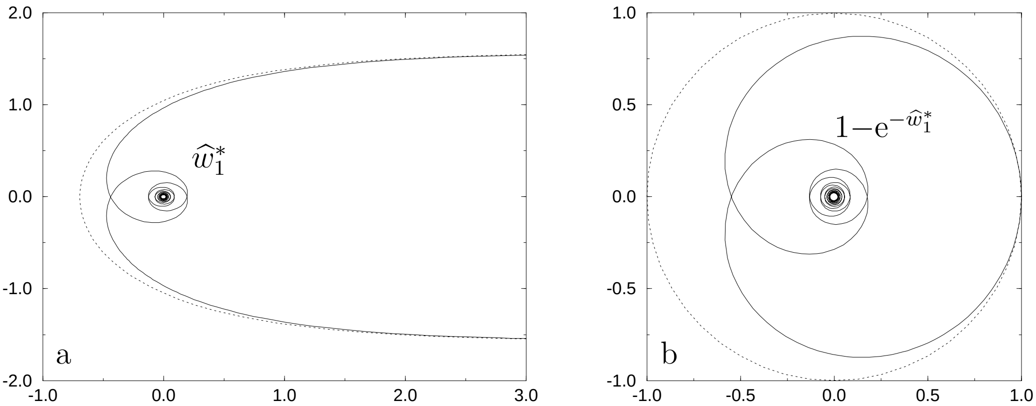

Figure A.1: (a) The region $D \subset \mathbb{C}$ delimited by the dotted line contains the curve $\{\widehat{w}^*(\xi)\,|\,\xi \in \mathbb{R}\}$ (solid line). (b) The curve $\{1-\mathrm{e}^{-\widehat{w}^*(\xi)}\,|\,\xi \in \mathbb{R}\}$ (solid line) is contained in the unit disk of $\mathbb{C}$.

Fix $0 < \theta \le 1$, and define $F_\theta : \mathbb{L}^- \to \mathbb{C}$ by $F_\theta(0) = 1$ and $F_\theta(\xi) = 1-\mathrm{e}^{-\theta\widehat{w}^*(\xi)}$ for $\xi \in \mathbb{L}^- \setminus \{0\}$. Then F_θ is continuous on $\mathbb{L}^-$, and analytic in the interior of $\mathbb{L}^-$. Moreover, F_θ is uniformly bounded, because $|F_\theta(\xi)| \le 1 + \exp(-\theta\,\mathrm{Re}(\widehat{w}^*(\xi)))$ and

$$\mathrm{Re}(\widehat{w}^*(\xi)) = \int_1^\infty \frac{1}{y}\,\mathrm{e}^{\xi_2 y}\cos(\xi_1 y)\,\mathrm{d}y \ge \int_{\pi/2}^\infty \frac{\cos(t)}{t}\,\mathrm{d}t \approx -0.472,$$

for all $\xi = \xi_1+\mathrm{i}\xi_2$ with $\xi_1 \in \mathbb{R}$ and $\xi_2 < 0$. Finally, since $|\widehat{w}^*(\xi)| \le \mathrm{E}_1(-\xi_2) \to 0$ as $\xi_2 \to -\infty$, it is clear that $|F_\theta(\xi)| \to 0$ as $\xi_2 \to -\infty$, uniformly in $\xi_1 \in \mathbb{R}$. Thus, by the maximum modulus principle and the Phragmen-Lindelöf theory (see e.g. [Ru87], Thm. 12.9), it is sufficient to show that (A.1) holds for all $\xi \in \mathbb{R} \setminus \{0\}$.

Let $D \subset \mathbb{C}$ be the open region defined by

$$D = \{x{+}\mathrm{i}y \in \mathbb{C} \mid |y| < \pi/2\,,\ x + \log(2\cos(y)) > 0\},$$

see Figure A.1. As is easily verified, $w \in D$ implies $|1{-}\mathrm{e}^{-w}| < 1$. Thus, all we need to show is that $\theta\widehat{w}^*(\xi) \in D$ for all $\xi \in \mathbb{R} \setminus \{0\}$. Since $0 \in D$ and D is convex, it is sufficient to prove this property for $\theta = 1$.

For $\xi > 0$ we define

$$x(\xi) = \int_\xi^\infty \frac{\cos(t)}{t}\,\mathrm{d}t \text{ and } y(\xi) = \int_\xi^\infty \frac{\sin(t)}{t}\,\mathrm{d}t. \tag{A.2}$$

Then $\widehat{w}^*(\xi) = \mathrm{E}_1(\mathrm{i}\xi) = x(\xi) - \mathrm{i}y(\xi)$ for $\xi > 0$ and $\widehat{w}^*(\xi) = x(|\xi|) + \mathrm{i}y(|\xi|)$ for $\xi < 0$. Moreover, $|y(\xi)| < \pi/2$ for all $\xi > 0$. Thus, it is enough to verify that $K(\xi) > 0$ for all $\xi > 0$, where

$$K(\xi) = x(\xi) + \log(2\cos(y(\xi))) \quad \text{for } \xi > 0.$$

We first observe that $K(\xi) > 0$ if $\xi > 0$ is sufficiently small. Indeed, in view of (4.18), we have the expansions

$$x(\xi) = -\log\xi - \gamma_{\mathrm{E}} + \mathcal{O}(\xi^2) \text{ and } y(\xi) = \frac{\pi}{2} - \xi + \mathcal{O}(\xi^3) \quad \text{for } \xi \searrow 0,$$

hence $K(\xi) \to \log 2 - \gamma_{\mathrm{E}} > 0$ as $\xi \searrow 0$.

We next show that $K(\xi) > 0$ for $0 < \xi \le \pi/2$. If not, there would exist $\xi \in (0, \pi/2]$ such that $K(\xi) = 0$ and $K'(\xi) \le 0$. In view of (A.2), $K'(\xi) \le 0$ if and only if $\sin(\xi)\sin(y(\xi)) \le \cos(\xi)\cos(y(\xi))$. Since $0 < y(\xi) < \pi/2$, this is equivalent to $\xi + y(\xi) \le \pi/2$, or $\cos(y(\xi)) \ge \sin(\xi)$. Therefore, $\xi \in (0, \pi/2]$ should satisfy $x(\xi) + \log(2\sin(\xi)) \le K(\xi) = 0$. But this is impossible, because $x(\xi) + \log(2\sin(\xi)) \to \log 2 - \gamma_{\mathrm{E}} > 0$ as $\xi \searrow 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Big(x(\xi) + \log(2\sin(\xi))\Big) = \frac{\xi - \sin(\xi)}{\xi\tan(\xi)} > 0 \quad \text{for } 0 < \xi < \pi/2.$$

It remains to show that $K(\xi) > 0$ for $\xi > \pi/2$. Let

$$\bar{x} = -x(\pi/2) \approx 0.472 \text{ and } \bar{y} = \max\{y(\pi/2), -y(\pi)\} = -y(\pi) \approx 0.281.$$

(See [AS72] for rigorous bounds on $x(\xi), y(\xi)$.) Using the definitions (A.2), it is easy to show that $|x(\xi)| \le \bar{x}$ and $|y(\xi)| \le \bar{y}$ for all $\xi \ge \pi/2$. Thus $x(\xi) + \log(2\cos(y(\xi))) \ge -\bar{x} + \log(2\cos(\bar{y})) > 0$ for $\xi \ge \pi/2$. This concludes the proof.

B Asymptotic behavior of the steady states

Fix $\theta \in (0, 1)$, and let $\eta = \eta^*_\theta : [1, \infty) \to \mathbb{R}$ be the solution of (4.3) with $\beta = \theta/q$. By Theorem 4.6, η is positive, strictly decreasing, and $\int_1^\infty \eta(y)\,\mathrm{d}y = 1$. The aim of this section is to prove that the limit

$$L(\theta) = \lim_{y\to\infty} y^{1+\theta}\eta(y) \tag{B.1}$$

exists (and is finite). This is especially easy in the particular case where $Q(z)=z$. Indeed, since $q=1$ and $\mathbb{Q}[\eta]=\eta$ in this case, it follows from (4.3) that

$$0 = y\eta'(y) + \eta(y) + \theta\eta(y{-}1) \geq y\eta'(y) + (1{+}\theta)\eta(y) \quad \text{for } y \geq 2,$$

hence $y \mapsto y^{1+\theta}\eta(y)$ is decreasing (and positive) for $y \geq 2$. In the general situation where $N = \deg Q > 1$, we need the following estimate:

Lemma B.1 *For all $y \geq N$,*

$$\mathbb{Q}[\eta](y) \geq \eta(y) Q'\Big(\int_1^{y/N} \eta(x)\,\mathrm{d}x\Big). \tag{B.2}$$

Proof. The only property of η that will be used in this proof is that η is nonnegative and non-increasing. Thus, by linearity and monotonicity, it is sufficient to prove (B.2) in the case where $Q(z)=z^j$ for some $j \in \mathbb{N}$, $j \geq 2$. For $a \geq 1$, we denote

$$\begin{aligned} D_j(a) &= \{(x_1,\ldots,x_j) \,|\, 1 \leq x_1,\ldots,x_j \leq a\} = [1,a]^j, \\ S_j(a) &= \{(x_1,\ldots,x_j) \,|\, 1 \leq x_1 \leq \cdots \leq x_j \leq a\}. \end{aligned}$$

Then, for $y \geq j$, we have

$$\begin{aligned} \mathbb{Q}[\eta](y) &= \int_{D_j(y+1-j)} \eta(x_1)\cdot\ldots\cdot\eta(x_j)\delta(x_1+\cdots+x_j-y)\,\mathrm{d}^j x \\ &= j!\int_{S_j(y+1-j)} \eta(x_1)\cdot\ldots\cdot\eta(x_j)\delta(x_1+\cdots+x_j-y)\,\mathrm{d}^j x, \end{aligned}$$

where δ denotes the Dirac measure. To obtain a lower bound, we replace $\eta(x_j)$ with $\eta(y)$ in the last integral, and we perform the (trivial) integration over x_j. We obtain

$$\mathbb{Q}[\eta](y) \geq (j!)\eta(y)\int_{R_{j-1}(y)} \eta(x_1)\cdot\ldots\cdot\eta(x_{j-1})\,\mathrm{d}^{j-1}x,$$

where $R_{j-1}(y) = \{(x_1,\ldots,x_{j-1}) \,|\, (x_1,\ldots,x_{j-1}, y{-}x_1{-}\ldots{-}x_{j-1}) \in S_j(y{+}1{-}j)\}$. Now, it is straightforward to verify that $R_{j-1}(y) \supset S_{j-1}(y/j)$. Thus

$$\begin{aligned} \mathbb{Q}[\eta](y) &\geq (j!)\eta(y)\int_{S_{j-1}(y/j)} \eta(x_1)\cdot\ldots\cdot\eta(x_{j-1})\,\mathrm{d}^{j-1}x \\ &= j\eta(y)\int_{D_{j-1}(y/j)} \eta(x_1)\cdot\ldots\cdot\eta(x_{j-1})\,\mathrm{d}^{j-1}x \\ &= j\eta(y)\Big(\int_1^{y/j}\eta(x)\,\mathrm{d}x\Big)^{j-1} = \eta(y)Q'\Big(\int_1^{y/j}\eta(x)\,\mathrm{d}x\Big). \end{aligned}$$

This concludes the proof. ∎

Combining (4.3) and Lemma B.1, we obtain the inequality

$$y\eta'(y) + \eta(y) + \frac{\theta}{q}\eta(y{-}1)\,Q'\Big(\int_1^{\frac{y-1}{N}}\eta(x)\,\mathrm{d}x\Big) \leq 0 \quad \text{for } y \geq N{+}1,$$

where $\eta(y{-}1)$ may also be replaced by $\eta(y)$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}y}(y^{1+\theta/2}\eta(y)) \leq \theta y^{\theta/2}\eta(y)\Big(\frac{1}{2} - \frac{1}{q}Q'\Big(\int_1^{\frac{y-1}{N}} \eta(x)\,\mathrm{d}x\Big)\Big) \quad \text{for } y \geq N{+}1.$$

Since $\int_1^\infty \eta(y)\,\mathrm{d}y = 1$ and $q = Q'(1)$, the right-hand side becomes negative for y sufficiently large. This shows that

$$\sup_{y\geq 1} y^{1+\theta/2}\eta(y) < \infty. \tag{B.3}$$

Similarly, for $y \geq N{+}1$,

$$\frac{\mathrm{d}}{\mathrm{d}y}(y^{1+\theta}\eta(y)) \leq y^{1+\theta}\eta(y)\Theta(y) \quad \text{with } \Theta(y) = \frac{\theta}{y}\Big(1 - \frac{1}{q}Q'\Big(\int_1^{\frac{y-1}{N}} \eta(x)\,\mathrm{d}x\Big)\Big) \geq 0. \tag{B.4}$$

It follows from (B.3) that $1 - \int_1^{(y-1)/N} \eta(x)\,\mathrm{d}x = \mathcal{O}(y^{-\theta/2})$ for $y \to \infty$, which yields $\Theta(y) = \mathcal{O}(y^{-1-\theta/2})$ and hence $\Theta \in L^1((N{+}1,\infty))$. Thus, the differential inequality (B.4) implies that the limit (B.1) exists.

C Bounds on the nonlinearity

In this section, we sketch the proofs of Lemmas 3.2 and 5.4. Without loss of generality, we assume here that $Q(z) = z^m$ for some $m \in \mathbb{N}_*$ (the general case follows by linearity). To bound the convolution products, we repeatedly use Young's inequality $\|f{*}g\|_p \leq \|f\|_p\|g\|_1$ where $f \in L^p$ and $g \in L^1$.

Proof of Lemma 3.2.
a) If $\eta \in L^1$, then $\mathbb{Q}[\eta] = \eta^{*m} \in L^1$ and $\|\mathbb{Q}[\eta]\|_1 \leq \|\eta\|_1^m = Q(\|\eta\|_1)$. If $\eta, \tilde\eta \in L^1$, then

$$\mathbb{Q}[\eta] - \mathbb{Q}[\tilde\eta] \;=\; (\eta{-}\tilde\eta) * \eta * \cdots * \eta \;+\; \ldots \;+\; \tilde\eta * \tilde\eta * \cdots * (\eta{-}\tilde\eta) \tag{C.1}$$

(m terms of m factors), hence $\|\mathbb{Q}[\eta] - \mathbb{Q}[\tilde\eta]\|_1 \leq mr^{m-1}\|\eta - \tilde\eta\|_1 = Q'(r)\|\eta - \tilde\eta\|_1$.
b) Assume now that $\eta \in L^p_\gamma \hookrightarrow L^1$. For all $y \geq 1$,

$$y^\gamma(T_1\mathbb{Q}[\eta])(y) = \int_{\mathbb{R}^m} \eta(x_1)\ldots\eta(x_m)(1{+}x_1{+}\ldots{+}x_m)^\gamma\,\delta(1{+}x_1{+}\ldots{+}x_m{-}y)\,\mathrm{d}^m x,$$

where δ denotes the Dirac measure. Due to the support property of η, only the values $x_1, \ldots, x_m \geq 1$ contribute to the integral. For such values, we have the estimate

$$(1{+}x_1{+}\ldots{+}x_m)^\gamma \leq C(x_1^\gamma{+}\ldots{+}x_m^\gamma), \tag{C.2}$$

where $C > 0$ depends on m, γ. Thus, $|y^\gamma(T_1\mathbb{Q}[\eta])|$ is bounded by a sum of m convolution products of the form $|y^\gamma\eta| * |\eta|^{*(m-1)}$. Taking the L^p norm and using Young's inequality, we obtain

$$\|T_1\mathbb{Q}[\eta]\|_{p,\gamma} \leq Cm\|\eta\|_1^{m-1}\|\eta\|_{p,\gamma} = CQ'(\|\eta\|_1)\|\eta\|_{p,\gamma}.$$

Finally, using the decomposition (C.1) and proceeding as above, we find

$$\begin{aligned}\|T_1\mathbb{Q}[\eta] - T_1\mathbb{Q}[\tilde\eta]\|_{p,\gamma} &\leq C(mr^{m-1}\|\eta - \tilde\eta\|_{p,\gamma} + m(m{-}1)r^{m-2}R\|\eta - \tilde\eta\|_1) \\ &\leq C(Q'(r) + RQ''(r))\|\eta - \tilde\eta\|_{p,\gamma},\end{aligned}$$

where $R = \max\{\|\eta\|_{p,\gamma}, \|\tilde{\eta}\|_{p,\gamma}\}$ and $r = \max\{\|\eta\|_1, \|\tilde{\eta}\|_1\}$. Since $r \leq R$ and $RQ''(R) \leq CQ'(R)$, this is the desired result. ∎

Proof of Lemma 5.4. The proof follows the same lines, except that (C.2) is replaced with a different estimate, which can be established by induction over m. If $\gamma \geq 1$ and $m \in \mathbb{N}_*$, there exists $C > 0$ such that, for all $x_1, \dots, x_m \geq 1$,

$$(1+x_1+\ldots+x_m)^\gamma \leq \sum_{i=1}^m \Big(x_i^\gamma + C x_i^{\gamma-1} \prod_{j\neq i} x_j \Big).$$

a) If $\gamma > 3/2$ and $\eta \in P_\gamma$, then

$$\begin{aligned} \|T_1\mathbb{Q}[\eta]\|_{2,\gamma} &\leq m\|\eta\|_1^{m-1}\|\eta\|_{2,\gamma} + Cm\|\eta\|_{1,1}^{m-1}\|\eta\|_{2,\gamma-1} \\ &= Q'(1)\|\eta\|_{2,\gamma} + CQ'(\|\eta\|_{1,1}^{m-1})\|\eta\|_{2,\gamma-1}. \end{aligned}$$

b) If $\eta, \tilde{\eta} \in P_\gamma$, let $M = \max\{\|\eta\|_{2,\gamma}, \|\tilde{\eta}\|_{2,\gamma}\}$, $r_1 = \max\{\|\eta\|_{1,1}, \|\tilde{\eta}\|_{1,1}\} \leq M$, and $r = \max\{\|\eta\|_1, \|\tilde{\eta}\|_1\} = 1$. Then

$$\begin{aligned} \|T_1\mathbb{Q}[\eta] - T_1\mathbb{Q}[\tilde{\eta}]\|_{2,\gamma} &\leq mr^{m-1}\|\eta - \tilde{\eta}\|_{2,\gamma} + m(m-1)r^{m-2}M\|\eta - \tilde{\eta}\|_1 \\ &\quad + Cmr_1^{m-1}\|\eta - \tilde{\eta}\|_{2,\gamma-1} + Cm(m-1)r_1^{m-2}M\|\eta - \tilde{\eta}\|_{2,1} \\ &= Q'(1)\|\eta - \tilde{\eta}\|_{2,\gamma} + Q''(1)M\|\eta - \tilde{\eta}\|_1 \\ &\quad + CQ'(r_1)\|\eta - \tilde{\eta}\|_{2,\gamma-1} + CQ''(r_1)M\|\eta - \tilde{\eta}\|_{2,1}. \end{aligned}$$

If $\gamma \geq 2$, the last three terms in the right-hand side can be bounded by $CQ'(M)\|\eta-\tilde{\eta}\|_{2,\gamma-1}$. ∎

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