SPECTRAL AND DYNAMICAL PROPERTIES OF RANDOM MODELS WITH NONLOCAL AND SINGULAR INTERACTIONS

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Abstract

We give a spectral and dynamical description of certain models of random Schrödinger operators on $L^2(I\!\!R^d)$ for which a modified version of the small moment method of Aizenman and Molchanov [?] can be applied. One family of models includes includes Schrödinger operators with random, nonlocal interactions constructed from a wavelet basis. The second family includes Schrödinger operators with random singular interactions randomly located on sublattices of $Z\!\!\!Z^d$, for $d = 1, 2, 3$. We prove that these models are amenable to Aizenman-Molchanov-type analysis of the Green's function, thereby eliminating the use of multiscale analysis. The basic technical result is an estimate on the expectation of small moments of the Green's function. Among our results, we prove a good Wegner estimate and the Hölder continuity of the integrated density of states, and spectral and dynamical localization at negative energies.

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1 Introduction and Main Results

In this paper, we treat two new interesting models in the theory of random Schrödinger operators. One model is a nonlocal, random interaction model, and the second is a random, singular interaction model. It turns out that, with an appropriate reduction for the second model, both models can be treated with the same techniques. The theoretical basis of our paper is the analysis of the following abstract model. We consider a family of random Schrödinger operators on a separable Hilbert space $\mathcal{H}$ having the form

Our main results on this family of operators (1) are stated precisely in section 2. We obtain *a priori* and *uniform* exponential bounds on the expectation of the small moments of the Green's functions in the small and large disorder regimes. These estimates are valid for an interval of energy I on which the unperturbed operator H_0 satisfies certain estimates. These bounds lead to a proof of exponential localization in the energy interval I. The main hypotheses on the resolvent of the unperturbed operator H_0 are as follows.

1. [Weak Disorder]. For any $0 < s < 1$, suppose there is a finite constant $K_s > 0$ such that

2. [Strong Disorder or High Energy]. For any $0 < s < 1$, suppose there is a finite constant $C_s > 0$ defined by

Our first application of this result is to a specific *random nonlocal model* on $\mathcal{H} = L^2(I\!\!R^d)$. The unperturbed operator is $H_0 = -\Delta$, and the P_n are orthogonal projections onto a basis of $L^2(I\!\!R^d)$ given by an orthonormal basis of wavelets. For this explicit model, we can verify hypothesis (3) with $\gamma = 0$, and prove localization at sufficiently negative energies. It appears that this model cannot be treated by the methods of MSA.

We call the second family of models *random singular interaction models.* These are formally defined by the family of Schrödinger operators

1.1 The Main Results

With these preparations, we can now state the main localization results for these two families of models. Specific statements are given in Theorem 3.5, Theorems 6.2 and 6.4, and Theorems 8.2 and 8.5.

A. Random nonlocal model.

Theorem 1.1. *Let* $H_\omega = -\Delta + \sum_n \lambda_n(\omega) P_n$ *be a random family of Schrödinger operators as given in (1) with wavelet interactions described in (41)–(44) of section 3. Under Hypotheses 3.1 and 3.2, there exists an energy* $E_1(\mu) > 0$ *so that* $\sigma(H_\omega) \cap (-\infty, -E_1(\mu)] \neq \emptyset$*, with probability one, and the model has only pure point spectrum almost surely in* $(-\infty, -E_0]$*.*

B. Random singular interaction model.

Theorem 1.2. *Let* $H_{\omega,\omega'}$ *be the family of random Schrödinger operators with singular interactions formally given in (77), and defined using the resolvent formula (81). We assume that the random variables satisfy Hypotheses 4.1 and 4.2. There exists a constant* $E_1(a) > 0$ *so that* $\Sigma \cap (-\infty, -E_1(a)]$ *is nonempty and pure point almost surely, with exponentially decaying eigenfunctions. The family of operators exhibits dynamical localization at all orders* $q \in \mathbb{N}$ *on* $I \equiv \Sigma \cap (-\infty, -E_1(a)]$*. That is, with probability one,*

We also consider the case when the impurities are located on random subsets of the lattice $\mathbb{Z}^{d-j}$ *of codimension* $j \geq 1$*, for* $d = 2, 3$*. In this case, the methods of scattering theory are used to prove that there is absolutely continuous spectrum at positive energies and localized states in an interval of negative energies with probability one.*

Theorem 1.3. *Suppose that the random set* $\Gamma(\omega')$ *is a subset of a sublattice* $\mathbb{Z}^{d-j}$ *of codimension* $j \geq 1$*. Let* $H_{\omega,\omega'}$ *be the corresponding random Schrödinger operator with singular interactions formally given in (77), and defined using the resolvent formula (81). We assume that the random variables satisfy Hypotheses 4.1 and 4.2. Let* Σ *be the almost sure spectrum, and let* Σ_{ac} *be the almost sure absolutely continuous spectrum. Then, we have that* $[0, \infty) \subset \Sigma_{ac}$*. Furthermore, there exists a constant* $E_1(a) > 0$ *so that*

$\Sigma \cap (-\infty, -E_1(a)]$ is nonempty and pure point almost surely, with exponentially decaying eigenfunctions. The model exhibits exponential localization in this energy interval as in Theorem 1.2.

1.2 Contents

The contents of this paper are as follows. In section 2, we prove the main exponential estimates on the expectation of the small powers of $\|P_n(H_\omega - E - i\epsilon)^{-1}P_m\|$, $m, n \in I$, for the abstract model (1). We prove a spectral averaging result and Kotani's trick for this family of operators. We then prove the localization results. In section 3, we apply these results to the random nonlocal model. We introduce the multidimensional wavelet basis and locate the negative part of the essential spectrum of this family with probability one. We prove the key estimate on the matrix elements of the resolvent of the Laplacian in this basis. The random singular interaction models are defined in section 4. The almost sure spectrum of the model is described in detail. Basic resolvent estimates are proved. Spectral averaging and the perturbation of singular spectrum results are proved in section 5. Exponential and dynamical localization are proved in section 6. The Wegner estimate, implying the Lipschitz continuity of the integrated density of states, is proved in section 7. In section 8, we discuss the case of scattering theory for random singular interactions with impurities randomly located on subsets of $\mathbb{Z}^{d-j}$ of codimension $j \geq 1$. Using the methods of scattering theory, we prove the persistence of absolutely continuous spectrum at positive energies, and the existence of an interval of pure point spectrum at negative energies, almost surely, for these models.

1.3 Related Results

*There are a few results directly related to this work. The idea of estimating the expectation of small moments of the Green's functions for lattice models is due to Aizenman and Molchanov [**?**], and its application to weak disorder is due to Aizenman [**?**]. These papers apply to lattice models only. Quite recently, Aizenman, Elgart, Naboko, Schenker, and Stolz [**?**] announced an extension of this method to a restricted family of random Anderson models in the continuum. It is not clear that these methods apply to the models treated in this paper. The methods of Aizenman and Molchanov have been used to study random Landau models on $L^2(\mathbb{R}^2)$ by Dorlas, Macris, and Pulé [**?**]. The results for the lattice case suffice for their analysis since they* *reduce the model to a lattice one using the eigenfunctions of the unperturbed Landau model.*

2 Exponential Localization for an Abstract Non-local Model

We begin with an abstract theorem concerning the exponential decay of the expectation of small moments of the Green's function for operators of the form $H_\omega = H_0 + V_\omega$, on a separable Hilbert space $\mathcal{H}$. We assume that we have control on the Green's function for the self-adjoint operator H_0. The potential V_ω is an Anderson-type potential of the form

2.1 Basic Resolvent Estimates

We prove the exponential decay of the Green's function given a priori estimates on the unperturbed operator H_0.

Theorem 2.1. *Suppose that H_0 is a self-adjoint operator on a separable Hilbert space $\mathcal{H}$, and that $\{P_n \mid n \in I\}$ is a countable family of one-dimensional orthogonal projections such that $\sum_n P_n = 1$. Let H_ω be a family of random Schrödinger operators given by*

3. **[Strong Disorder or High Energy].** *For any $0 < s < 1$, suppose there is a finite constant $C_s > 0$ defined by*

Then, under either condition, there exists a finite constant $D_s = D_s(\gamma, I) > 0$, so that for every $E \in I = [a, b]$,

Remarks.

(a) *To understand the notion of weak or strong disorder, we replace the potential V_ω by λV_ω. Carrying out the same calculations, we arrive at (12) and (14) with K_s and C_s replaced by $\lambda^s K_s$ and $\lambda^s C_s$, respectively. Then, we see that small λ improves (13), and large λ improves (15).*

(b) *The constants K_s and C_s depend on the distributions μ_n. We implicitly assume that the constants in (12) and (14) are uniform in n. An explicit expression for K_s is given in Proposition 6.1, cf. [?].*

(c) *For localization only, we may take $\gamma = 0$ in (13) and (15). We obtain the bound (16) with $\gamma = 0$. This suffices to prove localization.*

Proof.
1. **Weak Disorder.** *Let $E \in I$. We begin by writing the resolvent equation as*

$$\begin{aligned} P_n(H_\omega - E - i\epsilon)^{-1}P_m &= P_n(H_0 - E - i\epsilon)^{-1}P_m \\ &\quad - \sum_k P_n(H_0 - E - i\epsilon)^{-1}\lambda_k(\omega)P_k(H_\omega - E - i\epsilon)^{-1}P_m. \end{aligned} \tag{1}$$

Taking the norm of both sides, the s^{th}-power, and then the expectation, we obtain

$$\begin{aligned} &\mathbb{E}\{\|P_n(H_\omega - E - i\epsilon)^{-1}P_m\|^s\} \\ &\leq \|P_n(H_0 - E - i\epsilon)^{-1}P_m\|^s \\ &\quad + \sum_k \|P_n(H_0 - E - i\epsilon)^{-1}P_k\|^s \\ &\quad \times \mathbb{E}\{|\lambda_k(\omega)|^s\ \|P_k(H_\omega - E - i\epsilon)^{-1}P_m\|^s\}. \end{aligned} \tag{2}$$

We now use a decoupling inequality to estimate the expectation of the product of the random variables in (2). Let $P_n = |\phi_n\rangle\langle\phi_n|$, where ϕ_n is a normalized vector in the one-dimensional range of P_n. Let us define an operator with no interaction at site k by $H_{\omega,k} \equiv H_\omega - \lambda_k(\omega)P_k$. Comparing this operator with H_ω via the resolvent identity, we easily find

$$\begin{aligned} \|P_k(H_\omega - z)^{-1}P_m\|^s &= |\langle\phi_k, (H_\omega - z)^{-1}\phi_m\rangle|^s \\ &= \frac{|\langle\phi_k, (H_{\omega,k} - z)^{-1}\phi_m\rangle|^s}{|1 + \lambda_k(\omega)\langle\phi_k, (H_{\omega,k} - z)^{-1}\phi_k\rangle|^s}, \end{aligned} \tag{3}$$

*where we write $z = E + i\epsilon$. The decoupling inequality (**??**) now gives*

2. **Strong Disorder or High Energy.** *We begin with the basic operator equation satisfied by the resolvent*

*The resolvent estimate (**??**) is the key ingredient in the proof of exponential localization as presented in [**?**]. Since the projectors P_n are rank-one, a modification of the Simon-Wolff [**?**] argument will allow us to prove exponential localization on the energy interval $I = [a, b]$ of Theorem 2.1. The modification is necessary since the unperturbed operator H_0 is, in general, an arbitrary, unbounded, self-adjoint operator satisfying, of course, either condition (13) or (15) on $I = [a, b]$.*

2.2 Spectral Averaging for the Nonlocal Family

*We restate the spectral averaging theorem of [**?**] in its simplest setting that suffices to study the family $H_\omega = H_0 + V_\omega$, with the potential V_ω given in (10).*

Theorem 2.2. *Let $H_\lambda = H_0 + \lambda P$, $\lambda \in I\!R$, be a one-parameter family of self-adjoint operators on $\mathcal{H}$, with P a rank-one orthogonal projector. For each real, nonnegative function $g \in C_0^1(I\!R)$, there exists a finite constant $C > 0$, depending on $\|g^{(j)}\|_1$, for $j = 0, 1$, such that for all $E \in I\!R$, and for all $\phi \in \mathcal{H}$, we have*

*We note that (**??**) holds for each P_n with a constant uniform in n. There are two immediate consequences of Theorem 2.2. The first states that for any real, nonnegative function $g \in C_0^1(I\!R)$, and any Borel subset $J \subset I\!R$, we have*

2.3 Perturbation of Singular Spectra for the Nonlocal Family

The classical Simon-Wolff criterion [?] for the Anderson model on $\mathbb{Z}^d$ is the following. If for Lebesgue almost every $E \in (a,b)$, there is a set $\Omega_E \subset \Omega$ of full measure so that

For our model (11), the estimate analogous to (38) is the following. The completeness of the projectors P_n implies the identity

In order to use this result to prove the almost sure pure point spectrum in I, we first consider one-parameter perturbations obtained by varying one coupling constant. Consider variations $\omega \to \omega'$ for which $\lambda_j(\omega) = \lambda_j(\omega')$, for $j \neq k$, and $\lambda_k(\omega') - \lambda_k(\omega) \equiv \lambda$. We then have $H_{\omega'} = H_\omega + \lambda P_k$. We consider this Hamiltonian as the Hamiltonian $H_\lambda = H_0 + \lambda P$ treated in the previous subsection.

Theorem 2.3. *Assume that H_0 satisfies the estimate (40) for $E \in I_0$, for some $I_0 \subset I$ with $|I_0| = |I|$. Then, we have that for almost every $\lambda \in \mathbb{R}$, $\sigma_{ac}(H_\lambda) \cap I = \emptyset = \sigma_{sc}(H_\lambda) \cap I$, and the spectrum of H_λ in I is pure point with finitely degenerate eigenvalues.*

The argument that Theorem 2.3 implies that the spectrum of H_ω in I is pure point almost surely follows as in [?, ?]. The exponential decay of the corresponding eigenfunctions requires a separate argument, cf. [?]. We note that the model (10) need not be ergodic.

3 Random Nonlocal Model with Wavelet Interactions

We now consider a specific the random nonlocal model on $L^2(\mathbb{R}^d)$ for which the results of the previous section can be directly applied. For these models, the unperturbed operator $H_0 = -\Delta$ and the random potential V_ω, as in (10), is diagonal in a wavelet basis. In order to construct the projections P_j, we first recall the definition of wavelets in higher dimensions.

A wavelet in one dimension is a function ψ with the property that the collection of translated and diadically dilated functions $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x -$

$k) \mid j, k \in \mathbb{Z}\}$, *forms an orthonormal basis for* $L^2(\mathbb{R})$. *Associated with the wavelet* ψ *is the scaling function* ϕ. *The scaling function* ϕ *is used to construct the wavelet* ψ *through a procedure called multiresolution analysis (cf. [?, ?]).*

To define a wavelet in higher dimensions (as in [?], Proposition 5.2), we first start with a collection $\{\phi_1, \ldots, \phi_d, \psi_1, \ldots, \psi_d\}$ *of* $2d$ *functions on* $I\!R$ *of which the* ϕ_j *are scaling functions and the* ψ_j *are the associated wavelets constructed from the* ϕ_j. *We note that we may take all the* $\phi_j \equiv \phi, \quad \psi_j \equiv \psi, \quad j = 1, \ldots, d.$*, although this is not necessary. Let us define an index set* $F = \{c = (c_1, \ldots, c_d) \in \{0,1\}^d \setminus (0, 0, \ldots, 0)\}$. *For each* $c \in F$, *we define a function on* $I\!R^d$ *by*

$$\Psi_c(x) = \prod_{j=1}^{d} (\delta_{c_j,0}\phi_j + \delta_{c_j,1}\psi_j)(x_j), \quad c \in F. \tag{4}$$

Here, the $\delta_{c_j,k}$, *for* $c = (c_1, \ldots, c_d) \in F$ *and* $k = 0, 1$, *is the Kronecker delta. In the product, the function* $\phi_j(x_j)$ *is present if the index* c_j *is zero, and* $\psi_j(x_j)$ *is present otherwise. Note that there is at least one factor* ψ_j *in* Ψ_c *for any* $c \in F$. *We consider the set of diadic dilations and* $Z\!\!\!Z^d$*-lattice translations of these functions. We denote by* I *the countable index set* $I = F \times Z\!\!\!Z \times Z\!\!\!Z^d$. *An element* $\mathbf{n} \in I$ *is a triple* $\mathbf{n} = (c(\mathbf{n}), n_1, n_2)$. *The collection of dilated and translated functions*

In the following we shall, notationally, always refer to the collection of functions $\{\Psi_c \mid c \in F\}$ *simply as* Ψ, *and any property stated for* Ψ *is by definition to be take to be valid for each member of this collection. Thus a statement that the property* P *is valid for* $\widehat{\Psi}$ *means that* P *is valid for each of the Fourier transforms* $\widehat{\Psi_c}$, *for each* $c \in F$, *and so on.*

We assume the following conditions on the multivariable wavelet and the distribution of the random variables $\{\lambda_{\mathbf{n}}(\omega) \mid \mathbf{n} \in I\}$.

Hypothesis 3.1. *Let* Ψ *be a multivariable wavelet formed out of the scaling functions* $\phi_i, i = 1, \ldots, d$ *and the wavelets* $\psi_i, i = 1, \ldots, d$ *such that*

> *(a) the functions* $\widehat{\phi_j}, \widehat{\psi_j} \in \mathcal{C}^4(\mathbb{R}), \quad j = 1, \ldots, d;$
>
> *(b) the functions* $\widehat{\psi_j}$ *have compact support in* $I\!R \backslash 0, \quad j = 1, \ldots, d;$
>
> *(c) the functions* $\widehat{\phi}_j^{(\alpha)}$, *for* $|\alpha| \leq 4$, *decay rapidly;*

(d) the functions are normalized, $\int |\Psi|^2 \, dx = 1$.

Hypothesis 3.2. *Let* $I = F \times \mathbb{Z} \times \mathbb{Z}^d$, *and let* $\{\lambda_{\mathbf{n}}(\omega) \mid \mathbf{n} \in I\}$ *be independent and identically distributed random variables with their common probability distribution* μ *being absolutely continuous and of compact support in* $\mathbb{R}$.

Remarks:

(a) Any one-dimensional Lemarié-Meyer wavelet ψ, *and its related scaling function* ϕ, *satisfy Hypothesis 3.1. Typically, a Meyer wavelet can be constructed to be in the Schwartz class,* $\psi \in \mathcal{S}(\mathbb{R})$, *and its Fourier transform* $\hat{\psi}$ *is compactly supported in the set* $[-8\pi/3, -2\pi/3] \cup [2\pi/3, 8\pi/3]$. *The corresponding scaling function can also be chosen to satisfy* $\phi \in \mathcal{S}(\mathbb{R})$, *and so that* $\hat{\phi}$ *has compact support in* $[-4\pi/3, 4\pi/3]$, *cf. [**?**, **?**]. A large number of additional examples are constructed in the paper of Auscher, Weiss, and Wickerhauser [**?**].*

*(b) As discussed in [**?**], there is another method of constructing multidimensional wavelets using tensor products. We could also use this construction below. Wavelets constructed by tensoring one-dimensional wavelets have some undesirable support properties.*

As in section 1, we consider the operator $H_0 = -\Delta$, *and define the random family of nonlocal interaction Hamiltonians on* $L^2(\mathbb{R}^d)$ *by*

where

$P_{\mathbf{n}}$ *is the rank-one projection defined by*

$$H_\omega = -\Delta + V_\omega, \quad V_\omega = \sum_{\mathbf{n} \in I} \lambda_{\mathbf{n}}(\omega) P_{\mathbf{n}},$$

where $P_{\mathbf{n}}$ *is the rank-one projection defined by*

$$P_{\mathbf{n}} \equiv |\Phi_{\mathbf{n}}\rangle\langle\Phi_{\mathbf{n}}|, \quad \text{with } \mathbf{n} \in I. \tag{6}$$

Since μ *has compact support, each* $\lambda_{\mathbf{n}}(\omega)$ *takes values in a compact set, so the operator* V_ω *is bounded for each* ω. *Then, the operator* H_ω *is self-adjoint* *on the same domain as H_0. Although the operator H_ω is not ergodic, we can determine the essential spectrum using the action of the subgroup of dilations.*

Proposition 3.3. *Suppose that the support of μ is $[-M_0, 0]$. Then, the essential spectrum of H_ω is $[-M_0, \infty)$ almost surely.*

Proof. *We concentrate on the negative part of the essential spectrum. For any $E \in supp(\mu)$, the Borel-Cantelli theorem implies that for any $\epsilon > 0$, the event*

$\Omega(\epsilon, E) = \{\omega \mid |\lambda_{(c,k,0)}(\omega) - E| < \epsilon,$ *for infinitely-many* **negative** *integers k , $c \in F\}$,*

has measure one. Consequently, the set

$$alsohasmeasureone.Foreachrational$$

$r \in supp(\mu)$, and for each $l \in \mathbb{Z}^+$, there is a sequence s_l of **negative** *integers with $|\lambda_{(c,s_l,0)}(\omega) - r| < 1/l$, for all $\omega \in \Omega_0$. For each $l \in \mathbb{Z}^+$, we choose one element of s_l, so that $\{s_l\}$ is monotone decreasing, and define a sequence of vectors $f_l = \Phi_{(c,s_l,0)}$, with $\|f_l\| = 1$. In the lemma below, we prove a bound on $\|\Delta f_l\|$. Using these vectors f_l, we have,*

$$\Omega_0 = \bigcap_{l \in \mathbb{Z}^+} \bigcap_{r \in \mathbb{Q} \cap supp(\mu)} \Omega(1/l, r),$$

also has measure one. For each rational $r \in supp(\mu)$, and for each $l \in \mathbb{Z}^+$, there is a sequence s_l of **negative** *integers with $|\lambda_{(c,s_l,0)}(\omega) - r| < 1/l$, for all $\omega \in \Omega_0$. For each $l \in \mathbb{Z}^+$, we choose one element of s_l, so that $\{s_l\}$ is monotone decreasing, and define a sequence of vectors $f_l = \Phi_{(c,s_l,0)}$, with $\|f_l\| = 1$. In the lemma below, we prove a bound on $\|\Delta f_l\|$. Using these vectors f_l, we have,*

$$\begin{aligned} \|(H_\omega - r) f_l\| &\leq \|\Delta \Phi_{(c,s_l,0)}\| + \|(\lambda_{(c,s_l,0)}(\omega) - r) f_l\| \\ &\leq 2^{4 s_l} C + \frac{1}{l}, \end{aligned} \tag{8}$$

which obviously goes to zero as l goes to infinity, since s_l are **negative** *by choice. Thus, we produced a sequence f_l for which $\|(H_\omega - r) f_l\|$ goes to zero* *as l goes to ∞, for all $\omega \in \Omega_0$. Since the sequence $\{f_l\}$ is orthonormal, we have $r \in \sigma_{ess}(H_\omega),\ \ \omega \in \Omega_0$. Since this happens for all rationals in the support of μ, we see that $supp(\mu) \subset \sigma(H_\omega),\ \ \omega \in \Omega$.* $\square$

Lemma 3.4. *For any $\mathbf{n} \in I$, we have*

$$\|\Delta \Phi_{\mathbf{n}}\| \leq 2^{2n_1} C,$$

where C is a constant that is independent of $\mathbf{n}$.

Proof. *We recall equations (41) and (42) and the Hypothesis 3.1 on the functions ϕ_j and ψ_j that are used to define the functions $\Psi_{\mathbf{n}}$ and the orthonormal basis $\{\Phi_{\mathbf{n}}, \mathbf{n} \in I = F \times \mathbb{Z} \times \mathbb{Z}^d\}$. We also recall that the functions ψ_j have compact support in $\mathbb{R} \setminus \{0\}$, from which it follows that the following integral is finite for any polynomial P. In the following, the differential operator $P(\nabla)$ is defined using the (polynomial) symbol $P(\xi), \xi \in \mathbb{R}^d$ and the "hat" denotes Fourier transform.*

$$\begin{aligned}\langle \Phi_{\mathbf{m}}, P(\nabla)\Phi_{\mathbf{n}}\rangle = 2^{-(m_1-n_1)d/2} \int_{\mathbb{R}^d} & e^{i(2^{-(m_1-n_1)}m_2 - n_2)\cdot\xi} \\ & \times \overline{\widehat{\Psi_{c(\mathbf{m})}}}(2^{-(m_1-n_1)}\xi)\ \widehat{\Psi_{c(\mathbf{n})}}(\xi) P(2^{n_1}\xi)\ d\xi.\end{aligned} \tag{9}$$

Using the equation (9), where we take $\mathbf{m} = \mathbf{n}$ and the polynomial $P(\xi) = (\sum_j \xi_j^2)^2)$, we have

$$\begin{aligned}\|\Delta\Phi_{\mathbf{n}}\|^2 &= 2^{-(n_1-n_1)d/2} \int_{\mathbb{R}^d} e^{i(2^{-(n_1-n_1)}m_2-n_2)\cdot\xi}\overline{\widehat{\Psi_{c(\mathbf{n})}}}(2^{-(n_1-n_1)}\xi) \\ &\qquad \times\ \widehat{\Psi_{c(\mathbf{n})}}(\xi)(\sum_j (2^{n_1}\xi_j)^2)^2\ d\xi \\ &\leq 2^{4n_1} \int_{\mathbb{R}^d} |\widehat{\Psi_{c(\mathbf{n})}}(\xi)|\ |\widehat{\Psi_{c(\mathbf{n})}}(\xi)|(\sum_j (\xi_j)^2)^2\ d\xi \\ &\leq C 2^{4n_1},\end{aligned} \tag{10}$$

where we take

$$C = \sup_{c\in F} \int_{\mathbb{R}^d} |\widehat{\Psi}_c(\xi)|\ |\widehat{\Psi}_c(\xi)|(\sum_j (\xi_j)^2)^2\ d\xi,$$

which gives the lemma. $\square$

The main result is the following theorem.

Theorem 3.5. *Let H_ω be a family of random nonlocal Schrödinger operators as in equation (3), with the multivariable wavelet Ψ satisfying the Hypothesis 3.1, and the random variables $\lambda_{\mathbf{n}}(\omega), \mathbf{n} \in I$ satisfying Hypothesis 3.2. Then, there is a number $E_1(\mu) > 0$ such that the spectrum of H_ω in $(-\infty, -E_1(\mu)]$ is pure point for almost every ω.*

The proof of Theorem 3.5 follows the argument of section 2. It is necessary to prove either hypothesis (13) on the resolvent of H_0 or hypothesis (15) on H_0. We give a proof that hypothesis (13) is satisfied.

Proof of Theorem 3.5. *We have to verify the condition (13) of Theorem 2.1 when $H_0 = -\Delta$. Let us begin by estimating the matrix elements*

$$T(\mathbf{m}, \mathbf{n}, E) = \langle \Phi_{\mathbf{m}}, (-\Delta - E)^{-1} \Phi_{\mathbf{n}} \rangle, \quad \mathbf{n}, \mathbf{m} \in I, \quad E \in (-\infty, 0). \tag{11}$$

Since $-\Delta \geq 0$, the resolvent $(-\Delta - E)^{-1}$, for $E < 0$, is bounded. This implies that $T(\mathbf{m}, \mathbf{n}, E)$ is finite for all $\mathbf{n}, \mathbf{m} \in I$, $E < 0$. Our aim is to show that for suitable $1/4 < s < 1/2$, we actually have a $E(\mu) > 0$ such that

$$K_s \left(\sup_{m \in I} \sum_{n \in I} |T(\mathbf{m}, \mathbf{n}, E)|^s \right) < 1 \quad \textit{for} \quad E \in (-\infty, -E(\mu)), \tag{12}$$

*where K_s is a constant given in the inequality (**??**). Let $S_d = \{(2l+1)/2 \mid l = 0, 1, \ldots, 4d-1\}$, for $d \geq 1$. Given the estimate in the inequality (14), in Lemma 3.6 below, we have*

$$K_s \left(\sup_{m \in I} \sum_{n \in I} |T(\mathbf{m}, \mathbf{n}, E)|^s \right) \leq K_s \left[\frac{C(\Psi, s, d)}{\sqrt{|E|}} \max_{j \in S_d} \frac{1}{|E|^j} \right]. \tag{13}$$

This inequality makes it clear that for either K_s small enough, or for $|E|$ large, there is a critical $E(\mu) > 0$ such that for all $E < -E(\mu)$, the right hand side of (51) is smaller than 1. In order to see that the set of $E < 0$ for which these conditions hold, is nonempty, we need to examine the dependence of the constant K_s on the probability distribution μ. For this, we use estimates (154)–(155) of Proposition 6.1. For example, if $d\mu(\lambda) = \chi_{[-M_0, 0]}(\lambda) d\lambda$, it follows from (154)–(155) that $K_s \leq C_0 M_0^\tau$, for any $0 < \tau \leq 1$, where the constant C_0 is an absolute number. With upper bound (51), the inequality (13) is satisfied provided the energy $E < 0$ satisfies:

Lemma 3.6. *For any $E \in (-\infty, 0)$, there is a constant $C = C(\Psi, s, d)$, independent of E, such that*

$$\sup_{\mathbf{m}\in I} \sum_{\mathbf{n}\in I} |T(\mathbf{m},\mathbf{n},E)|^s \le \frac{C(\Psi,s,d)}{\sqrt{|E|}} \max_{j\in S_d} \frac{1}{|E|^j}, \tag{14}$$

where $S_d = \{(2l+1)/2 \mid l = 0, 1, \dots, 4d-1\}$.

Proof:

1. Recall that by definition $\Phi_{\mathbf{n}}(x) = 2^{n_1 d/2}\Psi_{c(\mathbf{n})}(2^{n_1}x - n_2)$, $(c(\mathbf{n}), n_1, n_2) \in I = F \times \mathbb{Z} \times \mathbb{Z}^d$, and similarly for $\mathbf{m} = (c(\mathbf{m}), m_1, m_2) \in I$. By Parseval's identity, we write the matrix element (11) in terms of the Fourier transform of the wavelets as

$$T(\mathbf{m},\mathbf{n},E) = \int_{\mathbb{R}^d} \overline{\widehat{\Phi_{\mathbf{m}}}}(\xi)\widehat{\Phi_{\mathbf{n}}}(\xi)\frac{1}{(\xi^2 - E)}\, d\xi, \tag{15}$$

where $\xi^2 = \sum_{j=1}^d \xi_j^2$. Using the definition of the Fourier transform and the definition of $\Phi_{\mathbf{n}}$, we see that

$$\begin{aligned}
\widehat{\Phi_{\mathbf{n}}}(\xi) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}} e^{-i\xi\cdot x}\Phi_{\mathbf{n}}(x)\, dx \\
&= \frac{2^{dn_1/2}}{(2\pi)^{d/2}} \int_{\mathbb{R}} e^{-i\xi\cdot x}\Psi_{c(\mathbf{n})}(2^{n_1}x - n_2)\, dx \\
&= 2^{-dn_1/2} e^{-i2^{-n_1}n_2\cdot\xi} \;\; \widehat{\Psi_{c(\mathbf{n})}}(2^{-n_1}\xi).
\end{aligned} \tag{16}$$

Then, the matrix element (11) becomes

$$T(\mathbf{m},\mathbf{n},E) = 2^{-(n_1+m_1)d/2} \int_{\mathbb{R}^d} e^{i(2^{-m_1}m_2 - 2^{-n_1}n_2)\cdot\xi}\; \overline{\widehat{\Psi_{c(\mathbf{m})}}}(2^{-m_1}\xi)\; \widehat{\Psi_{c(\mathbf{n})}}(2^{-n_1}\xi) \frac{1}{(\xi^2 - E)}\, d\xi. \tag{17}$$

Changing variables $\xi \to 2^{-n_1}\xi$ we find that the above equation is

$$T(\mathbf{m},\mathbf{n},E) = 2^{-(m_1-n_1)d/2} \int_{\mathbb{R}^d} e^{i(2^{-(m_1-n_1)}m_2 - n_2)\cdot\xi}\overline{\widehat{\Psi_{c(\mathbf{m})}}}(2^{-(m_1-n_1)}\xi)\; \widehat{\Psi_{c(\mathbf{n})}}(\xi) \frac{1}{(2^{2n_1}\xi^2 - E)}\, d\xi. \tag{18}$$