Some results on the phase structure of the two-dimensional Widom-Rowlinson model

Yasunari HIGUCHI and Masato TAKEI

1 Introduction

We study the phase structure of the two-dimensional (2D) lattice Widom-Rowlinson model. Let $\Omega = \{-1, 0, +1\}^{\mathbb{Z}^2}$ be the configuration space with product topology. The Borel σ-algebra of Ω is denoted by $\mathcal{F}$. For $\Lambda \subset \mathbb{Z}^2$, we consider $\Omega_\Lambda = \{-1, 0, +1\}^\Lambda$ and its Borel σ-algebra $\mathcal{F}_\Lambda$. We write $x \sim y$ if $x, y \in \mathbb{Z}^2$ are adjacent, namely $|x_1 - y_1| + |x_2 - y_2| = 1$. We say that x and y are $(*)$adjacent and write $x \overset{*}{\sim} y$ if $\max\{|x_1 - y_1|, |x_2 - y_2|\} = 1$. A configuration $\omega \in \Omega_\Lambda$ is said to be *feasible* if $\omega(x)\omega(y) \neq -1$ for all adjacent $x, y \in \Lambda$.

We write $\Lambda \subset\subset \mathbb{Z}^2$ if Λ is a finite subset of $\mathbb{Z}^2$. For $\Lambda \subset\subset \mathbb{Z}^2$ and a feasible boundary condition $\omega \in \Omega$, the *finite volume Gibbs distribution* $\mu^\omega_{\Lambda,\lambda,h}$ is defined by

$$\mu^\omega_{\Lambda,\lambda,h}(\sigma) = \frac{1}{Z^\omega_{\Lambda,\lambda,h}} 1_{\{\sigma*\omega:\text{feasible}\}} \prod_{x\in\Lambda} \lambda^{\sigma(x)^2} e^{h\sigma(x)}.$$

Here $\lambda > 0$ is a parameter called *activity*, and $h \in \mathbb{R}$ is a parameter which plays a similar role as the external field in the Ising model. The normalizing constant $Z^\omega_{\Lambda,\lambda,h}$ is called the *partition function*. The configuration $\sigma * \omega \in \Omega$ is defined by

$$\sigma * \omega(x) = \begin{cases} \sigma(x) & \text{if } x \in \Lambda, \\ \omega(x) & \text{if } x \in \Lambda^c. \end{cases}$$

A probability measure μ on $(\Omega, \mathcal{F})$ which satisfies the *DLR equation*

$$\mu\left(\cdot|\mathcal{F}_{\Lambda^c}\right)(\omega) = \mu^\omega_{\Lambda,\lambda,h}(\cdot) \quad \mu\text{-a.a.}\omega \quad (\Lambda \subset\subset \mathbb{Z}^2)$$

is said to be a *Gibbs measure with parameter* (λ, h). The set of all Gibbs measures with parameter (λ, h) is denoted by $\mathcal{G}(\lambda, h)$. It is well-known that

$\mathcal{G}(\lambda, h)$ is a non-empty compact convex set. We write $\mathcal{G}_{\rm ex}(\lambda, h)$ for the set of all extremal Gibbs measures. (For the general properties of Gibbs measures, we refer to [4] or [11].)

Russo[12] introduced the infinite cluster method for studying the phase structure of the 2D Ising model, which is the key step to the final answer([1], [9]). In [5], the structure of phases is described in terms of percolation and possible extensions are given. In this paper, we consider the 2D Widom-Rowlinson model. Although this model is generally thought to be similar to the 2D Ising model, the proof of [5] does not work.

We state our main results. Van den Berg and Steif conjectured that the hard-core lattice gas model on $\mathbb{Z}^d$ with parity-dependent activities has no phase transition, and Häggström proved it in the 2D case(see [6] §3.4 and [7]). In [6] §3.5, it is conjectured that the Widom-Rowlinson model on $\mathbb{Z}^d$ with asymmetric activities (i.e. $h \neq 0$) admits no phase transition. We expect that Häggström's method can be also adapted to the asymmetric Widom-Rowlinson model on $\mathbb{Z}^2$. Unfortunately, we cannot answer this question completely.

Theorem 1.1. For each $\lambda > 0$, there exists $h_c = h_c(\lambda) \geq 0$ such that

$$|h| > h_c \Longrightarrow |\mathcal{G}(\lambda, h)| = 1.$$

Especially, $h_c(\lambda) = 0$ when $|\mathcal{G}(\lambda, 0)| > 1$.

Now we turn to the symmetric case (i.e. $h = 0$). In [10], it is shown that Gibbs measures are unique when $\lambda < p_c/(1-p_c)$ and multiple when $\lambda > 8p_c/(1-p_c)$, where p_c denotes the critical probability of Bernoulli site percolation on $\mathbb{Z}^2$. Although our result is restricted to the large activity case, we can describe the structure of a good class of Gibbs measures.

Theorem 1.2. Assume that $h = 0$ and $\lambda > 8p_c/(1-p_c)$. Let μ_λ^+ and μ_λ^- be the limiting Gibbs measures with plus and minus boundary conditions, respectively.

(i) The limiting Gibbs measure with free boundary condition is equal to $(\mu_\lambda^+ + \mu_\lambda^-)/2$.

(ii) If $\mu \in \mathcal{G}(\lambda, 0)$ is either horizontally periodic or vertically periodic, then

$$\mu = \alpha\mu_\lambda^+ + (1-\alpha)\mu_\lambda^-$$

with some $\alpha \in [0,1]$.

Remark 1. The proof of Theorem 1.2(i) is valid for any dimension. Using the same argument, we can prove that for sufficient large λ the limit point of $\mu_{\Lambda,\lambda,0}^{\omega}$ with $\omega \geq 0$ or $\omega \leq 0$ is a mixture of μ_λ^+ and μ_λ^-.

The remainder of this paper is organized as follows. In section 2, we review important properties of Gibbs measures of Widom-Rowlinson model. The infinite cluster method introduced by Russo is explained in section 3. We give the proof of Theorem 1.1 in section 4. In section 5 and section 6, we concentrate on the symmetric case. We define the site-random cluster representation of the finite volume Gibbs distribution in section 5, which allows us to compare it with Bernoulli site percolation. The proof of Theorem 1.2 is given in section 6.

2 Preliminaries

For $\omega, \omega' \in \Omega$ and $\Lambda \subset \mathbb{Z}^2$, we write $\omega = \omega'$ on[off] Λ if $\omega(x) = \omega'(x)$ for all $x \in \Lambda[x \in \Lambda^c]$. Let $\partial\Lambda$ and $\partial^-\Lambda$ be outer and inner boundaries of Λ, respectively:

$$\partial\Lambda = \{y \notin \Lambda; y \sim x \text{ for some } x \in \Lambda\},$$
$$\partial^-\Lambda = \{x \in \Lambda; y \sim x \text{ for some } y \notin \Lambda\}.$$

A *cylinder function* is a function which is $\mathcal{F}_\Delta$-measurable for some $\Delta \subset\subset \mathbb{Z}^2$. For a cylinder function f, supp f denotes the smallest Δ such that f is

$\mathcal{F}_\Delta$-measurable, i.e.

$$\text{supp } f = \bigcap \{\Delta \subset\subset \mathbb{Z}^2; f \text{ is } \mathcal{F}_\Delta\text{-measurable}\}.$$

An event E is called a *cylinder event* if its indicator function 1_E is a cylinder function.

2.1 Strong Markov property

By definition, $\mu^\omega_{\Lambda,\lambda,h}$ enjoys the Markov property, namely $\mu^\omega_{\Lambda,\lambda,h}(\sigma)$ depends only on the values of ω on $\partial\Lambda$. Moreover we can state the strong Markov property as follows. Let $\mu \in \mathcal{G}(\lambda, h)$. We say that a $\mathbb{Z}^2$-valued random variable Γ is *determined from outside* if $\{\Gamma = \Lambda\} \in \mathcal{F}_{\Lambda^c}$ for any nonempty $\Lambda \subset\subset \mathbb{Z}^2$. We consider a σ-algebra

$$\mathcal{F}_{\Gamma^c} = \{A \in \mathcal{F}; A \cap \{\Gamma = \Lambda\} \in \mathcal{F}_{\Lambda^c} \text{ for any nonempty } \Lambda \subset\subset \mathbb{Z}^2\}.$$

Lemma 2.1 (strong Markov property). Each Gibbs measure μ enjoys the strong Markov property : if Γ is finite μ-a.s. and determined from outside, then

$$\mu\left(\cdot|\mathcal{F}_{\Gamma^c}\right)(\omega) = \mu^\omega_{\Gamma(\omega),\lambda,h}(\cdot) \quad \mu\text{-a.a.}\omega.$$

REMARK 2. Let A be a cylinder event. If $\Gamma(\omega) = \emptyset$, then we set $\mu^\omega_{\Gamma(\omega)}(A) = 1_A(\omega)$. If $\Gamma(\omega)$ contains infinitely many points, then we set $\mu^\omega_{\Gamma(\omega)}(A) = \mu(A)$.

The proof is elementary and we omit it.

2.2 Stochastic domination

First we state the Holley-FKG inequality for rather general settings.

Let Λ be a finite set and S be a finite subset of $\mathbb{R}$. We set $\tilde{\Omega}_\Lambda = S^\Lambda$. For $\sigma, \sigma' \in \tilde{\Omega}_\Lambda$, we write $\sigma \le \sigma'$ if $\sigma(x) \le \sigma'(x)$ for all $x \in \Lambda$. Let μ, μ' be probability measures on $\tilde{\Omega}_\Lambda$. We write $\mu \le \mu'$ if $\mu(f) \le \mu'(f)$ for any increasing function f on $\tilde{\Omega}_\Lambda$. For a probability measure μ on $\tilde{\Omega}_\Lambda$, we define $\tilde{\Omega}^\mu_\Lambda = \{\sigma \in \tilde{\Omega}_\Lambda; \mu(\sigma) > 0\}$. We say that μ is *nice* if there exists $M = M(\mu) \in$

$\tilde{\Omega}_\Lambda^\mu$ such that $\sigma \le M$ for all $\sigma \in \tilde{\Omega}_\Lambda^\mu$. For $\sigma, \sigma' \in \tilde{\Omega}_\Lambda$, we say $\sigma \sim \sigma'$ if there exists $x \in \Lambda$ such that $\sigma(x) \ne \sigma'(x)$ and $\sigma = \sigma'$ off x. We can define the connectedness of the subset of $\tilde{\Omega}_\Lambda$ with respect to the relation $\sim$. μ is called *irreducible* if $\tilde{\Omega}_\Lambda^\mu$ is connected in this sense.

Theorem 2.2.

(i) (*Holley's inequality*) Let μ, μ' be nice and irreducible probability measures. In addition we assume that $M(\mu) \le M(\mu')$. If for any $x \in \Lambda$, $a \in S$, $\eta, \eta' \in \tilde{\Omega}_{\Lambda \setminus \{x\}}$ such that $\eta \le \eta'$, $\mu(\sigma = \eta \text{ off } x) > 0$ and $\mu'(\sigma = \eta' \text{ off } x) > 0$,

$$\mu(\sigma(x) \ge a | \sigma = \eta \text{ off } x) \le \mu'(\sigma(x) \ge a | \sigma = \eta' \text{ off } x)$$

holds, then $\mu \le \mu'$.

(ii) (*the FKG inequality*) Let μ be a nice and irreducible probability measure on $\tilde{\Omega}_\Lambda$. If for any $x \in \Lambda$, $a \in S$, $\eta, \eta' \in \tilde{\Omega}_{\Lambda \setminus \{x\}}$ such that $\eta \le \eta'$, $\mu(\sigma = \eta \text{ off } x) > 0$ and $\mu(\sigma = \eta' \text{ off } x) > 0$,

$$\mu(\sigma(x) \ge a | \sigma = \eta \text{ off } x) \le \mu(\sigma(x) \ge a | \sigma = \eta' \text{ off } x)$$

is satisfied, then μ has positive correlations, i.e. $\mu(fg) \ge \mu(f)\mu(g)$ holds for increasing functions f, g on $\tilde{\Omega}_\Lambda$.

The proof of this theorem is obtained by a slight modification of the argument in [6] §4.2.

Now we return to the Widom-Rowlinson model. For $\omega, \omega' \in \Omega$, we write $\omega \le \omega'$ if $\omega(x) \le \omega'(x)$ for all $x \in \mathbb{Z}^2$, regarding $\{-1, 0, +1\} \subset \mathbb{R}$. Let μ and ν be probability measures on $(\Omega, \mathcal{F})$. We say $\mu \le \nu$ if $\mu(f) \le \nu(f)$ for any increasing cylinder function f on Ω. The finite Gibbs distribution in $\Lambda \subset\subset \mathbb{Z}^2$ with boundary condition $\omega \equiv +1$(resp. $0, -1$) is denoted by $\mu^+_{\Lambda,\lambda,h}$(resp. $\mu^0_{\Lambda,\lambda,h}, \mu^-_{\Lambda,\lambda,h}$).

Lemma 2.3. The finite Gibbs distributions have following properties :

(i) The FKG inequality holds for $\mu^\omega_{\Lambda,\lambda,h}$.

(ii) $\mu^\omega_{\Lambda,\lambda,h} \le \mu^{\omega'}_{\Lambda,\lambda,h}$ if $\omega \le \omega'$.

(iii) $\mu^\omega_{\Lambda,\lambda,h} \le \mu^\omega_{\Lambda,\lambda,h'}$ if $h \le h'$.

(iv) If $\Lambda \subset \Delta$, then $\mu^+_{\Lambda,\lambda,h} \ge \mu^+_{\Delta,\lambda,h}, \mu^-_{\Lambda,\lambda,h} \le \mu^-_{\Delta,\lambda,h}$.

Proof. Since the set of feasible configurations is connected, $\mu^{\omega}_{\Lambda,\lambda,h}$ is irreducible. It is clear that both $\mu^{\omega}_{\Lambda,\lambda,h}$ and $\mu^{\omega'}_{\Lambda,\lambda,h}$ are nice. Indeed,

$$M(\mu^{\omega}_{\Lambda,\lambda,h}) = \begin{cases} 0 & \text{on } \{x \in \partial^{-}\Lambda; \omega(y) = -1 \text{ for } y \in \partial\Lambda \text{ with } y \sim x\}, \\ +1 & \text{otherwise,} \end{cases}$$

and $M(\mu^{\omega'}_{\Lambda,\lambda,h})$ is similar. We note that $M(\mu^{\omega}_{\Lambda,\lambda,h}) \le M(\mu^{\omega'}_{\Lambda,\lambda,h})$ because $\omega \le \omega'$.

Fix any $x \in \Lambda$. For $\eta \in \tilde{\Omega}_{\Lambda\setminus\{x\}}$ such that $\eta * \omega$ is feasible, we can easily see that $\mu^{\omega}_{\Lambda,\lambda,h}(\sigma(x) = +1|\sigma = \eta \text{ off } x)$ is equal to

$$\begin{cases} 0 & \text{if } \eta * \omega(y) = -1 \text{ for some } y \sim x, \\ \frac{\lambda e^{h}}{\lambda e^{h}+1+\lambda e^{-h}} & \text{if } \eta * \omega(y) = 0 \text{ for all } y \sim x, \\ \frac{\lambda e^{h}}{\lambda e^{h}+1} & \text{otherwise.} \end{cases}$$

It turns out that this conditional probability is increasing in ω, η and h. Similarly, we can see that $\mu^{\omega}_{\Lambda,\lambda,h}(\sigma(x) \ge 0|\sigma = \eta \text{ off } x)$ is increasing in ω, η and h(but not in λ!). Hence (i)-(iii) follows from Theorem 2.2. (iv) is proved by standard application of (i). $\square$

Remark 3. Since the above conditional probability is not increasing in λ, the monotonicity of phase transition depends on the underlying graph. Examples are found in [2] and [8].

2.3 Extremal Gibbs measures

Let $\mu^{+}_{\lambda,h}$ and $\mu^{-}_{\lambda,h}$ be the limiting Gibbs measures of $\mu^{+}_{\Lambda,\lambda,h}$ and $\mu^{-}_{\Lambda,\lambda,h}$ as $\Lambda \nearrow \mathbb{Z}^2$. These exist by virtue of Lemma 2.3(iv). It is well-known that limiting Gibbs measures satisfy the DLR equation. Both $\mu^{+}_{\lambda,h}$ and $\mu^{-}_{\lambda,h}$ are invariant under any graph automorphism of $\mathbb{Z}^2$. It follows from Lemma 2.3(ii) that

$$\mu^{-}_{\lambda,h} \le \mu \le \mu^{+}_{\lambda,h}$$

for any $\mu \in \mathcal{G}(\lambda, h)$. From this, it is easy to see that $\mu^{+}_{\lambda,h}, \mu^{-}_{\lambda,h} \in \mathcal{G}_{\mathrm{ex}}(\lambda, h)$. Let $\mathcal{T} = \bigcap_{\Lambda\subset\subset\mathbb{Z}^2} \mathcal{F}_{\Lambda^c}$, which is called the *tail σ-algebra*. The following lemma is well-known.

Lemma 2.4. Following (i)-(iii) are equivalent.

(i) $\mu \in \mathcal{G}_{\rm ex}(\lambda, h)$

(ii) μ is tail-trivial, which means that $\mu(A) = 0$ or 1 for any $A \in \mathcal{T}$.

(iii) $\lim_{\Lambda \nearrow \mathbb{Z}^2} \mu^{\omega}_{\Lambda,\lambda,h} = \mu$ for μ-a.a.ω.

From this lemma, we can find that every extremal Gibbs measure satisfies the FKG inequality. It is also well-known that any Gibbs measure is uniquely represented as a convex combination of extremal Gibbs measures.

The following criterion of the uniqueness of Gibbs measure is useful(see [6] Theorem 4.17).

Proposition 2.5. Following (i)-(iii) are equivalent.

(i) $\mathcal{G}(\lambda, h)$ is a singleton.

(ii) $\mu^{+}_{\lambda,h} = \mu^{-}_{\lambda,h}$

(iii) For all $x \in \mathbb{Z}^2$, $\mu^{+}_{\lambda,h}(\sigma(x)) = \mu^{-}_{\lambda,h}(\sigma(x))$.

3 The infinite cluster method

Russo[12] created the infinite cluster method for determining the phase structure of the 2D Ising model. As in [5], we state his key results in the form of lemmata. In addition, we study the uniqueness of the infinite cluster under periodic Gibbs measures in section 3.6.

3.1 Basic concepts of percolation theory

A sequence $p = (x_1, \cdots, x_k)$ of distinct points of $\mathbb{Z}^2$ is a (finite) *path* from x_1 to x_k if $x_i \sim x_{i+1} (i = 1, \cdots, k-1)$. We similarly define an infinite path $p = (x_1, x_2, \cdots)$. We say p is a path in $S \subset \mathbb{Z}^2$ if $p \subset S$. A path p is called *circuit* if $x_k \sim x_1$. $C \subset \mathbb{Z}^2$ is said to be *connected* if for any $x, y \in C$ there exists a path in C from x to y. A *cluster* in $S \subset \mathbb{Z}^2$ is a maximal connected component of S. A cluster which contains infinitely many points is called an *infinite cluster*. A sequence $p = (x_1, \cdots, x_k)$ of distinct points of $\mathbb{Z}^2$ is a

$(*)$*path* from x_1 to x_k if $x_i \overset{*}{\sim} x_{i+1} (i = 1, \cdots, k-1)$. In the similar manner, we define a $(*)$circuit, a $(*)$cluster and $(*)$connectedness.

For $\omega \in \Omega$, we set

$$\begin{aligned} S^+(\omega) &= \{x \in \mathbb{Z}^2; \omega(x) = +1\}, \\ S^0(\omega) &= \{x \in \mathbb{Z}^2; \omega(x) = 0\}, \\ S^-(\omega) &= \{x \in \mathbb{Z}^2; \omega(x) = -1\}, \\ S^{0+}(\omega) &= \{x \in \mathbb{Z}^2; \omega(x) \geq 0\}, \\ S^{0-}(\omega) &= \{x \in \mathbb{Z}^2; \omega(x) \leq 0\}. \end{aligned}$$

A path in $S^+(\omega)$ is called a $(+)$path in ω. In the analogous way, we define a $(+)$circuit and a $(+)$cluster. We say that $x, y \in \mathbb{Z}^2$ is $(+)$connected in ω if there is a $(+)$path from x to y in ω. The event that x and y are $(+)$connected is denoted by $\{x \overset{+}{\longleftrightarrow} y\}$. For $C \subset \mathbb{Z}^2$, we write $\{x \overset{+}{\longleftrightarrow} C\}$ for the event that x and some point in C is $(+)$connected. A $(*)$path(resp. $(*)$circuit,$(*)$cluster) in $S^+(\omega)$ is called a $(+*)$path(resp. a $(+*)$circuit,a $(+*)$cluster). We call a prefix such as '$+*$' the *type* of this path. Let E^+ be the event that there exists an infinite $(+)$ cluster. The event that x belongs to an infinite $(+)$cluster is denoted by $\{x \overset{+}{\longleftrightarrow} \infty\}$. Let $I^+ = I^+(\omega) = \{x \in \mathbb{Z}^2; x \overset{+}{\longleftrightarrow} \infty \text{ in } \omega\}$, which is equal to the union of all infinite $(+)$clusters in ω. All notations will be used for $S^0(\omega), S^-(\omega), S^{0+}(\omega)$ and $S^{0-}(\omega)$ as well. Note that $E^+ \subset E^{+*} \subset E^{0+*}$ and so on.

3.2 Transformations of Ω

We consider the following transformations of Ω.

(i) The *translations* $\theta_s, s \in \mathbb{Z}^2$: which are defined by

$$(\theta_s \omega)(x) = \omega(x - s) \quad (x \in \mathbb{Z}^2)$$

for $\omega \in \Omega$. Particularly, let $\theta_{\text{hor}} = \theta_{(1,0)}$ and $\theta_{\text{vert}} = \theta_{(0,1)}$. The collection $(\theta_s)_{s \in \mathbb{Z}^2}$ is a group. For $a, b \in \mathbb{N}$, let $\mathbb{Z}^2(a, b) = \{(ak, bl) \in \mathbb{Z}^2; k, l \in \mathbb{Z}\}$. We say that $\mu \in \mathcal{G}(\lambda, h)$ is $((a, b)$-$)$*periodic* if it is invariant under the subgroup

$(\theta_s)_{s\in\mathbb{Z}^2(a,b)}$. In particular, it is called *translation-invariant* if this holds for $(a,b)=(1,1)$. We say that μ is *horizontally periodic* if it is invariant under $\theta_{(a,0)}$ for some $a\in\mathbb{N}$. Similarly, we define vertical periodicity.

(ii) The *spin-flip transformation* : For $\omega\in\Omega$, $T\omega\in\Omega$ is defined by

$$(T\omega)(x)=-\omega(x)(x\in\mathbb{Z}^2).$$

(iii) The *reflections* : For $k\in\mathbb{Z}$, let

$$R_{k,\mathrm{hor}}:\mathbb{Z}^2\ni x=(x_1,x_2)\mapsto(x_1,2k-x_2)\in\mathbb{Z}^2,$$
$$R_{k,\mathrm{vert}}:\mathbb{Z}^2\ni x=(x_1,x_2)\mapsto(2k-x_1,x_2)\in\mathbb{Z}^2.$$

3.3 Characterization of Gibbs measures by percolation

By the strong Markov property, the following lemma is easily obtained.

Lemma 3.1. (cf. [5] Lemma 2.1) Let $\mu\in\mathcal{G}(\lambda,h)$. If $\mu(E^{0+})=0$, then $\mu=\mu^-_{\lambda,h}$.

We need a variant of this lemma.

Proposition 3.2. Let $\mu\in\mathcal{G}(\lambda,h)$. If $\mu(E^{0*})=0$, then μ is a convex combination of $\mu^+_{\lambda,h}$ and $\mu^-_{\lambda,h}$.

Proof. Fix $\Lambda\subset\subset\mathbb{Z}^2$. By assumption, Λ is surrounded by either $(+)$circuit or $(-)$circuit μ-a.s. In such a case, we will say that Λ is surrounded by $(+/-)$circuit. For any $\varepsilon>0$, we can choose a large finite set $\Delta\supset\Lambda$ such that

$$\mu\left(\Lambda\text{ is surrounded by }(+/-)\text{circuit in }\Delta\right)>1-\varepsilon.$$

For each circuit C surrounding Λ in Δ, we consider the events

$$\begin{aligned}&A^+_{\Lambda,\Delta,C}\\&=\{C\text{ is the maximal }(+/-)\text{circuit surrounding }\Lambda\text{ in }\Delta\text{ and its type is }+\},\\&A^-_{\Lambda,\Delta,C}\\&=\{C\text{ is the maximal }(+/-)\text{circuit surrounding }\Lambda\text{ in }\Delta\text{ and its type is }-\},\end{aligned}$$

and

$$A^+_{\Lambda,\Delta} = \bigcup_C A^+_{\Lambda,\Delta,C},\, A^-_{\Lambda,\Delta} = \bigcup_C A^-_{\Lambda,\Delta,C},\, A^{+/-}_{\Lambda,\Delta} = A^+_{\Lambda,\Delta} \cup A^-_{\Lambda,\Delta},$$

where the union runs over all the circuit surrounding Λ in Δ. Clearly,

$$\mu\left(A^+_{\Lambda,\Delta}\right) + \mu\left(A^-_{\Lambda,\Delta}\right) = \mu\left(A^{+/-}_{\Lambda,\Delta}\right) > 1 - \varepsilon.$$

Let f be a nonnegative increasing function such that supp $f \subset \Lambda$. We have

$$\begin{aligned}
\mu(f) &= \mu\left(f \cdot 1_{A^+_{\Lambda,\Delta}}\right) + \mu\left(f \cdot 1_{A^-_{\Lambda,\Delta}}\right) + \mu\left(f \cdot 1_{(A^{+/-}_{\Lambda,\Delta})^c}\right) \\
&= \sum_C \left\{\mu\left(f \cdot 1_{A^+_{\Lambda,\Delta,C}}\right) + \mu\left(f \cdot 1_{A^-_{\Lambda,\Delta,C}}\right)\right\} + \mu\left(f \cdot 1_{(A^{+/-}_{\Lambda,\Delta})^c}\right).
\end{aligned}$$

The Markov property of μ implies that

$$\begin{aligned}
\mu(f) = \sum_C &\left\{\mu\left(\mu^+_{\mathrm{int}(C),\lambda,h}(f) \cdot 1_{A^+_{\Lambda,\Delta,C}}\right) + \mu\left(\mu^-_{\mathrm{int}(C),\lambda,h}(f) \cdot 1_{A^-_{\Lambda,\Delta,C}}\right)\right\} \\
&+ \mu\left(f \cdot 1_{(A^{+/-}_{\Lambda,\Delta})^c}\right),
\end{aligned}$$

where $\mathrm{int}(C)$ is the bounded $(*)$connected component of $\mathbb{Z}^2 \setminus C$. For any circuit C surrounding Λ, we note that

$$\mu^+_{\lambda,h}(f) \leq \mu^+_{\mathrm{int}(C),\lambda,h}(f) \leq \mu^+_{\Lambda,\lambda,h}(f)\ ,\ \mu^-_{\Lambda,\lambda,h}(f) \leq \mu^-_{\mathrm{int}(C),\lambda,h}(f) \leq \mu^-_{\lambda,h}(f).$$

So we have

$$\begin{aligned}
\mu(f) &\leq \sum_C \left\{\mu^+_{\Lambda,\lambda,h}(f)\mu\left(A^+_{\Lambda,\Delta,C}\right) + \mu^-_{\lambda,h}(f)\mu\left(A^-_{\Lambda,\Delta,C}\right)\right\} + \varepsilon||f||_\infty \\
&= \mu^+_{\Lambda,\lambda,h}(f)\mu\left(A^+_{\Lambda,\Delta}\right) + \mu^-_{\lambda,h}(f)\mu\left(A^-_{\Lambda,\Delta}\right) + \varepsilon||f||_\infty.
\end{aligned}$$

Similarly,

$$\mu(f) \geq \mu^+_{\lambda,h}(f)\mu\left(A^+_{\Lambda,\Delta}\right) + \mu^-_{\Lambda,\lambda,h}(f)\mu\left(A^-_{\Lambda,\Delta}\right) - \varepsilon||f||_\infty.$$

Take a sequence $\Delta \nearrow \mathbb{Z}^2$. Note that $A^{+/-}_{\Lambda,\Delta}$ is increasing in Δ. Since finite subsets of $\mathbb{Z}^2$ are countably many and $\mu\left(A^+_{\Lambda,\Delta}\right) \in [0,1]$, by a diagonal-sequence argument we can choose a subsequence of Δ such that $\mu\left(A^+_{\Lambda,\Delta}\right)$

converges for all $\Lambda \subset\subset \mathbb{Z}^2$. We write α_Λ for this limit. By letting $\Delta \nearrow \mathbb{Z}^2$ along this subsequence and $\varepsilon \searrow 0$, we have $\mu(A^+_{\Lambda,\Delta}) \to \alpha_\Lambda$, $\mu(A^-_{\Lambda,\Delta}) \to 1-\alpha_\Lambda$, and

$$\alpha_\Lambda \mu^+_{\lambda,h}(f) + (1-\alpha_\Lambda)\mu^-_{\Lambda,\lambda,h}(f) \le \mu(f) \le \alpha_\Lambda \mu^+_{\Lambda,\lambda,h}(f) + (1-\alpha_\Lambda)\mu^-_{\lambda,h}(f).$$

Next we take an increasing sequence $\Lambda \nearrow \mathbb{Z}^2$. As $\alpha_\Lambda \in [0,1]$, we can choose a suitable subsequence of Λ such that α_Λ converges to some $\alpha \in [0,1]$. By letting $\Lambda \nearrow \mathbb{Z}^2$ along this subsequence, we have

$$\mu(f) = \alpha\mu^+_{\lambda,h}(f) + (1-\alpha)\mu^-_{\lambda,h}(f)$$

for any nonnegative increasing f. Because both $\mu^+_{\lambda,h}$ and $\mu^-_{\lambda,h}$ are extremal in $\mathcal{G}(\lambda,h)$, the extremal decomposition theorem implies that α is unique and independent of the choice of subsequences. This completes the proof. $\square$

3.4 Flip-reflection domination

We assume that $h = 0$. In this case, the interaction is invariant under the *frip-refrection transformation* $R \circ T$, where R is any reflection. This implies that $\{\omega; \omega : \text{feasible}\} = \{\omega; R \circ T(\omega) : \text{feasible}\}$. Thus we can obtain the following lemma.

Lemma 3.3 (Flip-reflection domination). (cf. [5] Lemma 2.3)

Let $\mu \in \mathcal{G}(\lambda,0)$ and R be any reflection. If μ-a.a.ω any $\Lambda \subset\subset \mathbb{Z}^2$ is surrounded by $(*)$circuit which is R-invariant and on which $\omega \ge R \circ T(\omega)$, then we have $\mu \ge \mu \circ R \circ T$.

3.5 Percolation in half-planes

Let $n \in \mathbb{Z}$ and $i \in \{1,2\}$. A *half-plane* is the set of the form $\pi = \{x = (x_1,x_2) \in \mathbb{Z}^2; x_i \ge (\le) n\}$. The line $l = \{x = (x_1,x_2) \in \mathbb{Z}^2; x_i = n\}$ is called the *boundary line* of this half-plane. Let

$$\pi_{\text{up},n} = \{x \in \mathbb{Z}^2; x_2 \ge n\}\ ,\ \pi_{\text{down},n} = \{x \in \mathbb{Z}^2; x_2 \le n\}.$$

We simply write $\pi_{\text{up}}, \pi_{\text{down}}$ if $n = 0$. In the analogous way, $\pi_{\text{left},n}, \pi_{\text{right},n}, \pi_{\text{left}}$ and π_{right} are defined.

A path $p = (x_1, \cdots, x_k)$ is called a *half-circuit* of the half-plane π with boundary line l if $p \subset \pi$ and $p \cap l = \{x_1, x_k\}$. For a half plane π, let E_π^+ be the event that there exists an infinite $(+)$cluster in π. The union of infinite clusters in π is denoted by $I_\pi^+ = I_\pi^+(\omega) = \{x \in \pi; x \overset{+}{\longleftrightarrow} \infty \text{ in } \omega|_\pi\}$. When $\pi = \pi_{\text{up}}$, we write E_{up}^+ or I_{up}^+ for short. These notations will be used accordingly for infinite clusters of other types.

Lemma 3.4 (Shift lemma). (cf. [5] Lemma 3.4) Let $\mu \in \mathcal{G}(\lambda, h)$. Assume that two half-planes $\pi, \tilde{\pi}$ satisfy $\pi \supset \tilde{\pi}$(i.e. one half-plane is a translate of the other half-plane). Then $E_\pi^+ = E_{\tilde{\pi}}^+$ μ-a.s. This also holds for infinite clusters of any other types.

This lemma is proved by using so-called 'random Borel-Cantelli' argument(see [5]).

3.6 Percolation under periodic Gibbs measures

Proposition 3.5. Let $\lambda > 0$ and $h \in \mathbb{R}$. If $\mu \in \mathcal{G}(\lambda, h)$ is $((a, b)$-)periodic, then there is at most one infinite cluster of each type μ-a.s.

Proof. By the ergodic decomposition theorem([4] Chap.14), we can assume that μ is $(\theta_s)_{s \in \mathbb{Z}^2(a,b)}$-ergodic. We want to apply the Burton-Keane uniqueness theorem, but its proof requires the finite energy property to connect different clusters with positive probability. In spite of lack of the finite energy property in our case, this is still possible in the similar manner as noted in [7] and [5] for the hard-core lattice gas model. The $(\theta_s)_{s \in \mathbb{Z}^2(a,b)}$-ergodicity is sufficient to show that in a finite box there exist encounter points whose number has the same order as the volume of the box. Thus we can show the uniqueness of the infinite cluster. $\square$

By virtue of this proposition, we can establish the noncoexistence of several infinite clusters by using Zhang's argument.

Proposition 3.6 (Zhang's argument). The following statements hold.

(i) (cf. [6] Theorem 5.18) If $\mu \in \mathcal{G}(\lambda, h)$ is a periodic and rotation-invariant probability measure with positive correlations, then we have $\mu(E^+ \cap E^{0-*}) = 0$.

(ii) (cf. [5] Lemma 3.1) If $\mu \in \mathcal{G}(\lambda, 0)$ has positive correlations and is flip-reflection invariant(i.e. $\mu = \mu \circ R \circ T$ for any reflection R), then we have $\mu(E^+ \cap E^-) = 0$.

4 Number of phases : asymmetric case

In this section, we shall prove Theorem 1.1.

4.1 Differentiability of the pressure and uniqueness of Gibbs measures

We review the relation between the differentiability of the pressure and the uniqueness of Gibbs measures.

We set

$$p(\Lambda, \lambda, h, \omega) = \frac{1}{|\Lambda|} \log Z^{\omega}_{\Lambda,\lambda,h}.$$

Differentiating twice by h, we can see that $p(\Lambda, \lambda, h, \omega)$ is a convex function of h.

Lemma 4.1. Let S_n be a box in $\mathbb{Z}^2$ with side length n. The limit

$$P(\lambda, h) = \lim_{n \to \infty} p(S_n, \lambda, h, \omega)$$

exists and is independent of ω. $P(\lambda, h)$ is called the *pressure*. $P(\lambda, h)$ is also a convex function of h, therefore it is differentiable except at most countable h's.

Proof. By standard subadditive argument, we can show that $p(S_N, \lambda, h, 0)$ converges. We write $P(\lambda, h)$ for the limit. For an arbitrary boundary condition ω, we can see that $Z^0_{S_n,\lambda,h} \geq Z^{\omega}_{S_n,\lambda,h} \geq Z^0_{S_{n-2},\lambda,h}$ for all $n \geq 3$, which implies that $p(S_n, \lambda, h, \omega) \to P(\lambda, h)$ as $n \to \infty$. $\square$

The following result is well-known.

Theorem 4.2. ([3]) $|\mathcal{G}(\lambda, h)| = 1$ if and only if $P(\lambda, x)$ is differentiable at $x = h$.

Together with the preceding lemma, for each $\lambda > 0$, except at most countable h's, there is a unique Gibbs measure for (λ, h).

4.2 Proof of Theorem 1.1

We assume that $h > 0$. $h < 0$ case is treated analogously. First we remark that $\mu^+_{\lambda,0} \leq \mu^-_{\lambda,h}$ if $h > 0$.

Proposition 4.3. Let $\lambda > 0$. If $\mu^+_{\lambda,0} \neq \mu^-_{\lambda,0}$, then we have $|\mathcal{G}(\lambda, h)| = 1$ for all $h > 0$.

Proof. We can show that $\mu^+_{\lambda,0}(E^{0-}) = 0$ if $\mu^+_{\lambda,0} \neq \mu^-_{\lambda,0}$ (see Corollary 5.3 below). So we have $\mu^-_{\lambda,h}(E^{0-}) \leq \mu^+_{\lambda,0}(E^{0-}) = 0$. By Lemma 3.1, we can see that $\mu^-_{\lambda,h} = \mu^+_{\lambda,h}$. Proposition 2.5 gives the result. □

Next, we fix $\lambda > 0$ such that $\mu^+_{\lambda,0} = \mu^-_{\lambda,0}$. For the unique Gibbs measure $\mu_0 \in \mathcal{G}(\lambda, 0)$, we can show that $\mu_0(E^+ \cup E^-) = 0$ (see Proposition 5.2 below). Therefore, for arbitrary $\mu \in \mathcal{G}(\lambda, h)$ we have $\mu(E^-) \leq \mu^-_{\lambda,h}(E^-) \leq \mu_0(E^-) = 0$. We define

$$
\begin{aligned}
h_c^+ &= h_c^+(\lambda) = \inf\{h \geq 0; \mu^+_{\lambda,h}(E^+) = 1\}, \\
h_c^- &= h_c^-(\lambda) = \inf\{h \geq 0; \mu^-_{\lambda,h}(E^+) = 1\}.
\end{aligned}
$$

Because $\mu^+_{\lambda,h}(E^+) \geq \mu^-_{\lambda,h}(E^+)$, we have $h_c^+ \leq h_c^-$. When $h_c^+ < h_c^-$, $\mu^+_{\lambda,h}(E^+) = 1$ and $\mu^-_{\lambda,h}(E^+) = 0$ for all $h \in (h_c^+, h_c^-)$. This implies $\mu^+_{\lambda,h} \neq \mu^-_{\lambda,h}$ for uncountable h's, which is impossible. We can conclude $h_c^+ = h_c^-$, say h_c.

Proposition 4.4. If $\mu^+_{\lambda,0} = \mu^-_{\lambda,0}$, then $|\mathcal{G}(\lambda, h)| = 1$ for $h > h_c(\lambda)$.

Proof. When $h > h_c$, we have $\mu^-_{\lambda,h}(E^+) = 1$. It follows from Proposition 3.6(i) that $\mu^-_{\lambda,h}(E^{0-*}) = 0$. Lemma 3.1 again shows that $\mu^-_{\lambda,h} = \mu^+_{\lambda,h}$. □

5 Site random-cluster representation

Hereafter we assume that $h = 0$ and omit h. The *site random-cluster representation* of Widom-Rowlinson model is used in several papers; e.g. [2],[6],[8]. Here we introduce the site random-cluster representation of Gibbs distribution with an arbitrary boundary condition.

Fix $\Lambda \subset\subset \mathbb{Z}^2$. For $\xi \in \{0,1\}^\Lambda$, let

$$\tilde{S}^1(\xi) = \{x \in \Lambda; \xi(x) = 1\}\ ,\ \tilde{S}^0(\xi) = \{x \in \Lambda; \xi(x) = 0\}.$$

A path in $\tilde{S}^1(\xi)$ is called a (1)path in ξ. Analogously, we define a (1)circuit and a (1)cluster. We say that $x, y \in \mathbb{Z}^2$ are (1)connected if there is a (1)path from x to y in ξ. The event that x and y are (1)connected is denoted by $\{x \overset{1}{\longleftrightarrow} y\}$. For $C \subset \mathbb{Z}^2$, $\{x \overset{1}{\longleftrightarrow} C\}$ denotes the event that x and some point in C are (1)connected. Similarly, we define $(1*)$connectedness and so on.

Let $\omega \in \Omega$ be a feasible boundary condition. We set

$$W_\Lambda^+(\omega) = \{x \in \partial\Lambda; \omega(x) = +1\}\ ,\ W_\Lambda^-(\omega) = \{x \in \partial\Lambda; \omega(x) = -1\}.$$

For $\xi \in \{0,1\}^\Lambda$, let

$$1_{D(\omega,\xi)} = \begin{cases} 1 & \text{if there is no (1)path connecting } W_\Lambda^+(\omega) \text{ and } W_\Lambda^-(\omega) \text{ in } \xi, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\lambda > 0$. The *site random-cluster distribution* $R^\omega_{\Lambda,\lambda}$ is a probability measure on $\{0,1\}^\Lambda$ which is defined by

$$R^\omega_{\Lambda,\lambda}(\xi) = \frac{1}{\tilde{Z}^\omega_{\Lambda,\lambda}} 1_{D(\omega,\xi)} \prod_{x\in\Lambda} \lambda^{\xi(x)} \cdot 2^{k(\xi,\omega,\Lambda)} \quad (\xi \in \{0,1\}^\Lambda),$$

where $k(\xi,\omega,\Lambda)$ is the number of (1)clusters in ξ which touch neither $W_\Lambda^+(\omega)$ nor $W_\Lambda^-(\omega)$, and $\tilde{Z}^\omega_{\Lambda,\lambda}$ is a normalizing constant.

For any $x \in \Lambda$, we shall calculate the conditional probability $R^\omega_{\Lambda,\lambda}(\xi(x) = 1|\xi = \eta \text{ off } x)$, where $\eta \in \{0,1\}^{\Lambda\setminus\{x\}}$ such that $R^\omega_{\Lambda,\lambda}(\xi = \eta \text{ off } x) > 0$. We define $\eta_{x,s} \in \{0,1\}^\Lambda (s = 0,1)$ by

$$\eta_{x,s}(y) = \begin{cases} \eta(y) & \text{if } y \neq x, \\ s & \text{if } y = x. \end{cases}$$

Then we have

$$R^{\omega}_{\Lambda,\lambda}(\xi(x)=1|\xi=\eta \text{ off } x)=\frac{R^{\omega}_{\Lambda,\lambda}(\xi=\eta_{x,1})}{R^{\omega}_{\Lambda,\lambda}(\xi=\eta_{x,1})+R^{\omega}_{\Lambda,\lambda}(\xi=\eta_{x,0})}.$$

By definition,

$$R^{\omega}_{\Lambda,\lambda}(\xi=\eta_{x,0})=\frac{1}{\tilde{Z}^{\omega}_{\Lambda,\lambda}}\prod_{y\in\Lambda\setminus\{x\}}\lambda^{\eta(y)}\cdot 2^{k(\eta_{x,0},\omega,\Lambda)},$$

$$R^{\omega}_{\Lambda,\lambda}(\xi=\eta_{x,1})=\frac{\lambda}{\tilde{Z}^{\omega}_{\Lambda,\lambda}}1_{D(\omega,\eta_{x,1})}\prod_{y\in\Lambda\setminus\{x\}}\lambda^{\eta(y)}\cdot 2^{k(\eta_{x,1},\omega,\Lambda)}.$$

Thus we have

$$\frac{R^{\omega}_{\Lambda,\lambda}(\xi=\eta_{x,1})}{R^{\omega}_{\Lambda,\lambda}(\xi=\eta_{x,0})}=\lambda\cdot 1_{D(\omega,\eta_{x,1})}\cdot 2^{k(\eta_{x,1},\omega,\Lambda)-k(\eta_{x,0},\omega,\Lambda)}.$$

The values of $1_{D(\omega,\eta_{x,1})}$ and $k(\eta_{x,1},\omega,\Lambda)-k(\eta_{x,0},\omega,\Lambda)$ are closely related to the number of (1)clusters which contain the sites adjacent to x. The number of such (1)clusters is denoted by N, and the number of ones which touch neither $W^+_{\Lambda}(\omega)$ nor $W^-_{\Lambda}(\omega)$ is denoted by n. It is clear that $0\le n\le N\le 4$. We define $\kappa(\eta,\omega,x,\Lambda)$ as follows: If one (1)cluster containing sites adjacent to x touch $W^+_{\Lambda}(\omega)$ and another touch $W^-_{\Lambda}(\omega)$, then we set $\kappa(\eta,\omega,x,\Lambda)=-\infty$. Otherwise, we set

$$\kappa(\eta,\omega,x,\Lambda)=\begin{cases}1-n & \text{if } N=n,\\ -n & \text{if } N>n.\end{cases}$$

Noting that $R^{\omega}_{\Lambda,\lambda}(\xi=\eta_{x,1})/R^{\omega}_{\Lambda,\lambda}(\xi=\eta_{x,0})=\lambda\cdot 2^{\kappa(\eta,\omega,x,\Lambda)}$, we have

$$R^{\omega}_{\Lambda,\lambda}(\xi(x)=1|\xi=\eta \text{ off } x)=\frac{\lambda\cdot 2^{\kappa(\eta,\omega,x,\Lambda)}}{\lambda\cdot 2^{\kappa(\eta,\omega,x,\Lambda)}+1}.$$

REMARK 4. By the definition of κ, it turns out that $R^{\omega}_{\Lambda,\lambda}$ does not satisfy the conditions of Theorem 2.2.

Lemma 5.1 (Site random-cluster representation). The finite volume Gibbs distribution $\mu^{\omega}_{\Lambda,\lambda}$ is related to the site random-cluster distribution $R^{\omega}_{\Lambda,\lambda}$ as follows.

(i) First we pick $Y \in \{0,1\}^\Lambda$ according to $R^\omega_{\Lambda,\lambda}$. For $x \in \Lambda$ with $Y(x) = 0$, we set $X(x) = 0$. For each (1)cluster C of Y, we assign $+1$ or -1 to this cluster by tossing a fair coin. There are two exceptional cases when C touches $W^+_\Lambda(\omega) \cup W^-_\Lambda(\omega)$. If C is connected to $W^+_\Lambda(\omega)$, then we set $X \equiv +1$ on C. If C is connected to $W^-_\Lambda(\omega)$, then we set $X \equiv -1$ on C. Then, the distribution of $X \in \Omega_\Lambda$ is $\mu^\omega_{\Lambda,\lambda}$.

(ii) We choose $X \in \Omega_\Lambda$ according to $\mu^\omega_{\Lambda,\lambda}$ and set $Y(x) = X(x)^2$ for each $x \in \Lambda$. Then, the distribution of $Y \in \{0,1\}^\Lambda$ is $R^\omega_{\Lambda,\lambda}$.

The proof is straightforward and we omit it. Note that the distribution of $S^0(\sigma)$ with respect to $\mu^\omega_{\lambda,\Lambda}$ is equal to the distribution of $\tilde{S}^0(\sigma^2)$ with respect to $R^\omega_{\lambda,\Lambda}$. For example, we have $\mu^\omega_{\Lambda,\lambda}(x \overset{0*}{\longleftrightarrow} y) = R^\omega_{\Lambda,\lambda}(x \overset{0*}{\longleftrightarrow} y)$ for any $x, y \in \Lambda$.

Using site random-cluster representation, we can prove the following characterization of the phase transition of Widom-Rowlinson model in terms of percolation.

Proposition 5.2. (cf. [6] Theorem 4.17,8.13,[8] §3) Suppose $\lambda > 0$. Following (i)-(v) are equivalent.

(i) $\mathcal{G}(\lambda)$ is a singleton.

(ii) $\mu^+_\lambda = \mu^-_\lambda$

(iii) $\mu^+_\lambda(\sigma(x) = +1) = \mu^-_\lambda(\sigma(x) = +1)$ for all $x \in \mathbb{Z}^2$.

(iv) $\lim\limits_{\Lambda \nearrow \mathbb{Z}^2} R^+_{\Lambda,\lambda}(x \overset{1}{\longleftrightarrow} \partial\Lambda) = 0$ for all $x \in \mathbb{Z}^2$.

(v) $\mu^+_\lambda(x \overset{+}{\longleftrightarrow} \infty) = 0$ for all $x \in \mathbb{Z}^2$.

REMARK 5. This equivalence holds in Widom-Rowlinson model not only on $\mathbb{Z}^2$ but also on arbitrary infinite connected graph.

Corollary 5.3. If $\lambda > 0$ and $\mu^+_\lambda \neq \mu^-_\lambda$, then $\mu^+_\lambda(E^{0-*}) = \mu^-_\lambda(E^{0+*}) = 0$.

Proof. If $\mu^+_\lambda \neq \mu^-_\lambda$, then we have $\mu^+_\lambda(E^+) = \mu^-_\lambda(E^-) = 1$ by Proposition 5.2. We get the conclusion from Proposition 3.6(i). $\square$

Let P_p denote the Bernoulli probability measure on $\{0,1\}^{\mathbb{Z}^2}$ with density p. For $x \in \Lambda$ and $\eta \in \{0,1\}^{\Lambda\setminus\{x\}}$ such that $R^{\omega}_{\Lambda,\lambda}(\xi = \eta \text{ off } x) > 0$, we can see that $-\infty \leq \kappa(\eta,\omega,x,\Lambda) \leq 1$. Holley's inequality implies that $R^{\omega}_{\Lambda,\lambda} \leq P_{\frac{2\lambda}{2\lambda+1}}$. Moreover, if $\omega \in \Omega$ satisfies $\omega \geq 0$ or $\omega \leq 0$ on $\partial\Lambda$, then $-3 \leq \kappa(\eta,\omega,x,\Lambda) \leq 1$. Thus we obtain the following lemma.

Lemma 5.4. If a feasible boundary condition $\omega \in \Omega$ satisfies $\omega \geq 0$ or $\omega \leq 0$ on $\partial\Lambda$, then we have

$$P_{\frac{\lambda}{\lambda+8}} \leq R^{\omega}_{\Lambda,\lambda} \leq P_{\frac{2\lambda}{2\lambda+1}}.$$

6 Number of phases : symmetric,large activity case

In this section, we shall prove Theorem 1.2.

6.1 Preliminary results

Proposition 6.1. ([10]) When $\lambda > 8p_c/(1-p_c)$, phase transition occurs, namely $\mu^+_\lambda \neq \mu^-_\lambda$.

Proof. Here we give the proof based on Proposition 5.2. When $\lambda/(\lambda+8) > p_c$(i.e. $\lambda > 8p_c/(1-p_c)$), $P_{\frac{\lambda}{\lambda+8}}(0 \stackrel{1}{\longleftrightarrow} \infty) = \theta > 0$. For $\Lambda \subset\subset \mathbb{Z}^2$, it follows from Lemmata 5.1 and 5.4 that

$$\begin{aligned}
\mu^+_{\Lambda,\lambda}(0 \stackrel{+}{\longleftrightarrow} \partial\Lambda) \geq \mu^0_{\Lambda,\lambda}(0 \stackrel{+}{\longleftrightarrow} \partial\Lambda) \\
&= \frac{1}{2} R^0_{\Lambda,\lambda}(0 \stackrel{1}{\longleftrightarrow} \partial\Lambda) \\
&\geq \frac{1}{2} P_{\frac{\lambda}{\lambda+8}}(0 \stackrel{1}{\longleftrightarrow} \partial\Lambda) \geq \frac{1}{2} P_{\frac{\lambda}{\lambda+8}}(0 \stackrel{1}{\longleftrightarrow} \infty) = \frac{\theta}{2} > 0.
\end{aligned}$$

By letting $\Lambda \nearrow \mathbb{Z}^2$, we have $\mu^+_\lambda(0 \stackrel{+}{\longleftrightarrow} \infty) \geq \theta/2 > 0$. It follows from this and Proposition 5.2 that $\mu^+_\lambda \neq \mu^-_\lambda$. □

When activity is large, we can determine the limiting Gibbs measure with free boundary condition.

Proposition 6.2. When $\lambda > 8p_c/(1-p_c)$,

$$\mu_\lambda^0 = \lim_{\Lambda \nearrow \mathbb{Z}^2} \mu_{\Lambda,\lambda}^0 = \frac{1}{2}\left(\mu_\lambda^+ + \mu_\lambda^-\right).$$

Proof. Take a sequence $\Lambda \nearrow \mathbb{Z}^2$. By taking the suitable subsequence $\{\Lambda_n\}$, $\mu_{\Lambda_n}^0$ converges to a probability measure on Ω, say μ_λ^0, as $n \to \infty$.

We shall prove $\mu_\lambda^0(E^{0*}) = 0$ when $\lambda > 8p_c/(1-p_c)$. Let p_c^* be the critical probability of infinite $(*)$cluster of Bernoulli site percolation on $\mathbb{Z}^2$. It is well-known that $p_c + p_c^* = 1$([13]). Now, as $1 - \lambda/(\lambda+8) < p_c^*$, there is no infinite $(0*)$cluster $P_{\frac{\lambda}{\lambda+8}}$-a.s. Fix $x \in \mathbb{Z}^2$. For any $\varepsilon > 0$, we can choose large N so that $P_{\frac{\lambda}{\lambda+8}}(x \overset{0*}{\longleftrightarrow} \partial\Lambda_n) < \varepsilon$ for all $n \geq N$. By Lemmata 5.1 and 5.4, for $m > n \geq N$ we have

$$\mu_{\Lambda_m,\lambda}^0(x \overset{0*}{\longleftrightarrow} \partial\Lambda_n) = R_{\Lambda_m,\lambda}^0(x \overset{0*}{\longleftrightarrow} \partial\Lambda_n) \leq P_{\frac{\lambda}{\lambda+8}}(x \overset{0*}{\longleftrightarrow} \partial\Lambda_n) < \varepsilon.$$

By letting $m \to \infty$, $n \to \infty$ and $\varepsilon \searrow 0$, we have $\mu_\lambda^0(x \overset{0*}{\longleftrightarrow} \infty) = 0$ for all $x \in \mathbb{Z}^2$. Thus $\mu_\lambda^0(E^{0*}) = 0$.

By Proposition 3.2, $\mu_\lambda^0 = \alpha\mu_\lambda^+ + (1-\alpha)\mu_\lambda^-$ for some coefficient $\alpha \in [0,1]$. We note that $\mu_{\Lambda_n,\lambda}^0(A) = \mu_{\Lambda_n,\lambda}^0 \circ T(A)$ for each n and any $A \in \mathcal{F}_{\Lambda_n}$. By letting $n \to \infty$, we have $\mu_\lambda^0(A) = \mu_\lambda^0 \circ T(A)$. This implies that $\alpha = 1/2$. We can conclude that $\mu_{\Lambda,\lambda}^0$ converges to $(\mu_\lambda^+ + \mu_\lambda^-)/2$, independent of the choice of the subsequence of $\Lambda \nearrow \mathbb{Z}^2$. $\square$

Proposition 6.3. Suppose that $\lambda > 8p_c/(1-p_c)$. If $\mu \in \mathcal{G}(\lambda, 0)$ satisfies that $\mu(E^{0*}) > 0$, then $\mu(E^+ \cap E^-) > 0$.

Proof. Without loss of generality, we can assume that $\mu \in \mathcal{G}_{\mathrm{ex}}(\lambda, 0)$ and $\mu(E^{0*}) = 1$. We shall show that $\mu(E^+ \cap E^-) = 1$.

Suppose that $\mu(E^+) = 0$, which implies that any finite set Λ of $\mathbb{Z}^2$ is surrounded by $(0-*)$circuit μ-a.s. On the other hand, since $\lambda > 8p_c/(1-p_c)$, for $x \in \mathbb{Z}^2$ and $\varepsilon > 0$, we can choose large $\Lambda \subset\subset \mathbb{Z}^2$ containing x such that $P_{\frac{\lambda}{\lambda+8}}(x \overset{0*}{\longleftrightarrow} \partial\Lambda) < \varepsilon$. As Λ is surrounded by a $(0-*)$circuit μ-a.s., we can choose large $\Delta \subset\subset \mathbb{Z}^2$ such that with μ-probability$> 1-\varepsilon$ there is such a $(0-*)$circuit in Δ. Let Γ be the region surrounded by the maximal

$(0-*)$circuit in Δ if it exists. Otherwise we set $\Gamma = \emptyset$. Because Γ is determined from outside, we can show by using the strong Markov property of μ that

$$\begin{aligned}&\mu(x \overset{0*}{\longleftrightarrow} \partial\Lambda)\\ &= \mu(\mu^{\omega}_{\lambda,\Gamma(\omega)}(x \overset{0*}{\longleftrightarrow} \partial\Lambda)1_{\{\Gamma(\omega)\neq\emptyset\}}) + \mu(\{x \overset{0*}{\longleftrightarrow} \partial\Lambda\} \cap \{\Gamma(\omega) = \emptyset\}).\end{aligned}$$

By Lemmata 5.1 and 5.4, we have

$$\mu^{\omega}_{\lambda,\Gamma(\omega)}(x \overset{0*}{\longleftrightarrow} \partial\Lambda) = R^{\omega}_{\lambda,\Gamma(\omega)}(x \overset{0*}{\longleftrightarrow} \partial\Lambda) \leq P_{\frac{\lambda}{\lambda+8}}(x \overset{0*}{\longleftrightarrow} \partial\Lambda) < \varepsilon.$$

Thus we have $\mu(x \overset{0*}{\longleftrightarrow} \partial\Lambda) < \varepsilon + \varepsilon = 2\varepsilon$. By letting $\Delta \nearrow \mathbb{Z}^2, \varepsilon \searrow 0$ and $\Lambda \nearrow \mathbb{Z}^2$, we can see that $\mu(x \overset{0*}{\longleftrightarrow} \infty) = 0$. Since x is arbitrary, we can conclude $\mu(E^{0*}) = 0$, which is a contradiction. Thus we have $\mu(E^+) = 1$.

Analogously, we can show that $\mu(E^-) = 1$. □

6.2 Periodic phases

When λ is large, we can get the complete description of periodic Gibbs measures.

Theorem 6.4. If $\lambda > 8p_c/(1-p_c)$, then any periodic $\mu \in \mathcal{G}(\lambda, 0)$ is a mixture of μ^+_λ and μ^-_λ.

Before proving this, we prepare a lemma. We say $(\pi, \tilde{\pi})$ is a pair of *conjugate half-planes* if half-planes $\pi, \tilde{\pi}$ share only a common boundary line. An associated pair of infinite clusters $(I^{0+*}_\pi, I^{0+*}_{\tilde{\pi}})$ or $(I^{0-*}_\pi, I^{0-*}_{\tilde{\pi}})$ is called a *butterfly*. In particular, a butterfly in $(\pi_{\text{left}}, \pi_{\text{right}})$ is called a *horizontal butterfly*. A *vertical butterfly* is the one in $(\pi_{\text{up}}, \pi_{\text{down}})$.

Lemma 6.5 (Butterfly lemma). (cf. [5] Lemma 3.1) Suppose that $\lambda > 8p_c/(1-p_c)$ and $\mu \in \mathcal{G}(\lambda, 0)$. If $\mu(E^{0*}) > 0$, then there exists at least one butterfly with positive probability.

Proof. By the extremal decomposition theorem, there exists $Q \in \mathcal{G}_{\text{ex}}(\lambda, 0)$ such that $Q(E^{0*}) = 1$. By Proposition 6.3, $Q(E^+ \cap E^-) = 1$. If Q-a.s. there is no butterfly, then it turns out that Q is flip-reflection invariant. Because this is impossible by Proposition 3.6(ii), we can see that there exists at least one butterfly Q-a.s. This gives the result. $\square$

We can prove Theorem 6.4 by using Proposition 3.2 and the following proposition.

Proposition 6.6. If $\lambda > 8p_c/(1-p_c)$, then $\mu(E^{0*}) = 0$ for any periodic $\mu \in \mathcal{G}(\lambda, 0)$.

Proof. By the ergodic decomposition theorem, it is sufficient to show that $\mu(E^{0*}) = 0$ for ergodic μ. So we assume that μ is ergodic.

Suppose that $\mu(E^{0*}) = 1$. By Proposition 6.3, we have $\mu(E^+ \cap E^-) > 0$. By butterfly lemma, we can assume that there is a vertical $(0+*)$ butterfly with positive probability. We can find a large square $\Lambda \subset\subset \mathbb{Z}^2$ such that with positive probability Λ intersects $I^{0+*}_{\text{up}}, I^{0+*}_{\text{down}}$ and I^-. Without loss of generality, we can assume that I^- leaves on the right between I^{0+*}_{up} and I^{0+*}_{down} with positive probability. For $k \in \mathbb{Z}$, let $A_k = \{(k,0) \in I^{0+*}_{\text{up}} \cap I^{0+*}_{\text{down}}, (k+1,0) \in I^-\}$ and A_∞ be the event that A_k occurs for infinitely many $k \in \mathbb{Z}$. By changing the configuration in Λ suitably, we have $\mu(A_0) > 0$. Poincaré's recurrence theorem([4] Lemma(18.15)) shows that $\mu(A_\infty) = 1$. But on A_∞ there exist infinitely many infinite $(-)$clusters. This contradicts Proposition 3.5. Consequently $\mu(E^{0*}) = 0$. $\square$

6.3 1-periodic phases : proof of Theorem 1.2

Let $\mu \in \mathcal{G}(\lambda, 0)$. We say that an infinite cluster in a half-plane has the *line toucing property* if the cluster touches the boundary line of the half-plane infinitely many times μ-a.s.

We define $\pm \in \Omega$ by

$$\pm(x) = \begin{cases} +1 & \text{if } x_2 > 0, \\ 0 & \text{if } x_2 = 0, \\ -1 & \text{if } x_2 < 0. \end{cases}$$

It follows from Lemma 2.3(iv) that $\mu^{\pm}_{\text{up}} = \lim_{\Lambda\text{up}\nearrow\pi\text{up}} \mu^{\pm}_{\Lambda\text{up}}$ exists and is θ_{hor}-invariant.

Lemma 6.7. (cf. [5] Lemma 4.2) $\mu^{\pm}_{\text{up}}(E^{0+*}_{\text{up}}) = 0$ when $\lambda > 8p_c/(1-p_c)$.

This lemma is proved by using Theorem 6.4 and flip-refrection domination. Now we are ready to derive the line touching property of infinite clusters of several types. But note that the same argument as in the Ising model do not give the line touching property of the infinite clusters of types $+$,$+*$,0,$0*$,$-$ and $-*$.

Lemma 6.8 (Line touching lemma). (cf. [5] Lemma 4.1) Let $\lambda > 8p_c/(1-p_c)$ and $\mu \in \mathcal{G}(\lambda, 0)$. The infinite $(0+)$cluster in any half-plane π have the line toucing property μ-a.s. if it exists. The same holds for infinite clusters of type $0+*$ or $0-$ or $0-*$.

Corollary 6.9. Suppose $\lambda > 8p_c/(1-p_c)$ and $\mu \in \mathcal{G}(\lambda, 0)$. In an arbitrary half plane π, there exists at most one infinite $(+)$cluster μ-a.s. The same holds for infinite clusters of types $+*$ or $-$ or $-*$.

Lemma 6.10 (Orthogonal butterflies). (cf. [5] Lemma 4.3) Let $\lambda > 8p_c/(1-p_c)$ and $\mu \in \mathcal{G}(\lambda, 0)$. If $\mu(E^{0*}) > 0$, then there exist both horizontal butterflies and vertical butterflies μ-a.s.

Proof. We can see that $\mu(E^+ \cap E^-) > 0$ by Proposition 6.3. By the extremal decomposition theorem, $Q(E^+ \cap E^-) = 1$ for some $Q \in \mathcal{G}_{\text{ex}}(\lambda, 0)$. By butterfly lemma, there exist at least one butterfly Q-a.s.

Assume that there is a vertical $(0+*)$butterfly but no horizontal butterfly, for example. In this case, $Q = Q \circ R_{k,\text{vert}} \circ T$ for any $k \in \mathbb{Z}$. Therefore Q is horizontally periodic. Fix $n \in \mathbb{N}$. By shift lemma, we have $Q(E^{0+*}_{\text{up},n} \cap E^{0+*}_{\text{down},-n}) = 1$. For $k \in \mathbb{Z}$, we set

$$A^n_k = \left\{ \omega \in \Omega; \begin{array}{c} (k,n) \in I^{0+*}_{\text{up},n}, (k,-n) \in I^{0+*}_{\text{down},-n}, \\ \omega(k,l) = 0 \text{ for } -(n-1) \le l \le n-1 \end{array} \right\}$$

and $A_\infty^n = \{A_k^n$ occurs for infinitely many $k \in \mathbb{Z}\}$. We can easily see that $Q(A_0^n) > 0$. Poincaré's recurrence theorem and tail-triviality of Q imply that $Q(A_\infty^n) = 1$ for all n. Thus we have $Q\left(\bigcap_{n=1}^\infty A_\infty^n\right) = 1$. If for some n there is an infinite $(-)$cluster in $\pi_{\mathrm{up},n}$, Poincaré's recurrence theorem again shows that infinitely many infinite $(-)$clusters appear, which contradicts Corollary 6.9. Hence for any n there is a unique infinite $(0+*)$cluster in $\pi_{\mathrm{up},n}$. Similarly, the infinite $(0+*)$cluster in $\pi_{\mathrm{down},-n}$ is also unique. We can find that any finite region in $\mathbb{Z}^2$ is surrounded by $(0+*)$circuit in $\omega \in \bigcap_{n=1}^\infty A_\infty^n$, which contradicts $Q(E^+ \cap E^-) = 1$.

Consequently, both vertical butterflies and horizontal butterflies exist Q-a.s., which implies that this occurs with positive μ-probability. $\square$

Proof of Theorem 1.2(ii). By the ergodic decomposition theorem, we can assume that μ is horizontally ergodic and satisfies $\mu(E^{0*}) = 1$. Because at least one vertical butterfly must exist, as in the proof of Lemma 6.10, we can show that $\mu(E^+ \cap E^-) = 0$. This is a contradiction, which implies that $\mu(E^{0*}) = 0$. Together with Proposition 3.2, we can find that μ is a mixture of μ_λ^+ and μ_λ^-. $\square$

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Department of Mathematics
Faculty of Science
Kobe University
Rokko, Kobe 657-8501, Japan
higuchi@math.kobe-u.ac.jp

Department of Information and Media Science
Graduate School of Science and Technology
Kobe University
Rokko, Kobe 657-8501, Japan

takei@math.kobe-u.ac.jp