AVERAGING THEOREM FOR QUASILINEAR HAMILTONIAN PDEs IN ARBITRARY SPACE DIMENSIONS

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Abstract. We study the dynamics of quasilinear Hamiltonian wave equations with Dirichlet boundary conditions in an n–dimensional parallepided. We prove an averaging theorem according to which the solution corresponding to an arbitrary small amplitude smooth initial datum remains arbitrarily close to a finite dimensional torus up to very long times. We expect the result to be valid for a very general class of quasilinear Hamiltonian equations.

1. Introduction

In this paper we study the dynamics of Hamiltonian partial differential equation of quasilinear type (in the sense of Kato [1,2]) in arbitrary space dimension. To be definite we will concentrate on a nonlinear wave equation in an n–dimensional parallelepiped $\mathcal{R}$ with Dirichlet boundary conditions (see eq.(2.1) below): under a nonresonance condition that will be proved to have probability 1 we will construct infinitely many approximate integrals of motion, and deduce that solutions corresponding to small amplitude smooth initial data remain $\mathcal{O}(\epsilon^M)$ close to a finite dimensional torus up to times $\mathcal{O}(\epsilon^{-2})$, ϵ being the norm of the initial datum and M an arbitrary integer.

The proof is obtained by generalizing to PDEs a classical algorithm of construction of integrals of motion [3,4,5,6], which is commonly used in celestial mechanics (we will recall it in sect. 3). To generalize such an algorithm to PDEs we need a nonresonance condition for the frequencies of the linearized system. Here we will use a new diophantine type condition (see eq. (2.13) below) that will be proved to be fulfilled for almost all values of the parameters of the system. Such condition allows to treat small denominators as "derivatives" (see lemma 4.5 below), in a way similar to that used by Shatah in [7] and by Craig in [8]. Once this is done, an almost straightforward application of the classical algorithm allows to construct the candidate approximate integrals of motion, namely functions whose Lie derivative is small in a neighbourhood of the origin of a sufficiently smooth Sobolev space. Then we have to prove that, up to the times we are interested in, such functions remain approximately constant. This is obtained by showing that solution with sufficiently small initial datum at most double their size up to such times, and therefore they do not leave the considered neighbourhood of the origin. In turn this is proved by using Kato's theory with the addition of explicit estimates on the existence times and on the size of the solution. We expect the techniques just described to be applicable to a very general class of equations.

We recall that while many results have been obtained on the dynamics of semilinear Hamiltonian PDEs in one space [9, 10, 11, 12, 13, 14, 15, 16, 17], very little is known for equations in higher space dimensions and of general quasilinear type. In particular, in higher dimensional equations only persistence of periodic orbits [18] (see also [19]) has been proved in quite general models, while persistence of finite dimensional invariant tori is known only in some special cases [20, 21]. For what concerns other solutions, averaging methods have been used to study the dynamics corresponding to initial data in which the energy is initially essentially concentrated on finitely many linear modes [22]. However the methods of [22] give no informations for general initial data and for quasilinear equations. We point out that in the latter case (where the perturbation involves derivatives) one cannot use Gronwall lemma to estimate the difference between the true and the approximate solution, and this is the main difficulty related to this kind of equations. We point out that also KAM theory allows to deal with some quasilinear equations [23,24], but, due to well known results on the formation of singularities in quasilinear equations [25], one cannot expect KAM theory to be applicable to general quasilinear models.

Finally we recall the papers [26,27,28] where averaging techniques have been used together with a priori estimates in order to construct approximate solutions of some semilinear PDEs by a method closely related to the present one.

Plan of the paper. In sect. 2 we state the main results of the paper. In sect. 3 we recall the formal algorithm of construction of the integrals of motion in Hamiltonian systems. In sect. 4 we proof the main result of the paper. In sect. 5 we show that the diophantine condition we use is fulfilled with probability one. Finally in the appendix we use Kato's theory to estimate the size of the solution of the nonlinear wave equation.

2. Statement of the main result

Having fixed a vector $a = (a_1, ..., a_n) \in \mathbb{R}^n$ fulfilling $a_i > 0$, consider the n dimensional parallelepiped $\mathcal{R}$ with sides of length $\pi/\sqrt{a_i}$, namely

$$\mathcal{R} := \left\{ \mathbf{x} \equiv (x_1, ..., x_n) \in \mathbb{R}^n \ : \ 0 < x_i < \frac{\pi}{\sqrt{a_i}} \right\}$$

and the nonlinear wave equation

$$\begin{aligned} &u_{tt} - \Delta u + mu - b_{ij}(u, \nabla u)\partial_i\partial_j u + g(u, \nabla u) = 0 \ , \quad \mathbf{x} \in \mathcal{R} \\ &u\big|_{\partial\mathcal{R}} = 0 \ , \end{aligned} \tag{2.1}$$

where we used the summation convention for the indexes $i, j = 1, ..., n$, we denoted by $\nabla u \equiv (\partial_1 u, ..., \partial_n u)$ the derivatives of u with respect to the space variables, and b_{ij}, g are functions of class C^∞ in a neighbourhood of the origin. Moreover we assume that the system is Hamiltonian, i.e. that there exists a C^∞ function $W = W(u, s_1, ..., s_n)$ such that

$$b_{ij}(u, s_1, ..., s_n) = \frac{\partial^2 W}{\partial s_i \partial s_j}(u, s_1, ..., s_n) \ , \quad g = \frac{\partial W}{\partial u} - \frac{\partial^2 W}{\partial u \partial s_k}\partial_k u \ ,$$

so that the Hamiltonian function of the system is given by

$$H(u, v) = \int_{\mathcal{R}} \left[\frac{v^2}{2} + \frac{\|\nabla u\|^2}{2} + \frac{mu}{2} + W(u, \nabla u) \right] d^n\mathbf{x} \ , \tag{2.2}$$

where $v = \dot u$ is the momentum conjugated to u, and we denoted by $\|\,.\,\|$ the euclidean norm of a vector of $\mathbb{R}^n$.

Finally we assume that W has a zero of order $p + 2$ at the origin, namely that

$$|W(u, s)| \le C\left(|u| + \|s\|\right)^{p+2} \quad , \quad p \ge 1 \ ,$$

and that the potential W is even in each of the arguments, namely

$$W((-)^{c_0}u, (-)^{c_1}s_1, ..., (-)^{c_n}s_n) = W(u, s_1, ..., s_n) \ , \quad \forall (c_0, c_1, ..., c_n) \in (Z_2)^{n+1} \ , \tag{2.3}$$

from which in particular it follows $p \ge 2$.

We remark that from (2.3) it follows that

$$\begin{aligned} b_{ij}((-)^{c_0}u, (-)^{c_1}s_1, ..., (-)^{c_n}s_n) &= (-)^{c_i + c_j} b_{ij}(u, s_1, ..., s_n) \ , \\ g((-)^{c_0}u, (-)^{c_1}s_1, ..., (-)^{c_n}s_n) &= (-)^{c_0}(u, s_1, ..., s_n) \ , \end{aligned} \tag{2.4}$$

two conditions that will be required to ensure the smoothness of the solution of (2.1)*.

For example one can take $W(u) = u^4/4$ which gives the standard semilinear Klein Gordon equation

$$u_{tt} - \Delta u + mu + u^3 = 0 \ .$$

Other choices are possible, for example one can take $W(\nabla u) = \|\nabla u\|^4/4$ or $W(u, \nabla u) = u^2\|\nabla u\|^2$ that give

$$\begin{gathered} u_{tt} - \left(1 + \|\nabla u\|^2\right)\Delta u + mu - 2(\partial_i u)(\partial_k u)\partial_k\partial_i u = 0, \\ u_{tt} - (1 + 2u^2)\Delta u + mu - 2\|\nabla u\|^2 u = 0 \end{gathered} \tag{2.5}$$

respectively, which are true quasilinear equations.

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* In particular it ensures that the compatibility conditions [2] of the nonlinear system reduce to those of the linear system.

Expand u in Fourier series in the space variable, namely write

$$u(\mathbf{x},t) = \sum_{\mathbf{j}\equiv(j_1,\dots,j_n)} u_{\mathbf{j}}(t) \prod_{i=1}^{n} a_i^{1/4} \sin(\sqrt{a_i} j_i x_i) \ , \tag{2.6}$$

with

$$\mathbf{j} \equiv (j_1, ..., j_n) \in \bar{\mathbf{N}}^n := \left\{ \mathbf{k} \equiv (k_1, ..., k_n) \in \bar{\mathbf{Z}}^n \ : \ k_i \geq 1 \ , \ i = 1, ..., n \right\} \ ,$$

so that (2.1) is converted into the infinite system

$$\ddot{u}_{\mathbf{j}} + \omega_{\mathbf{j}}^2 u_{\mathbf{j}} = f_{\mathbf{j}}(u) \ , \ \mathbf{j} \in \bar{\mathbf{N}}^n$$

where f is of order at least $p+1$ in u, and

$$\omega_{\mathbf{j}} := \sqrt{a_1 j_1^2 + a_2 j_2^2 + ... + a_n j_n^2 + m} \ . \tag{2.7}$$

In the linear approximation ($f_{\mathbf{j}} \equiv 0$) the solution is almost periodic and moreover all the actions

$$I_{\mathbf{j}}(u,v) := \frac{v_{\mathbf{j}}^2 + \omega_{\mathbf{j}}^2 u_{\mathbf{j}}^2}{2\omega_{\mathbf{j}}} \ , \tag{2.8}$$

as well as all their linear combinations are constant of motion.

To study the nonlinear system we first introduce a topology in the phase space. This is done by defining the phase spaces $\mathcal{F}_s$ of the functions $(u,v) \in H^{s+1} \oplus H^s$ fulfilling the compatibility conditions

$$\begin{aligned} (-\Delta)^j u\big|_{\partial\mathcal{R}} &= 0 \ , \quad 0 \leq j \leq \left[\frac{s}{2}\right] \ , \\ (-\Delta)^j v\big|_{\partial\mathcal{R}} &= 0 \ , \quad 0 \leq j \leq \left[\frac{s+1}{2}\right] - 1 \ . \end{aligned}$$

In $\mathcal{F}_s$ we will use the norm

$$\|(u,v)\|_s^2 := \int_R \left((D^{s+1}u)^2 + (D^s v)^2 \right) d^n\mathbf{x} \ , \quad D := (-\Delta)^{1/2} \ .$$

Existence and uniqueness of the solution the Cauchy problem is obtained by applying Kato's theory (see [2]). Precisely, denoting by (u_0, v_0) the initial data for u and v, one has

Theorem 2.1. *Let b_{ij} and g be C^∞ functions fulfilling (2.4). Let $(u_0,v_0) \in \mathcal{F}_s$, where $s \geq [n/2]+2$. Then there exists $T>0$ and a unique solution*

$$u \in C^0((-T,T);H^{s+1}) \cap C^1((-T,T);H^s) \cap ... \cap C^{s+1}((-T,T);H^0) \tag{2.9}$$

of (2.1) with $u(0) = u_0$ and $\partial_t u(0) = v_0$.

It is also possible to estimate (from below) the existence time for small amplitude solution and to give a first estimate of the solution. In the statement of the corresponding result and in the rest of the paper C *will denote a (large) constant independent of all the relevant quantities. Its value changes through the paper.*

Theorem 2.2. *Consider the Cauchy problem for equation (2.1), and assume that b_{ij} and g are of class C^∞ and fulfill (2.4). Fix an arbitrary $s > n$ and an arbitrary $R_1 > 1$. Then there exist positive ϵ_*, C such that if the initial datum fulfills*

$$\epsilon := \left\|(u_0, v_0)\right\|_s \le \epsilon_* \ , \tag{2.10}$$

then the above theorem 2.1 holds with

$$T = \frac{1}{C\epsilon^p} \ , \tag{2.11}$$

and moreover the solution fulfills

$$\sup_{|t| \le 1/C\epsilon^p} \left\|(u(t), \dot{u}(t))\right\|_s \le \epsilon R_1 \ . \tag{2.12}$$

We think that this theorem should be well known, however, since it is crucial for our developments and we were not able to find a proof in literature, we will give a prove of it in the appendix.

We are now going to state our main result. To this end we need the frequencies to fulfill a diophantine type condition. Fix a set $\underline{\mathbf{j}} := (\mathbf{j}^{(1)}, ..., \mathbf{j}^{(N)})$ with $\mathbf{j}^{(l)} \in \bar{\mathbf{N}}^n$, then, given a choice σ of N signs, define the vector $\mathbf{k} := \sum_{l=1}^N \pm e_{\mathbf{j}^{(l)}}$ where $e_{\mathbf{j}^{(l)}} \in \mathbf{Z}^{\bar{\mathbf{N}}^n}$ is the vector with all components equal to zero but the $\mathbf{j}^{(l)}$–th which is equal to 1, so that

$$\mathbf{k} \cdot \omega = \pm\omega_{\mathbf{j}^{(1)}} \pm \omega_{\mathbf{j}^{(2)}} \pm ... \pm \omega_{\mathbf{j}^{(N)}}$$

We consider only choices of σ such that $\mathbf{k} \neq 0$. Such choices will be called *allowed choices.* We assume that $\forall N$ there exist positive γ and τ such that

$$\left|\omega_{\mathbf{j}^{(1)}} \pm \omega_{\mathbf{j}^{(2)}} \pm ... \pm \omega_{\mathbf{j}^{(N)}}\right| \ge \frac{\gamma}{\left\|\mathbf{j}^{(1)}\right\|^\tau \left\|\mathbf{j}^{(2)}\right\|^\tau ... \left\|\mathbf{j}^{(N)}\right\|^\tau} \ , \tag{2.13}$$

holds for any allowed choice of the signs and of the indixes.

The validity of such a condition for the frequencies (2.7) will be ensured by a suitable choice of the parameters a_i and m (see theorem 2.7 and corollary 2.8 below).

In the statement of the main theorem we will use again the notation (2.8) for the actions of the linearized system. Consider the Cauchy problem for equation (2.1) with initial conditions $u(x, 0) = u_0(x)$ and $\dot{u}(x, 0) = v_0(x)$: the following theorem holds

Theorem 2.3. *Fix $M \ge 4$ and an integer $r \ge 0$, and assume that the frequencies fulfill (2.13), then there exist positive constants s' and ϵ_* and infinitely many functions $\Phi^{\mathbf{j}}$ defined and analytic in a neighbourhood $\mathcal{U}$ of the origin of $\mathcal{F}_{r+s'}$, such that if the initial datum fulfills*

$$\epsilon := \left\|(u_0, v_0)\right\|_{r+s'} \le \epsilon_* \tag{2.14}$$

then along the corresponding solution one has

$$\sup_{|t|\leq 1/C\epsilon^p} |\Phi^{\mathbf{j}}(t)-\Phi^{\mathbf{j}}(0)| \leq \frac{C\epsilon^M}{\|\mathbf{j}\|^{2r}} \ , \tag{2.15}$$

moreover, one has

$$|I_{\mathbf{j}}(u,v)-\Phi^{\mathbf{j}}(u,v)| \leq C\frac{\|(u,v)\|_{r+s'}^{p+2}}{\|\mathbf{j}\|^{2r}} \tag{2.16}$$

for all (u,v) in the considered neighbourhood.

Corollary 2.4. *Fix $M'\geq 4$ and an integer $s\geq 0$ and assume that the frequencies fulfill (2.13), then there exist positive constants s'' and ϵ_* such that the following holds true: if the initial datum is small enough, namely if*

$$\epsilon := \|(u_0,v_0)\|_{s+s''} \leq \epsilon_* \tag{2.17}$$

then there exists a finite dimensional torus $\mathbf{T}_0$ smoothly embedded in $\mathcal{F}_s$ with the property that

$$\sup_{|t|\leq 1/C\epsilon^p} d_s(\zeta(t),\mathbf{T}_0) \leq C\epsilon^{M'} \tag{2.18}$$

where d_s is the distance in $\mathcal{F}_s$, and $\zeta(t) := (u(t),v(t))$ is the solution of (2.1).

Remark 2.5. *The technique of proof applies to all quasilinear Hamiltonian equations (in the sense of Kato) with discrete imaginary spectrum fulfilling (2.13). Moreover, one has to assume that the compatibility conditions*[2] *of the linearized systems coincide with those of the nonlinear system.*

Remark 2.6. *The strongest limitation of the present result rests in its time of validity. Indeed, in the finite dimensional case, and also in the case of semilinear equations in one space dimension*[16]*, a similar description of the dynamics has been proved to hold up to times of order ϵ^{-M} for any M. Due to results of the kind of [25] we expect the time (2.11) to be optimal unless one restricts the considered class of equations. We have no ideas on how this should be done and on the actual time of validity of (2.18) in semilinear equations.*

We now discuss the validity of the nonresonance condition (2.13). To this end fix the vector $a=(a_1,...,a_n)$ to be diophantine, namely such that there exist $\gamma_1>0$ and $\bar{\tau}$ with the property that*

$$|a_1k_1+...+a_nk_n| \equiv |a\cdot k| \geq \frac{\gamma_1}{|k|^{\bar{\tau}}} \ , \quad \forall k\in \mathbf{Z}^n\setminus\{0\} \ . \tag{2.19}$$

* We recall that the subset of $\mathbb{R}^n$ formed by the vectors a for which this is true has full measure.

We will prove that, having fixed a positive m_1, for most values of $m \in \mathcal{I} := [0, m_1]$, the condition (2.13) holds. To be precise fix $\gamma > 0$, τ, $\underline{\mathbf{j}} := (\mathbf{j}^{(1)}, ..., \mathbf{j}^{(N)})$ and any choice σ of N signs, and define

$$\mathcal{R}^{\sigma}_{\underline{\mathbf{j}}}(\gamma, \tau) := \left\{ m \in \mathcal{I} \ : \ \left| \omega_{\mathbf{j}^{(1)}} \pm \omega_{\mathbf{j}^{(2)}} \pm ... \pm \omega_{\mathbf{j}^{(N)}} \right| < \frac{\gamma}{\left\| \mathbf{j}^{(1)} \right\|^{\tau} \left\| \mathbf{j}^{(2)} \right\|^{\tau} ... \left\| \mathbf{j}^{(N)} \right\|^{\tau}} \right\} \ . \tag{2.20}$$

Theorem 2.7. *Fix N and $\tau > Nn + 2N^2(2\bar{\tau} + 2)$, then one has*

$$\left| \bigcup_{\underline{\mathbf{j}}, \sigma} \mathcal{R}^{\sigma}_{\underline{\mathbf{j}}}(\gamma, \tau) \right| \leq \gamma^{1/N} C |\mathcal{I}| \ , \tag{2.21}$$

where $|\mathcal{M}|$ denotes the Lebesgue measure of the set $\mathcal{M}$ and $\underline{\mathbf{j}}$ varies in $(\bar{\mathbf{N}}^n)^N$ and σ over all allowed choices of signs.

This theorem will be proved in sect. 5.

Corollary 2.8. *There exists a set $\mathcal{J} \subset \mathcal{I}$ with measure $m_1 = |\mathcal{I}|$, with the property that, if $m \in \mathcal{J}$ then the frequencies (2.7) fulfill (2.13) with suitable constants γ, τ.*

3. The classical algorithm

Here we recall the classical algorithm for constructing the formal integrals of motion of a Hamiltonian system. Consider a Hamiltonian system of the form

$$H = \sum_{k \geq 0} H_k \tag{3.1}$$

where H_k is a homogeneous polynomial of degree $k+2$ and H_0 is of the form $\sum_l \omega_l (p_l^2 + q_l^2)/2$ with $\omega = (\omega_1, ..., \omega_N)$ which is a nonresonant vector, namely such that $\omega \cdot k \neq 0$ for any $k \in \mathbf{Z}^N \setminus \{0\}$. Let $I = \sum_j c_j (p_j^2 + q_j^2)/2$ be a linear combination of the actions, we aim to construct an integral of motion of the form $\Phi := I + \sum_{l \geq 1} \Phi_l$ with Φ_l a homogeneous polynomial of degree $l + 2$. Clearly Φ has to fulfill the equation $\{H, \Phi\} = 0$. Inserting the Taylor expansions of H and of Φ and equating terms of the same degree we get the recursive system

$$L_{H_0} \Phi_l = \Psi_l \ , \ l \geq 1 \ , \quad L_{H_0} := \{H_0, .\} \tag{3.2}$$

where Ψ_l is determined by

$$\begin{aligned}\Psi_1 &= \{I, H_1\} \\ \Psi_l &= \sum_{j=1}^{l-1} \{\Phi_j, H_{l-j}\} + \{I, H_l\}\end{aligned} \tag{3.3}$$

So, provided one is able to solve the homological equation (3.2), the formal integral of motion can be computed.

In order to solve the homological equation we remark that at the l-th step (3.2) appears as a linear equation in the finite dimensional space of the polynomials of degree $l+2$. So, if one is able to diagonalize the linear operator L_{H_0}, then equation (3.2) is easily solved. To this end we introduce the variables

$$\xi_j := \frac{1}{\sqrt{2}}(q_j - \mathrm{i}p_j) \ , \quad \eta_j := \frac{1}{\sqrt{2}}(q_j + \mathrm{i}p_j) \ , \tag{3.4}$$

which give the unperturbed Hamiltonian the form $H_0 = \sum \omega_l \xi_l \eta_l$ (and the symplectic form becomes $\mathrm{i}d\xi_l \wedge d\eta_l$), so that one has

$$L_{H_0}\xi^j\eta^k = [\mathrm{i}(k-j)\cdot\omega] \ \xi^j\eta^k \ , \quad \xi^j\eta^k \equiv \xi_1^{j_1}...\xi_N^{j_N}\eta_1^{k_1}...\eta_N^{k_N}$$

i.e. the monomial $\xi^j\eta^k$ form the basis of the space of polynomials on which L_{H_0} is diagonal. So, (3.2) is easily solvable provided Ψ_l has no component on the kernel of L_{H_0}, a fact that can be easily verified recursively in the case where $H_k(p,q)$ depends only on q*.

As is well known the series defining the integral of motion are in general divergent, so, following [6], we stop the construction at a given order and we estimate the Poisson bracket of the so obtained function with the Hamiltonian. To this end remark that, defining

$$\Phi^{(M)} := I + \sum_{l=1}^{M} \Phi_l \ ,$$

one has

$$\dot{\Phi}^{(M)} = \left\{H, \Phi^{(M)}\right\} = \left\{\sum_{k=M+1}^{\infty} H_k, \Phi^{(M)}\right\} + \sum_{\substack{k,j=0 \\ k+j>M}}^{M} \{H_k, \Phi_j\} \tag{3.5}$$

where we denoted for simplicity $\Phi_0 := I$. In particular the quantity (3.5) has a zero of order $M+3$ at the origin, so it follows that, in a ball of radius ϵ it is smaller that $C\epsilon^{M+3}$. Finally in order to obtain a long time estimate of the solution one uses the function $\Phi^{(M)}$ as a Lyapunov function in order to control that the domain of construction of such a function is not left by the solution up to the considered times.

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* Indeed, a component on the Kernel of L_{H_0} is always even in the momenta p, while from the construction of Ψ_k such a function always turns out to be skewsymmetric in p (see [6]).

In the infinite dimensional case the situation is more difficult essentially because of three problems: the first is the solution of the homological equation (3.2), indeed one has that even in the absence of exact resonances the linear combination with integer coefficients of the frequencies accumulate to zero at any finite order. The second problem is constituted by the fact that the vector field of a Hamiltonian function is typically only densely defined: as a consequence one has that the domain of the Poisson Bracket of two functions is only dense in the domain of the original functions. The third problem is that one cannot use the function $\Phi^{(M)}$ as a Lyapunov function since it does not have the so called potential well property, i.e. it does not provide a good estimate of the norm of the solution. These are the problems we will solve in the next section.

4. Proof theorem 2.3

First we give the Hamiltonian (2.2) the form we need. It is useful to scale the independent variable $\mathbf{x}$ by defining $\tilde{x}_i = x_i\sqrt{a_i}$, so that the equation turns out to be defined in

$$\mathcal{Q} := \{\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n \ : \ 0 < x_i < \pi\}$$

and takes the form

$$u_{tt} - a_{ij}\partial_i\partial_j u + mu + g(u, \nabla u) = 0 \tag{4.1}$$

with

$$a_{ij}(u, \nabla u) = a_j\delta_{ij} + b_{ij}(u, \nabla u)$$

and redefined functions b_{ij}, g; in what follows we will omit the tildes from $\mathbf{x}$. Moreover it is useful to study eq.(4.1) in $\mathbf{T}^n \equiv \mathbb{R}^n/2\pi\mathbf{Z}^n$: the original problem being obtained by restricting the initial data to the submanifold $\mathcal{A}$ of the functions which are skewsymmetric in each variable, namely that fulfill

$$u((-)^{c_1}x_1, ..., (-)^{c_n}x_n) = (-)^{c_1+...+c_n}u(x_1, ..., x_n) \ .$$

Clearly one has $\mathcal{F}_s \simeq \mathcal{A} \cap (H^{s+1}(\mathbf{T}^n) \times H^s(\mathbf{T}^n))$. Due to the symmetry properties (2.4), the subspace $\mathcal{A}$ is invariant under the dynamics of (4.1) in $\mathbf{T}^n$. For our aim it will be enough to consider (4.1) in a space of functions with zero average; this is what we will do. To work in $\mathbf{T}^n$ allows us to use the Fourier expansions on the basis of the exponentials, and this greatly simplifies the analytic estimates that we will need. We point out that one could avoid this extension but the resulting formalism would be more complicated.

Expand u and v in Fourier series:

$$u(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \sum_{\mathbf{j}\in\bar{\mathbf{Z}}^n} q_{\mathbf{j}}\sqrt{\omega_{\mathbf{j}}} e^{i\mathbf{j}\cdot\mathbf{x}} \ , \quad v(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \sum_{\mathbf{j}\in\bar{\mathbf{Z}}^n} \frac{p_{\mathbf{j}}}{\sqrt{\omega_{\mathbf{j}}}} e^{i\mathbf{j}\cdot\mathbf{x}}$$

where $\bar{\mathbf{Z}}^n = \mathbf{Z}^n \backslash \{0\}$, $\omega_{\mathbf{j}}$ is still given by (2.7), and $q_{\mathbf{j}}$ turns out to be conjugated to $p_{-\mathbf{j}}$. Then we pass to complex variables $\xi_{\mathbf{j}}$ and $\eta_{\mathbf{j}}$ by

$$\xi_{\mathbf{j}} := \frac{1}{\sqrt{2}}\left(q_{\mathbf{j}} - \mathrm{i}p_{\mathbf{j}}\right) \ , \quad \eta_{\mathbf{j}} := \frac{1}{\sqrt{2}}\left(q_{\mathbf{j}} + \mathrm{i}p_{\mathbf{j}}\right) \ , \tag{4.2}$$

in terms of which the symplectic form becomes $\mathrm{i}d\xi_{\mathbf{j}} \wedge d\eta_{-\mathbf{j}}$. We define the topology in the phase space as follows. Consider the space ℓ^2_s of the complex sequences $\xi \equiv \{\xi_{\mathbf{j}}\}_{j\in\bar{\mathbf{Z}}^n}$ such that

$$\|\xi\|^2_{\ell^2_s} := \sum_{\mathbf{j}\in\bar{\mathbf{Z}}^n} \|\mathbf{j}\|^{2s} \left|\xi_{\mathbf{j}}\right|^2 < \infty \ , \quad \|\mathbf{j}\| := \sqrt{j_1^2 + ... + j_n^2}$$

and define

$$\mathcal{P}_s := \ell^2_s \oplus \ell^2_s \ni \zeta \equiv \left(\{\xi_{\mathbf{j}}\}_{\mathbf{j}\in\bar{\mathbf{Z}}^n}, \{\eta_{\mathbf{j}}\}_{\mathbf{j}\in\bar{\mathbf{Z}}^n}\right) \ ;$$

this is the family of phase spaces we will use. In $\mathcal{P}_s$ we will use the norm $\|\zeta\|^2_s = \|\xi\|^2_{\ell^2_s} + \|\eta\|^2_{\ell^2_s}$, and will denote by $\mathcal{B}_s(R)$ the ball of radius R centered at the origin of $\mathcal{P}_s$. For the Hamiltonian vector field of a Hamiltonian function H we will use the notation X_H, so that the Hamilton equations are $\dot{\zeta} = X_H(\zeta)$. Explicitly they are given by

$$\frac{d}{dt}\xi_{\mathbf{j}} = \mathrm{i}\frac{\partial H}{\partial \eta_{-\mathbf{j}}} \ , \quad \frac{d}{dt}\eta_{\mathbf{j}} = -\mathrm{i}\frac{\partial H}{\partial \xi_{-\mathbf{j}}} \ .$$

Then the Hamiltonian (2.2) takes the form

$$H := H_0 + f \tag{4.3}$$

with

$$H_0(\xi,\eta) := \sum_{\mathbf{j}\in\bar{\mathbf{Z}}^n} \omega_{\mathbf{j}}\xi_{\mathbf{j}}\eta_{-\mathbf{j}} \ , \tag{4.4}$$

and f of class C^∞ on $\mathcal{P}_s$ for any $s > n/2 + 1/2$. Actually f has more properties: we are now going to give the corresponding definitions.

Definition. *Consider a polynomial $h(\xi,\eta)$ homogeneous of degree n_1 in ξ and of degree n_2 in η, namely*

$$h(\xi,\eta) = \sum_{\underline{\mathbf{k}},\underline{\mathbf{j}}} h_{\underline{\mathbf{k}},\underline{\mathbf{j}}}\xi_{\mathbf{k}^{(1)}}...\xi_{\mathbf{k}^{(n_1)}}\eta_{\mathbf{j}^{(1)}}...\eta_{\mathbf{j}^{(n_2)}} \tag{4.5}$$

where $\underline{\mathbf{k}} = (\mathbf{k}^{(1)}, ..., \mathbf{k}^{(n_1)})$, $\underline{\mathbf{j}} = (\mathbf{j}^{(1)}, ..., \mathbf{j}^{(n_2)})$. We will say that it satisfies the **zero momentum condition** *if*

$$h_{\underline{\mathbf{k}},\underline{\mathbf{j}}} \neq 0 \Longrightarrow \mathbf{k}^{(1)} + \mathbf{k}^{(2)} + ... + \mathbf{k}^{(n_1)} + \mathbf{j}^{(1)} + ... + \mathbf{j}^{(n_2)} = 0 \ .$$

Definition. *Consider a polynomial $h(\xi,\eta)$ homogeneous of degree n_1 in ξ and of degree n_2 in η, fulfilling the zero momentum condition; the quantity*

$$|h|_r := \sup_{\underline{\mathbf{k}},\underline{\mathbf{j}}} \frac{|h_{\underline{\mathbf{k}},\underline{\mathbf{j}}}|}{\left\|\mathbf{k}^{(1)}\right\|^r \left\|\mathbf{k}^{(2)}\right\|^r ... \left\|\mathbf{k}^{(n_1)}\right\|^r \left\|\mathbf{j}^{(1)}\right\|^r \left\|\mathbf{j}^{(2)}\right\|^r ... \left\|\mathbf{j}^{(n_2)}\right\|^r} \tag{4.6}$$

will be called **modulus of order** r **of** h*.*

Definition. *Given a homogeneous polynomial h of degree N, we will write it as*

$$h = \sum_{n_1+n_2=N} h^{n_1,n_2}$$

with h^{n_1,n_2} homogeneous of degree n_1 in ξ and of degree n_2 in η. Assume that h^{n_1,n_2} satisfy the zero momentum condition, then

$$|h|_r := \sum_{n_1+n_2=N} |h^{n_1,n_2}|_r$$

is called modulus of order r of h; if such a quantity is finite for some r then we will say that h is of class $\mathcal{C}_r$ and we will write $h \in \mathcal{C}_r$.

Proposition 4.1. *Let H_k be the homogeneous polynomial of degree $k+2$, $k \geq 1$ in the Taylor expansion of the Hamiltonian (2.2), then $H_k \in \mathcal{C}_{1/2}$.*

Proof. Clearly one has

$$H_k(u) = \int_{\mathbf{T}^n} \sum_{|\mathbf{j}|+l=k+2} f_{\mathbf{j},l} u^l (\nabla u)^{\mathbf{j}} d^n\mathbf{x} \ , \quad (\nabla u)^{\mathbf{j}} \equiv (\partial_1 u)^{j_1} ... (\partial_n u)^{j_n}$$

We concentrate on the term with $l=0$, that contains all the difficulties. Such a term is the linear combination of finitely many terms of the form

$$\begin{aligned} &\int_{\mathbf{T}^n} \partial_{j_1} u ... \partial_{j_{k+2}} u d^n\mathbf{x} \\ &= \frac{1}{(2\pi)^{nk/2}} \mathrm{i}^{k+2} \sum_{\mathbf{l}^{(1)},...,\mathbf{l}^{(k+2)}} \int_{\mathbf{T}^n} l_{j_1} ... l_{j_{k+2}} \frac{q_{\mathbf{l}^{(1)}} ... q_{\mathbf{l}^{(k+2)}}}{\sqrt{\omega_{\mathbf{l}^{(1)}} ... \omega_{\mathbf{l}^{(k+2)}}}} e^{\mathrm{i}(\mathbf{l}^{(1)}+...+\mathbf{l}^{(k+2)})\cdot\mathbf{x}} d^n\mathbf{x} \\ &= \frac{1}{(2\pi)^{nk/2}} \mathrm{i}^{k+2} \sum_{\mathbf{l}^{(1)}+...+\mathbf{l}^{(k+2)}=0} l_{j_1} ... l_{j_{k+2}} \frac{q_{\mathbf{l}^{(1)}} ... q_{\mathbf{l}^{(k+2)}}}{\sqrt{\omega_{\mathbf{l}^{(1)}} ... \omega_{\mathbf{l}^{(k+2)}}}} \end{aligned}$$

which shows that it satisfies the zero momentum condition and that, due to the asymptotic of the frequencies, the coefficients of the monomials $q_{\mathbf{l}^{(1)}} ... q_{\mathbf{l}^{(k+2)}}$, when divided by $\left\|\mathbf{l}^{(1)}\right\|^{1/2} ... \left\|\mathbf{l}^{(k+2)}\right\|^{1/2}$ form a bounded sequence. Moreover it is clear that such properties are not modified by the introduction of the complex variables ξ,η, and therefore the statement holds. □

Now we would like to apply the algorithm of the previous section to construct the approximate integrals of motion, but previously it has to be slightly modified. This is due to the fact that, when we extended the system to the torus we introduced many oscillators with the same frequency as the original ones. As a consequence, in the case of periodic boundary conditions, the operator L_{H_0} turns out to have a quite large kernel and the equation (3.2) is in general not solvable. So, given a polynomial h on the phase space, we split it as

$$h = \tilde{h} + \bar{h}$$

with $\tilde{h}$ in the range of L_{H_0} and $\bar{h}$ in the kernel of L_{H_0}. Then we fix an index $\mathbf{j}_0 \in \bar{\mathbf{N}}^n$ and define $I := \|\mathbf{j}_0\|^{2r} I_{\mathbf{j}_0}$: the modified algorithm of construction of the integrals of motion is given by the recursive system

$$L_{H_0}\Phi_l = \tilde{\Psi}_l \ , \ l \geq 1 \ , \tag{4.7}$$

where Ψ_l is determined by

$$\begin{aligned} \Psi_1 &= \{I, H_1\} \\ \Psi_l &= \sum_{j=1}^{l-1} \{\Phi_j, H_{l-j}\} + \{I, H_l\} \end{aligned} \tag{4.8}$$

Remark 4.2. *One has $\bar{\Psi}_l|_{\mathcal{A}} = 0$; indeed due to the properties (2.4), one has*

$$\{H_0, \Psi_l\}\,|_{\mathcal{A}} = \{H_0|_{\mathcal{A}}, \Psi_l|_{\mathcal{A}}\}$$

and therefore $\Psi_l|_{\mathcal{A}}$ is in the kernel of the restriction of L_{H_0} to $\mathcal{A}$. But on this space the frequencies are nonresonant, and therefore the kernel of L_{H_0} restricted to $\mathcal{A}$ fulfills the properties described in the previous section. In particular it contains only functions which are even in the variables p, while, by the same discussion of the previous section $\Psi_l|_{\mathcal{A}}$, is always odd in p.

Finally to compute the Lie derivative of the approximate integral it is useful not to expand H in series, so we fix once for ever the order $M \geq 2$ that we want to reach, and we split the Hamiltonian into an effective and an inessential part according to

$$H = \sum_{k=0}^{M} H_k + \mathcal{R}$$

where $\mathcal{R}$ is the remainder of the expansion; we compute the Lie derivative of $\Phi^{(M)}$ and we restrict to $\mathcal{A}$. We thus get (in virtue of remark 4.2)

$$\dot{\Phi}^{(M)}\big|_{\mathcal{A}} = \left\{\mathcal{R}, \Phi^{(M)}\right\}\Big|_{\mathcal{A}} + \sum_{\substack{k,j=1 \\ k+j>M}}^{M} \{H_k, \Phi_j\}|_{\mathcal{A}} \tag{4.9}$$

which is the formula we will use to compute the variation in time of $\Phi^{(M)}$.

Remark that also in this case the algorithm is composed by some building blocks which are the computation of Poisson Brackets and the solution of the Homological equation; so, first of all we study the behavior of the classes $\mathcal{C}_r$ under these operations. In order to estimate (4.9) we will also need an estimate of the vector field of a function of class $\mathcal{C}_r$.

We will denote by Π_N the set of the homogeneous polynomial of degree N.

Lemma 4.3. *Let $h \in \mathcal{C}_r \cap \Pi_N$, then, for any s' such that $s'+r > n/2$, one has*

$$\sup_{\zeta \in \mathcal{B}_{2r+s'}(R)} \left\| X_h(\zeta) \right\|_{s'} \leq C|h|_r R^{N-1} \ .$$

Proof. Consider first the case where h is homogeneous of degree n_1 in ξ and of degree n_2 in η. Consider the functional

$$\check{h}(\check{\xi}, \check{\eta}) := \int_{\mathbf{T}^n} [D^r \check{\xi}(\mathbf{x})]^{n_1} [D^r \check{\eta}(\mathbf{x})]^{n_2} d^n \mathbf{x} \ , \quad D \equiv (-\Delta)^{1/2}$$

where

$$\check{\xi}(\mathbf{x}) := \frac{1}{(2\pi)^{n/2}} \sum_{\mathbf{j}} \xi_{\mathbf{j}} e^{i\mathbf{j}\cdot\mathbf{x}} \ , \quad \check{\eta}(\mathbf{x}) := \frac{1}{(2\pi)^{n/2}} \sum_{\mathbf{j}} \eta_{\mathbf{j}} e^{i\mathbf{j}\cdot\mathbf{x}} \ .$$

Now, exploiting the property of algebra of the Sobolev space H^s with $s > n/2$ one has that the vector field of such a functional is an analytic function from $\mathcal{P}_{2r+s'}$ to $\mathcal{P}_{s'}$. Then, for any pair s, s'' in the allowed range of values, one has

$$\sup_{\zeta \in \mathcal{B}_{s''}(R)} \left\| X_h(\zeta) \right\|_s \leq \sup_{\zeta \in B_{s''}(R)} \left\| X_{\check{h}}(\zeta) \right\|_s \ .$$

This can be easily seen by writing the vector fields of h and $\check{h}$ in Fourier coordinates and writing explicitly the above norms (for more details, in a similar situation, see [16] proof of lemma 5.1).

Then in order to obtain the statement of the lemma one has to estimate the norm of the vector field of $\check{h}$. Working in the $\mathbf{x}$ space one has that the $\check{\xi}$ component $[X_{\check{h}}]_{\check{\xi}}$ of $X_{\check{h}}$ is given by

$$[X_{\check{h}}]_{\check{\xi}}(\mathbf{x}) = \mathrm{i}\, n_2 \left\{ D^r \left[\left(D^r \check{\xi} \right)^{n_1} \left(D^r \check{\eta} \right)^{n_2 - 1} \right] \right\} (\mathbf{x})$$

Its H^s norm is easily estimated giving the thesis in the considered case. The general case easily follows. $\square$

Lemma 4.4. *Consider two homogeneous polynomial $h \in \mathcal{C}_r \cap \Pi_{N_1}$, $g \in \mathcal{C}_{r'} \cap \Pi_{N_2}$, $N_1, N_2 \leq M+2$ then one has $\{h, g\} \in \mathcal{C}_{r+r'}$, and*

$$\left| \{h, g\} \right|_{r+r'} \leq C |h|_r |g|_{r'} \ .$$

Proof. Consider first the case where h depends only on ξ and g depends only on η. Denote $\varphi := \{h, g\}$, then

$$\varphi(\xi,\eta) = \mathrm{i}\sum_{\mathbf{t}} \frac{\partial h}{\partial \xi_{\mathbf{t}}}\frac{\partial g}{\partial \eta_{-\mathbf{t}}} \ , \tag{4.10}$$

and therefore

$$\varphi(\xi,\eta) = \sum_{\mathbf{k}^{(2)},\dots,\mathbf{k}^{(N_1)},\mathbf{j}^{(2)},\dots,\mathbf{j}^{(N_2)}} \varphi_{\mathbf{k}^{(2)},\dots,\mathbf{k}^{(N_1)},\mathbf{j}^{(2)},\dots,\mathbf{j}^{(N_2)}} \xi_{\mathbf{k}^{(2)}},\dots,\xi_{\mathbf{k}^{(N_1)}}\eta_{\mathbf{j}^{(2)}}\dots\eta_{\mathbf{j}^{(N_2)}}$$

with

$$\begin{aligned}\varphi_{\mathbf{k}^{(2)},\dots,\mathbf{k}^{(N_1)},\mathbf{j}^{(2)},\dots,\mathbf{j}^{(N_2)}} &= N_1N_2\sum_{\mathbf{j}^{(1)}} h_{-\mathbf{j}^{(1)},\mathbf{k}^{(2)},\dots,\mathbf{k}^{(N_1)}} g_{\mathbf{j}^{(1)},\mathbf{j}^{(2)},\dots,\mathbf{j}^{(N_2)}} \\ &= N_1N_2 h_{-\mathbf{j}^{(1)},\mathbf{k}^{(2)},\dots,\mathbf{k}^{(N_1)}} g_{\mathbf{j}^{(1)},\mathbf{j}^{(2)},\dots,\mathbf{j}^{(N_2)}} \delta^{\mathbf{k}^{(2)}+\dots+\mathbf{k}^{(N_1)}}_{\mathbf{j}^{(1)}} \delta^{\mathbf{j}^{(2)}+\dots+\mathbf{j}^{(N_2)}}_{-\mathbf{j}^{(1)}} \ .\end{aligned}$$

where we used the standard notation for the Kronecker δ and for the last equality we used the zero momentum condition. So the modulus of order $r+r'$ of φ is given by

$$\begin{aligned}\frac{|\varphi|_{r+r'}}{N_1N_2} &= \sup_{\substack{\mathbf{k}^{(2)},\dots,\mathbf{k}^{(N_1)}\\ \mathbf{j}^{(2)},\dots,\mathbf{j}^{(N_2)}}} \frac{|h_{-\mathbf{j}^{(1)},\mathbf{k}^{(2)},\dots,\mathbf{k}^{(N_1)}}|\,|g_{\mathbf{j}^{(1)},\mathbf{j}^{(2)},\dots,\mathbf{j}^{(N_2)}}|\,|\delta^{\mathbf{k}^{(2)}+\dots+\mathbf{k}^{(N_1)}}_{\mathbf{j}^{(1)}} \delta^{\mathbf{j}^{(2)}+\dots+\mathbf{j}^{(N_2)}}_{-\mathbf{j}^{(1)}}}{\left\|\mathbf{k}^{(2)}\right\|^{r+r'}\dots\left\|\mathbf{k}^{(N_1)}\right\|^{r+r'}\left\|\mathbf{j}^{(2)}\right\|^{r+r'}\dots\left\|\mathbf{j}^{(N_2)}\right\|^{r+r'}} \\
&\leq \sup_{\substack{\mathbf{k}^{(2)},\dots,\mathbf{k}^{(N_1)}\\ \mathbf{j}^{(2)},\dots,\mathbf{j}^{(N_2)}}} \left[\frac{|h|_r\left\|\mathbf{k}^{(2)}+\dots+\mathbf{k}^{(N_1)}\right\|^{r'}\left\|\mathbf{k}^{(2)}\right\|^{r}\dots\left\|\mathbf{k}^{(N_1)}\right\|^{r}}{\left\|\mathbf{k}^{(2)}\right\|^{r+r'}\dots\left\|\mathbf{k}^{(N_1)}\right\|^{r+r'}}\right. \\
&\qquad\left.\times\frac{|g|_{r'}\left\|\mathbf{j}^{(2)}+\dots+\mathbf{j}^{(N_2)}\right\|^{r}\left\|\mathbf{j}^{(2)}\right\|^{r'}\dots\left\|\mathbf{j}^{(N_1)}\right\|^{r'}}{\left\|\mathbf{j}^{(2)}\right\|^{r+r'}\dots\left\|\mathbf{j}^{(N_2)}\right\|^{r+r'}}\right] \\
&\leq |h|_r|g|_{r'}\sup_{\mathbf{k}^{(2)},\dots,\mathbf{k}^{(N_1)}}\frac{\left\|\mathbf{k}^{(2)}+\dots+\mathbf{k}^{(N_1)}\right\|^{r'}}{\left\|\mathbf{k}^{(2)}\right\|^{r'}\dots\left\|\mathbf{k}^{(N_1)}\right\|^{r'}}\sup_{\mathbf{j}^{(2)},\dots,\mathbf{j}^{(N_2)}}\frac{\left\|\mathbf{j}^{(2)}+\dots+\mathbf{j}^{(N_2)}\right\|^{r}}{\left\|\mathbf{j}^{(2)}\right\|^{r}\dots\left\|\mathbf{j}^{(N_2)}\right\|^{r}} \ .\end{aligned}$$

The suprema are bounded and therefore the thesis follows in the considered case. The general case easily follows. □

Lemma 4.5. *Assume that the frequencies fulfill the nonresonance condition*

$$|\omega_{\mathbf{k}^{(1)}}\pm\omega_{\mathbf{k}^{(2)}}\pm\dots\pm\omega_{\mathbf{k}^{(M+2)}}| \geq \frac{\gamma}{\left\|\mathbf{k}^{(1)}\right\|^{\tau}\left\|\mathbf{k}^{(2)}\right\|^{\tau}\dots\left\|\mathbf{k}^{(M+2)}\right\|^{\tau}} \ , \tag{4.11}$$

for some positive γ and τ and any choice of the signs and of the indexes such that the left hand side of (4.11) does not vanishes identically; Let $f\in\mathcal{C}_r\cap\Pi_N$, $N\leq M+2$ be a function with no components on Ker(L_{H_0}), then the homological equation

$$L_{H_0}\chi = f$$

has a solution $\chi \in \mathcal{C}_{r+\tau} \cap \Pi_k$ *fulfilling the estimate*

$$|\chi|_{r+\tau} \leq C|f|_r \ .$$

Proof. Assume for simplicity that f is homogeneous of degree N_1 in ξ and of degree N_2 in η and expand f as

$$f(\xi,\eta) = \sum_{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(N_1)},\mathbf{k}^{(1)},\ldots,\mathbf{k}^{(N_2)}} f_{\underline{\mathbf{j}},\underline{\mathbf{k}}} \xi_{\mathbf{j}^{(1)}} \ldots \xi_{\mathbf{j}^{(N_1)}} \eta_{\mathbf{k}^{(1)}} \ldots \eta_{\mathbf{k}^{(N_2)}} \ ;$$

the coefficients $\chi_{\underline{\mathbf{j}},\underline{\mathbf{k}}}$ of the analogous expansion for χ are given by

$$\chi_{\underline{\mathbf{j}},\underline{\mathbf{k}}} := \frac{\mathrm{i} f_{\underline{\mathbf{j}},\underline{\mathbf{k}}}}{\omega_{\mathbf{j}^{(1)}} + \ldots + \omega_{\mathbf{j}^{(N_1)}} - \omega_{\mathbf{k}^{(1)}} - \ldots - \omega_{\mathbf{k}^{(N_2)}}}$$

Since $N_1 + N_2 \leq M + 2$ the denominator can be controlled by (4.11), and therefore the statement is a simple application of the definition of class $\mathcal{C}_r$. $\square$

We conclude by estimating the Lie derivative of $\Phi^{(M)}$.

Lemma 4.6. *Let $I \in \mathcal{C}_r$, and consider the function $\Phi^{(M)} = I + \sum_{l=1}^{M} \Phi_k$ with Φ_k generated by the above construction. Then one has $\Phi^{(M)} \in \mathcal{C}_{M(1/2+\tau)+r}$ and there exists R_* and s' such that, for any $R < R_*$*

$$\sup_{B_{r+s'}(R)} \left| \left\{ H, \Phi^{(M)} \right\} \big|_{\mathcal{A}} \right| \leq C R^{M+3} \tag{4.12}$$

$$\sup_{B_{r+s'}(R)} \left| \Phi^{(M)} - I \right| \leq C R^{p+2} \tag{4.13}$$

Proof. It is easily seen recursively that $\Phi_k \in \mathcal{C}_{k(1/2+\tau)+r}$ and that its modulus of order $k(1/2+\tau)+r$ is bounded. To obtain the estimate (4.12) recall that by remark 4.2 the Lie derivative of $\Phi^{(M)}$ when restricted to $\mathcal{A}$ is given by (4.9). The second term of this equation is the sum of finitely many terms of the form $\{H_k, \Phi_l\}$ with $k + l > M$ and $k, l \leq M$ which are easily estimated using lemma 4.4 and lemma 4.3. We study in detail only the first term, namely

$$\left\{ \mathcal{R}, \Phi^{(M)} \right\} \ . \tag{4.14}$$

The quantity (4.14) is estimated by

$$\|X_{\Phi^{(M)}}(\xi,\eta)\|_s \, \|d\mathcal{R}(\xi,\eta)\|_{\mathcal{P}_s \to \mathcal{P}_s} \ ; \tag{4.15}$$

provided $s > n/2 + 1/2$ the quantity $\|d\mathcal{R}(\xi,\eta)\|_{\mathcal{P}_s \to \mathcal{P}_s}$ can be estimated using the standard Lagrange formula for the remainder of the Taylor series of a smooth function (recall that under this condition H is of class C^∞). So $\|d\mathcal{R}(\xi,\eta)\|_{\mathcal{P}_s \to \mathcal{P}_s} \leq C R^{M+2}$. Concerning

$X_{\Phi^{(M)}}(\xi,\eta)$, it is the sum of finitely many terms. The less smooth one is X_{Φ_M} which maps $\mathcal{P}_{2r'+s}$ to $\mathcal{P}_s$, where $r' = M(r+\tau)+1/2$, and s is such that $r'+s > n/2+1/2$. So take $s = n/2+1$, $s' = (2M-1)r + 2M\tau + 2 + n/2$ and it follows that (4.15) is bounded by a constant time R^{M+3} in the considered domain. Concerning the estimate (4.13), it can be obtained by remarking that in the expansion of $\Phi^{(M)}$ the first correction to I is of order $p+2$. □

Proof of theorem 2.3. Define $\Phi^{\mathbf{j}_0} := \frac{\Phi^{(M)}}{\|\mathbf{j}_0\|^{2r}}$. To obtain the proof just use theorem 2.2 together with the estimate (4.12). Indeed given in initial datum fulfilling (2.10) in the space $\mathcal{F}_s$ with $s = r + s'$ then, up to the times we are interested in, the solution fulfills (2.12). Take now $R := \epsilon R_1$ and use the estimate (4.12) this gives exactly (2.15) provided one takes the M of this formula equal to the M of (4.12) plus 3. The estimate (2.16) immediately follows from (4.13). □

Proof of corollary 2.4. To define the torus we proceed as follows: First remark that due to the two inequalities (2.15) and (2.16) one has, up to the considered times

$$I_{\mathbf{j}}(t) \leq \frac{C\epsilon^{p+2}}{\|\mathbf{j}\|^{2r}} \ .$$

Fix N, and split the solution $\zeta(t) \equiv (u(t), v(t))$ into a part $\zeta_f(t)$ which is the projection on the Fourier modes with index $\mathbf{j}$ with $\|\mathbf{j}\| \leq N$ and $\zeta_\infty(t)$ which is the projection on the remaining modes. Then one has

$$d_s(\zeta(t), \zeta_f(t)) = \sqrt{\sum_{\|\mathbf{j}\|>N} \|\mathbf{j}\|^{2s} I_{\mathbf{j}}(t)} \leq \epsilon^{(p+2)/2} \sqrt{\sum_{\|\mathbf{j}\|>N} \frac{C}{\|\mathbf{j}\|^{2(r-s)}}} .$$

Provided $r > s + n/2$ this tends to zero as $N \to \infty$. It follows that provided N is large enough such a quantity can be bounded by $C\epsilon^{M'}$. So, consider the projection $\zeta_f(0)$ of the initial datum on the modes with $\|\mathbf{j}\| \leq N$ and define

$$\mathbf{T}_0 := \left\{ \zeta \in \mathcal{F}_s \ : \ \begin{array}{c} \Phi^{\mathbf{j}}(\zeta_f(0)) = \Phi^{\mathbf{j}}(\zeta) \ , \ \|\mathbf{j}\| \leq N \\ I_{\mathbf{j}}(\zeta) = 0 \ , \ \|\mathbf{j}\| > N \end{array} \right\} .$$

It can be easily seen that this is a smooth torus. Then the thesis follows provided the indexes are chosen in a suitable way. □

5. Measure Estimates

In this section we will prove theorem 2.7.

Lemma 5.1. *For any $K \le N$, let $\mathbf{j}^{(1)},...,\mathbf{j}^{(K)}$ be K different indexes belonging to $\bar{\mathbf{N}}^n$; then one has*

$$\left|\begin{matrix} \omega_{\mathbf{j}^{(1)}} & \omega_{\mathbf{j}^{(2)}} & . \; . \; . & \omega_{\mathbf{j}^{(K)}} \\ \frac{d\omega_{\mathbf{j}^{(1)}}}{dm} & \frac{d\omega_{\mathbf{j}^{(2)}}}{dm} & . \; . \; . & \frac{d\omega_{\mathbf{j}^{(K)}}}{dm} \\ . & . & . \; . \; . & . \\ . & . & . \; . \; . & . \\ \frac{d^{K-1}\omega_{\mathbf{j}^{(1)}}}{dm^{K-1}} & \frac{d^{K-1}\omega_{\mathbf{j}^{(2)}}}{dm^{K-1}} & . \; . \; . & \frac{d^{K-1}\omega_{\mathbf{j}^{(K)}}}{dm^{K-1}} \end{matrix}\right| \ge \frac{C}{\left\|\mathbf{j}^{(1)}\right\|^{K(\bar\tau+2)-1}\left\|\mathbf{j}^{(2)}\right\|^{K(\bar\tau+2)-1}...\left\|\mathbf{j}^{(K)}\right\|^{K(\bar\tau+2)-1}} \; . \tag{5.1}$$

Proof. First remark that, by explicit computation one has

$$\frac{d^N\omega_i}{dm^N} = \frac{(2N-3)!}{2^{N-2}(N-2)!2^N}\frac{(-)^{N+1}}{(\mu_{\mathbf{j}}+m)^{N-\frac{1}{2}}} \; . \tag{5.2}$$

where $\mu_{\mathbf{j}} := \sum_l a_l j_l^2$. Substituting (5.2) in the l.h.s. of (5.1) we get the determinant to be estimated. To obtain the estimate factorize from the $l-th$ column the term $(\mu_{\mathbf{j}^{(l)}}+m)^{1/2}$, and from the $j-th$ row the term $\frac{(2j-3)!}{2^{j-2}(j-2)!2^j}$. Forgetting the inessential powers of -1, we obtain that the determinant to be estimated is given by

$$\left[\prod_{l=1}^{K}\omega_{\mathbf{j}^{(l)}}\right]\left[\prod_{j=1}^{K-1}\frac{(2j-3)!}{2^{j-2}(j-2)!2^j}\right] \times \left|\begin{matrix} 1 & 1 & 1 & . \; . \; . & 1 \\ x_{\mathbf{j}^{(1)}} & x_{\mathbf{j}^{(2)}} & x_{\mathbf{j}^{(3)}} & . \; . \; . & x_{\mathbf{j}^{(K)}} \\ x^2_{\mathbf{j}^{(1)}} & x^2_{\mathbf{j}^{(2)}} & x^2_{\mathbf{j}^{(3)}} & . \; . \; . & x^2_{\mathbf{j}^{(K)}} \\ . & . & . & . \; . \; . & . \\ . & . & . & . \; . \; . & . \\ . & . & . & . \; . \; . & . \\ x^{K-1}_{\mathbf{j}^{(1)}} & x^{K-1}_{\mathbf{j}^{(2)}} & x^{K-1}_{\mathbf{j}^{(3)}} & . \; . \; . & x^{K-1}_{\mathbf{j}^{(K)}} \end{matrix}\right| \tag{5.3}$$

where we denoted by $x_{\mathbf{j}} := (\mu_{\mathbf{j}}+m)^{-1} \equiv \omega_{\mathbf{j}}^{-2}$. The last determinant is a Vandermonde determinant whose value is given by

$$\prod_{l<N\le K}(x_{\mathbf{k}^{(l)}} - x_{\mathbf{j}^{(N)}}) \qquad (5.4) \; .$$

Now one has

$$|x_{\mathbf{k}} - x_{\mathbf{j}}| = \left|\frac{1}{\mu_{\mathbf{k}}+m} - \frac{1}{\mu_{\mathbf{j}}+m}\right| = \frac{|\mu_{\mathbf{k}}-\mu_{\mathbf{j}}|}{\omega^2_{\mathbf{k}}\omega^2_{\mathbf{j}}} \tag{5.5}$$

Using the assumption that a is diophantine the numerator is estimated by

$$|\mu_{\mathbf{k}}-\mu_{\mathbf{j}}| = \left|\sum_l a_l(k_l^2 - j_l^2)\right| \ge \frac{\gamma_1}{\left(\sum_l |k_l^2 - j_l^2|\right)^{\bar\tau}} \; ,$$

from the estimate

$$\sum_l |k_l^2 - j_l^2| < C \left\|\mathbf{k}\right\|^2 \left\|\mathbf{j}\right\|^2$$

and the asymptotic of the frequencies one has that the l.h.s. of (5.5) is larger than

$$\frac{C}{\left\|\mathbf{k}\right\|^{2\bar{\tau}+2} \left\|\mathbf{j}\right\|^{2\bar{\tau}+2}} ,$$

and therefore, denoting $y_{\mathbf{k}} := \left\|\mathbf{k}\right\|^{-(2\bar{\tau}+2)}$, (5.4) is estimated by

$$\prod_{l=2}^{K}\prod_{N=1}^{l-1} C y_{\mathbf{k}^{(l)}} y_{\mathbf{k}^{(N)}} = C^{\sum_{l=2}^{K}(l-1)} \prod_{l=2}^{K}\left(y_{\mathbf{k}^{(l)}}^{l-1} \prod_{N=1}^{l-1} y_{\mathbf{k}^{(N)}} \right) = C \prod_{l=1}^{K} y_{\mathbf{k}^{(l)}}^{K} ,$$

from which the thesis immediately follows. □

The rest of the proof follows very closely the proof of theorem 6.16 of [16]. We repeat the main steps for completeness.

Lemma 5.2. *(Proposition of Appendix B in [29]) Let $u^{(1)}, ..., u^{(K)}$ be K independent vectors with $\left\|u^{(i)}\right\|_{\ell^1} \leq 1$. Let $w \in \mathbb{R}^K$ be an arbitrary vector, then there exist $i \in [1, ..., K]$, such that*

$$|u^{(i)} \cdot w| \geq \frac{\left\|w\right\|_{\ell^1} \det(u^{(i)})}{K^{3/2}} ,$$

where $\det(u^{(i)})$ is the determinant of the matrix formed by the components of the vectors $u^{(i)}$.

For the proof see [29].

Corollary 5.3. *Let $w \in \mathbb{R}^{\bar{\mathbf{N}}^n}$ be a vector with only K components different from zero, namely those with index $\mathbf{k}^{(1)}, ..., \mathbf{k}^{(K)}$; assume $K \leq N$. Then, for any $m \in \mathcal{I}$ there exists, an index $i \in [0, ..., K-1]$ such that*

$$\left| w \cdot \frac{d^i\omega}{dm^i}(m) \right| \geq C \frac{\left\|w\right\|_{\ell^1}}{\prod_{l=1}^{K} \left\|\mathbf{k}^{(l)}\right\|^{\beta}}$$

where ω is the frequency vector, and $\beta = N(2\bar{\tau}+2)$.

Proof. The proof is obtained by considering the vectors

$$u^{(i)} := \begin{cases} \frac{1}{\left\|\frac{d^i\omega}{dm^i}\right\|}\frac{d^i\omega}{dm^i} & \text{if} \quad \left\|\frac{d^i\omega}{dm^i}\right\| > 1 \\ \frac{d^i\omega}{dm^i} & \text{if} \quad \left\|\frac{d^i\omega}{dm^i}\right\| \leq 1 \end{cases}$$

and applying lemma 5.2. For more details see [16] proof of corollary 6.10. □

Lemma 5.4. *(Lemma 8.4 of [30]) Let $g : \mathcal{I} \to \mathbb{R}$ be N times differentiable, and assume that*

(1) $\forall m \in \mathcal{I}$ there exists $i \leq N-1$ such that $\frac{d^i g}{dm^i}(m) > d$

(2) there exists A such that $|g^{(i)}(m)| \leq A$ $\forall m \in \mathcal{I}$, and $\forall i$ with $1 \leq i \leq N$.

Define

$$\mathcal{I}_h := \{m \in \mathcal{I} \ : \ |g(m)| \leq h\} \ ,$$

then

$$\frac{|\mathcal{I}_h|}{|\mathcal{I}|} \leq \frac{A}{d} 2(2+3+...+r+2d^{-1})h^{1/N}$$

For the proof see [30] and [31].

By combining lemma 5.4 and lemma 5.3 we get the following

Lemma 5.5. *For any choice of $\underline{\mathbf{k}}$ and any acceptable choice σ of the signs, one has*

$$\left|\mathcal{R}^{\sigma}_{\underline{\mathbf{k}}}(\gamma,\tau)\right| \leq |\mathcal{I}| C \frac{\gamma^{1/N}}{\prod_{l=1}^{N} \left\|\mathbf{k}^{(l)}\right\|^{\delta}} \tag{5.6}$$

with $\delta = -2\beta + \tau/N$.

Proof of theorem 2.7. By (5.6) one has

$$\left| \bigcup_{\mathbf{k}^{(1)}\in\bar{\mathbf{N}}^n} \bigcup_{\mathbf{k}^{(2)}\in\bar{\mathbf{N}}^n} \bigcup_{\mathbf{k}^{(N)}\in\bar{\mathbf{N}}^n} \bigcup_{\sigma} \mathcal{R}^{\sigma}_{\underline{\mathbf{k}}} \right| \leq \sum_{\mathbf{k}^{(1)},\ldots,\mathbf{k}^{(N)},\sigma} \left|\mathcal{R}^{\sigma}_{\underline{\mathbf{k}}}\right| \leq C\gamma^{1/N}|\mathcal{I}|2^N \left(\sum_{\mathbf{k}\in\bar{\mathbf{N}}^n} \frac{1}{\|\mathbf{k}\|^{\delta}} \right)^N ,$$

from which, remarking that the condition on τ ensures exactly $\delta > n$, one has that the series is convergent and therefore the thesis. □

Appendix A.

In this appendix we prove theorem 2.2. First of all we introduce scaled variables $\tilde{u}$ by $u = \epsilon\tilde{u}$, so that equation (4.1) is converted into

$$u_{tt} - a^{\epsilon}_{ij}\partial_i\partial_j u + mu = \epsilon^p g_{\epsilon}(u, \nabla u) \tag{A.1}$$

where we omitted tildes, and used the notations

$$a^{\epsilon}_{ij}(u,\nabla u) = a_i\delta_{ij} + \epsilon^p b^{\epsilon}_{ij}(u,\nabla u) \ ;$$

in what follows we will omit the index ϵ for g, a_{ij} and b_{ij}.

As in sect. 4 we consider equation (A.1) in $\mathbf{T}^n$; moreover we will assume that the initial datum has zero average and that the vector field lives invariant the space of functions with zero average (this is ensured by (2.4)). Since we are interested in the case of Dirichlet boundary condition this is not restrictive.

Theorem 2.2 amounts to an estimate of the existence time of the solution. To obtain the wanted estimate we write equation (A.1) as an evolution equation in $X := H^s \times H^{s-1} \ni (u, v)$, namely

$$\partial_t \zeta = A(\zeta)\zeta + f(\zeta)$$

with $\zeta \equiv (u, v)$, $f(\zeta) \equiv (0, \epsilon^p g(u, \nabla u))$ and

$$A(\zeta) := \begin{pmatrix} 0 & \mathbb{1} \\ \mathcal{A}(u) & 0 \end{pmatrix}, \quad \mathcal{A}(u) := a_{ij}(u, \nabla u)\partial_i \partial_j - m$$

Following Kato (see [1]) we will also introduce the space $Y \equiv H^{s+1} \times H^s$.

Moreover, for any fixed $w \in C^0(I; H^{s+1}) \cap C^1(I; H^s)$ with $I := [-T, T]$, we will consider the family of linear operators

$$A^w(t) \equiv \begin{pmatrix} 0 & \mathbb{1} \\ \mathcal{A}(w(t)) & 0 \end{pmatrix}$$

and the linear time dependent equation

$$\dot{\zeta} = A^w(t)\zeta \tag{A.2}$$

in X; it is well known that the corresponding evolution operator $U(t, s)$ exists*. We recall that if one denotes by M, β the stability constants of the operator family $A^w(t)$ (for the definition see e.g. [1]) then

$$\sup_{0 \le t \le s \le T} \|U(t, s)\|_{X \to X} \le M e^{\beta T} .$$

Lemma A.1. *Fix $M_1 > 0$ and, for any $T > 0$ and any function $w \in C^0(I; H^{s+1}) \cap C^1(I; H^s)$ fulfilling*

$$\sup_{t \in I} \|(w(t), \dot{w}(t))\|_Y \le M_1 \tag{A.3}$$

consider the linear differential equation (A.2). Assume that the corresponding evolution operator fulfills

$$\sup_{0 \le s \le t \le T} \|U(t, s)\|_{Y,Y} \le M e^{C\epsilon^p T} \tag{A.4}$$

with suitable constants independent of ϵ and T. Then there exists C such that for any initial datum fulfilling

$$\|(u_0, v_0)\|_Y \le 1/C$$

the existence time for the corresponding solution of (A.1) is larger than $1/(C\epsilon^p)$ and moreover one has the estimate

$$\sup_{0 \le t \le 1/C\epsilon^p} \|(u(t), \dot{u}(t))\|_Y \le M_1 \tag{A.5}$$

* We recall that this is defined by the property that the solution of (A.2) fulfilling $\zeta(s) = \zeta_0$ is given $\zeta(t) = U(t, s)\zeta_0$.

Proof. Let u be the solution of the Cauchy problem for (A.1), then by standard continuation argument it can be continued at least until

$$\|(u(t), \dot{u}(t))\|_Y < M_1 \tag{A.6}$$

holds. Let $\bar{T}$ be the first time at which (A.6) is violated, then one has $\left\|(u(\bar{T}), \dot{u}(\bar{T}))\right\|_Y = M_1$. So, $\zeta(t) := (u(t), \dot{u}(t))$ fulfills the "linear" equation

$$\partial_t \zeta = A^u(t)\zeta + f^u(t), \quad 0 \le t \le \bar{T} \tag{A.7}$$

where, obviously $f^u(t) := (0, \epsilon^p g(u(t), \nabla u(t)))$, and the estimate (A.4) holds until time $\bar{T}$. By theorem 2 of [1] (which follows from the formula of variation of constants) the solution of (A.7) satisfies the estimate

$$M_1 = \left\|\zeta(\bar{T})\right\|_Y \le M e^{C\epsilon^p \bar{T}} \left(\|\zeta_0\|_Y + C\epsilon^p \bar{T}\right)$$

which, provided $\|\zeta_0\|_Y$ is small enough, implies $\bar{T} > 1/C\epsilon^p$ with a suitable C. □

So the problem is reduced to the estimate the stability constants of the family of linear operators $A^w(t)$ in Y.

We fix an arbitrary $w \in C^0(I; H^{s+1}) \cap C^1(I; H^s)$. We begin by estimating the stability constants of the operator family $A^w(t)$ in X. To this end it is useful to make reference to the the second order equation

$$\ddot{u} = \mathcal{A}^w(t)u \tag{A.8}$$

where $\mathcal{A}^w(t) := a_{ij}(w(t), \nabla w(t))\partial_i\partial_j$. We remark that the corresponding first order equation is (A.2), and that, since $w(t)$ is fixed with the smoothness properties (2.9) and fulfills the estimates (A.3) one has $b_{ij} \in C^0(I; H^s) \cap C^1(I; H^{s-1})$ with bounded norms. The proof will be based on proposition 9.3 of [2], so first of all we recall its statement.

Proposition A.2. *[Proposition 9.3 of [2]]. Assume that:*
(i'a) H and H' have structures of real Hilbert spaces, with inner products $\langle\ .\ \rangle_t$ and $\langle\ .\ \rangle'_t$ depending on $t \in I$, such that the norms induced by them are equivalent to the standard norms $\|\ \|_H$ and $\|\ \|_{H'}$, respectively. These inner products are Lipschitz continuous in t, in the sense that $|\langle u, v\rangle_t - \langle u, v\rangle_s| \le c|t - s|\, \|u\|_H \|v\|_H$, and similarly for $\langle\ .\ \rangle'_t$.
(i'b) $\{\mathcal{A}(t)\}$ is a family of (unbounded) linear operators in H, such that $\mathcal{A}(t) = \mathcal{A}_S(t) + \mathcal{A}_R(t)$, where $\mathcal{A}_S(t)$ is selfadjoint and uniformly positive definite with respect to inner product $\langle\ .\ \rangle_t$, while $\mathcal{A}_R(t)$ is uniformly bounded (with bound K, say) from H' to H. Moreover $D(\mathcal{A}_S(t)) \subset H'$ and

$$\langle \mathcal{A}_S(t)u, v\rangle_t = \langle u, v\rangle'_t \qquad (u \in D(\mathcal{A}_S(t)), v \in H')\ . \tag{A.9}$$

Then the family

$$A(t) := \begin{pmatrix} 0 & \mathbb{1} \\ \mathcal{A}(t) & 0 \end{pmatrix}$$

is a stable family in $H' \oplus H$ with stability constants M, $\beta := 2c + K$, where M depends on the relation between the equivalent norms $\| \ \|_H$ and $\langle \ , \ \rangle_0^{1/2}$.

Lemma A.3. *The evolution operator of the equation (A.2) fulfills the estimate*

$$\sup_{0 \le s \le t \le T} \|U(s,t)\|_{X,X} \le M \mathrm{e}^{C\epsilon^p T}$$

with constants M, C independent of T and ϵ.

Proof. We define the spaces $H := H^{s-1}$, and $H' := H^s$. The standard scalar product in H^{s-1} is defined by

$$\langle u, v\rangle_H \equiv \langle u, v\rangle_{H^{s-1}} \equiv \int (D^{s-1}u)(D^{s-1}v) \ , \quad D := (-\Delta)^{1/2}$$

and the integral is taken over the whole torus. The standard scalar product in H' is defined in the same way with s in plase of $s-1$. Define the scalar product $\langle \ , \ \rangle_t := \langle \ , \ \rangle_H$ (independent of t), and the operator $\mathcal{A}_S$

$$\mathcal{A}_S u := -D^{-(s-1)} \left(\partial_j \left(a_{jk} \partial_k (D^{s-1}u) \right) \right) + mu \ .$$

Then $\langle \ , \ \rangle'_t$ is defined by (A.9). It is easy to see that the operator $\mathcal{A}_S$ is uniformly elliptic (just use positivity of a_{ij}). We will study the relation of this operator with $\tilde{\mathcal{A}} := \partial_j(a_{jk}\partial_k) + m$, who in turn is related to $\mathcal{A}^w(t)$ by

$$\tilde{\mathcal{A}} u = \mathcal{A}u + (\partial_j a_{jk})(\partial_k u) \ .$$

Remark that the second term defines an operator whose norm, as an operator from H' to H, is bounded by a constant time ϵ^p.

We now compare the operators $\tilde{\mathcal{A}}$ and $\mathcal{A}_S$. For simplicity we restrict to the simplest case where $s - 1 = 2r$, i.e. s is odd*. One has

$$\mathcal{A}_S = \tilde{\mathcal{A}} + D^{-(s-1)} \left[\tilde{\mathcal{A}}, D^{s-1} \right] = \tilde{\mathcal{A}} + D^{-(s-1)} \partial_j [a_{jk}\partial_k, D^{s-1}] \ .$$

So, it is enough to compute $[a_{jk}\partial_k, (\partial_l \partial_l)^r]u$ (summation convention!). To this end consider

$$(\partial_l \partial_l)^r \left(a_{jk} \partial_k u \right) = \partial_{l_1} \partial_{l_1} ... \partial_{l_r} \partial_{l_r} \left(a_{jk} \partial_k u \right)$$

which is the sum of terms of the form

$$c_\alpha \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} a_{jk} \right) \left(\frac{\partial^{|\beta|}}{\partial x^\beta} u \right)$$

* The general case can be dealt with using the operators S_i defined in the proof of lemma A.4

where α and β are multindices fulfilling $|\alpha|+|\beta| = 2r$. The only term with $|\beta| = 2r$ is the one where the Laplacians act entirely on u so it disappears when computing the commutator. It is easy to see that all the other terms define operators which are bounded as operators from H' to H, provided $s > [n/2]$. And moreover the norms of such operators are bounded by a constant times ϵ^p. Applying the remaining two operators (namely $D^{-(s-1)}$ and ∂_j) one has that

$$\mathcal{A} = \mathcal{A}_S + \mathcal{A}_R \ ,$$

with $\mathcal{A}_R : H' \to H$ having norm bounded by a constant times ϵ^p. Finally, it is very easy to see that

$$|\, \langle u, v\rangle_t' - \langle u, v\rangle_s' \,| \le C\epsilon^p \, \|u\|_{H^s} \, \|v\|_{H^s} \, |t - s| \ .$$

So, proposition A.2 gives the result. □

To estimate the linear flow in Y we construct an isomorphism $S : Y \to X$ such that $SA^wS^{-1} = A^w + \epsilon^p B$ with an operator $B : X \to X$ uniformly bounded.

We define

$$S := \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \qquad (A.10) \ .$$

Lemma A.4. *One has*

$$SA^wS^{-1} = A^w + \epsilon^p B$$

with B uniformly bounded as an operator from X to X

Proof. One has

$$SA^wS^{-1} = \begin{pmatrix} 0 & \mathbb{1} \\ D\mathcal{A}^wD^{-1} & 0 \end{pmatrix} \ ,$$

so it is enough to prove that

$$D\mathcal{A}^wD^{-1} = \mathcal{A}^w + \epsilon^p \mathcal{B}$$

with $\mathcal{B} : H^s \to H^{s-1}$ uniformly bounded. To prove this fact remark that

$$\epsilon^p \mathcal{B} = [D, \mathcal{A}^w]D^{-1} = [D, \epsilon^p b_{ij}\partial_i\partial_j]D^{-1}$$

so the result is equivalent $[D, b_{ij}\partial_i\partial_j] : H^{s+1} \to H^{s-1}$ continuously. But $[D, b_{ij}\partial_i\partial_j]u = [D, b_{ij}]\partial_i\partial_j u$, so it is enough the prove that $\forall b \in H^s$ one has $[D, b] : H^{s-1} \to H^{s-1}$ continuously. To prove this fact we compute the commutator in the Fourier space. Denoting by a "hat" the (space) Fourier coefficients of a function one has

$$\begin{aligned} \{[D, b]u\}^{\wedge}(\mathbf{k}) &= \|\mathbf{k}\| \sum_{\mathbf{l}} \hat{b}_{\mathbf{k}-\mathbf{l}} \hat{u}_{\mathbf{l}} - \sum_{\mathbf{l}} \hat{b}_{\mathbf{k}-\mathbf{l}} \|\mathbf{l}\| \, \hat{u}_{\mathbf{l}} \\ &= \sum_{\mathbf{l}} \hat{b}_{\mathbf{k}-\mathbf{l}} \hat{u}_{\mathbf{l}} \left(\|\mathbf{k}\| - \|\mathbf{l}\|\right) \end{aligned}$$

The last brackett can be rewritten in the form

$$\frac{k_1^2 + ... + k_n^2 - l_1^2 - ... - l_n^2}{\|\mathbf{k}\| + \|\mathbf{l}\|} = (k_1 - l_1)\frac{k_1 + l_1}{\|\mathbf{k}\| + \|\mathbf{l}\|} + ... + (k_n - l_n)\frac{k_n + l_n}{\|\mathbf{k}\| + \|\mathbf{l}\|} \ .$$

Define now the bilinear operators S_i by

$$\left\{S_i(b,u)\right\}^\wedge(\mathbf{k}) := \sum_{\mathbf{l}} \hat{b}_{\mathbf{k}+\mathbf{l}}\hat{u}_{\mathbf{l}}\frac{k_i+l_i}{\|\mathbf{k}\|+\|\mathbf{l}\|} \ ,$$

then one has

$$[D,b]u = \sum_i S_i(\partial_i b, u) \ .$$

We prove now that $S_i : H^{s-1}\times H^{s-1}\to H^{s-1}$ continuously, and from this the thesis will immediately follow. Since S_i is bilinear it is enough to prove that

$$\left\|S_i(v,u)\right\|_{H^{s-1}} \leq C\left\|v\right\|_{H^{s-1}}\left\|u\right\|_{H^{s-1}} \ .$$

One has

$$\begin{aligned}
\left\|S_i(v,u)\right\|^2_{H^{s-1}} &= \sum_{\mathbf{k}}\|\mathbf{k}\|^{2(s-1)}\left(\sum_{\mathbf{l}}\hat{v}_{\mathbf{k}-\mathbf{l}}\hat{u}_{\mathbf{l}}\frac{k_i+l_i}{\|\mathbf{k}\|+\|\mathbf{l}\|}\right)^2 \\
&\leq \sum_{\mathbf{k}}\|\mathbf{k}\|^{2(s-1)}\left(\sum_{\mathbf{l}}\left|\hat{v}_{\mathbf{k}-\mathbf{l}}\hat{u}_{\mathbf{l}}\right|\right)^2\left[\sup_{\mathbf{k},\mathbf{l}}\left|\frac{k_i+l_i}{\|\mathbf{k}\|+\|\mathbf{l}\|}\right|\right]^2 \qquad (A.11)\\
&= C\sum_{\mathbf{k}}\|\mathbf{k}\|^{2(s-1)}\left(\sum_{\mathbf{l}}\left|\hat{v}_{\mathbf{k}-\mathbf{l}}\right|\left|\hat{u}_{\mathbf{l}}\right|\right)^2 \ .
\end{aligned}$$

Define now $u^+(\mathbf{x}) := \sum_{\mathbf{k}}|\hat{u}_{\mathbf{k}}|e^{i\mathbf{k}\cdot\mathbf{x}}$, and similarly for v^+. Then one has $\left\|u^+\right\|_{H^{s-1}} = \left\|u\right\|_{H^{s-1}}$, and therefore the square root of the last line of (A.11) is given by

$$C\left\|u^+v^+\right\|_{H^{s-1}} \leq C\left\|u^+\right\|_{H^{s-1}}\left\|v^+\right\|_{H^{s-1}} = C\left\|u\right\|_{H^{s-1}}\left\|v\right\|_{H^{s-1}}$$

(with a different C), which is the thesis. □

Then the estimate (3.3) of [1] gives

Corollary A.5. *The evolution operator of equation (A.2) fulfills the estimate*

$$\sup_{0\leq s\leq t\leq T}\left\|U(s,t)\right\|_{Y,Y} \leq M e^{C\epsilon^p T}$$

with constants M, C independent of T and ϵ.

¿From this, recalling lemma A.1, theorem 2.2 immediately follows.

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