PRECISE ASYMPTOTICS OF SMALL EIGENVALUES OF REVERSIBLE DIFFUSIONS IN THE METASTABLE REGIME

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Abstract: We investigate the connection between metastability and the spectrum near zero corresponding to the elliptic, second-order, differential operator

$$L_\epsilon \equiv -\epsilon\Delta + \nabla F \cdot \nabla, \qquad \epsilon > 0$$

with $F : \mathbb{R}^d \to \mathbb{R}$. For generic F to each local minimum of F there corresponds an eigenvalue of L_ϵ which converges to zero exponentially fast as $\epsilon \downarrow 0$. Modulo errors of exponentially small order in ϵ this eigenvalue is given as the inverse of the expected metastable relaxation time. The corresponding eigenstate, which may be viewed as a metastable state, is highly concentrated in the basin of attraction of this trap.

Keywords: eigenvalue problem, diffusion process, metastability, relaxation time, potential theory

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1. Introduction

We address in this work the problem of characterizing - in terms of potential theoretic quantities - the low lying spectrum of the following second-order, elliptic differential operator

$$L_\epsilon \equiv -\epsilon\Delta + \nabla F \cdot \nabla, \qquad \epsilon > 0 \tag{1.1}$$

where the precise conditions on $F : \mathbb{R}^d \to \mathbb{R}$ are given in 1.2. We continue the work started in [BEGK3] and generalize the analysis of [BEGK2] from the discrete to the continuous state space setting. To each local attractor x of ∇F there corresponds a simple eigenvalue λ_x of L_ϵ which is exponentially small in ϵ. Modulo in ϵ exponentially small errors this eigenvalue equals the inverse of the expected time of the diffusion generated by L_ϵ of a transition from the basin of attraction to the more stable attractors. With the same precision an eigenfunction corresponding to λ_x is constant in the basin of attraction of x and exponentially small in those basins which correspond to attractors y satisfying $\lambda_y < \lambda_x$. The results obtained in [BEGK3] then yield in terms of F the leading order asymptotic behavior of these eigenvalues. Moreover, below some threshold of order ϵ^N no other eigenvalues occur.

Metastability in random dynamical systems is an intensively studied phenomenon. A Markov process in the metastable regime, roughly speaking, exhibits quasi-invariant sets of the state space, which may be viewed as metastable states, in which the process is captured for long time periods. For systems with discrete state space in this regime in [GM] and [GS] as well as [KMST] and [CGOV] the authors study different aspects in this area. Concerning systems with continuous state space in this regime, we refer the reader to [FW] and [W2], where the authors develop a large deviation technique for diffusion processes to study spectral and dynamical properties. From the point of view of asymptotic expansions in the small parameter $\epsilon > 0$ we mention [FJ] and [KMST]. In [GS], [D1-3] and in [M], where the full description of the low lying spectrum is accomplished, the authors investigate properties of the spectrum of the generator of the dynamical system that are connected to metastability. Unfortunately, these approaches encounter the following shortcomings. Generally speaking, rigorous asymptotic expansions, though giving sharp error estimates, suffer from strong regularity assumptions. On the other hand L^2 - methods as applied in [M] and [D1-3] as well as large deviation theory lead to rough error estimates. In [BEGK2] and in [E2] we establish the characterization of the low lying spectrum in the context of Markov chains in the metastable regime. A key idea of [BEGK2] and [E1,2] for irreversible chains is to analyze metastability from the dynamical or from the spectral point of view by potential theoretic methods. In this way we considerably improve the range of applicability as well as the quality of the error estimates. In [E1,2] this aspect is particularly emphasized. Here we concentrate on the main technical complications which do not exist in systems with finite, discrete state space.

The technical tool to connect spectral to potential theory already appears in [W1] or in [Pi], relying on work of [Z], and was rediscovered in [BEGK2]. [W1] contains a description of the spectrum in terms of the underlying Markovian process while in [Z] and [Pi] the analytical counterpart is used to investigate criticality of elliptic operators. This characterization is far more transparent for processes with discrete state space as is demonstrated in [BEGK2] and in [E1,2]. The fact that a point in discrete space can be visited by the process with strictly positive probability, i.e. has strictly positive capacity, might be seen as a main reason for this difference. In continuous state spaces small balls are the equivalent of points in discrete spaces. This choice entails the disadvantage that a function a priori may change its sign on a small ball. Using level sets of functions instead of balls, one quickly runs into technical complications whose solutions go beyond the questions we are addressing. The approach presented in the previously mentioned references naturally requires to work in the L^∞ - context. We hence follow the strategy to first establish rather strong pointwise L^∞ - estimates. Compared to [BEGK2] and [M] a second complication arises from the fact that the state space is non-compact. A treatment of the analogous problem concerning Markov chains can be found in [E2]. There is a well established L^2 - theory of weighted estimates of solutions of second-order elliptic differential equations as developed in [A] or in [HS1-2] involving a small parameter from which pointwise bounds on solutions also can be obtained. We show that the methods introduced in [BEGK3] already suffice to prove the same kind of estimates.

The a priori input enables us to relate small eigenvalues of L_ϵ to the capacity matrix as introduced in [J]. For generic F the analysis of this matrix then is a straightforward generalization of that in [BEGK2] and [E2] for Markov chains. Let us mention that this matrix representation also can be used to treat the degenerate situation, where small eigenvalues are (asymptotically) not simple. It turns out that to each small eigenvalue there corresponds a quasi-invariant set and a time scale, which roughly speaking equals the expected time the process generated by L_ϵ is captured this set. These time scales are defined in terms of capacities and the invariant measure of the process. As is shown in [BEGK1] and [E2] in the discrete state space setting they determine the long time behavior of the process in a precise manner. These kind of results were extended in [BEGK3] to the diffusion process generated by L_ϵ. They will serve as a crucial tool to investigate small eigenvalues of L_ϵ.

We now recall the main potential theoretic background. Fix disjoint, nonempty sets $A, B \subset \mathbb{R}^d$ such that their boundaries are piecewise, continuously differentiable. A set with the latter property henceforth will be referred to as a *regular* set. The λ - capacity of the capacitor (A, B) is given by

$$\mathrm{cap}_A^\lambda(B) \equiv \epsilon \int_{\partial A} e^{-F/\epsilon} \partial_n h_{A,B}^\lambda \, d\sigma - \lambda \int_A e^{-F/\epsilon} \, dy \tag{1.2}$$

where n is the unit normal at $\partial A \cup \partial B$ pointing towards $A \cup B$. Here $h_{A,B}^\lambda$ denotes the electrostatic equilibrium potential of the capacitor, i.e. the solution of the Dirichlet problem

$$(L_\epsilon - \lambda) h(x) = g(x) \quad x \in \Gamma, \qquad h(x) = f(x) \quad x \in \partial\Gamma \tag{1.3}$$

where $\Gamma \equiv \mathbb{R}^d \backslash A \backslash B$, $g \equiv 0$ and $f \equiv \mathbb{1}_A$. We shall use the convention that $h_{A,B}^\lambda$ takes the value one on A and zero on B. To functions h satisfying (1.3) for some f and $g \equiv 0$ we sometimes refer to as λ-*harmonic* functions. The communication height between sets A and B is defined by

$$\hat{F}(A, B) \equiv \inf_{\substack{c:[0,1]\to\mathbb{R}^d \\ c(0)\in A, c(1)\in B}} \max F(c[0,1]) \tag{1.4}$$

where the infimum is taken over all continuous curves. Furthermore, for a finite set of points $I \cup x$ such that $B_{I\cup x}$ is a disjoint union of open balls, where

$$B_J \equiv \bigcup_{y\in J} B(y, \epsilon/4), \qquad J \subset \mathbb{R}^d \tag{1.5}$$

we introduce

$$A_{x,I} \equiv \{ y \in \mathbb{R}^d \mid \hat{F}(y, x) < \hat{F}(y, I\backslash x) \} \tag{1.6}$$

In analogy to [E2] we define the time scales

$$T_{x,I} \equiv \frac{\int_{A_{x,I}} e^{-F/\epsilon} \, dy}{\mathrm{cap}_{B_x}^0 (B_{I\backslash x})} \tag{1.7}$$

We recall from Theorem [3.1 in BEGK3] the classical Eyring formula for the capacity.

Theorem 1.1: *Fix regular, disjoint, nonempty sets A and B. Assume that there is only one solution of $F(z^*) = \hat{F}(A, B) > \max F(A \cup B) + R\epsilon \log(1/\epsilon)$ such that z^* is a critical point of F. If the Hessian at z^* of F is nondegenerate, then for some R*

$$\mathrm{cap}_A^0(B) = (1 + \mathcal{O}(1)\epsilon \log(1/\epsilon)) \frac{(2\pi)^{d/2-1} |\lambda^*|}{\sqrt{|\det \mathrm{Hess}\, F(z^*)|}} \epsilon^{d/2} e^{-\hat{F}(A,B)/\epsilon} \tag{1.8}$$

where λ^* is the unique, negative eigenvalue of the Hessian at z^*. The modulus of the Landau symbol is dominated by a constant $C \equiv C(d,F)$.

Let $\mathcal{M}$ denote the set of *local minima* of F. For $x \in \mathcal{M}$ and $I \subset \mathcal{M} \backslash x$ nonempty with nondegenerate Hessian at x we obtain from (1.8) that the time scale introduced above satisfies

$$T_{x,I} = (1 + \mathcal{O}(1)\epsilon \log(1/\epsilon)) \sqrt{\frac{|\det \mathrm{Hess}\, F(z^*)|}{|\lambda^*| \det \mathrm{Hess}\, F(x)}} e^{(\hat{F}(x,I) - F(x))/\epsilon} \tag{1.9}$$

Let us remark that the conditions under which (1.9) holds are not at all borderline of our approach. In fact, the only condition we need is that either $T_{x,I\backslash x} >> T_{y,I\backslash y}$ or $T_{x,I\backslash x} << T_{y,I\backslash y}$, where $x, y \in \mathcal{M}$, $x \neq y$, $I \subset \mathcal{M}$. In particular, as long as this condition is valid we can deal with all kinds of degeneracies of F in the relevant regions. This just leads to different asymptotic behaviors of $T_{x,I}$. We refer the reader to [E1,2] where in the context of Markov chains precise, minimal conditions on the time scales are given. As we only focus on the new technical complications in the continuous state space setting we do not aim at the most general conditions under which the analysis works.

Let us now describe the main results of this paper. We have to introduce some more notation. Let us define for $y \in \mathbb{R}^d$

$$\delta(y) \equiv \delta_{F,\epsilon}(y) \equiv \sup\big\{\delta > 0 \,|\, 8\delta \sup_{B(y,8\epsilon\delta)} |\nabla F| < 1\big\} \tag{1.10}$$

For a subset $\Sigma \subset \mathbb{R}^d$ and a function $r : \mathbb{R}^d \to (0,\infty)$ we introduce its r - neighborhood by

$$\Sigma^r \equiv \{x \in \mathbb{R}^d \mid \mathrm{dist}(x,\Sigma) < r(x)\} \tag{1.11}$$

In the sequel we impose the following conditions on F.

Assumption 1.2: *$F : \mathbb{R}^d \to \mathbb{R}$ is continuously differentiable. There are constants $c > 0$ and C_0 satisfying $\inf_{\{F > C_0\}} |\nabla F| \geq c$. There are $\epsilon_0 > 0$ and C_1 such that*

$$\frac{\sup_{c^{4\epsilon\delta}} |\nabla F|}{\inf_{c(0,l)} |\nabla F|} < \frac{C_1 l}{|c(l) - c(0)|} \tag{1.12}$$

for all $0 < \epsilon < \epsilon_0$ and every flow line $c : [0,l] \to \mathbb{R}^d$ of F of euclidean length l satisfying $F(c(l)), F(c(0)) > C_0$ and $|c(l) - c(0)| \leq 4\epsilon\delta(x)$, where $\delta : \mathbb{R}^d \to (0,\infty)$ is defined in (1.10), $c^{4\epsilon\delta}$ is defined in (1.11) and where $x = (1/2)c(0) + (1/2)c(l)$.

The condition $\inf_{\{F > C_0\}} |\nabla F| \geq c$ can be relaxed to the existence of a subset $\tilde{\mathcal{M}}$ of the set of local minima $\mathcal{M}$ of F with the property that $\min_{x \in \tilde{\mathcal{M}}} T_{x,\mathcal{M}\backslash\tilde{\mathcal{M}}} << \sup_{y \in \mathcal{M}\backslash\tilde{\mathcal{M}}} T_{x,\tilde{\mathcal{M}}}$.

In particular, F may have infinite many local minima where the minima in $\mathcal{M}\backslash\tilde{\mathcal{M}}$ are not as "deep" as those in $\tilde{\mathcal{M}}$.

The first result, stated in 4.2 and referred to as the sharp uncertainty principle, is strikingly reminiscent of the uncertainty priniciple in quantum mechanics. We recall that the tunneling time of a quantum mechanical particle moving in a double well potential approximately is given by the inverse of the spectral gap. Let $\lambda(\Omega)$ be the principal eigenvalue of the Dirichlet operator L_ϵ^Ω with zero boundary conditions on $\mathbb{R}^d\backslash\Omega$, where Ω is an open, regular set.

Theorem 1.3: *Assume that F satisfies 1.2. Then there exists $N \equiv N(d)$ such that for all $\rho > N\epsilon\log(1/\epsilon)$ satisfying $T_{x,I} = T_I \equiv \max_{y\in\mathcal{M}\backslash I} T_{y,I} \geq e^{-\rho/\epsilon}T_{I\cup x}$ it follows*

$$\lambda(\mathbb{R}^d\backslash B_I) = (1+\mathcal{O}(1)\epsilon^{-N}e^{-\rho/\epsilon})\frac{1}{T_I} = (1+\mathcal{O}(1)\epsilon^{-N}e^{-\rho/\epsilon})\frac{1}{\mathbb{E}^x[\tau_{B_I}]} \tag{1.13}$$

where $\mathbb{E}^x[\tau_{B_I}]$ is the expected time to reach B_I starting from x of the diffusion generated by L_ϵ. Here the modulus of the Landau symbol is dominated by a constant $C \equiv C(d,N,F)$.

We are also able to analyze all other exponentially small eigenvalues and relate them to the metastable structure given by F. Let us define $\rho \equiv \rho(F,\epsilon)$ by

$$e^{\rho/\epsilon} \equiv \max\left\{ \frac{T_{x,I\backslash x}}{T_{y,I\backslash y}} \,\middle|\, x,y\in\mathcal{M}, x\neq y, I\subset\mathcal{M}\right\} \tag{1.14}$$

We then have (see 4.6 for a more detailed version)

Theorem 1.4: *Assume that F satisfies 1.2. There exists $N \equiv N(d)$ and $C \equiv C(d,F,N)$ such that for all $\rho > N\epsilon\log(1/\epsilon)$ the following holds.*

(i) For every $x\in\mathcal{M}$ there exists a simple eigenvalue λ_x of L_ϵ such that

$$\lambda_x = (1+\mathcal{O}(1)\epsilon^{-N}e^{-\rho/\epsilon})\lambda(\mathbb{R}^d\backslash B_{\mathcal{M}(x)}) \tag{1.15}$$

where $\mathcal{M}(x) \equiv \{y\in\mathcal{M} \,|\, F(y) < F(x)\}$.

(ii) Let $\mathcal{M}_x \equiv \{y\in\mathcal{M} \,|\, \lambda_y < \lambda_x\}$. For every eigenfunction ϕ_x corresponding to λ_x there exists a set $\tilde{\mathcal{M}}_x$ such that $B(y,\sqrt{\epsilon})\cap\tilde{\mathcal{M}}_x$ is a singleton such that for all $z\in\{F<C_0\}$

$$\phi_x(z) = (1+\mathcal{O}(1)\epsilon^{-N}e^{-\rho/\epsilon})h^0_{B_x,B_{\tilde{\mathcal{M}}_x}}(y) + \mathcal{O}(1)\epsilon^{-N}e^{-\rho/\epsilon}h^0_{B_{\tilde{\mathcal{M}}_x},\cup x}(z) \tag{1.16}$$

Here the Landau symbols are bounded by C in absolute value.

(iii)

$$\sigma(L_\epsilon) \cap [0, \epsilon^N) = \{\lambda_x \,|\, x \in \mathcal{M}\} \tag{1.17}$$

(1.15) in combination with (1.13) relates exponentially small eigenvalues to the metastable structure of the diffusion generated by L_ϵ. Furthermore, under the conditions required for (1.9) we have determined the leading asymptotic in (1.13).

Let us finally describe the organization of the paper. Using Harnack and Hölder type estimates, in chapter two we derive analogous estimates for a priori non-positive harmonic functions. As a result we gain in Lemma 2.3 pointwise control on the oscillation of eigenfunctions corresponding to small eigenvalues in terms of suprema over suitable small balls close to the local minima of F. In chapter three we prove bounds of those suprema by exploiting the strong drift of the diffusion towards local minima of F. The a priori input then gives precise control of eigenfunctions in compact sets. As soon as we have established this structural information we are in a position to generalize the analysis developed in the discrete state space setting to the diffusion setting. In particular, in chapter four we relate the low lying spectrum to the capacity matrix as introduced in [J] and derive the asymptotic information in terms of the time scales introduced in (1.7).

2. Regularity of λ-harmonic functions changing sign

This chapter is devoted to the following simple idea. A solution ϕ of the eigenvalue problem

$$(L_\epsilon - \lambda)\phi(x) = 0 \quad x \in \Omega, \qquad \phi(x) = 0 \quad x \in \partial\Omega \tag{2.1}$$

in a regular, open set Ω with small energy λ cannot create large oscillations everywhere in a region where F is small. We start with

A priori bounds on principal eigenvalues: On several occasions we shall meet the following obvious representation formula. The Poisson-Dirichlet problem (1.3) for a regular, open set Γ in a compactly contained, regular, open set $\Sigma \subset\subset \Gamma$ is given by

$$h(x) = G^\lambda_\Sigma g(x) + H^\lambda_\Sigma h(x) \quad x \in \Sigma \tag{2.2}$$

where $H^\lambda_\Sigma f$ is the $(L_\epsilon - \lambda)$ - harmonic extension of h to Σ and where

$$G^\lambda_\Sigma \equiv (L^\Sigma_\epsilon - \lambda)^{-1} \tag{2.3}$$

is the resolvent of the operator L^Σ_ϵ with Dirichlet conditions on $\partial\Sigma$. Recall the definition of the λ - equilibrium potential $h^\lambda_{A,B}$, A, B closed and regular, $\lambda \geq 0$, introduced in (1.3). Furthermore, let $w^\lambda_{A,B}$ be the solution of the Poisson's problem (1.3) with $f \equiv 0$ and $g \equiv h^\lambda_{A,B}$. We also shall use the convention $h^\lambda_A \equiv h^\lambda_{A,A}$ and $w^\lambda_A \equiv w^\lambda_{A,A}$. Define for $K \subset (A \cup B)^c$

$$s^\lambda_K(A, B) \equiv \sup_K \frac{w^\lambda_{A,B}}{h^\lambda_{A,B}} \tag{2.4}$$

We abbreviate

$$s^\lambda(A, B) \equiv s^\lambda_{(A\cup B)^c}(A, B), \qquad s^\lambda_K(A) \equiv s^\lambda_K(A, B \equiv A), \qquad s^\lambda_{A^c} \equiv s^\lambda_{A^c}(A, B \equiv A) \tag{2.5}$$

and denote the principal eigenvalue of the Dirichlet operator L^Σ_ϵ by

$$\lambda(\Sigma) \equiv \inf \sigma(L^\Sigma_\epsilon) \tag{2.6}$$

We refer to the following lower bound on the principal eigenvalue as the uncertainty principle.

Lemma 2.1: *Let Σ be a bounded, regular, open set. Then for all regular, closed sets A, B, such that $A \cup B = \Sigma^c$ it follows*

$$\lambda(\Sigma) \geq \frac{1}{s^0(A, B)} \tag{2.7}$$

Proof: The following variational characterization of Donsker and Varadhan is well-known, see [DV] or Theorem [7.1 in P].

$$\lambda(\Sigma) = \inf_{\substack{f \in C^1(\overline{\Sigma}) \\ f|\partial\Sigma = 0, \int_\Sigma f^2 = 1}} \sup_{\substack{u \in C^2(\overline{\Sigma}) \\ u(x) > 0, x \in \Sigma}} \int_\Sigma \frac{L_\epsilon u(x)}{u(x)} f(x)^2 dx \tag{2.8}$$

To obtain (2.7), we simply insert $u \equiv w^0_{A,B}$ and use $L_\epsilon u = h^0_{A,B}$ on Σ. $\diamondsuit$

From the well-known Min-Max Principle we obtain the following as we shall see in 4.2 sharp upper bound.

Lemma 2.2: *Let Σ be a regular, open set such that* $\mathrm{dist}(\mathcal{M} \cap \Sigma, \partial\Sigma) > \rho$ *for some* $\rho > 0$. *Then for some $C \equiv C(d, F|\Sigma, \rho)$ and all $x \in \mathcal{M} \cap \Sigma$*

$$\lambda(\Sigma) \leq (1 + Ce^{-\beta/\epsilon}/\epsilon) \frac{\mathrm{cap}^0_{B(x,\epsilon)}(\Sigma^c)}{\int_{A^\beta_{x,\Sigma^c}} e^{-F/\epsilon}\, dx} \tag{2.9}$$

where we have defined $A^\beta_{x,\Sigma^c} \equiv \{y \in \Sigma \,|\, \hat{F}(y,x)) < \hat{F}(y,\Sigma^c) - \beta\}$ for all $\beta > 0$ such that $B(x,\rho/2) \subset A$. Here $\hat{F}$ denotes the communication height introduced in (1.4).

Proof: Insert $u \equiv h^0_{B(x,\epsilon),\Sigma^c}$ with $x \in \mathcal{M} \cap \Sigma$ and by convention $h^0_{B(x,\epsilon),\Sigma^c} \equiv 1$ on $B(x,\epsilon)$ into

$$\lambda(\Sigma) = \inf_{u \in W^{1,2}_0(\Sigma) \setminus 0} \frac{\int_\Sigma e^{-F/\epsilon} |\nabla u|^2\, dx}{\int_\Sigma e^{-F/\epsilon} |u|^2\, dx} \tag{2.10}$$

to obtain

$$\lambda(\Sigma) \leq \frac{\mathrm{cap}^0_{B(x,\epsilon)}(\Sigma^c)}{\int_{A^\beta_{x,\Sigma^c}} e^{-F/\epsilon} (h^0_{B(x,\epsilon),\Sigma^c})^2\, dx} \tag{2.11}$$

Invoking Corollary [4.8 in BEGK3], we derive $h^0_{\Sigma^c, B(x,\epsilon)}(y) \leq Ce^{-\beta/\epsilon}/\epsilon$ for some $C \equiv C(d, F|A^\beta_{x,\Sigma^c})$ and all $y \in A^\beta_{x,\Sigma^c} \setminus B(x, 2\epsilon)$. This estimate in combination with the Maximum Principle shows $h^0_{B(x,\epsilon),\Sigma^c}(y) \geq 1 - Ce^{-\beta/\epsilon}/\epsilon$ for all $y \in A^\beta_{x,\Sigma^c}$. This establishes (2.9). $\diamondsuit$

Uniform regularity estimates for λ-harmonic functions changing sign: We are now in a position to turn the idea mentioned in the beginning of this chapter into

Lemma 2.3: *Let $h \in W^{2,n}(\mathbb{R}^d)$ be a strong solution of the equation $(L_\epsilon - \lambda)h = 0$ in $B(x, 2\sqrt{\epsilon})$, where $x \in \mathcal{M}$ and $0 \leq \lambda \leq \epsilon$. Then there exists $\tilde{x} \in B(x, \sqrt{\epsilon})$ such that h does not change sign in $B(\tilde{x}, \epsilon)$.*

Let Γ, $\Sigma\subset\Gamma$, be a regular, open sets. Let $g\in L^\infty_{\rm loc}(\mathbb{R}^d)$ and let h be a non-negative, strong solution of the equation $(L_\epsilon-\lambda)h=g$ in Γ. Assume that there is $0<r<1/2$ and $B(x,2\sqrt{\epsilon})\subset\Sigma$ such that $(1-r)\sup_\Sigma h\le\sup_{B(x,\epsilon)}h$. Then for all $0\le\lambda<\lambda(\Gamma\backslash B(x,\epsilon))$ there is $C\equiv C(d,F|B(x,2\sqrt{\epsilon}))$ such that

$$\operatorname{osc}_\Sigma h\le\big(4r+C\epsilon^{d/2}\lambda+4\sup_{\Sigma\backslash B(x,\epsilon)}h^\lambda_{\mathbb{R}^d\backslash\Gamma,B(x,\epsilon)}\big)\sup_{B(x,\epsilon)}h+C\epsilon^{d/2}\sup_{B(x,2\sqrt{\epsilon})}|g| \tag{2.12}$$

Having established positivity of eigenfunctions in vicinities of the local minima of F, we may use strong pointwise regularity such as the local (Boundary) Maximum Principle Theorem [9.20 in GT] (Theorem [9.26 in GT]), the Harnack's inequality Theorem [9.22 in GT] and the (Boundary) Hölder's estimates Corollary [9.24 in GT] (Corollary [9.28 in GT]).

Recall the definition (1.10) of the radius $\delta(x)$, $x\in\mathbb{R}^d$. Clearly, $\epsilon\delta$ only depends on $|\nabla F|/\epsilon$ and that $8\epsilon\delta(y)\sup_{B(y,4\epsilon\delta(y))}|\nabla F|/\epsilon=1$. Combination of Harnack's and Hölder's principles gives

Theorem 2.4: *Fix $0<\rho\le\epsilon$. Let $0\le h\in W^{2,n}(\mathbb{R}^d)$ be a strong, non-negative solution of the equation $(L_\epsilon-\lambda)h=0$ in $B(x,2\sqrt{\rho})$, where $x\in\mathcal{M}$ and $0\le\lambda\le1$. Then there exists $C=C(F|B(x,2\sqrt{\rho}))$ and $\alpha=\alpha(F|B(x,2\sqrt{\rho}))>0$ such that for all $0<r<\sqrt{\rho}$*

$$\operatorname{osc}_{B(x,r)}h\le C(r/\sqrt{\rho})^\alpha\inf_{B(x,r)}h \tag{2.13}$$

Assume that $0\le h\in W^{2,n}(\mathbb{R}^d)$ is a strong, non-negative solution of the equation $(L_\epsilon-\lambda)h=f$ in Σ, where Σ is an open, regular set, $0\le\lambda\le1$ and f is in $L^d(\Sigma)$. There are constants $C=C(d)$ and $\alpha=\alpha(d)>0$ such that for all $x\in\Sigma$ and all $0<\rho<\epsilon\delta(x)$ satisfying $B(x,4\rho)\subset\Sigma$ and all $0<r<\rho$

$$\operatorname{osc}_{B(x,r)}h\le C(r/\rho)^\alpha\big(\inf_{B(x,r)}h+||f||_{L^n(B(x,r)\cap\Sigma)}\big) \tag{2.14}$$

For $x\in\partial\Sigma$ let V_x be the exterior cone at x. We still have for some constant $C\equiv C(d,V_x)$ and $\alpha\equiv\alpha(d,V_x)>0$ and all $0<r<\rho<\epsilon\delta(x)$

$$\begin{aligned}\operatorname{osc}_{B(x,r)\cap\Sigma}h\le&C(r/\rho)^\alpha\big(\operatorname{osc}_{B(x,\rho)\cap\Sigma}h+||f-\lambda h||_{L^n(B(x,r)\cap\Sigma)}\big)\\&+C\operatorname{osc}_{B(x,\sqrt{r\rho})\cap\partial\Sigma}h\end{aligned} \tag{2.15}$$

where $\operatorname{osc}_{B(x,r)\cap\partial\Sigma}h\equiv\limsup_{x\to B(x,r)\cap\partial\Sigma}h-\liminf_{x\to B(x,r)\cap\partial\Sigma}h$.

We also need the boundary Harnack's inequality, which is a consequence of Theorem [8.0.1 in P].

Theorem 2.5: *Assume that ∇F is locally Lipschitz continuous. Let Σ be an open set with uniformly Lipschitz continuous boundary. There exist $C \equiv C(d)$, $\rho \equiv \rho(d) > 0$ and a function $R : \partial\Sigma \to (0,\infty)$, $R \leq \delta$ with the following properties. Fix $z \in \partial\Sigma$ and write $\partial\Sigma \cap B_z = \operatorname{graph}\chi$ for some ball B_z around z and some function χ. Fix $0 \leq r \leq \rho$ and let $0 < u, v \in W^{2,n}(\Sigma)$ be positive solutions of $L_\epsilon h = 0$ in $\Sigma \cap B_z \cap B(z, 8r\epsilon R(z))$ and $h = 0$ on $\partial\Sigma \cap B_z \cap B(z, 8r\epsilon R(z))$. Then*

$$\frac{u(x)}{v(x)} \leq C\frac{u(y)}{v(y)}, \qquad x, y \in \Sigma \cap B_z \cap B(z, r\epsilon R(z)) \tag{2.16}$$

Proof: Denote by $1/\gamma(z)$ the best Lipschitz constant of χ at z in $B(z, 8\epsilon)$ and let $1/\beta(z)$ be the best Lipschitz constant of ∇F in $B(z, 8\epsilon)$. Define $R(z) \equiv \min(\beta(z), \gamma(z), \delta(z))$, where $\delta(z)$ is given in (1.10). Let us introduce the function $\tilde u(\tilde x) \equiv u(x)$, $\tilde x \equiv (x-z)/(\epsilon R(z))$, and likewise $\tilde v$. Furthermore, let $\tilde L \equiv -\Delta + \tilde b \cdot \nabla$, where $\tilde b(\tilde x) \equiv \tilde b_{\epsilon R(z)}(\tilde x) \equiv \nabla F(x)$. Fix $r > 0$ and let u and v be L_ϵ - harmonic in $B(z, 8r\epsilon R(z)) \cap B_z \cap \Sigma$, vanishing identically on $B(z, 8r\epsilon R(z)) \cap B_z \cap \partial\Sigma$. We then compute $\tilde L \tilde u = \tilde L \tilde v = 0$ in $B(0, 8r) \cap \tilde B_z \cap \tilde\Sigma$, where $\tilde\Sigma \equiv \tilde\Sigma_{\epsilon R(z)} \equiv \{\tilde x \,|\, x \in \Sigma\}$ and likewise $\tilde B_z$, and clearly $\tilde u = \tilde v = 0$ on $B(0, 8r) \cap \tilde B_z \cap \partial\tilde\Sigma$. Note that by definition of $R(z)$ under this transformation, the best Lipschitz constant of $\tilde\chi(\tilde x) \equiv \chi(x)$ at z and the best Lipschitz constant of $\tilde b$ in $B(0, 8)$ are bounded by one. Moreover, the supremum norm of $\tilde b$ in $B(0, 8)$ is that of ∇F in $B(z, 8\epsilon R(z))$ and hence is bounded by one by definition of $\delta(z)$. The boundary Harnack's principle Theorem [8.0.1 in P] applied to $D \equiv B(0, 8) \cap \tilde B_z \cap \tilde\Sigma$ gives the existence of $C \equiv C(d)$ and $\rho \equiv \rho(d) > 0$ such that $\tilde u(x)/\tilde v(x) \leq C\tilde u(y)/\tilde v(y)$ for all $0 < r \leq \rho$ and $\tilde x, \tilde y \in B(0, r) \cap \tilde B_z \cap \tilde\Sigma$. $\diamondsuit$

Several times in the sequel we shall use the following obvious consequence of (2.2).

$$\sup_K |h| \leq s^0_K(\partial\Sigma) \sup_\Sigma (\lambda|h| + |g|) + \sup_K H^0_\Sigma |h|, \qquad K \subset \Sigma \subset \Gamma \tag{2.17}$$

The following consequence of Theorem 2.4 serves for uniform a priori control in $\mathbb{R}^d$. It is a crucial ingredient for the proof of growth estimates of λ - harmonic functions in Proposition 3.6.

Lemma 2.6: *Assume 1.2. Let $c : [0, l] \to \mathbb{R}^d$ be a flow line of euclidean length l such that $F(c(l)) > F(c(0)) > C_0$, where C_0 is specified in 1.2. Let $0 \leq h \in W^{2,n}(B^c)$ satisfy*

$(L_\epsilon - \lambda)h \geq 0$, $0 \leq \lambda \leq 1$, *in* $c^{4r\epsilon\delta}$, $0 < r \leq 1$. *Then there is* $C \equiv C(d)$ *such that*

$$h(c(0)) \geq e^{-C(F(c(l))-F(c(0)))/(r\epsilon)} h(c(l)) \tag{2.18}$$

There is $C \equiv C(d)$ *such that for all* $\gamma > 0$, $\alpha > C_0$ *and* $0 \leq \lambda < \lambda(\mathbb{R}^d \backslash \Sigma)$

$$w^\lambda_{\mathbb{R}^d\backslash\Sigma} \leq Ce^{C\gamma/\epsilon} \sup_\Sigma h^\lambda_{\mathbb{R}^d\backslash\Sigma}, \qquad \Sigma \equiv \{\alpha < F < \alpha + \gamma\} \tag{2.19}$$

Proof: Let $y_0 = c(0), \ldots, y_k$ be the uniquely determined sequence of points on the flow line such that $|y_i - y_{i-1}| = 2r\epsilon\delta(x_i)$, where x_i is the center of the ball B_i containing y_i and y_{i-1} in its boundary and $c(l) \in B(x_k, r\epsilon\delta(x_k))$. Applying the Harnack's inequality Theorem [9.22 in GT] to each ball $B_i \subset B(y_{i-1}, 4r\epsilon\delta(y_{i-1})) \subset c^{4r\epsilon\delta}$, we derive the existence of a constant $C = C(d)$ such that

$$h(c(l)) \leq h(c(0))e^{Ck} \tag{2.20}$$

Let $c(s_i) = y_i$, $0 \leq i \leq k$, and choose $\tilde{x}_i \in c(s_{i-1}, s_i)$ satisfying $\sup_{c(s_{i-1},s_i)^{4r\epsilon\delta}} |\nabla F| = \sup_{B(\tilde{x}_i, 4\epsilon\delta(\tilde{x}_i)))} |\nabla F|$. By Hypothesis 1.2 we now conclude

$$\begin{aligned} F(c(l)) - F(c(0)) \geq & \sum_{i=1}^k (s_{i-1} - s_i) \inf_{c(s_{i-1},s_i)} |\nabla F| \\ \geq & (r\epsilon/C_1) \sum_{i=1}^k \delta(\tilde{x}_i) \sup_{B(\tilde{x}_i, 4\epsilon\delta(\tilde{x}_i)))} |\nabla F| = \frac{kr\epsilon}{8C_1} \end{aligned} \tag{2.21}$$

where we have used the remark after (1.10). This proves the former assertion.

For the proof of the upper bound on the exit time we fix $0 < r \equiv r(d) < 1/2$ such that $Cr^\alpha(1 + C\lambda(4r\epsilon\delta(x))^d) < 1/2$ in (2.15). Here we use $\sup_\Sigma \delta \leq 1/c$ since $\inf_\Sigma |\nabla F| \geq c$ by 1.2. Define recursively $\Omega_0 \equiv \Sigma$, $\Gamma_0 \equiv \{F = \alpha + \gamma\}$ and for $k > 0$

$$\Omega_k \equiv \{x \in \Omega_{k-1} \,|\, \mathrm{dist}(x, \partial\Gamma_{k-1}) \geq r\epsilon\delta(x)\}, \qquad \Gamma_k \equiv \partial\Omega_k \backslash \{F = \alpha\} \tag{2.22}$$

Let K be the minimal number such that $\Omega_{K+1} = \emptyset$. Choose k_0 such that $\sup_\Sigma w^\lambda = \sup_{\Omega_{k_0}} w^\lambda$, where $w^\lambda \equiv w^\lambda_{\Sigma^c}$. The choice of r in (2.15) ensures that for all $x \in \Omega_k$ and $0 \leq k \leq K$

$$w_k(x) \equiv w^\lambda_{\Omega_k^c}(x) \leq (1/2) \sup_{B(x,4r\epsilon\delta(x))\cap\Omega_k} w^\lambda_{\Omega_k^c} + C(4r\epsilon\delta(x))^d \sup_{B(x,4r\epsilon\delta(x))\cap\Omega_k} h^\lambda_{\Omega_k^c} \tag{2.23}$$

Consequently, w_k is bounded by $2C(4r\epsilon\delta(x))^d \sup_{B(x,4r\epsilon\delta(x))\cap\Omega_k} h^{\lambda}_{\Omega_k^c}$ if it attains its maximal value at some $x \in \Omega_k \backslash \Omega_{k-2}$. It hence suffices to consider $k_0 > 1$. The definition of r implies $\sup_{\Gamma_k} h_{k-1} < 1/2$ for all $1 \le k \le K$ by (2.15) for $h_k \equiv h^0_{\Gamma_{k-1},\{F\le\alpha\}} \le 1$. Application of (2.17) to w in $\Gamma \equiv \Omega_{k_0-1}$ for $K \equiv \Omega_{k_0}$ then gives

$$\sup_{\Omega_{k_0}} w^{\lambda} \le s^{\lambda}_{\Omega^c_{k_0-1}} + \sup_{\Gamma_{k_0}} h_{k_0-1} \sup_{\Gamma_{k_0-1}} w^{\lambda} \tag{2.24}$$

so that $\sup_{\Sigma} w^{\lambda} \le 2s_0$, where $s_0 \equiv s^{\lambda}_{\Omega^c_{k_0-1}}$. We now iterate this procedure. For, choose k_1 such that $s^{\lambda}_{\Omega^c_{k_0-1}} = \sup_{\Omega_{k_1}} w^{\lambda}_{k_0-1}$. We already argued that $k_1 > k_0$ by (2.23) is the only case to be considered. As in (2.24) we thus obtain $s_0 = s^{\lambda}_{\Omega^c_{k_0-1}} = \sup_{\Omega_{k_1}} w^{\lambda}_{k_0-1} \le 2s_1$, where $s_1 \equiv s^{\lambda}_{\Omega^c_{k_1-1}}$, from (2.17) applied to $h \equiv w^{\lambda}_{k_0-1}$ in $\Gamma \equiv \Omega_{k_1-1}$ for $K \equiv \Omega_{k_1}$. Inductively, one proves $s_{l-1} \le 2s_l$, where $s_l \equiv s^{\lambda}_{\Omega^c_{k_l-1}}$ as long as $k_l \equiv \max\{k \ge k_{l-1} \,|\, s^{\lambda}_{\Omega^c_{k_{l-1}-1}} = \sup_{\Omega_k} w^{\lambda}_{k_{l-1}-1}\}$ is such that $k_l > k_{l-1}$. Let L be the minimal number such that $k_{L+1} = k_L$. We hence conclude $\sup_{\Sigma} w^{\lambda} \le 2^L s_L \le 2^K 2C(4r\epsilon/c)^d \sup_{\Omega_{k_L-1}} h^{\lambda}_{\Omega^c_{k_L-1}}$. Analogous to (2.21) we have $K \le C\gamma/\epsilon$ for some $C = C(d)$. $\diamondsuit$

As already pointed out, the problem is that a priori we cannot apply Theorem 2.4 to an eigenfunction ϕ. However, by combination of Theorem 2.4 with the Poisson's representation formula we still can control the regularity of ϕ. We recall from e.g. [BEGK3] that the $(L_\epsilon - \lambda)$ - harmonic extension $H^{\lambda}_{\Sigma} f$, Σ regular and open, of a function $f \in L^{\infty}(\partial\Sigma)$ is given by

$$\begin{aligned} h(x) = H^{\lambda}_{\Sigma} f(x) &= -\,\epsilon \int_{\partial\Sigma} f(y) \partial_{n(y)} G^{\lambda}_{\Sigma}(y,x) e^{(F(x)-F(y))/\epsilon} d\sigma(y) \\ &= -\,\epsilon \int_{\partial\Sigma} f(y) \partial_{n(y)} G^{\lambda}_{\Sigma}(x,y) d\sigma(y) \end{aligned} \tag{2.25}$$

where $n(y)$ denotes the outer unit normal at $y \in \partial\Sigma$ and the normal derivative is taken from inside of Σ. Here we have used that $e^{-F(x)/\epsilon} G^{\lambda}_{\Sigma}(x,y)$ is symmetric in x and y, that the normal derivative at $\partial\Sigma$ exists and that $G^{\lambda}_{\Sigma}(x,y)$ vanishes on the boundary.

Proof of Lemma 2.3: By standard comparison arguments with the ordinary Laplace operator in $B \equiv B(x,R)$ as can be found e.g. in the proof of Theorem [2.1.(i), P] one finds $\delta = \delta(F|B) > 0$ and $C(F|B)$ such that

$$s^0_B \le C \quad \text{for} \quad R \equiv \delta\sqrt{\epsilon} \tag{2.26}$$

where s^0_B is defined in (2.5). For the convenience of the reader we shall formulate the details of the proof in our situation. Define $v_R(y) \equiv (R^2 - |y-x|^2)/(2d\epsilon)$ for $|y-x| \le R$. We compute $-\epsilon\Delta v_R = 1$ for $|y-x| \le R$. Since $|\nabla F(y) \cdot \nabla v_R(y)| \le \sup_B |\nabla F||y-$

$x|/(d\epsilon) \le \delta \sup_B |\nabla F|/(d\sqrt{\epsilon})$ and since $x \in \mathcal{M}$, it follows $L_\epsilon v_R \ge 1/2$ for $\delta \equiv \sup\{r > 0 \,|\, r \sup_{B(x,r\sqrt{\epsilon})} |\nabla F| \le d\sqrt{\epsilon}/2\} > 0$ and $|y-x| \le R$ so that $v_R(y) \ge (1/2) w_B(y)$ in B.

Recall the notion of the principal eigenvalue $\lambda(\Sigma)$ of the Dirichlet operator L_ϵ^Σ introduced in (2.6). Since the uncertainty principle Lemma (2.7) tells us $\lambda(B) \ge 1/s_B^0$, the condition on λ ensures that G_B^λ exists and that h satisfies (2.25). Choose a ball $\tilde{B} \subset B$ of radius $0 < \rho < \epsilon$ such that $\sup_B |h| = \sup_{\tilde{B}} |h|$. Since $-\partial_{n(y)} G_B^\lambda(x,y)$ is a positive strong solution for every $y \in \partial\tilde{B}$, we may apply (2.15) for and obtain for some $C \equiv C(d, F|B)$ and all $y_0, y_1, y_2 \in \tilde{B}$,

$$\begin{aligned} |h(y_1) - h(y_2)| \le & \epsilon \int_{\partial B} |h(z)| |\partial_{n(z)} G_B^\lambda(y_1, z) - \partial_{n(z)} G_B^\lambda(y_2, z)| d\sigma(z) \\ \le & C(\rho/\sqrt{\epsilon})^\alpha \sup_{\partial B} |h| \epsilon \int_{\partial B} -\partial_{n(z)} G_B^\lambda(y_0, z) d\sigma(z) \\ \le & C(\rho/\sqrt{\epsilon})^\alpha \sup_{\tilde{B}} |h| \sup_B h_{\tilde{B}}^\lambda \end{aligned} \tag{2.27}$$

where $h_B^\lambda \equiv H_B^\lambda \mathbb{1}_{\partial B}$. Applying (2.2) to h_B^λ and $\Sigma \equiv B$, we obtain

$$\sup_B h_B^\lambda \le \lambda s_B^0 \sup_B h_B^\lambda + 1 \tag{2.28}$$

Combination of (2.28) with (2.27) implies

$$|h(y_1) - h(y_2)| \le \frac{C(\rho/\sqrt{\epsilon})^\alpha \sup_{\tilde{B}} |h|}{1 - \lambda s_B^0} \tag{2.29}$$

The bounds on λ and s_B^0 show that the denominator can be absorbed in the constant. We thus have proven for some C and $\alpha > 0$ only depending on $F|B$ that for all $y_1, y_2 \in \tilde{B}$

$$\sup_{y_1, y_2 \in \tilde{B}} |h(y_1) - h(y_2)| \le C(\rho/\sqrt{\epsilon})^\alpha \sup_{\tilde{B}} |h| \tag{2.30}$$

Now let us assume that there is $y \in \tilde{B}$ such that $h(y) = 0$. We apply (2.17) for $\Sigma \equiv \tilde{B}$ and deduce from (2.30), using the condition on $\tilde{B}$

$$\sup_{\tilde{B}} |h| \le \lambda s_{\tilde{B}}^0 \sup_{\tilde{B}} |h| + C\epsilon^{\alpha/2} \sup_{\tilde{B}} |h| \tag{2.31}$$

It follows that h vanishes identically in $\tilde{B}$ for small $\epsilon > 0$ and hence by analytic continuation everywhere in $\mathbb{R}^d$.

For the proof of (2.12) let $c_n \equiv (7C\epsilon^\alpha)^n (C\epsilon^\alpha + C|B|(2\lambda + \sup_B |g| / \sup_{\tilde{B}} h))$, where $B \equiv B(x, 2\sqrt{\epsilon})$. We claim the existence of $C \equiv C(d, F|B)$, $\alpha \equiv \alpha(d, F|B) > 0$, such that for all n the inequality $c_{n-1} > M \equiv \max(r, C|B|(2\lambda + \sup_B |g|/\sup_{\tilde{B}} h), \sup_{\Sigma\setminus\tilde{B}} h^\lambda_{\mathbb{R}^d\setminus\Gamma,\tilde{B}})$, $\tilde{B} \equiv B(x,\epsilon)$, implies

$$\operatorname{osc}_{\tilde{B}} h \le c_n \sup_{\tilde{B}} h \tag{2.32}$$

For $n \equiv 1$ this is nothing more than (2.13). Assume (2.32) for some $n \ge 1$. It follows from $h \ge 0$ and (2.2) applied to h for $K \equiv \Sigma\setminus\tilde{B}$ in $\Sigma \equiv \Gamma\setminus\tilde{B}$ - in slight abuse of notation - that

$$\inf_\Sigma h \ge 0 + (1-c_n)(1 - \sup_{\Sigma\setminus\tilde{B}} h^\lambda_{\mathbb{R}^d\setminus\Gamma,\tilde{B}}) \sup_{\tilde{B}} h \ge (1-c_n)(1-r)(1 - \sup_{\Sigma\setminus\tilde{B}} h^\lambda_{\mathbb{R}^d\setminus\Gamma,\tilde{B}}) \sup_\Sigma h \tag{2.33}$$

where we use the convention that $h^\lambda_{\mathbb{R}^d\setminus\Gamma,\tilde{B}} \equiv 0$ in $\tilde{B}$. Thus

$$\begin{aligned}\operatorname{osc}_\Sigma h \le&(c_n + r + \sup_{\Sigma\setminus\tilde{B}} h^\lambda_{\mathbb{R}^d\setminus\Gamma,\tilde{B}}) \sup_\Sigma h \le ((c_n + r + \sup_{\Sigma\setminus\tilde{B}} h^\lambda_{\mathbb{R}^d\setminus\Gamma,\tilde{B}})/(1-r)) \sup_{\tilde{B}} h \\ \le& 2(c_n + r + \sup_{\Sigma\setminus\tilde{B}} h^\lambda_{\mathbb{R}^d\setminus\Gamma,\tilde{B}}) \sup_{\tilde{B}} h\end{aligned} \tag{2.34}$$

From (2.13) again we hence obtain

$$\operatorname{osc}_{\tilde{B}} h \le C\epsilon^\alpha (2(c_n + r + \sup_{\Sigma\setminus\tilde{B}} h^\lambda_{\mathbb{R}^d\setminus\Gamma,\tilde{B}}) + C|B|(\lambda/(1-r) + \sup_B |g|/\sup_{\tilde{B}} h) \sup_{\tilde{B}} h \le c_{n+1} \sup_{\tilde{B}} h \tag{2.35}$$

since $c_n > M$. Choosing n maximal in (2.32), from (2.34) we obtain the estimate since $c_n \le M$. $\diamondsuit$

3. Growth estimates at infinity

The strong drift of ∇F towards the local minima of F shows that the influence of the values of a solution ϕ of (2.1) at infinity on its values in compact sets can be neglected.

A priori bounds on conditioned, expected exit times from bounded sets: In this section we prove an estimate on the suprema $s_K^0(A,B)$ for regular, closed sets A and B with bounded complement of their union.

For the sake of convenience we set

$$T_J \equiv \max_{y\in\mathcal{M}\backslash J} T_{x,J}, \qquad J\subset\mathcal{M}, J\neq\mathcal{M} \tag{3.1}$$

where the time scale $T_{x,J}$ is defined in (1.7). In the case $J\equiv\mathcal{M}$ we use the convention that $T_{\mathcal{M}}\equiv 1/\epsilon^{d-1}$. For every finite set of points $I\subset\mathbb{R}^d$ we set

$$B_I \equiv B_I(\epsilon/4) \equiv \bigcup_{x\in I} B(x,\epsilon/4), \qquad I\subset\mathbb{R}^d, \qquad \min_{x,y\in I, x\neq y} \operatorname{dist}(x,y) > 2\epsilon \tag{3.2}$$

We then have

Lemma 3.1: *Fix disjoint, regular, nonempty, closed sets $A,B\subset\mathbb{R}^d$ such that $B_I\subset A$ and $B_J\subset B$, where $I\equiv\mathcal{M}\cap A$ and $J\equiv\mathcal{M}\cap B$. There are $N\equiv N(d)$ and $C=C(d)$ such that*

$$s_K^0(A,B) \leq C\epsilon^{-N}(T_{I\cup J} + |\mathbb{R}^d\backslash A\backslash B|) \tag{3.3}$$

We start with the following bound on the Green's function $G_\Sigma^0(x,y)$ defined in (2.3).

Lemma 3.2: *For all regular, open, bounded sets Γ there exists $C=C(d)$ such that for all $x,y\in\Gamma$ and all $0<\rho<\delta(y)$ satisfying $|x-y|>\rho\epsilon$ and $\operatorname{dist}(x\cup y,\partial\Gamma)\geq 4\rho\epsilon$*

$$G_\Gamma^0(x,y) \leq \frac{C h_{B(y,\rho\epsilon),\Gamma^c}(x) e^{-F(y)/\epsilon}}{\operatorname{cap}_{B(y,\rho\epsilon)}(\Gamma^c)} \tag{3.4}$$

For $|x-y|<\rho\epsilon$, $0<\rho\leq\delta(x)\epsilon$ and $\operatorname{dist}(x,\partial\Gamma)>4\rho\epsilon$ we have

$$G_\Gamma^0(x,y) \leq \frac{C}{\epsilon} G^\Delta(|x-y|) + \frac{C e^{-F(x)/\epsilon}}{\operatorname{cap}_{B(x,\rho\epsilon)}(\Gamma^c)} \tag{3.5}$$

where G^Δ is the Green's function of the Laplace operator in $\mathbb{R}^d$.

Proof: Let $h_y \equiv h^0_{B(y,\rho\epsilon),\Gamma^c}$. The second Green's formula as e.g. in [(2.8) in BEGK3] for $\Gamma \equiv \Gamma \backslash \overline{B}(y,\rho\epsilon) \backslash \overline{B}(x,r)$ and for $\Gamma \equiv B(y,\rho\epsilon)$ shows for all $0 < r < |x-y| - \rho\epsilon$

$$
\begin{aligned}
\epsilon \int_{\partial B(y,\rho\epsilon)} e^{-F/\epsilon} G_\Gamma(\cdot,x) \partial_n h_y d\sigma =& - \epsilon \int_{\partial B(x,r)} h_y \partial_n G_\Gamma(\cdot,x) + \epsilon \int_{\partial B(y,\rho\epsilon)} e^{-F/\epsilon} \partial_n G_\Gamma(\cdot,x) d\sigma \\
=& - e^{\mathcal{O}(1)(r/(|x-y|-\rho\epsilon))^\alpha} h_y(x) e^{-F(x)/\epsilon} \epsilon \int_{\partial\Gamma} \partial_n G_\Gamma(x,\cdot) d\sigma \\
=& e^{-F(x)/\epsilon} h_y(x)
\end{aligned}
\tag{3.6}
$$

where ∂_n is the normal derivative taken from the interior with respect to the outer unit normal at the boundary. The last equation uses (2.25) and the fact that $r > 0$ can be chosen arbitrarily small. Invoking the Harnack's inequality Corollary [9.25 in GT] on the left-hand side of (3.6), we thus have for some $C = C(d)$

$$
G_\Gamma(y,x) \operatorname{cap}_{B(y,\rho\epsilon)}(\Gamma^c) \le C e^{-F(x)/\epsilon} h_y(x) \tag{3.7}
$$

(3.4) now follows from the symmetry of $G_\Gamma(y,x) e^{F(x)/\epsilon}$ in x and y.

For the proof of (3.5) we first observe that considering $y \equiv x$ in (3.6) a similar calculation gives for all $0 < \rho \le 1$

$$
\epsilon \int_{\partial B(x,\rho\epsilon)} e^{-F/\epsilon} G_\Gamma(\cdot,x) \partial_n h_x = e^{-F(x)/\epsilon} \tag{3.8}
$$

Analogous to (3.7) we find C independent of ϵ such that for all $y \in \partial B(x,\rho\epsilon)$

$$
G_\Gamma(y,x) \le C \frac{e^{-F(x)/\epsilon}}{\operatorname{cap}_{B(x,\rho\epsilon)}(\Gamma)} \tag{3.9}
$$

For, choose a sequence of points $y_0 = y, \dots, y_k = z \in \partial B(x, 2\rho\epsilon)$ such that $\rho\epsilon/100 < |y_i - y_{i+1}| < \rho\epsilon/3$ and balls B_i of radii $\rho\epsilon/3$ such that $y_{i-1}, y_i \in B_i$. Applying the Harnack's inequality to each ball, we derive $G_\Gamma(y_i,x)/G_\Gamma(y_{i+1},x) \le C$ for some $C \equiv C(d)$ and $0 < \rho \le \delta(x)$. Since the arclength of a ball depends linearly on the distance, we get that k is bounded independent of ϵ and $y, z \in \partial B(x, 2\rho\epsilon)$ and thus

$$
G_\Gamma(y,x)/G_\Gamma(z,x) \le C^k \tag{3.10}
$$

from which (3.9) follows. Assume first that $\Gamma = B(x,\rho\epsilon)$. Invoking the Dirichlet's principle for the capacity, we derive for $0 \le r \le \rho$

$$
\operatorname{cap}_{B(x,r\epsilon)}(B(x,\rho\epsilon)^c) \le \epsilon C e^{-F(x)/\epsilon} \operatorname{cap}^{\Delta}_{B(x,r\epsilon)}(B(x,\rho\epsilon)^c) \tag{3.11}
$$

where cap^Δ denotes the capacity with respect to the Laplace's operator. Since G^Δ is rotationally invariant, it follows $\mathrm{cap}^\Delta_{B(x,r\epsilon)}(B(x,\rho\epsilon)^c) = 1/(G^\Delta(\rho\epsilon) - G^\Delta(\epsilon)) \geq 1/(G^\Delta(\rho\epsilon)$. Combination of (3.11) and (3.9) shows

$$G_{B(x,\rho\epsilon)}(y,x) \leq \frac{C}{\epsilon} G^\Delta(|x-y|) \qquad |x-y| < \rho\epsilon \tag{3.12}$$

To obtain the full estimate we note that the function $G_\Omega(\cdot,x) - G_{B(x,\rho\epsilon)}(\cdot,x) - h$ is a weakly L_ϵ-harmonic function in $B(x,\rho\epsilon)$ and equals zero on $\partial B(x,\rho\epsilon)$, where h is the solution of the Dirichlet's problem in $B(x,\rho\epsilon)$ with boundary values $G_{B(x,\rho\epsilon)}(\cdot,x)$. By (3.9) and (3.12) we thus have proven for $|x-y| < \rho\epsilon$

$$G_\Gamma(y,x) \leq \frac{C}{\epsilon} G^\Delta(|x-y|) + \frac{Ce^{-F(x)/\epsilon}}{\mathrm{cap}_{B(x,\rho\epsilon)}(\Gamma)} \tag{3.13}$$

which gives (3.5). $\diamondsuit$

For later purpose we notice that the definition of δ in (1.10) implies

$$\inf_\Sigma F - \epsilon \leq \inf_{\Sigma^{\epsilon\delta}} F \leq \sup_{\Sigma^{\epsilon\delta}} F \leq \sup_\Sigma F + \epsilon \tag{3.14}$$

Indeed, fix arbitrary $x \in \Sigma$ and let $y \in \partial B(x, \epsilon\delta(x))$. We then obtain, using (1.10),

$$F(y) - F(x) = \int_0^1 \nabla F((1-t)x + ty) \cdot (y-x) dt \leq |x-y|/\delta(x) = \epsilon \tag{3.15}$$

and $F(y) - F(x) \geq -\epsilon$ by replacing the roles of x and y.

Proof of 3.1: Applying (2.16), respectively (2.15), to $w^0_{A,B}$, respectively $h^0_{A,B}$, with the obvious choice $z \in \partial B$, respectively $z \in \partial A$, we may assume that $x \in \mathbb{R}^d \backslash \tilde{A} \backslash \tilde{B}$, where $\tilde{A} \equiv A \cup (\partial A)^{R\epsilon}$, $\tilde{B} \equiv B \cup (\partial B)^{R\epsilon}$ and where $R : \partial A \cup \partial B \to (0,\infty)$ is as in 2.5. Let $\epsilon r : \mathbb{R}^d \backslash A \backslash \tilde{B} \to (0,\infty)$ be the maximum of $\epsilon\delta$ and the distance from B. We may assume that R is bounded by one. We now may write using (3.5) for $\Gamma \equiv \mathbb{R}^d \backslash A \backslash B$ and all $x \in \mathbb{R}^d \backslash \tilde{A} \backslash \tilde{B}$

$$\begin{aligned}
\frac{w^0_{A,B}(x)}{h^0_{A,B}(x)} =& \int_{\mathbb{R}^d \backslash A \backslash B} \frac{G^0_{\mathbb{R}^d \backslash A \backslash B}(x,y) h^0_{A,B}(y)}{h^0_{A,B}(x)} dy \\
\leq& C \int_{\tilde{B} \backslash B} \frac{G^0_{\mathbb{R}^d \backslash A \backslash B}(x,y) h^0_{A,B}(y)}{h^0_{A,B}(x)} dy \\
&+ \frac{C}{\epsilon} + \frac{Ce^{-F(x)/\epsilon}}{\mathrm{cap}_{B(x,\epsilon r(x))}(A \cup B)} |B(x,\epsilon)| \\
&+ C \int_{\substack{\mathbb{R}^d \backslash A \backslash \tilde{B} \\ y: |y-x| > \epsilon}} \frac{h^0_{B(y,r(y)\epsilon), A\cup B}(x) h^0_{A,B}(y)}{h^0_{A,B}(x)\, \mathrm{cap}_{B(y,r(y)\epsilon)}(A \cup B)} e^{-F(y)/\epsilon} dy
\end{aligned} \tag{3.16}$$

Using the Harnack's inequality for $h^0_{A,B}$ on $B(y,\epsilon r(y))$, we observe for $x \in (A \cup B \cup B(y,\epsilon r(y)))^c$ that

$$h^0_{B(y,\epsilon r(y)),A\cup B}(x) h^0_{A,B}(y) \leq C H^0_{(A\cup B\cup B(y,\epsilon r(y)))^c} 1\!\!1_{\partial B(y,\epsilon r(y))} h_{A,B}(x) \leq C h^0_{A,B}(x) \quad (3.17)$$

By (2.16) for $u \equiv G^0_{\mathbb{R}^d\setminus A\setminus B}(x,\cdot)$ and $v \equiv h^0_{A,B}$ and by (3.4) once more we have for all $y \equiv z + tR(z)\epsilon n(z)$, $z \in \partial B$, $0 < t \leq 1$, the bound

$$\begin{aligned} G^0_{\mathbb{R}^d\setminus A\setminus B}(x,y) h^0_{A,B}(y) \leq & C G^0_{\mathbb{R}^d\setminus A\setminus B}(x, z + R(z)\epsilon n(z)) h^0_{A,B}(z + R(z)\epsilon n(z)) \\ \leq & C \frac{h^0_{B(z+R(z)\epsilon n(z),R(z)\epsilon/4),A\cup B}(x) h^0_{A,B}(z + R(z)\epsilon n(z))}{\mathrm{cap}_{B(z+R(z)\epsilon n(z),R(z)\epsilon/4)}(A\cup B)^c} \end{aligned} \quad (3.18)$$

where n is the outer unit normal vector field at the boundary of B. We hence may apply the Harnack's inequality again to the right-hand side of (3.18), proving

$$G^0_{\mathbb{R}^d\setminus A\setminus B}(x,y) h^0_{A,B}(y) \leq C \frac{h^0_{A,B}(x)}{\mathrm{cap}_{B(z+R(z)\epsilon n(z),R(z)\epsilon/4)}(A\cup B)^c} \quad (3.19)$$

Invoking Proposition [4.7 in BEGK3], we have for all $0 < \rho < 1$

$$\mathrm{cap}_{B(y,\rho\epsilon)}(A \cup B) \geq e^{-\hat{F}(y,A\cup B))/\epsilon} (\rho\epsilon)^d / (C\rho\epsilon) \quad (3.20)$$

Inserting (3.19) and (3.20) into the integrals on the right-hand side of (3.16), we thus may bound the right-hand side by C times

$$\begin{aligned} & \frac{1}{\epsilon} + \frac{1}{\epsilon} e^{(\hat{F}(x,I\cup J)) - F(x))/\epsilon} \\ & + |\{F = \hat{F}(\cdot, A\cup B))\}\setminus A\setminus B| \\ & + |\{\mathrm{dist}(\cdot, B) \leq R\epsilon\}| \sup_{z\in\partial B, 0<t<1} e^{|F(z+tR(z)\epsilon n(z))) - F(z+R(z)\epsilon n(z))|/\epsilon} \\ & + \frac{1}{\epsilon^{d-1}} \int_{\{F<\hat{F}(\cdot,A\cup B))\}\setminus A\setminus \tilde{B}} e^{(\hat{F}(y,I\cup J)) - F(y))/\epsilon} dy \end{aligned} \quad (3.21)$$

Since $R(z) \leq \delta(z)$, (3.14) tells us that the supremum appearing in the fourth term is bounded by ϵ. We readily verify (3.3) by computation of a Laplace type integral. $\diamondsuit$

Random walk expansion: We now want to sharply compare in compact sets eigenfunctions to linear combinations of electrostatic equilibrium potentials $h^0_{A,B} \equiv H^0_{(A\cup B)^c} 1\!\!1_A$. More precisely, for a suitable choice of $(\Omega_k)_{0\leq k\leq L+1}$ of regular, open regions

$$\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \subset\subset \Omega_{L+1} \subset \Omega \quad (3.22)$$

defined below in (3.38)-(3.39) we want to approximate for small $\lambda \geq 0$ in $\Omega_1 \backslash \Omega_0$ the solution ϕ of (2.1) with the solution h^λ of the Dirichlet problem

$$(L_\epsilon - \lambda)h(x) = 0 \quad x \in \Sigma\backslash\Gamma, \qquad h(x) = f(x) \quad x \in \partial\Gamma, \qquad h(x) = 0 \quad x \in \partial\Sigma \tag{3.23}$$

where $\lambda \equiv 0$, $\Sigma \equiv \Omega_{L+1}$, $\Gamma \equiv \Omega_0$ and $f(x) \equiv \phi(x)$, $x \in \partial\Omega_0$. Generally speaking, for $\lambda > 0$, ϕ and h^0 are not close to each other everywhere in unbounded regions Ω even if $\Sigma \equiv \Omega$. In fact, we allow Ω to be equal to $\mathbb{R}^d$ and in this case h^0 stays bounded while in general ϕ is unbounded near infinity. However, exploiting the drift of F towards the local minima, we can show that ϕ is close to h^0 in bounded regions $\Omega_1\backslash\Omega_0$ independent of ϵ and containing all relevant local minima. For similar problems in discrete space we refer the reader to [E2].

We identify H^λ_Σ as an operator on $\mathbb{R}^d$ via $H^\lambda_\Sigma = \mathbb{1}_{\Sigma\cup\partial\Sigma} H^\lambda_\Sigma \mathbb{1}_{\partial\Sigma}$ and likewise for functions a priori defined on Σ, which by definition take the value zero outside Σ.

Lemma 3.3: *Fix regular, bounded, open sets $(\Omega_k)_{0\leq k\leq L+1}$ satisfying (3.22), where Ω is a regular, open set, and define $\Gamma_k \equiv \Omega_{k+1}\backslash\overline{\Omega}_{k-1}$ for $1 \leq k \leq L$. Let h^λ be a solution of (3.23) for $\Sigma \equiv \Omega_{L+1}$ and non-negative $0 \leq f \in L^\infty(\mathbb{R}^d)$. We then have for $x \in \Gamma_1$*

$$h^\lambda(x) = \sum_{T=0}^{\infty} \sum_{\substack{\omega:1\sim 1\\ |\omega|=2T}} \prod_{t=1}^{2T-1} H^\lambda_{\Gamma_{\omega_t}} \mathbb{1}_{\partial\Omega_{\omega_{t+1}}} H^\lambda_{\Gamma_1} \mathbb{1}_{\partial\Omega_0} f(x) \tag{3.24}$$

where $\omega : k \sim l$ denotes a sequence $\omega \equiv (\omega_1, \dots, \omega_T)$ such that $\omega_t \in \{1, \dots, L\}$ for $1 \leq t \leq T$, $\omega_1 = k$, $\omega_T = l$, $\omega_t \neq 1$ for $1 < t < T$ and $|\omega_t - \omega_{t-1}| = 1$ for $2 \leq t \leq T$. We have set $|\omega| \equiv T$.

Proof: One proves by induction for $x \in \Gamma_l$, $1 \leq l \leq L$ that

$$\begin{aligned} h^\lambda(x) = &\sum_{\substack{\omega:l\sim 1\\ |\omega|\leq K}} \prod_{t=1}^{|\omega|-1} H^\lambda_{\Gamma_{\omega_t}} \mathbb{1}_{\partial\Omega_{\omega_{t+1}}} H^\lambda_{\Gamma_1} \mathbb{1}_{\partial\Omega_0} f(x) \\ &+ \sum_{\substack{\omega:\omega_1=l,|\omega|=K+1\\ \omega_k\neq 1, 2\leq k\leq K+1}} \prod_{t=1}^{K} H^\lambda_{\Gamma_{\omega_t}} \mathbb{1}_{\partial\Omega_{\omega_{t+1}}} h^\lambda(x) \end{aligned} \tag{3.25}$$

By positivity (3.25) already implies that both terms on the right-hand side are convergent in the limit $K \to \infty$. For each $2 \leq k \leq L+1$ set $\eta_k \equiv (k, k-1, \dots, 1)$. Since both functions in the following are positive on $\partial\Omega_{\omega_{K+1}}$ for a sequence ω with the properties listed below the second sum in (3.25) and since there are only L possible choices for ω_{K+1}, by compactness there is $C \equiv C(d, F|\Omega_{L+1}, \lambda, (\Omega_k)_{1\le k\le L+1}, \epsilon, f)$

$$\max_{\partial\Omega_{\omega_{K+1}}} h^\lambda / \min_{\partial\Omega_{\omega_{K+1}}} \prod_{t=1}^{|\eta_{\omega_{K+1}}|-3} H^\lambda_{\Gamma_{\eta_t}} \mathbb{1}_{\partial\Omega_{\eta_{t+2}}} H^\lambda_{\Gamma_1} \mathbb{1}_{\partial\Omega_0} f < C \tag{3.26}$$

We thus obtain that the last term on the right-hand side of (3.25) is bounded by

$$C \sum_{\substack{\omega:\omega_1=1,\omega_{K+1}\neq 1\\ |\omega|=K+1}} \prod_{t=1}^{K} H^\lambda_{\Gamma_{\omega_t}} \mathbb{1}_{\partial\Omega_{\omega_{t+1}}} \prod_{t=1}^{|\eta_{\omega_{K+1}}|-2} H^\lambda_{\Gamma_{\eta_t}} \mathbb{1}_{\partial\Omega_{\eta_{t+1}}} H^\lambda_{\Gamma_1} \mathbb{1}_{\partial\Omega_0} f(x) \tag{3.27}$$

which is a term appearing in the sum over sequences $\omega : 1 \sim 1$ of length at least $K+1$ on the right-hand side of (3.24). Since this sum is convergent, it follows that this sequence tends to zero in the limit $K \to \infty$ and (3.24) follows. $\diamondsuit$

Laplace transforms: In the sequel we prove two type of estimates on Laplace transforms $h^\lambda_{A,B}$ away from its first pole $\lambda(\mathbb{R}^d \backslash A \backslash B)$ in terms of the maximal conditioned expected exit time from $\mathbb{R}^d \backslash A \backslash B$. The first estimate provides us with control of Laplace transforms which is uniform in volume.

Lemma 3.4: *There is $C \equiv C(d)$ such that for all $\alpha < \beta - 2\epsilon$ and all $\lambda > 0$ satisfying $\lambda \max_{\gamma=\alpha,\beta} s^0_{\{\alpha<F<\beta\}}(\{F=\gamma\}) < 1/2$*

$$\frac{H^\lambda_{\{\alpha<F<\beta\}} \mathbb{1}_{\{F=\gamma\}}(x)}{H^0_{\{\alpha<F<\beta\}} \mathbb{1}_{\{F=\gamma\}}(x)} \le 1 + \lambda C e^{C(\beta-\alpha)/\epsilon} \tag{3.28}$$

Proof: (2.7) and the condition on λ show that G^λ_Σ and H^λ_Σ, where $\Sigma \equiv \{\alpha < F < \beta\}$, exist. (2.2) applied to $h \equiv H^\lambda_\Sigma f - H^0_\Sigma f$ and $f \equiv \mathbb{1}_{\{F=\gamma\}}$ yields for all $x \in \Sigma$

$$\begin{aligned} \frac{H^\lambda_\Sigma f(x)}{H^0_\Sigma f(x)} - 1 =& \lambda \frac{1}{H^0_\Sigma f(x)} G^0_\Sigma \left(H^0_\Sigma f \left(\frac{H^\lambda_\Sigma f}{H^0_\Sigma f} - 1 \right) \right)(x) + \lambda \frac{G^0_\Sigma H^0_\Sigma f(x)}{H^0_\Sigma f(x)} \\ \le& \lambda s^0_\Sigma(\{F=\gamma\}) \sup_\Sigma \left(\frac{H^\lambda_\Sigma f}{H^0_\Sigma f} - 1 \right) + \lambda s^0_\Sigma(\{F=\gamma\}) \end{aligned} \tag{3.29}$$

Taking the supremum on the left-hand side and assuming it is finite, we have proven (3.28). Simply by continuity at the boundary, the supremum stays finite near boundary points $x_0 \in \partial\Sigma$ such that $f(x_0) > 0$. Since $H^0_\Sigma f$ takes its minimal value at zero by the Hopf Maximum Principle Theorem [3.2.5 in P], it follows that

$$\lim_{\Sigma \ni x \to x_0} \frac{H^\lambda_\Sigma f(x) - 0}{H^0_\Sigma f(x) - 0} = \frac{\partial_{n(x_0)} H^\lambda_\Sigma f(x_0)}{\partial_{n(x_0)} H^0_\Sigma f(x_0)} < \infty \tag{3.30}$$

We thus have proven

$$\sup_{\Sigma} \frac{H^{\lambda}_{\Sigma} f}{H^{0}_{\Sigma} f} \leq 1 + 2\lambda \max_{\gamma=\alpha,\beta} s^{0}_{\Sigma}(\{F = \gamma\}) \tag{3.31}$$

It remains to estimate the supremum on the right-hand side of (3.31). We first note that $\{F = \gamma\}^{\epsilon\delta} \subset \{F \leq \gamma + \epsilon\}$ by (3.14). By the boundary Harnack's inequality (2.16) we then conclude $s^{0}_{\Sigma}(\{F = \gamma\}) \leq C s^{0}_{\Sigma \backslash \{F=\tilde{\gamma}\}^{\epsilon\delta/8}}(\{F = \gamma\})$, where $\tilde{\gamma} = (\alpha \cup \beta) \backslash \gamma$. Using (2.15) in combination with (2.18) for $h \equiv h^{0}_{\{F=\gamma\},\Sigma^c}$, we observe that there exists $C \equiv C(d)$ such that $h^{0}_{\{F=\gamma\},\Sigma^c}(x) \geq (1/C) e^{-C(\beta-\alpha)/\epsilon}$ for all $x \in \Sigma \backslash \{F = \tilde{\gamma}\}^{\epsilon\delta/8}$. Invoking (2.19), we thus obtain $s^{0}_{\Sigma}(\{F = \gamma\}) \leq C e^{C(\beta-\alpha)/\epsilon}$. By (3.31) the proof is complete. $\diamondsuit$

The estimate given in the previous lemma becomes rough when α and β are fixed while ϵ tends to zero. In the following lemma we consider the case where the region is fixed.

Lemma 3.5: *Fix a regular, closed, disjoint, nonempty sets A and B with bounded complement $\mathbb{R}^d \backslash A \backslash B$. Assume that $B_I \subset A$ and $B_J \subset B$, where $I \equiv \mathcal{M} \cap A$ and $J \equiv \mathcal{M} \cap B$. Assume that $0 \leq \lambda s^0(A, B) \leq 1/2$. Then for some $N \equiv N(d)$ and $C \equiv C(d)$ all $x \notin A \cup B$*

$$\frac{h^{\lambda}_{A,B}(x)}{h^{0}_{A,B}(x)} \leq 1 + \lambda C \epsilon^{-N} (T_{I \cup J} + |\mathbb{R}^d \backslash A \backslash B|) \tag{3.32}$$

Moreover,

$$\frac{w^{\lambda}_{A,B}(x)}{w^{0}_{A,B}(x)} \leq 1 + \lambda C \epsilon^{-N} (T_{I \cup J} + |\mathbb{R}^d \backslash A \backslash B|) \tag{3.33}$$

Proof: The proof of (3.32) is almost identical to that of Lemma 3.4, except that we estimate $s^0(A, B)$ via (3.3). For the proof of (3.33) we apply (1.3) to $h \equiv w^{\lambda}_{A,B} - w^{0}_{A,B}$ and obtain for all $x \notin A \cup B$

$$\begin{aligned} \frac{w^{\lambda}_{A,B}(x)}{w^{0}_{A,B}(x)} - 1 =& \frac{\lambda}{w^{0}_{A,B}(x)} G^{0}_{\mathbb{R}^d \backslash A \backslash B} \left(h^{0}_{A,B} \frac{w^{0}_{A,B}}{h^{0}_{A,B}} \frac{w^{\lambda}_{A,B}}{w^{0}_{A,B}} \right)(x) \\ &+ \frac{1}{w^{0}_{A,B}(x)} G^{0}_{\mathbb{R}^d \backslash A \backslash B} \left(h^{0}_{A,B} \left(\frac{h^{\lambda}_{A,B}}{h^{0}_{A,B}} - 1 \right) \right)(x) \\ \leq& \lambda s^{0}_{A,B} \sup_{\mathbb{R}^d \backslash A \backslash B} \frac{w^{\lambda}_{A,B}}{w^{0}_{A,B}} + \sup_{\mathbb{R}^d \backslash A \backslash B} \frac{h^{\lambda}_{A,B}}{h^{0}_{A,B}} - 1 \end{aligned} \tag{3.34}$$

By the Hopf's Maximum Principle we again may take the supremum over $\mathbb{R}^d \backslash A \backslash B$ in this inequality. The assertion thus follows from (3.33) and (3.3). $\diamondsuit$

We now turn to the investigation of Laplace transforms in the non-compact case. Exploiting the strong drift towards the local minima of F, we can prove

Proposition 3.6: *Assume that F satisfies hypothesis 1.2 and let Ω be a regular, open, connected set Ω independent of $\epsilon > 0$ and containing $\{F < C_0\}$, where C_0 is defined in 1.2. Fix $I \subset \mathcal{M}$ and let Ω_0 be a union of $|I|$ balls $B_y \equiv B(\tilde{y}, \epsilon/4) \subset B(y, \sqrt{\epsilon})$, $y \in I$. There is $N \equiv N(d)$ such that for all $C_2 > C_0$ and $R > 0$ we find $C \equiv C(d, F|\{F < C_2\}, |\{F < C_2\}|, R, \rho)$ with the following property. For all $0 \leq \lambda \leq \epsilon^N / T_I$, all non-negative $f \in L^\infty(\mathbb{R}^d) \backslash 0$ satisfying $\sup_{\partial B_y} f \leq R \inf_{\partial B_y} f$ and all $x \in \Gamma \equiv \Omega \backslash \{F \geq C_0\} \backslash \overline{\Omega}_0$ it follows*

$$H^\lambda_{\Omega \backslash \overline{\Omega}_0} \mathbb{1}_{\partial\Omega_0} f(x) \leq (1 + C\epsilon^{-N}(\lambda T_I + e^{-(C_2 - C_0)/\epsilon})) H^0_\Gamma \mathbb{1}_{\partial\Omega_0} f(x) \tag{3.35}$$

and for all $y \in I$

$$w^\lambda_{B_y, B_I \cup \Omega^c}(x) \leq (1 + C\epsilon^{-N}\lambda T_I) w^0_{B_y, B_I \cup \Omega_2^c}(x) + C \sup_{\Gamma_1} w^0_{B_y, B_I \cup \Omega_2^c} \epsilon^{-N} e^{-(C_2 - F(x))/\epsilon} \tag{3.36}$$

Before we turn to the proof of this Proposition, we note the following a priori lower bound on principal eigenvalues in unbounded domains, which is an immediate consequence of (3.35).

Corollary 3.7: *In the situation of the previous Proposition we have*

$$\lambda(\Omega \backslash \overline{B}_{\tilde{I}}) \geq \frac{\epsilon^{-N}}{T_I}, \qquad B_{\tilde{I}} \equiv \bigcup_{y \in I} B_y \tag{3.37}$$

Let us now specify about the choice of $(\Omega_k)_{0 \leq k \leq L+1}$ for which we want to apply the random walk representation written in (3.24). Depending on Ω and $I \subset \mathcal{M}$ we define Ω_0 as in Proposition 3.6, i.e.

$$\Omega_0 \equiv \bigcup_{y \in I} B_y, \qquad B_y \equiv B(\tilde{y}, \epsilon/4) \tag{3.38}$$

For $\gamma > 0$ let $(z_k)_{k \geq 1}$ be a sequence in $\overline{\Omega}_0^c$ such that $\inf_{k \geq 2} F(z_{k+1}) - F(z_k) \geq \gamma > 0$ and such that $F(z_1) = C_0$ and $F(z_2) = C_2 \geq C_0 - \gamma$. We define

$$\Omega_k \equiv \{z \in \Omega \,|\, F(z) < F(z_k)\} \tag{3.39}$$

The following consequence of Proposition [4.7 in BEGK3] can be used to show that the random walk expansion (3.24) converges exponentially fast.

Lemma 3.8: *There is $C \equiv C(d)$ such that for all $\alpha < \beta + \epsilon \log \epsilon$ and all regular, open sets $\Sigma \subset K \subset \Gamma$ satisfying $K \subset \{F \leq \alpha\}$ and $\{F < \beta\} \subset \Gamma$*

$$||\mathbb{1}_{K \backslash \overline{\Sigma}} H^0_{\Gamma \backslash \overline{\Sigma}} \mathbb{1}_{\partial\Gamma}|| = \sup_{K \backslash \overline{\Sigma}} h^0_{\Gamma^c, \Sigma} \leq C e^{-(\beta - \alpha)/\epsilon} / \epsilon \tag{3.40}$$

Proof: It follows from Propositions [4.3 and 4.7 in BEGK3] the existence of $C(d)$ such that for all $x \in K$

$$h^0_{\Gamma^c,\Sigma}(x) \leq C\frac{\text{cap}_{B(x,\delta(x)\epsilon)}(\Gamma^c)}{\text{cap}_{B(x,\delta(x)\epsilon)}(\Gamma^c \cup \Sigma)} \leq \frac{C}{\delta(x)\sup_{B(x,\epsilon\delta(x))}|\nabla F|}e^{-(\beta-\alpha)/\epsilon}/\epsilon \tag{3.41}$$

(3.40) holds since $\delta(x)\sup_{B(x,\epsilon\delta(x))}|\nabla F| = 1/8$. $\diamondsuit$

Proof of Proposition 3.6: Choose $\tilde{N}$ and define $\gamma \equiv \tilde{N}\epsilon\log(1/\epsilon) + \epsilon\log T_I$

$$h^\lambda_L(x) \equiv H^\lambda_{\Gamma_1}1\!\!1_{\partial\Omega_0}f(x) + \sum_{T=1}^{\infty}\sum_{\substack{\omega:1\sim 1\\ |\omega|=2T}}\prod_{t=1}^{2T-1}H^\lambda_{\Gamma_{\omega_t}}1\!\!1_{\partial\Omega_{\omega_{t+1}}}H^\lambda_{\Gamma_1}1\!\!1_{\partial\Omega_0}f(x) \tag{3.42}$$

For each sequence $\omega : 1 \sim 1$ of length $2T > 0$ there are precisely T steps of the form $(k, k+1)$, $1 \leq k < L$. Furthermore, for some universal constant C there are at most $(2T)!/(2T!^2) \leq C4^T/\sqrt{T}$ such sequences. Hence, combination of (3.40) with (3.28) for $\Gamma \equiv \Gamma_k$, $k \geq 3$, and (3.32) for $\Gamma \equiv \Gamma_1, \Gamma_2$ gives

$$\begin{aligned}\sup_{\Gamma_1} h^\lambda_L \leq &\big(1 + \lambda C\epsilon^{-N}(T_I + |\{F \leq F(z_2)\}|)\big)\sup_{\partial\Omega_0} f \\ &+ \big(1 + \lambda C\epsilon^{-N}(T_I + |\{F \leq F(z_2)\}|)\big)^2 \sup_{\partial\Omega_0} f \\ &\times \sum_{T=1}^{\infty} C4^T(1 + C\lambda e^{C\gamma/\epsilon})^{2(T-1)}C^T\epsilon^{1-T}e^{-\gamma(T-1)/\epsilon}/\sqrt{T}\end{aligned} \tag{3.43}$$

where we have used that $\epsilon^{-N} + |\{F(z_1) \leq F \leq F(z_2)\}| < Ce^{C\gamma/\epsilon}$ for some $\tilde{N} \equiv \tilde{N}(d)$ and some constant $C \equiv C(d, |\{F \leq F(z_2)\}|)$. Choosing $N \equiv N(d)$ appropriately, we may estimate $C\lambda e^{C\gamma/\epsilon}$ by one for all $0 \leq \lambda < \epsilon^N T_I$. (3.43) then tells us for some constant $C \equiv C(d, |\{F \leq F(z_2)\}|)$

$$\sup_{\Gamma_1} h^\lambda \leq C\sup_{\partial\Omega_0} f \tag{3.44}$$

Since h^λ_L is a bounded, analytic continuation of $\{\text{Re}\,\lambda \leq 0\} \ni \lambda \mapsto H^\lambda_{\Omega_{L+1}\backslash\overline{\Omega}_0}1\!\!1_{\partial\Omega_0}f$ by Lemma 3.3, we have proven $h^\lambda_L = H^\lambda_{\Omega_{L+1}\backslash\overline{\Omega}_0}1\!\!1_{\partial\Omega_0}f$ for all $0 \leq \lambda \leq \epsilon^{-N}/T_I$ and for all L. We now use the equation

$$h^\lambda_L = H^\lambda_{\Gamma_1}1\!\!1_{\partial\Omega_0}f + H^\lambda_{\Gamma_1}1\!\!1_{\partial\Omega_2}h^\lambda_L \tag{3.45}$$

in Γ_1 so that by the condition on f, (3.44) and (3.32) in combination with (3.3) for $A \equiv \overline{\Omega}_0$ and $B \equiv \Omega^c_2$ for some $C \equiv (d, F|\{F < C_2\}, |\{F < C_2\}|, R)$ and all $x \in \overline{\Omega}_1\backslash\overline{\Omega}_0$

$$\begin{aligned}\frac{h^\lambda_L(x)}{H^\lambda_{\Gamma_1}1\!\!1_{\partial\Omega_0}f(x)} \leq &1 + C\frac{h^\lambda_{\Omega^c_2,\overline{\Omega}_0}(x)}{h^\lambda_{\overline{\Omega}_0,\Omega^c_2}(x)} \leq 1 + C\frac{h^0_{\Omega^c_2,\overline{\Omega}_0}(x)}{1 - h^0_{\Omega^c_2,\overline{\Omega}_0}(x)} \\ \leq &1 + Ce^{-(F(z_2)-F(z_1))/\epsilon}/\epsilon\end{aligned} \tag{3.46}$$

where we have used Corollary [4.7 in BEGK3] in the latter inequality. Since the constant in (3.46) does not depend on L, $h^\lambda_L(x)$ converges increasingly to a $(L_\epsilon - \lambda)$ - harmonic function on $\Omega\backslash\overline{\Omega}_0$ which equals $1\!\!1_{\partial\Omega_0} f$ on $\partial\Omega \cup \partial\Omega_0$ and thus equals $H^\lambda_{\Omega\backslash\overline{\Omega}_0} 1\!\!1_{\partial\Omega_0} f$. To derive the result, it remains to compare $H^\lambda_{\Gamma_1} 1\!\!1_{\partial\Omega_0} f$ with $H^0_{\Gamma_1} 1\!\!1_{\partial\Omega_0} f$. But (3.32) once more for $A \equiv \overline{\Omega}_0$ and $B \equiv \Omega^c_2$ shows after possibly increasing N for some $C \equiv C(d)$, some $\hat N \equiv \hat N(d)$ and all $x \in \Gamma_1$

$$\frac{H^\lambda_{\Gamma_1} 1\!\!1_{\partial\Omega_0} f(x)}{H^0_{\Gamma_1} 1\!\!1_{\partial\Omega_0} f(x)} \le 1 + C\epsilon^{-\hat N}\lambda T_I \tag{3.47}$$

again by the condition on λ . This proves the first equation in (3.35).

Differentiation with respect to λ of (3.45) for $f \equiv 1\!\!1_{B_x}$ in combination with

$$\dot H^\lambda_\Sigma = G^\lambda_\Sigma H^\lambda_\Sigma, \qquad \Sigma \quad \text{regular, open} \tag{3.48}$$

which follows from taking derivatives in (1.3), yields

$$\begin{aligned} w^\lambda_{B_x, B_I\cup\Omega^c_{L+1}}(x) = {} & w^\lambda_{B_x,B_I\cup\Omega^c_2}(x) + \sum_{T=1}^\infty \sum_{\substack{\omega:1\sim 1\\ |\omega|=2T}} \prod_{t=1}^{2T-1} H^\lambda_{\Gamma_{\omega_t}} 1\!\!1_{\partial\Omega_{\omega_{t+1}}} w^\lambda_{B_x,B_I\cup\Omega^c_2}(x) \\ & + \sum_{T=1}^\infty \sum_{\substack{\omega:1\sim 1\\ |\omega|=2T}} \sum_{s=1}^{2T-1} \prod_{t=1}^{s-1} H^\lambda_{\Gamma_{\omega_t}} 1\!\!1_{\partial\Omega_{\omega_{t+1}}} G^\lambda_{\Gamma_{\omega_s}} H^\lambda_{\Gamma_{\omega_s}} 1\!\!1_{\partial\Omega_{\omega_{s+1}}} \\ & \circ \prod_{t=s+1}^{2T-1} H^\lambda_{\Gamma_{\omega_t}} 1\!\!1_{\partial\Omega_{\omega_{t+1}}} H^\lambda_{\Gamma_1} 1\!\!1_{B_x}(x) \end{aligned} \tag{3.49}$$

For $2 \le k \le L$ we estimate by (3.28), (2.18), (2.19) for some $C \equiv C(d)$ and $0 \le f \in L^\infty(\mathbb{R}^d)$ on $\partial\Omega_k$

$$\begin{aligned} G^\lambda_{\Gamma_k} H^\lambda_{\Gamma_k} 1\!\!1_{\partial\Omega_{k\pm1}} f(x) \le & \sup_{\partial\Omega_k} h^0_{\partial\Omega_{k\pm1},\partial\Gamma_k} C \frac{\sup_{\partial\Omega_k} w^\lambda_{\partial\Omega_{k\pm1},\partial\Gamma_k}}{\inf_{\partial\Omega_k} h^0_{\partial\Omega_{k\pm1},\partial\Gamma_k}} \sup_{\partial\Omega_{k\pm1}} f \\ \le & Ce^{C\gamma/\epsilon} \sup_{\partial\Omega_k} h^0_{\partial\Omega_{k\pm1},\partial\Gamma_k} \sup_{\partial\Omega_{k\pm1}} f \end{aligned} \tag{3.50}$$

By (3.28) and (3.40) again and by (3.50), a similar argumentation as in (2.26) leads to

$$\begin{aligned} \sup_{\Gamma_1} w^\lambda_{B_x,B_I\cup\Omega^c_{L+1}} \le & \sup_{\Gamma_1} w^\lambda_{B_x,B_I\cup\Omega^c_2} \\ & + \left(1 + \lambda C\epsilon^{-N}(T_I + |\{F \le F(z_2)\}|)\right) \sup_{\Gamma_1} w^\lambda_{B_x,B_I\cup\Omega^c_2} \\ & \times \sum_{T=1}^\infty C4^T (1 + C\lambda e^{C\gamma/\epsilon})^{2(T-1)} C^T e^{-\gamma(T-1)/\epsilon}/(\sqrt{T}\epsilon) \\ & + \left(1 + \lambda C\epsilon^{-N}(T_I + |\{F \le F(z_2)\}|)\right) \\ & \times \sum_{T=1}^\infty \sqrt{T} C e^{C\gamma/\epsilon}(1 + C\lambda e^{C\gamma/\epsilon})^{2(T-1)} C^T \epsilon^{1-T} e^{-\gamma(T-1)/\epsilon} \end{aligned} \tag{3.51}$$

for some $C \equiv C(d, |\{F \leq F(z_2)\}|)$. (3.51) implies for large $\tilde{N}$

$$\sup_{\Gamma_1} w^{\lambda}_{B_x, B_I \cup \Omega^c_{L+1}} \leq 2 \sup_{\Gamma_1} w^{\lambda}_{B_x, B_I \cup \Omega^c_2} \tag{3.52}$$

where we have used $\sup_{\Gamma_1} w^{\lambda}_{B_x, B_I \cup \Omega^c_{L+1}} \geq 1$. (1.3) in combination with (3.52) and (3.40) gives for all $y \in \Omega_1 \backslash B_I$

$$\begin{aligned} w^{\lambda}_{B_x, B_I \cup \Omega^c}(y) =& w^{\lambda}_{B_x, B_I \cup \Omega^c_2}(y) + H^{\lambda}_{\Gamma_1} \mathbb{I}_{\partial \Omega_2} w^{\lambda}_{B_x, B_I \cup \Omega^c}(y) \\ \leq& w^{\lambda}_{B_x, B_I \cup \Omega^c_2}(y) + C \sup_{\Gamma_1} w^{\lambda}_{B_x, B_I \cup \Omega^c_2} e^{-(C_2 - F(y))/\epsilon}/\epsilon \end{aligned} \tag{3.53}$$

The second assertion in (3.35) now follows form (3.3) in combination with (3.33). $\diamondsuit$

As an immediate corollary from Proposition 3.6 we have the following relation between eigenfunctions with small eigenvalue and equilibrium potentials. Recall that we have defined for $\beta > 0$

$$A^{\beta}_{I,J} \equiv \{y \in \mathbb{R}^d \mid \hat{F}(y, I)) \leq \hat{F}(y, J \backslash I)) - \beta\} \tag{3.54}$$

Corollary 3.9: *Choose $N \equiv N(d)$ and $C \equiv C(d, F|\{F < C_2\}, |\{F < C_2\}|)$ as in 3.6 and let ϕ be a solution of (2.1) such that $0 \leq \lambda \leq \epsilon^N / T_{\tilde{I}}$, where $\tilde{\mathcal{M}}$ consists of $|\mathcal{M}|$ points such that $\tilde{\mathcal{M}} \cap B(y, \sqrt{\epsilon})$ is a singleton for all $y \in \mathcal{M}$ and such that - according to Lemma 2.3 - ϕ does not change sign in $B(y, \epsilon)$ for all $y \in \tilde{I}$. For large C_2 it follows for all $x \in \Gamma$*

$$\phi(x) = \sum_{y \in \tilde{I}} \phi_y(x), \qquad \phi_y(x) \equiv H^{\lambda}_{\Omega \backslash \overline{\Omega}_0} \mathbb{I}_{\partial B_y} \phi(x) = \big(1 + \mathcal{O}(1) \epsilon^{-N} \lambda T_{\tilde{I}}\big) H^0_{\Gamma} \mathbb{I}_{\partial B_y} \phi(x) \tag{3.55}$$

where the modulus of the Landau symbol is dominated by C.

Moreover, let $x \in \tilde{I}$ be such that $\sup_{\partial B_x} |\phi| = \sup_{\partial B_{\tilde{I}}} |\phi|$. Then for some $N \equiv N(d)$ and all $\beta > N \epsilon \log(1/\epsilon)$ there is a constant $C \equiv C(d, F|\{F < C_2\}, |\{F < C_2\}|)$ dominating the modulus of the Landau symbol in the estimate

$$\phi(y) = (1 + \mathcal{O}(1) \epsilon^{-N} (\lambda T_{\tilde{I}} + e^{-\beta/\epsilon})) \phi(x), \qquad y \in A^{\beta}_{x, \tilde{I}} \tag{3.56}$$

Proof: We have to prove the last equation in (3.55). We may assume that ϕ is normalized such that it is positive on B_y. Since ϕ is nonnegative in B_y, the Harnack's inequality ensures the existence of $C \equiv C(d, F|B(y, \sqrt{\epsilon}))$ such that $\sup_{B_y} \phi \leq C \inf_{B_y} \phi$. (3.35) for $f \equiv \mathbb{I}_{\partial B_y} \phi$ is the last equation in (3.55).

For the proof of (3.56) we first note that Corollary [4.8 in BEGK3] in combination with (3.55) and the condition on x implies for all $y \in \tilde{I}\backslash x$ and all $z \in \Sigma \equiv A^{\beta}_{x,\tilde{I}}$

$$|\phi_y(z)| \leq 2h^0_{B_y,B_{\tilde{I}}}(z) \sup_{\partial B_x} |\phi| \leq Ce^{-\beta/\epsilon}|\phi(x)|/\epsilon \tag{3.57}$$

where we have used the Harnack's inequality to replace the supremum in the latter inequality. Furthermore, for all $z \in \Sigma$

$$|\phi_x(z)| \geq (1/2)h^0_{B_x,B_{\tilde{I}}}(z) \inf_{B_x} |\phi_x| \geq (1/3)|\phi(x)| \tag{3.58}$$

In particular, ϕ does not change sign in Σ. In view of (3.55) again and (3.57) we now may choose $r \equiv C\epsilon^{-N}(\lambda T_{\tilde{I}} + e^{-\beta/\epsilon})$ in (2.12) applied to ϕ. Since λ is smaller than the error terms written above, (3.56) follows from (2.12). $\diamondsuit$

4. Small eigenvalues

In this chapter we derive precise asymtotics of the exponentially small eigenvalues of L_ϵ in a regular, open set Ω independent of ϵ and containing $\{F < C_0\}$, where C_0 was introduced in 1.2. We first relate these eigenvalues to the capacity matrix defined in (4.2). In the last section we show that for generic F that they are exponentially close to certain principal eigenvalues. In the following section we therefor study principal eigenvalues in detail.

Sharp uncertainty principle: In the sequel we want to derive necessary conditions on small eigenvalues by relating them to a matrix which in leading order equals the capacity matrix introduced in [J]. Namely, fix an eigenvalue $0 \le \lambda < \epsilon$ with corresponding eigenfunction ϕ. For each $x \in \mathcal{M}$ fix $\tilde{x} \in B(x, \sqrt{\epsilon})$ such that ϕ does not change sign in $B(\tilde{x}, \epsilon)$. The existence of $\tilde{x}$ is guaranteed by Lemma 2.3. Let $\tilde{\mathcal{M}}$ be the collection of all such $\tilde{x}$ and put

$$B_{\tilde{I}} \equiv \bigcup_{x \in \tilde{I}} B(x, \epsilon/4), \qquad \tilde{I} \subset \tilde{\mathcal{M}} \tag{4.1}$$

We define $C_{\tilde{I}}(\lambda, \phi)$ to be the matrix with entries

$$C_{\tilde{I}}(\lambda, \phi)_{yz} \equiv \epsilon \int_{\partial B_z} e^{-F/\epsilon} \frac{\phi}{\phi(z)} \partial_n h^\lambda_{y,\tilde{I}} \, d\sigma - \delta_{zy} \lambda \int_{B_y} e^{-F/\epsilon} \frac{\phi}{\phi(y)} \, dx \tag{4.2}$$

where $y, z \in \tilde{I}$. For the sake of convenience we henceforth write shorthand for $\tilde{I}, \tilde{J} \subset \tilde{\mathcal{M}}$

$$h^\lambda_{\tilde{I},\tilde{J}} \equiv h^\lambda_{B_{\tilde{I}}, B_{\tilde{J}} \cup \Omega^c}, \qquad \mathrm{cap}^\lambda(\tilde{I}, \tilde{J}) \equiv \mathrm{cap}^\lambda_{B_{\tilde{I}}}(B_{\tilde{J}} \cup \Omega^c), \qquad \lambda_{\tilde{I}} \equiv \lambda(\Omega \backslash B_{\tilde{I}}) \tag{4.3}$$

Note that the choice of $\tilde{y} \in B(y, \sqrt{\epsilon})$, $\tilde{y} \in \tilde{I}$, $y \in \mathcal{M}$, a priori depends on ϕ.

Lemma 4.1: *Let $0 \le \lambda < \lambda_{\tilde{I}}$ for some $\tilde{I} \subset \tilde{\mathcal{M}}$. Then $\lambda \in \sigma(L^\Omega_\epsilon)$ implies $\det C_{\tilde{I}}(\lambda, \phi) = 0$. Moreover, the vector $\vec{\phi} \equiv (\phi(\tilde{y}))_{\tilde{y} \in \tilde{I}}$ solves the system $C_{\tilde{I}}(\lambda, \phi)\vec{\phi} = 0$.*

Proof: This characterization is a consequence of the second Green's formula applied to $B_{\tilde{I}}$ and $\Omega \backslash \overline{B}_{\tilde{I}}$ showing for all $y \in \tilde{I}$

$$\begin{aligned} 0 &= \int_\Omega e^{-F/\epsilon} h^\lambda_{y,\tilde{I}} (L - \lambda) \phi \, dx \\ &= \sum_{z \in \tilde{I}} \int_{\partial B_z} e^{-F/\epsilon} \phi \, \partial_n h^\lambda_{y,\tilde{I}} \, d\sigma - \lambda \int_{B_y} e^{-F/\epsilon} \phi \, dx, \end{aligned} \tag{4.4}$$

where we have used that the normal derivative of ϕ at $\partial B_{\tilde{I}}$ taken from inside $\Omega \backslash \overline{B}_{\tilde{I}}$ equals the negative of the normal derivative taken from inside $B_{\tilde{I}}$. $\diamondsuit$

Lemma 4.1 can be used to analyze principal eigenvalues leading to the sharp uncertainty principle 4.2. As in [BEGK2] or [E2] one proves $\lambda_{\tilde I} < \lambda_{\tilde I \cup x}$ for $x \in \tilde{\mathcal{M}} \backslash \tilde I$ so that from 4.1 it follows

$$\epsilon \int_{\partial B_x} e^{-F/\epsilon} \phi_{\tilde I} \partial_n h^{\lambda_{\tilde I}}_{x,\tilde I}\, d\sigma - \lambda_{\tilde I} \int_{B_x} e^{-F/\epsilon} \phi_{\tilde I}\, dy = 0 \tag{4.5}$$

where $\phi_{\tilde I}$ is the principal eigenfunction of L^Ω_ϵ such that $\phi_{\tilde I}(x) = 1$. This equation implies

Theorem 4.2: *There exists $N \equiv N(d)$ and $C \equiv C(d,F)$ with the following properties. For nonempty, properly contained $\tilde I \subset \tilde{\mathcal{M}}$ and $x \in \mathcal{M}\backslash \tilde I$ such that $T_{x,\tilde I} = T_{\tilde I} < \epsilon^N T_{\tilde I \cup x}$ we have*

$$\lambda_{\tilde I} = \left(1 + \mathcal{O}(1)\epsilon^{-N} \frac{T_{\tilde I \cup x}}{T_{\tilde I}}\right) \frac{\operatorname{cap}^0(x, \tilde I)}{\int_{A_{x,\tilde I}} e^{-F/\epsilon}\, dy} \tag{4.6}$$

where $A_{x,\tilde I} \equiv A^0_{x,\tilde I}$ was defined in (3.54) and where the modulus of the Landau symbol is dominated by C.

Proof: Taylor's formula shows for all $y \in B^r_x \backslash B_x$, $0 < r < \epsilon$, and some $0 < \lambda_0 \equiv \lambda_0(y) < \lambda_{\tilde I}$

$$h^{\lambda_{\tilde I}}_{x,\tilde I}(y) = h^0_{x,\tilde I}(y) + \lambda_{\tilde I} w^0_{x,\tilde I}(y) + \lambda^2_{\tilde I} \dot w^{\lambda_0}_{x,\tilde I}(y)/2 \tag{4.7}$$

By Chauchy's inequality and (3.37) we may bound for some universal C

$$\dot w^{\lambda_0}_{x,\tilde I}(y) \le (C/(\lambda_1 - \lambda_0)) w^{\lambda_1}_{x,\tilde I}(y) \le C\epsilon^{-N} T_{\tilde I \cup x} w^{\lambda_1}_{x,\lambda_1}(y) \tag{4.8}$$

for some $\lambda_1 < e^N / T_{\tilde I \cup x}$ not depending on y so that from (4.7) it follows

$$h^{\lambda_{\tilde I}}_{x,\tilde I}(y) = h^0_{x,\tilde I}(y) + \lambda_{\tilde I} w^0_{x,\tilde I}(y) + \mathcal{O}(1)\lambda_{\tilde I} \frac{T_{\tilde I \cup x}}{T_{\tilde I}} w^{\lambda_1}_{x,\tilde I}(y) \tag{4.9}$$

where we have used (3.37) once more. From (4.9) and (4.7) we obtain for $y \in \partial B_x$ the double side estimate

$$\frac{\lambda_{\tilde I}}{C\epsilon^N} \frac{T_{\tilde I \cup x}}{T_{\tilde I}} \partial_n w^{\lambda_1}_{x,\tilde I}(y) \le \partial_n h^{\lambda_{\tilde I}}_{x,\tilde I}(y) - \partial_n h^0_{x,\tilde I}(y) - \lambda_{\tilde I} \partial_n w^0_{x,\tilde I}(y) \le 0 \tag{4.10}$$

where the normal derivative is taken from outside B_x. We want to estimate for $\lambda \equiv 0, \lambda_1$

$$\begin{aligned} \epsilon \int_{\partial B_x} e^{-F/\epsilon} h^\lambda_{B_x, \mathbb{R}^d \backslash B^{\sqrt\epsilon}_x} \partial_n w^\lambda_{x,\tilde I}\, d\sigma = &- \int_{B^{\sqrt\epsilon}_x \backslash B_x} e^{-F/\epsilon} h^\lambda_{B_x, \mathbb{R}^d \backslash B^{\sqrt\epsilon}_x}\, dy \\ &+ \epsilon \int_{\partial B^{\sqrt\epsilon}_x} e^{-F/\epsilon} w^\lambda_{x,\tilde I} \partial_n h^\lambda_{B_x, \mathbb{R}^d \backslash B^{\sqrt\epsilon}_x}\, d\sigma \end{aligned} \tag{4.11}$$

where the latter equality uses Green's second formula. By (3.36) $w^{\lambda}_{x,\tilde{I}}$ is bounded by

$$(1 + C\lambda T_{\tilde{I}\cup x}) w^{0}_{x,\tilde{I}\cup\Sigma^c} + Ce^{-(C_2-F)/\epsilon} \sup_{F\leq C_0} w^{0}_{x,\tilde{I}\cup\Sigma^c}/\epsilon \tag{4.12}$$

where in slight abuse of notation $w^{\lambda}_{x,\tilde{I}\cup\Sigma^c} \equiv w^{\lambda}_{B_x,B_{\tilde{I}}\cup\Sigma^c}$ for $\Sigma \equiv \{F < C_2\}$ and some $C \equiv C(d, F|\{F \leq C_2\}, |\{F \leq C_2\}|)$. Combination of (4.12) with (3.32), (3.33) and (3.3) we derive from (4.11) for some C as before and some $N \equiv N(d)$

$$\begin{aligned}\epsilon \int_{\partial B_x} e^{-F/\epsilon} \partial_n w^{\lambda}_{x,\tilde{I}}\, d\sigma \geq& (1 - C\epsilon^{-N}\lambda T_{\tilde{I}\cup x})\epsilon \int_{\partial B_x} e^{-F/\epsilon}\partial_n w^{0}_{x,\tilde{I}\cup\Sigma^c}\, d\sigma \\ &- C\epsilon^{-N} \operatorname{cap}^{0}_{B_x}(\mathbb{R}^d\backslash B^{\sqrt{\epsilon}}_x) T_{\tilde{I}\cup x} e^{-(C_2-F(x))/\epsilon}/\epsilon\end{aligned} \tag{4.13}$$

Again by Green's second formula we compute for $\beta > N\epsilon\log(1/\epsilon)$

$$\begin{aligned}-\epsilon \int_{\partial B_x} e^{-F/\epsilon} h^{0}_{x,\tilde{I}} \partial_n w^{0}_{x,\tilde{I}\cup\Sigma^c}\, d\sigma =& \int_{\Sigma\backslash B_{\tilde{I}\cup x}} e^{-F/\epsilon} h^{0}_{x,\tilde{I}}\, dy \\ =& \sum_{y\in\tilde{I}} \int_{A^{\beta}_{y,x}\backslash B_y} e^{-F/\epsilon} h^{0}_{x,\tilde{I}}\, dz + \int_{A^{\beta}_{x,\tilde{I}}\backslash B_x} e^{-F/\epsilon} h^{0}_{x,\tilde{I}}\, dz \\ &+ \int_{\Sigma\backslash\bigcup_{y\in\tilde{I}} A^{\beta}_{y,x}\backslash A^{\beta}_{x,\tilde{I}}} e^{-F/\epsilon} h^{0}_{x,\tilde{I}}\, dz\end{aligned} \tag{4.14}$$

Since F is bounded below by $\hat{F}(x,\tilde{I})) - \beta$ on $\Sigma\backslash\bigcup_{y\in\tilde{I}} A^{\beta}_{y,x}\backslash A^{\beta}_{x,\tilde{I}}$ we simply bound the last integral in (4.14) by

$$\left|\Sigma\backslash\bigcup_{y\in\tilde{I}} A^{\beta}_{y,x}\backslash A^{\beta}_{x,\tilde{I}}\right| e^{-(\hat{F}(x,\tilde{I}))-\beta)/\epsilon} \tag{4.15}$$

For the integrals in the sum on the right-hand side by Corollary [4.8 in BEGK3] we may bound $h^{0}_{x,\tilde{I}}$ on $B^{\epsilon}_y\backslash\overline{B}_y$, $y \in \tilde{I}$, by

$$h^{0}_{x,\tilde{I}} = H^{0}_{B^{\epsilon}_y\backslash\overline{B}_y} \mathbb{1}_{\partial B^{\epsilon}_y} h^{0}_{x,\tilde{I}} \leq Ce^{-(\hat{F}(x,\tilde{I}))-F(y))/\epsilon}/\epsilon \tag{4.16}$$

and on $A^{\beta}_{y,x}\backslash B^{\epsilon}_y$ by $Ce^{-(\hat{F}(x,\tilde{I}))-F)/\epsilon}/\epsilon$, so that

$$\int_{A^{\beta}_{y,x}\backslash B_y} e^{-F/\epsilon} h^{0}_{x,\tilde{I}}\, dz \leq C\big|A^{\beta}_{y,x}\backslash B_y\big| e^{-F(z^*(x,\tilde{I}))/\epsilon}/\epsilon \tag{4.17}$$

Concerning the second term on the right-hand side of (4.14), we again use Corollary [4.8 in BEGK3] $h^{0}_{B_{\tilde{I}},B}$, $B \equiv B_x\backslash(\mathbb{R}^d\backslash B_x)^{\epsilon/100}$, on $A^{\beta}_{x,\tilde{I}}\backslash B_x$ in combination with Green's second formula showing that for some $N \equiv N(d)$ and some $C \equiv C(d, F|A^{\beta}_{x,\tilde{I}}, |A^{\beta}_{x,\tilde{I}}|)$

$$\begin{aligned}\int_{A^{\beta}_{x,\tilde{I}}\backslash B_x} e^{-F/\epsilon} h^{0}_{x,\tilde{I}}\, dz + \int_{B_x} e^{-F/\epsilon}\, dz \geq& \int_{A^{\beta}_{x,\tilde{I}}\backslash B_x} e^{-F/\epsilon}(1 - h^{0}_{B_{\tilde{I}},B})\, dz + \int_{B_x} e^{-F/\epsilon}\, dz \\ \geq& (1 - C\epsilon^{-N}/T_{\tilde{I}}) \int_{A^{\beta}_{x,\tilde{I}}} e^{-F/\epsilon}\, dz\end{aligned} \tag{4.18}$$

Inserting (4.18), (4.17) and (4.15) into (4.14) and the result into (4.13), we derive

$$\epsilon\int_{\partial B_x} e^{-F/\epsilon}\partial_n w^\lambda_{x,\tilde I}\,d\sigma + \int_{B_x} e^{-F/\epsilon}\,dz \geq -\left(1 + C\lambda T_{\tilde I\cup x} + C\frac{\epsilon^{-N}+e^{\beta/\epsilon}}{T_{\tilde I}}\right)\int_{A^\beta_{x,\tilde I}} e^{-F/\epsilon}\,dz \\ - C\epsilon^{-N}T_{\tilde I\cup x}e^{-C_2/\epsilon} \tag{4.19}$$

where we have used $\mathrm{cap}^0_{B_x}(\mathbb{R}^d\backslash B_x^{\sqrt{\epsilon}}) \leq Ce^{-F(x)/\epsilon}/\epsilon^{d-1}$ following from Proposition [4.7 in BEGK3]. Using (4.11) in combination with (4.19) in (4.5) we now conclude

$$\mathrm{cap}^0(x,\tilde I) = \left(1 + \mathcal{O}(1)\epsilon^{-N}\left(\frac{1+e^{\beta/\epsilon}}{T_{\tilde I}} + e^{-(C_2-F(x))/\epsilon} + \frac{T_{\tilde I\cup x}}{T_{\tilde I}}\right)\right)\lambda_{\tilde I}\int_{A^\beta_{x,\tilde I}} e^{-F/\epsilon}\,dz \tag{4.20}$$

where we have replaced $\phi_{\tilde I}$ by $1+\mathcal{O}(1)e^{-N}(1/T_{\tilde I}+e^{-(C_2-F(x))/\epsilon}))$ in view of (3.56) and (2.9). Since C_2 can be made arbitrarily large, the proof is completed by choosing $\beta \equiv N\epsilon\log(1/\epsilon)$ and that $A^\beta_{x,\tilde I}$ may be replaced by $A^0_{x,\tilde I}$ without harm. $\diamondsuit$

(4.6) implies the following intuitive sharp uncertainty principle.

Corollary 4.3: *In the situation of the previous theorem, we have*

$$\lambda_{\tilde I} = \left(1 + \mathcal{O}(1)\epsilon^{-N}\frac{T_{\tilde I\cup x}}{T_{\tilde I}}\right)\frac{1}{\mathbb{E}^x\left[\tau_{B_{\tilde I}\cup\Omega^c}\right]} \tag{4.21}$$

where $\mathbb{E}^x\left[\tau_{B_{\tilde I}\cup\Omega^c}\right]$ *is the expected time of the first visit of* $B_{\tilde I}\cup\Omega^c$ *of the diffusion generated by* L_ϵ *and starting in* x.

Proof: We first note that by (2.17) for $\Sigma \equiv \Omega\backslash\{F \geq C_0\}$ and $\Gamma \equiv \Omega\backslash\{F \geq C_2\}$, $C_2 > C_0$,

$$s^0_{\Sigma\backslash\overline{B}_{\tilde I\cup x}}(B_{\tilde I}\cup\Omega^c) \leq s^0_{\Sigma\backslash\overline{B}_{\tilde I\cup x}} + \sup_{\Sigma} h^0_{\Gamma^c,B_{\tilde I\cup x}} s^0_{\partial\Gamma}(B_{\tilde I}\cup\Omega^c) + s^0_{\partial B_x}(B_{\tilde I}\cup\Omega^c) \tag{4.22}$$

Since for large C_2 the first term on the right-hand side of (4.22) is bounded by $C\epsilon^{-N}T_{\tilde I\cup x}$ for some $N \equiv N(d)$ and $C \equiv C(d, F|\{F < C_2\}, |\{F < C_2\}|)$ by (3.36) in combination with (3.3), after possibly increasing C_2 we obtain $s^0_{\Sigma\backslash\overline{B}_{\tilde I\cup x}}(B_{\tilde I}\cup\Omega^c) \leq C\epsilon^{-N}T_{\tilde I\cup x} + s^0_{\partial B_x}(B_{\tilde I}\cup\Omega^c)$. Invoking [(6.1) in BEGK3] in combination with the fact that the nominator in this estimate is computed in (4.18), we conclude

$$s^0_{\Sigma\backslash\overline{B}_{\tilde I\cup x}}(B_{\tilde I}\cup\Omega^c) \leq \left(1 + C\epsilon^{-N}\frac{T_{\tilde I\cup x}}{T_{\tilde I}}\right)s^0_{\partial B_x}(B_{\tilde I}\cup\Omega^c) \tag{4.23}$$

It follows from (2.12), (4.23) and [(6.1) in BEGK3] that

$$\sup_{B_x} w^0_{I\cup\Omega^c} \leq \left(1 + C\epsilon^{-N}\frac{T_{\tilde I\cup x}}{T_{\tilde I}}\right)\inf_{B_x} w^0_{I\cup\Omega^c} \tag{4.24}$$

It follows from the Green's second formula and (4.24) that modulo the error term appearing above

$$\begin{aligned} w^0_{I\cup\Omega^c}(x)\,\mathrm{cap}^0(x,I) =&\epsilon\int_{\partial B_x} e^{-F/\epsilon} w^0_{I\cup\Omega^c}\partial_n h^0_{x,I}\,d\sigma \\ =&-\epsilon\int_{\partial B_x} e^{-F/\epsilon}\partial_{-n} w^0_{I\cup\Omega^c}\,d\sigma+\int_{\Omega\setminus\overline{B}_x} e^{-F/\epsilon} h^0_{x,I}\,dy \\ =&\int_\Omega e^{-F/\epsilon} h^0_{x,I}\,dy \end{aligned} \tag{4.25}$$

The latter integral was computed in (4.18) so that the assertion follows from (4.6). $\diamondsuit$

We want to analyze the diagonal entries of the matrix $C_{\tilde I}(\lambda,\phi)$ in more detail.

Lemma 4.4: *In the situation of the previous theorem there are $N\equiv N(d)$ and $C\equiv C(d,F)$ satisfying for all $0\le\lambda\le\lambda_0$, $\lambda_0\equiv\epsilon^{-N}/T_{\tilde I\cup x}$,*

$$\begin{aligned} C_x(\lambda,\phi_{\tilde I})_{xx} =&-\left(1+\mathcal{O}(1)\epsilon^{-N}\lambda T_{\tilde I\cup x}\right) \\ &\left(\lambda-\lambda_{\tilde I}-(\lambda-\lambda_{\tilde I})^2\mathcal{O}(1)\epsilon^{-N}T_{\tilde I\cup x}\right)\int_{A_{x,\tilde I}} e^{-F/\epsilon}\,dy \end{aligned} \tag{4.26}$$

Proof: Performing a Taylor expansion at $\lambda\equiv\lambda_{\tilde I}$ to second order of the Laplace transform on the left-hand side of (4.26), we compute similarly to (4.10) using (4.5)

$$\begin{aligned} C_x(\lambda_{\tilde I},\phi_{\tilde I})_{xx} =&(\lambda-\lambda_{\tilde I})\left(\epsilon\int_{\partial B_x} e^{-F/\epsilon}\phi_{\tilde I}\partial_n w^{\lambda_{\tilde I}}_{x,\tilde I}\,d\sigma-\int_{B_x} e^{-F/\epsilon}\phi_{\tilde I}\,dx\right) \\ &+(\lambda-\lambda_{\tilde I})^2\epsilon\int_0^1 s\,ds\int_{\partial B_x} d\sigma e^{-F/\epsilon}\phi_{\tilde I}\partial_n \dot w^{(1-s)\lambda_{\tilde I}+s\lambda}_{x,\tilde I} \end{aligned} \tag{4.27}$$

Analogous to (4.9) by Chauchy's formula and (3.37) on ∂B_x we have the estimate

$$\partial_n \dot w^{(1-s)\lambda_{\tilde I}+s\lambda}_{x,\tilde I}\ge (C/(\lambda_1-\lambda))\partial_n w^{\lambda_1}_{x,\tilde I}\ge(100C/\lambda_0)\partial_n w^{\lambda_1}_{x,\tilde I} \tag{4.28}$$

for some universal constant C, $(1/100)\lambda_0\le\lambda_1<\lambda_0$ and all $0\le\lambda<(1/200)\lambda_0$. Moreover, (4.19) remains valid for $\lambda\equiv\lambda_I,\lambda_1$ so that analogously to (4.20) for C_2 large the last term on the right-hand side of (4.27) is of order

$$(\lambda-\lambda_{\tilde I})^2\epsilon^{-N}T_{\tilde I\cup x}\int_{A^\beta_{x,\tilde I}\setminus B_x} e^{-F/\epsilon}\,dy \tag{4.29}$$

while the first term can be estimated by

$$(\lambda-\lambda_{\tilde I})\left(1+\mathcal{O}(1)\epsilon^{-N}\left(\lambda_{\tilde I}T_{\tilde I\cup x}+\frac{1+e^{\beta/\epsilon}}{T_{\tilde I}}\right)\right)\int_{A^\beta_{x,\tilde I}} e^{-F/\epsilon}\,dx \tag{4.30}$$

In view of (3.37) from (4.30) and (4.29) for some $N \equiv N(d)$ and $\beta \equiv N\epsilon \log(1/\epsilon)$ the assertion follows. $\diamondsuit$

Small eigenvalues: We now turn to the main result of this paper. Namely, we will show how the capacity matrix introduced in (4.2) can be used to analyze the spectrum of L_ϵ^Ω near zero. The proof of Theorem 4.6 proceeds close to the line of arguments of the analogous assertion in [E2] or in [BEGK2]. In particular, for proofs, which are straightforward generalizations, we refer the reader to the counterparts therein. Let us briefly outline the strategy of the proof. For each $I \equiv \overline{\mathcal{M}}_x$, $x \in \mathcal{M}$ we look for a solution of the equation appearing in 4.1 near the ground state energy of the associated Dirichlet operator $L_\epsilon^{\Omega \setminus B_{\tilde{\mathcal{M}}_x}}$. We then show that these solutions are the only candidates for eigenvalues below ϵ^N for some N and that they have to be simple in case. Since from e.g. [S] or [CFKS] we already know that there are $|\mathcal{M}|$ many eigenvalues, these candidates in fact constitute all eigenvalues below ϵ^N.

In addition to the notation introduced in (3.1) let us define

$$\mathcal{M}(x) \equiv \{y \in \mathcal{M} \,|\, F(y) < F(x)\}, \qquad T_x \equiv T_{x,\mathcal{M}(x)}, \qquad x \in \mathcal{M} \tag{4.31}$$

in case that $\mathcal{M}(x) \neq \emptyset$. We us the convention $T_x \equiv \infty$ for $\mathcal{M}(x) \equiv \emptyset$.

Assumption 4.5: *F is generic in the sense that $\rho > N\epsilon \log(1/\epsilon)$ for some $N \equiv N(d)$, where ρ was introduced in (1.14).*

In particular, it follows as in [BEGK2] or [E2] that $\mathcal{M} \ni x \mapsto T_x$ is injective. We hence obtain an ordering of $\mathcal{M}$ via

$$x < y \quad \text{if and only if} \quad T_x > T_y \tag{4.32}$$

Let us define

$$\mathcal{M}_x \equiv \{y \in \mathcal{M} \,|\, y < x\}, \qquad \overline{\mathcal{M}}_x \equiv \{y \in \mathcal{M} \,|\, y \leq x\} \tag{4.33}$$

We also shall need

$$\mathcal{T}_x \equiv \min_{y \in \mathcal{M}_x} \min_{z \in \mathcal{M}_x \setminus y} T_{y,z} \tag{4.34}$$

The result states

Theorem 4.6: *Set $j_0 \equiv |\mathcal{M}|$. For some $N \equiv N(d)$ there is $C \equiv C(d,F)$ dominating all moduli of the Landau symbols appearing below. Assume 1.2 and 4.5. There are j_0 simple eigenvalues $\lambda_x < \lambda_y$, $x, y \in \mathcal{M}$, $x < y$, satisfying*

$$\sigma(L_\epsilon^\Omega) \cap [0, \epsilon^N) = \{\lambda_x \,|\, x \in \mathcal{M}\} \tag{4.35}$$

For every $x \in \mathcal{M}$, $x \neq \min \mathcal{M}$, we have $\mathcal{T}_x \geq e^{\rho/\epsilon} T_x$ and $T_x \geq e^{\rho/\epsilon} \max_{y \in \mathcal{M} \setminus \overline{\mathcal{M}}_x} T_y$, where ρ is defined in (1.14). Furthermore, there exists a set $\tilde{\mathcal{M}}_x$ of $|\mathcal{M}_x|$ points such that $\tilde{\mathcal{M}}_x \cap B(y, \sqrt{\epsilon})$, $y \in \mathcal{M}_x$, is a singleton, and

$$\lambda_x = \left(1 + \mathcal{O}(1)\epsilon^{-N} \left(\frac{T_x}{\mathcal{T}_x} + \frac{\max_{y \in \mathcal{M} \setminus \overline{\mathcal{M}}_x} T_y}{T_x}\right)\right) \lambda_{\tilde{\mathcal{M}}_x} \tag{4.36}$$

where we use the notation introduced in (4.1) and (4.3) for the principal eigenvalue $\lambda_{\tilde{I}}$. Moreover, every eigenfunction ϕ_x corresponding to λ_x satisfies for all $z \in \{F < C_0\} \cap \Omega$

$$\phi_x(z) = (1 + \mathcal{O}(1)\epsilon^{-N} e^{-\rho/\epsilon}) h^0_{x, \tilde{\mathcal{M}}_x}(y) + \sum_{y \in \tilde{\mathcal{M}}_x} \mathcal{O}(1)\epsilon^{-N} \frac{T_x}{T_{y,x}} h^0_{y, \tilde{\mathcal{M}}_x \cup x}(z) \tag{4.37}$$

Combination of 4.2 with (4.36) and (1.9) yields the following

Corollary 4.7: *In the situation of 4.6 we have for all $x \in \mathcal{M}$*

$$\lambda_x = (1 + \mathcal{O}(1)\epsilon^{-N} e^{-\rho/\epsilon}) \frac{1}{T_{\tilde{\mathcal{M}}(x)}} \tag{4.38}$$

In particular, under the conditions of 1.1 it follows

$$\lambda_x = (1 + \mathcal{O}(1)\epsilon \log(1/\epsilon)) |\lambda^*| \sqrt{\frac{\det \operatorname{Hess} F(x)}{|\det \operatorname{Hess} F(z^*)|}} e^{-(\hat{F}(x,I) - F(x))/\epsilon} \tag{4.39}$$

We have to introduce some more notation. Fix a set $\tilde{\mathcal{M}}$ of cardinality $|\mathcal{M}|$ such that $B(x, \sqrt{\epsilon}) \cap \tilde{\mathcal{M}}$ is a singleton. $\tilde{\mathcal{M}}$ inherits the ordering of $\mathcal{M}$ in an obvious way. Let us define $\tilde{\mathcal{E}}_x \equiv \infty$ for $x \equiv \min \tilde{\mathcal{M}}$ and for $x \in \tilde{\mathcal{M}} \setminus \min \tilde{\mathcal{M}}$ set

$$\tilde{\mathcal{E}}_x \equiv \min_{y \in \tilde{\mathcal{M}}_x} T_{y, \overline{\tilde{\mathcal{M}}_x} \setminus y} \tag{4.40}$$

The first lemma actually is a special case of Lemma [4.5 in E2] or Lemma [5.3 in BEGK2].

Lemma 4.8: *For all $x \in \tilde{\mathcal{M}}$ it follows*

$$T_x = T_{\tilde{\mathcal{M}}_x} = T_{x, \tilde{\mathcal{M}}_x} \tag{4.41}$$

For $x \in \tilde{\mathcal{M}} \setminus \min \tilde{\mathcal{M}}$ we have

$$\tilde{\mathcal{T}}_x \geq \tilde{\mathcal{E}}_x \geq e^{\rho/\epsilon} T_x \tag{4.42}$$

Moreover, for $x, y \in \tilde{\mathcal{M}}$, $y < x$,

$$\max_{z \in \tilde{\mathcal{M}} \setminus \overline{\tilde{\mathcal{M}}_x}} T_{z, \overline{\tilde{\mathcal{M}}_x} \setminus y} \leq e^{-\rho/\epsilon} T_{y, \overline{\tilde{\mathcal{M}}_x} \setminus y} \tag{4.43}$$

In particular, for $y \in \tilde{\mathcal{M}}_x$

$$T_{y, \overline{\tilde{\mathcal{M}}_x} \setminus y} = T_{\overline{\tilde{\mathcal{M}}_x} \setminus y} \tag{4.44}$$

We now analyze the possible solutions of the capacity matrix introduced in (4.2).

Henceforth, we assume for some $x \in \mathcal{M} \setminus \min \mathcal{M}$ that λ_x is an eigenvalue of L_ϵ^Ω satisfying for some $\alpha > 0$

$$e^{\alpha/\epsilon} / \mathcal{E}_x < \lambda_x < e^{-\alpha/\epsilon} T_{\overline{\mathcal{M}_x}} \tag{4.45}$$

Furthermore, ϕ_x denotes a corresponding eigenfunction.

For $y \in \overline{\mathcal{M}_x}$ according to 2.3 we choose a set $\overline{\tilde{\mathcal{M}}_x}$ of $|\overline{\mathcal{M}_x}|$ points each of them lying in one ball $B(y, \sqrt{\epsilon} - \epsilon)$, $y \in \mathcal{M}_x$, such that ϕ_x does not change sign in $B(\tilde{y}, \epsilon)$. (3.56) implies that ϕ_x does not change sign in $B(x, \epsilon/4)$. Indeed, assume that $|\phi_x|$ attains its supremum in some ball $B(y, \epsilon/4)$, where $y \in \tilde{I} \equiv \tilde{\mathcal{M}}_x$, $y \notin B(x, \sqrt{\epsilon})$ in 3.9. It follows from (3.56) in combination with the upper bound in (4.45) that ϕ_x does not change sign in $A^\beta_{y, \overline{\tilde{\mathcal{M}}_x}}$ for arbitrary but fixed $\beta > 0$. Choosing $u \equiv |\phi_x|$ restricted to $\overline{A}^\beta_{y, \overline{\tilde{\mathcal{M}}_x}}$ in (2.8) for $\Sigma \equiv A^\beta_{y, \overline{\tilde{\mathcal{M}}_x}}$, it follows $\lambda(\Sigma) \geq \lambda_x$. On the other hand we obtain from (2.9) in combination with Proposition [4.7 in BEGK3] that $\lambda(\Sigma) \leq \epsilon^{-N} e^{\beta/\epsilon} T_{y, \overline{\tilde{\mathcal{M}}_x} \setminus y} \leq \epsilon^{-N} e^{\beta/\epsilon} / \tilde{\mathcal{E}}_x$. Since $\tilde{\mathcal{E}}_x \leq \epsilon^N \mathcal{E}_x$, we thus have derived a contradiction to the lower bound in (4.45) for $\beta < \alpha$. In view of (3.56) we now may assume that $\overline{\tilde{\mathcal{M}}_x} \cap B(x, \sqrt{\epsilon}) = x$. Note that the various time scales in the error estimates appearing in 4.6 may be replaced without harm by those defined with respect to $\tilde{\mathcal{M}}_x$ instead. In addition, the replacement of $\overline{\mathcal{M}_x}$ by $\overline{\tilde{\mathcal{M}}_x}$ changes α only by an amount of order $N\epsilon \log(1/\epsilon)$. Therefore,

In the sequel we identify $\mathcal{M}_x$ with the set $\tilde{\mathcal{M}}_x$.

Let $C_x \equiv \operatorname{diag}(e^{F(y)/\epsilon})_{y \in \overline{\mathcal{M}_x}} C_{\overline{\mathcal{M}_x}}(\lambda, \phi_x)$, where $C_{\overline{\mathcal{M}_x}}(\lambda, \phi_x)$ is the matrix defined in (4.2) for $\tilde{I} \equiv \overline{\mathcal{M}_x}$. Note that by Green's second formula and $h^\lambda_{z, \overline{\mathcal{M}_x}} | \partial B_z = 1$, B_z defined in (4.1), it follows that the latter matrix is symmetric. Hence

$$\begin{aligned} &\begin{pmatrix} \mathcal{K}_x(\lambda) & -\vec{g}_x(\lambda) \\ -(\operatorname{diag}(e^{(F(x)-F(y))/\epsilon})_{y \leq x} \vec{g}_x(\lambda))^t & e^{F(x)/\epsilon} C_{\overline{\mathcal{M}_x}}(\lambda, \phi_x)_{xx} \end{pmatrix} \equiv C_x(\lambda) = \\ &\left(\epsilon \int_{\partial B_z} e^{-(F-F(z))/\epsilon} \frac{\phi_x}{\phi_x(z)} \partial_n h^\lambda_y \, d\sigma - \delta_{zy} \lambda \int_{B_y} e^{-(F-F(y))/\epsilon} \frac{\phi_x}{\phi_x(y)} \, du \right)_{zy \in \overline{\mathcal{M}_x}} \end{aligned} \tag{4.46}$$

During the rest of the chapter we write shorthand $h_y^\lambda \equiv h_{y,\overline{\mathcal{M}}_x}^\lambda$. Let us furthermore define

$$\mathcal{N}_x \equiv \mathcal{D}_x - \mathcal{K}_x \tag{4.47}$$

where

$$\mathcal{D}_x \equiv \operatorname{diag}\left(\epsilon \int_{\partial B_y} e^{-(F-F(y))/\epsilon} \frac{\phi_x}{\phi_x(y)} \partial_n h_y^\lambda \, d\sigma - \lambda \int_{B_y} e^{-(F-F(y))/\epsilon} \frac{\phi_x}{\phi_x(y)} \, du \right)_{y<x} \tag{4.48}$$

Equipped with the ultrametric structure in the form written in Lemma 4.8 and the control of Laplace transforms and of eigenfunctions obtained in the previous chapter one simply can write a Neumann series for $\mathbb{1} - \mathcal{D}_x(\lambda)^{-1}\mathcal{N}_x(\lambda)$ for λ near $\lambda_{\mathcal{M}_x}$ proving invertibility of $\mathcal{K}_x(\lambda)$. We then compute

$$\det C_x = \det \begin{pmatrix} \mathcal{K}_x & 0 \\ -(\operatorname{diag}(e^{(F(x)-F(y))/\epsilon})_{y\le x}\vec{g}_x)^t & G_x \end{pmatrix} = G_x \det \mathcal{K}_x \tag{4.49}$$

where

$$G_x \equiv C_x(\cdot)_{xx} - (\operatorname{diag}(e^{(F(x)-F(y))/\epsilon})_{y\le x}\vec{g}_x)^t \cdot \mathcal{K}_x^{-1}\vec{g}_x \tag{4.50}$$

This follows by simply adding the column vector

$$\begin{pmatrix} \mathcal{K}_x \\ -(\operatorname{diag}(e^{(F(x)-F(y))/\epsilon})_{y\le x}\vec{g}_x)^t \end{pmatrix} \mathcal{K}_x^{-1}\vec{g}_x$$

(which clearly is a linear combination of the first columns of C_x) to the last column in C_x. From this representation we obtain that λ_x is very close to $\lambda_{\mathcal{M}_x}$. We begin with

Lemma 4.9: *There is $N \equiv N(d)$ such that for all $\alpha > 0$ and some $C \equiv C(d, F, \alpha)$ dominating the supremum norms of the Landau symbols appearing below and all*

$$e^{\alpha/\epsilon}/\mathcal{E}_x < \lambda < e^{-\alpha/\epsilon}/T_{\overline{\mathcal{M}}_x} \tag{4.51}$$

the inverse of $\mathcal{K}_x(\lambda)$ exists. More precisely, uniformly in λ

$$(\mathcal{K}_x(\lambda)^{-1}\vec{g}_x(\lambda))_y = \mathcal{O}(1)\frac{1}{\epsilon^N \lambda T_{y,x}}, \qquad y \le x \tag{4.52}$$

Moreover, we obtain

$$\lambda \in \sigma(L_\epsilon) \qquad \Longleftrightarrow \qquad G_x(\lambda) = 0 \tag{4.53}$$

where $G_x(\lambda)$ is defined in (4.50).

Proof: Formally we obtain

$$\mathcal{K}_x(\lambda)^{-1} = \left(\mathbb{I} - \mathcal{D}_x(\lambda)^{-1}\mathcal{N}_x(\lambda)\right)^{-1}\mathcal{D}_x(\lambda)^{-1} = \sum_{s=0}^{\infty}(\mathcal{D}_x(\lambda)^{-1}\mathcal{N}_x(\lambda))^s\mathcal{D}_x(\lambda)^{-1} \tag{4.54}$$

In order to make sense out of this calculation and to extract the exponential decay estimate written in (4.52) out of the sum a straightforward computation for $s \in \mathbb{N}\backslash 0$ gives the random walk representation

$$(\mathcal{D}_x(\lambda)^{-1}\mathcal{N}_x(\lambda))^s\mathcal{D}_x(\lambda)^{-1} = \Bigg(\sum_{\substack{\omega:y\to z\\ |\omega|=s}}\prod_{t=1}^{|\omega|}\frac{C_x(\lambda)_{\omega_{t-1}\omega_t}}{C_x(\lambda)_{\omega_{t-1}\omega_{t-1}}}\frac{1}{C_x(\lambda)_{zz}}\Bigg)_{yz,y,z<x} \tag{4.55}$$

where for $J\subset\mathcal{M}_x$ we write shorthand $\omega : y \to J$ for a sequence $\omega = (\omega_0, \dots, \omega_T)$ such that $\omega_0 = y$, $\omega_T \in J$, $\omega_t \in \overline{\mathcal{M}}_x\backslash J$ and $\omega_{t-1} \neq \omega_t$ for all $t = 1, \dots, T$. $|\omega|$ denotes the length T of the sequence. By means of (4.44) we may apply (4.26) for $x \equiv y < x$ - in slight abuse of notation - and $\tilde{I} \equiv \overline{\mathcal{M}}_x\backslash y$ and conclude using (4.6) in combination with Proposition [4.7 in BEGK3] that for some $C \equiv C(d, F)$ and $N \equiv N(d)$ and all λ satisfying (4.51)

$$C_x(\lambda)_{yy} \geq (1/C)\int_{A_{x,\mathcal{M}_x}} e^{-(F-F(y))/\epsilon}\, dy(\lambda-\lambda_{\overline{\mathcal{M}}_x\backslash y})\big(1+(\lambda-\lambda_{\overline{\mathcal{M}}_x\backslash y})\mathcal{O}(1)T_{\overline{\mathcal{M}}_x}\big) = (1/C)\lambda\epsilon^N \tag{4.56}$$

In addition, (3.35) for $I \equiv \mathcal{M}_x$ in combination with the upper bound in (4.51) proves for some $C \equiv C(d, F)$ and all $y, z \in \overline{\mathcal{M}}_x$, $y \neq z$

$$\partial_n h_z^\lambda \geq C\partial_n h_z^0 \geq C\partial_n h_{z,y}^0 = -C\partial_n h_{y,z}^0 \tag{4.57}$$

on ∂B_y. Harnack's inequality applied to $\phi_x|B(y,\epsilon)$ and Corollary [4.8 in BEGK3] shows for some $N \equiv N(d)$

$$\begin{aligned} -C_x(\lambda)_{yz} =& -\epsilon\int_{\partial B_y} e^{-(F-F(y))/\epsilon}\frac{\phi_x}{\phi_x(y)}\partial_n h_z^\lambda\, d\sigma \\ \leq& C\epsilon\int_{\partial B_y} e^{-(F-F(y))/\epsilon}\partial_n h_{y,z}^0\, d\sigma \\ \leq& C\epsilon^{-N}/T_{y,z} \end{aligned} \tag{4.58}$$

Now fix $y \in \overline{\mathcal{M}}_x$, $z \in \overline{\mathcal{M}}_x\backslash y$ and a sequence $\omega \equiv (\omega_0, \dots, \omega_T) : z \to y$ such that $\omega_{t+1} \neq \omega_t$ for $t = 0, \dots, T-1$, $\omega_t \in \mathcal{M}_x$ for $t = 1, \dots, T-1$. We observe that there is $1 \leq t_0 \leq T$ such that

$$\hat{F}(\omega_{t_0-1}, \omega_{t_0}) \geq \hat{F}(z, y) \tag{4.59}$$

For, if we assume that the contrary is true for all $t_0 = 1, \dots, T-1$ it follows from ultrametricity of $\mathcal{M}_x \times \mathcal{M}_x \ni (y, z) \mapsto \hat{F}(y, z)$ that $\hat{F}(\omega_1, y) = \hat{F}(z, y)$. By the same argument this implies $\hat F(\omega_2,y)=\hat F(z,y)$ and so forth. We conclude $\hat F(\omega_{T-1},\omega_T)=\hat F(\omega_{T-1},y)=\hat F(z,y)$ so that $t_0\equiv T$ does the job. Choose t_0 satisfying (4.59). Combining this with the triviality $T_{y,z}\geq \mathcal{E}_x$ for $y,z\in\overline{\mathcal{M}}_x$, $y\neq z$, $y<x$, where $\mathcal{E}_x$ is defined in (4.40), we obtain

$$\prod_{t=1}^{|\omega|} T_{\omega_{t-1},\omega_t}=\prod_{t=1}^{t_0-1} T_{\omega_t,\omega_{t-1}} e^{\hat F(\omega_{t_0-1},\omega_{t_0})-F(z))/\epsilon}\prod_{t=t_0+1}^{|\omega|} T_{\omega_{t-1},\omega_t}\geq T_{z,y}\mathcal{E}_x^{|\omega|-1} \tag{4.60}$$

Combination of (4.60), (4.58) and (4.56) tells us that the computations in (4.54) are justified and that for $y<x$

$$\begin{aligned}(\mathcal{K}_x(\lambda)^{-1}\vec g_x(\lambda))_y=&\sum_{\omega:y\to x}\prod_{t=1}^{|\omega|}\frac{C_x(\lambda)_{\omega_t\omega_{t-1}}}{C_x(\lambda)_{\omega_{t-1}\omega_{t-1}}}=\sum_{\omega:y\to x}\mathcal{O}(1)\frac{1}{\epsilon^N\lambda T_{y,x}}\left(\mathcal{O}(1)\frac{1}{\epsilon^N\lambda\mathcal{E}_x}\right)^{|\omega|-1}\\ =&\mathcal{O}(1)\frac{1}{\epsilon^N\lambda T_{y,x}}\sum_{t=1}^{\infty}(|\mathcal{M}_x|-1)^{t-1}\epsilon^{-N(t-1)}e^{-\rho(t-1)/\epsilon}=\mathcal{O}(1)\frac{1}{\epsilon^N\lambda T_{y,z}}\end{aligned} \tag{4.61}$$

We thus obtain (4.52).

(4.53) then is a direct consequence of (4.49) and 4.1 for $\tilde I\equiv\overline{\mathcal{M}}_x$. $\diamondsuit$

We are searching for solutions λ near $\lambda_{\mathcal{M}_x}$ of the equation appearing in (4.53). We want to apply Lagrange's Theorem to this equation (see [WW]) which tells us the following: Fix a point $a\in\mathbb{C}$ and an analytic function Ψ defined on a domain containing the point a. Assume that there is a contour in the domain surrounding a such that on this contour the estimate $|\Psi(\zeta)|<|\zeta-a|$ holds. Then the equation

$$\zeta=a+\Psi(\zeta) \tag{4.62}$$

has a unique solution in the interior of the contour. Furthermore, the solution can be expanded in the form

$$\zeta=a+\sum_{n=1}^{\infty}(n!)^{-1}\partial_\zeta^{n-1}\Psi(a)^n. \tag{4.63}$$

We are in a position to prove Theorem 4.6.

Proof of Theorem 4.6: The equation in (4.53) can be written as

$$-C_x(\lambda)_{xx}+\Phi_j(\zeta)=0, \tag{4.64}$$

where we have set $\zeta\equiv\lambda\int_{A^\beta_{x,\mathcal{M}_x}}e^{-F/\epsilon}\,du/\operatorname{cap}^0(x,\mathcal{M}_x)$ and

$$\Phi_j(\zeta)\equiv\sum_{y<x}C_x(\lambda)_{xy}(\mathcal{K}_x(\lambda)^{-1}\vec g_x(\lambda))_y \tag{4.65}$$

Fix constants arbitrary $\alpha > 0$ and let us denote by U_x the interval of all ζ such that

$$e^{\alpha/\epsilon} T_x/\mathcal{E}_x < \zeta < e^{-\alpha/\epsilon} T_x/T_{\overline{\mathcal{M}}_x} \tag{4.66}$$

Defining $\zeta_{\mathcal{M}_x} \equiv \lambda_{\mathcal{M}_x} \operatorname{cap}^0(x, \mathcal{M}_x)/\int_{A^\beta_{x,\mathcal{M}_x}} e^{-F/\epsilon}\, du$, $\beta > 0$ small, it follows $\zeta_{\mathcal{M}_x} = e^{\mathcal{O}(1)}$ from 4.2 and we may apply (4.26) for all $\zeta \in U_x$ to obtain for some $N \equiv N(d)$

$$-C_x(\lambda)_{xx} = \frac{\int_{A^\beta_{x,\mathcal{M}_x}} e^{-F/\epsilon}\, du}{\operatorname{cap}^0(x, \mathcal{M}_x)} e^{\mathcal{O}(1)} \epsilon^{-N} \big(\zeta - \zeta_{\mathcal{M}_x} + (\zeta - \zeta_{\mathcal{M}_x})^2 R_x(\zeta)\big) \tag{4.67}$$

where

$$R_x(\zeta) = \mathcal{O}(1) e^{\rho/\epsilon} T_{\overline{\mathcal{M}}_x} \frac{\operatorname{cap}^0(x, \mathcal{M}_x)}{\int_{A^\beta_{x,\mathcal{M}_x}} e^{-F/\epsilon}\, du} = \mathcal{O}(1) \epsilon^{-N} e^{\rho/\epsilon} \frac{T_{\overline{\mathcal{M}}_x}}{T_x} \tag{4.68}$$

by Proposition [4.7 in BEGK3]. In view of (4.67) it follows that (4.64) is equivalent to

$$\zeta = \zeta_{\mathcal{M}_x} + \Psi_x(\zeta) \tag{4.69}$$

for some function Ψ_x satisfying

$$\Psi_x(\zeta) = \frac{\operatorname{cap}^0(x, \mathcal{M}_x)}{\int_{A^\beta_{x,\mathcal{M}_x}} e^{-F/\epsilon}\, du} \epsilon^{-N} e^{\mathcal{O}(1)} \Phi_x(\zeta) + (\zeta - \zeta_{\mathcal{M}_x})^2 R_x(\zeta) \tag{4.70}$$

Furthermore, (4.58) shows for all $\zeta \in U_x$ and all $y < x$

$$-C_x(\lambda)_{xy} = \mathcal{O}(1) \frac{1}{T_{x,y}} \tag{4.71}$$

Thus for all $|\zeta - \zeta_{\mathcal{M}_x}| = 1$ we deduce from (4.6) and (4.52) that for all $\zeta \in U_x$

$$\frac{\operatorname{cap}^0(x, \mathcal{M}_x)}{\int_{A^\beta_{x,\mathcal{M}_x}} e^{-F/\epsilon}\, du} \epsilon^{-N} |\Phi_x(\zeta)| \leq \sum_{y<x} \frac{\epsilon^{-N} e^{\rho/\epsilon} T_x^2}{T_{x,y} T_{y,x}} \leq \frac{\epsilon^{-N} e^{\rho/\epsilon} T_x}{\mathcal{T}_x} \tag{4.72}$$

By means of (4.68) and (4.72) it follows for $|\zeta - \zeta_{\Sigma_{j-1}}| = 1$

$$|\Psi_x(\zeta)| \leq \frac{\epsilon^{-N} e^{\rho/\epsilon} T_x}{\mathcal{T}_x} + \frac{\epsilon^{-N} e^{\rho/\epsilon} T_{\overline{\mathcal{M}}_x}}{T_x} \tag{4.73}$$

Since $\mathcal{T}_x \geq \mathcal{E}_x$, in view of (4.42) we may apply Lagrange's Theorem to (4.69) giving the existence of a unique solution $\zeta_x = \lambda_x \int_{A^\beta_{x,\mathcal{M}_x}} e^{-F/\epsilon}\, du/\operatorname{cap}^0(x, \mathcal{M}_x)$ of (4.64) satisfying $|\tilde{\zeta}_j - \zeta_{\mathcal{M}_x}| < 1$. We rewrite (4.69) in the form

$$\zeta_x = \zeta_{\mathcal{M}_x} + \mathcal{O}(1) \left(\frac{\epsilon^{-N} e^{\rho/\epsilon} T_x}{\mathcal{T}_x} + \frac{\epsilon^{-N} e^{\rho/\epsilon} T_{\overline{\mathcal{M}}_x}}{T_x} \right) \tag{4.74}$$

Since from invertibility of $\mathcal{K}_x(\lambda_x)$ it follows that the kernel of $C_x(\lambda_x)$ is at most one-dimensional, 4.1 implies that λ_x is simple. Using (4.6) for $\tilde{I} \equiv \mathcal{M}_x$ from (4.74) we derive that (4.36) holds. Moreover, using $\lambda_x < \lambda_{\overline{\mathcal{M}_x}}$, which follows from (4.6) and (2.7) in combination with (3.3) from 4.1 we conclude that

$$(\phi_x(y))_{y<x} = \phi_x(x)\mathcal{K}_x(\lambda_x)^{-1}\vec{g}_x(\lambda_x) \tag{4.75}$$

Hence from (4.52) and $\lambda_x = e^{\mathcal{O}(1)}\lambda_{\mathcal{M}_x}$ we obtain from (3.55) and (3.56) that (4.37) is satisfied. Now it is very easy to finish the theorem. In view of 4.8 and 4.5 choosing $\alpha < \liminf_{\epsilon\downarrow 0}\rho$ the union of the intervals described in (4.51) contains an interval of the form $[0,\epsilon^N)$, $N \equiv N(d)$. Noting that [S] after possibly increasing N gives the existence of $|\mathcal{M}|$ eigenvalues in this interval, we obtain (4.35). $\diamondsuit$

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