SPARSE POTENTIALS WITH FRACTIONAL HAUSDORFF DIMENSION

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Abstract. We construct non-random bounded discrete half-line Schrödinger operators which have purely singular continuous spectral measures with fractional Hausdorff dimension (in some interval of energies). To do this we use suitable sparse potentials. Our results also apply to whole line operators, as well as to certain random operators. In the latter case we prove and compute an exact dimension of the spectral measures.

1. Introduction

In the present paper, we consider discrete half-line Schrödinger operators H_ϕ on $\ell^2(\mathbb{Z}^+) = \ell^2(\{1, 2, \dots\})$, given by

$$(H_\phi u)(x) \equiv u(x+1) + u(x-1) + V(x)u(x) \tag{1.1}$$

for $x \in \mathbb{Z}^+$, with the potential V and boundary condition $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2}]$

$$u(0)\cos(\phi) + u(1)\sin(\phi) = 0. \tag{1.2}$$

Here (1.2) defines $u(0)$, which then enters in (1.1) for $x = 1$. Hence H_0 is the *Dirichlet operator* with $u(0) = 0$, and $H_\phi = H_0 - \tan(\phi)\delta_1$ where δ_1 is the delta function at $x = 1$. $H_{\pi/2}$ is the *Neumann operator* with $u(1) = 0$. All these are rank one perturbations of H_0.

A function u on $\mathbb{Z}^+ \cup \{0\}$ is a *generalized eigenfunction* of the above operators for energy E and boundary condition ϕ if

$$u(x+1) + u(x-1) + V(x)u(x) = Eu(x) \tag{1.3}$$

for $x \in \mathbb{Z}^+$ and (1.2) holds. Since such u is uniquely given by its values at $x = 0, 1$, the space of generalized eigenfunctions for any energy is 2-dimensional. The 2×2 unimodular matrix $T_E(x, y)$ which takes $\binom{u(y+1)}{u(y)}$ to $\binom{u(x+1)}{u(x)}$ whenever u is a generalized eigenfunction for energy E, is called the *transfer matrix* for E. It is immediate that

$$T_E(x, y) = \prod_{j=y+1}^{x} T_E(j, j-1) = \prod_{j=y+1}^{x} \begin{pmatrix} E - V(j) & -1 \\ 1 & 0 \end{pmatrix}.$$

We denote $T_E(x) \equiv T_E(x, 0)$.

We let μ_ϕ be the spectral measures of the above operators. The aim of this paper is to construct a (non-random) bounded potential V such that these measures are purely singular continuous and have fractional (not 0 or 1) Hausdorff dimension in some interval of energies. We consider the sparse potential with equal barriers $V_{v,\gamma}$ given by (1.6) below, with $v \neq 0$ and $\gamma \geq 2$. Here is our main result:

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Theorem. Let H_ϕ be the discrete Schrödinger operator on $\mathbb{Z}^+$ with potential $V_{v,\gamma}$ given by (1.6), and boundary condition ϕ. Let μ_ϕ be its spectral measure. For any closed interval of energies $J \subset (-2,2)$ there is $v_0 > 0$ and $\gamma_0 \in \mathbb{N}$ such that if $0 < |v| < v_0$ and $\gamma \geq \gamma_0 v^{-2}$ is an integer, then for any ϕ, the measure μ_ϕ has fractional Hausdorff dimension in J.

This is proved in Section 5 as *Theorem 5.1.* From the rest of our results we would like to single out *Theorems 1.4, 6.3* (random case) and *7.1.*

Our motivation is a paper by Jitomirskaya-Last [4], which relates the power growth/decay of eigenfunctions and the Hausdorff dimension of spectral measures. We will apply ideas from [5] and use sparse potentials, which allow us to control this growth. We mention that [4] also provides an example of potentials with the above properties, but these are unbounded (and hence so are the operators).

First we recall some basic facts about dimension of sets and measures. If $S \subseteq \mathbb{R}$ and $\alpha \in [0,1]$, then the α-dimensional Hausdorff measure of S is

$$h^\alpha(S) \equiv \lim_{\delta \to 0} \left[\inf_{\delta\text{-covers}} \sum_{n=1}^{\infty} |I_n|^\alpha \right].$$

Here a δ-cover is a covering of S by a countable set of intervals I_n of lengths at most δ. Notice that h^0 is the counting measure and h^1 the Lebesgue measure. For any S there is a number $\alpha_S \in [0,1]$ such that $h^\alpha(S) = 0$ if $\alpha > \alpha_S$, and $h^\alpha(S) = \infty$ if $\alpha < \alpha_S$. This α_S is the *dimension* of S.

If μ is a measure on $\mathbb{R}$, we say that μ is *α-continuous* if it is absolutely continuous with respect to h^α, and μ is *α-singular* if it is singular to h^α. Hence α-continuous measures do not give weight to sets S with $h^\alpha(S) = 0$ (e.g., to sets S such that $\dim(S) < \alpha$), and α-singular measures are supported on sets S with $h^\alpha(S) = 0$ (and so $\dim(S) \leq \alpha$). We say that μ has *fractional Hausdorff dimension* in some interval I if $\mu(I \cap \cdot)$ is α-continuous and $(1-\alpha)$-singular for some $\alpha > 0$. Finally, μ has *exact (local) dimension* in I if for any $E \in I$ there is an $\alpha(E)$, and for any $\varepsilon > 0$ there is $\delta > 0$ such that $\mu((E-\delta, E+\delta) \cap \cdot)$ is both $(\alpha(E)-\varepsilon)$-continuous and $(\alpha(E)+\varepsilon)$-singular. We do not prove an exact dimension for our measures μ_ϕ (corresponding to (1.6)), but we do it for the random potential case which we consider in Section 6.

In the present paper, we will sometimes say that α-continuous measures have dimension at least α and that α-singular measures have dimension at most α.

We will mainly use two results from [4] (*Corollaries 4.4* and *4.5*) which relate eigenfunction growth and spectral dimension. Here, however, these results will be restated in terms of the *EFGP transform* of eigenfunctions (*Propositions 1.2* and *1.3* below), rather than in terms of the eigenfunctions themselves.

The EFGP transform (R, θ) of an eigenfunction u with energy $E \in (-2,2)$ is a Prüfer-type transform which makes the growth/decay of u more transparent. It is defined as follows. We let $k \in (0, \pi)$ be such that $E = 2\cos(k)$ and set

$$\begin{aligned} u(x) - \cos(k)u(x-1) &= R(x)\cos(\theta(x)), \\ \sin(k)u(x-1) &= R(x)\sin(\theta(x)). \end{aligned} \tag{1.4}$$

These equations define $R(x) > 0$ and $\theta(x) \pmod{2\pi}$ uniquely, and we write $u \sim (R, \theta)$.

If we set

$$\overline{\theta}(x) \equiv \theta(x) + k \qquad \text{and} \qquad v_k(x) \equiv -\frac{V(x)}{\sin(k)},$$

then (1.3) becomes (see [5])

$$\begin{aligned} \cot\bigl(\theta(x+1)\bigr) &= \cot\bigl(\overline{\theta}(x)\bigr) + v_k(x), \\ \frac{R(x+1)^2}{R(x)^2} &= 1 + v_k(x)\sin\bigl(2\overline{\theta}(x)\bigr) + v_k(x)^2\sin^2\bigl(\overline{\theta}(x)\bigr). \end{aligned} \tag{1.5}$$

Notice that (1.5) only determines $\theta(x+1) \pmod{\pi}$. There are two ways to deal with this. The first is to examine (1.4) more closely and conclude (as in [6]) that $\operatorname{sgn}\bigl(\sin(\theta(x+1))\bigr) = \operatorname{sgn}\bigl(\sin(\overline{\theta}(x))\bigr)$, and if this is 0, then $\operatorname{sgn}\bigl(\cos(\theta(x+1))\bigr) = \operatorname{sgn}\bigl(\cos(\overline{\theta}(x))\bigr)$. This fact and (1.5) determine $\theta(x+1) \pmod{2\pi}$ uniquely, but we will not need this extra condition here.

The second way is to realize that $\sin(2\theta)$, $\sin^2(\theta)$ and $\cot(\theta)$ all have period π, and so this ambiguity in θ does not affect the values of R, which are of main interest to us. Notice also, that once $u(0)$ and $u(1)$ are fixed and k is varied, $u(x)$ is a polynomial in E of degree x. Therefore $u(x)$, and by (1.4) also $R(x)$ and $\theta(x)$ (viewed as a function on the unit circle), are well-defined C^∞ functions of k. Moreover, there is completely no ambiguity in $\frac{\partial}{\partial k}\theta(x)$, which will be of significant importance in our considerations.

The relations (1.5) are of interest to us for two reasons. The first is that they are particularly useful when dealing with *sparse potentials*, which we will consider here. Sparse potentials are non-zero only at sites $x = x_n$ such that $x_n - x_{n-1} \to \infty$. This is because on the gaps — the intervals where the potential is zero — the propagation of R and θ is especially transparent. Namely, R is constant and θ increments by k when passing from x to $x+1$.

The second reason is that (1.5) provides a good control of the growth of R, which is the same as the growth of u in the sense of the following lemma. Let us define

$$\|u\|_L^2 \equiv \sum_{x=1}^{L} u(x)^2.$$

Then we have

Lemma 1.1. *There are constants $c_1, c_2 > 0$ depending only on $k \in (0,\pi)$ such that if $u \sim (R,\theta)$ is any generalized eigenfunction for energy $2\cos(k)$ and $L \geq 2$, then*

$$c_1\|u\|_L \leq \|R\|_L \leq c_2\|u\|_L.$$

Remark. The proof shows that c_1 and c_2 can be chosen uniformly for $k \in I$, with I any closed sub-interval of $(0,\pi)$.

Proof. From (1.4) we have

$$\begin{aligned} R(n)^2 &= u(n)^2 + u(n-1)^2 - 2\cos(k)u(n)u(n-1) \\ &\in \Bigl[d_1\left(u(n)^2 + u(n-1)^2\right), d_2\left(u(n)^2 + u(n-1)^2\right)\Bigr] \end{aligned}$$

with $d_i = 1 + (-1)^i\cos(k)$. Hence

$$d_1\|u\|_L^2 \leq \|R\|_L^2 \leq 2d_2\bigl(\|u\|_L^2 + u(0)^2\bigr).$$

The result follows from the fact that

$$u(0)^2 = \left[(2\cos(k) - V(1))u(1) - u(2)\right]^2 \le \left[(2\cos(k) - V(1))^2 + 1\right]\left[u(1)^2 + u(2)^2\right]. \quad \square$$

Let us denote by $u_{\phi,k} \sim (R_{\phi,k}, \theta_{\phi,k})$ the generalized eigenfunction for energy $E = 2\cos(k)$ satisfying the boundary condition ϕ. We are now ready to state, in terms of R rather than u, the abovementioned results from [4]. These will be our main tools for proving fractional dimension of measures.

Proposition 1.2 ([4])**.** *Let $0 < \alpha \le 1$ and let A be a Borel set of energies. If for every $E \in A$ and every generalized eigenfunction $u \sim (R, \theta)$ for energy E*

$$\overline{\lim_{L\to\infty}} \frac{\|R\|_L^2}{L^{2-\alpha}} < \infty,$$

then for any ϕ the restriction $\mu_\phi(A \cap \cdot)$ is α-continuous.

This says that if all eigenfunctions for all energies in some support of μ_ϕ have a small power growth, then μ_ϕ cannot be very singular.

Proposition 1.3 ([4])**.** *Let $0 < \alpha \le 1$ and let A be a Borel set of energies. If for every $E \in A$*

$$\underline{\lim_{L\to\infty}} \frac{\|R_{\phi,k}\|_L^2}{L^{\alpha}} = 0,$$

where k is such that $E = 2\cos(k)$, then the restriction $\mu_\phi(A \cap \cdot)$ is α-singular.

An eigenfunction v for energy E is called a *subordinate solution* if

$$\lim_{L\to\infty} \frac{\|v\|_L}{\|u\|_L} = 0$$

for any other eigenfunction u with the same energy. The Gilbert-Pearson subordinacy theory [3] shows that μ_ϕ is supported off the set of energies for which a subordinate solution exists, but does not satisfy the boundary condition ϕ. Hence *Proposition 1.3* says that the existence of a power decaying eigenfunction (which is then by standard arguments the subordinate solution) for all energies in some support of μ_ϕ implies certain singularity of μ_ϕ.

Moreover, *Lemma 2.1* below shows that the existence of a power growing eigenfunction u implies (for the potential (1.6)) the existence of a power decaying subordinate solution v, and the power of decay of v is the same as the power of growth of u. Thus we only need to estimate the power of *growth* of eigenfunctions.

We will specifically concentrate on the generalized eigenfunctions with Dirichlet boundary condition $\phi = 0$. Let us denote by u_k the eigenfunction for energy $E = 2\cos(k)$ with $k \in (0, \pi)$, such that $u_k(0) = 0$ and $u_k(1) = 1$. Let $u_k \sim (R_k, \theta_k)$, and let $\overline{\theta}_k(x) = \theta_k(x) + k$. Notice that $R_k(1) = 1$. Recall that $R_k(x)$, $\theta_k(x)$ and $\overline{\theta}_k(x)$ are C^∞ functions of k.

As we mentioned before, we will use sparse potentials, which are non-zero only for $x \in \{x_n\}_{n=1}^\infty$. It turns out that the set of possible candidates is quite small. Firstly, for our considerations we will need to have a good control of $\frac{\partial}{\partial k}\overline{\theta}_k(x_n)$, in order to estimate the long-run behavior of R_k using (1.5). This turns out to be hard if $\{x_n\}_{n=1}^\infty$ grows sub-geometrically and so we will not consider this case. On the other hand we have

Theorem 1.4. *Let $x_n \in \mathbb{N}$ be an increasing sequence, $a_n \in \mathbb{R}$ and let μ_ϕ be the spectral measure for the discrete Schrödinger operator on $\mathbb{Z}^+$ with boundary condition ϕ and potential*

$$V(x) = \begin{cases} a_n & x = x_n \text{ for some } n, \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) *if $\overline{\lim}\, x_n/x_{n+1} < 1$ and $a_n \to 0$, then the dimension of μ_ϕ is 1 everywhere in $(-2,2)$;*
(2) *if $x_n/x_{n+1} \to 0$ and $\sup |a_n| < \infty$, then the dimension of μ_ϕ is 1 everywhere in $(-2,2)$.*

Remark. It is known [5, 6] that in (2) the type of the spectrum depends on $\sum_{n=1}^\infty a_n^2$. If this is finite, then we have purely a.c. spectrum, whereas if it is infinite, we have purely s.c. spectrum.

Proof. (1) Let $I \subset (0,\pi)$ be a closed interval. For $k \in I$ let $w_k \sim (P_k, \psi_k)$ be the generalized eigenfunction with energy $E = 2\cos(k)$ such that $w_k(0) = 1$ and $w_k(1) = 0$.

There are $\gamma > 1$ and $n_0 \in \mathbb{N}$ such that $x_n/x_{n-1} > \gamma$ and $x_n > \gamma^n$ for all $n > n_0$. Since $a_n \to 0$, for each $\varepsilon > 0$ there are $c_1, c_2 > 0$ such that $R_k^2(L) \geq c_1(1-\varepsilon)^{n+1}$ and $P_k^2(L) \leq c_2(1+\varepsilon)^n$ for $L \in [x_n + 1, x_{n+1}]$ and $k \in I$. This follows from (1.5). Also, $L - x_{n-1} \geq c_0 L$ with $c_0 = 1 - \frac{1}{\gamma} > 0$ if $n > n_0$. Let $\beta < 1$. Then for $n > n_0$

$$\frac{\|R_k\|_L^2}{\|P_k\|_L^{2\beta}} \geq \frac{(L - x_{n-1})c_1(1-\varepsilon)^n}{(Lc_2(1+\varepsilon)^n)^\beta} \geq cL^{1-\beta}\alpha^n \geq c\left(\gamma^{1-\beta}\alpha\right)^n$$

with $\alpha = (1-\varepsilon)/(1+\varepsilon)^\beta$. We can choose ε small enough so that $\gamma^{1-\beta}\alpha > 1$ and we obtain

$$\lim_{L\to\infty} \frac{\|R_k\|_L^2}{\|P_k\|_L^{2\beta}} = \infty.$$

Then *Theorem 1.2* from [4] implies $\frac{2\beta}{1+\beta}$-continuity of μ_0. Since this is true for any $\beta < 1$ and for any pair of generalized eigenfunctions with energy $E = 2\cos(k)$, as well as for any I, the result follows.

(2) We know that $|a_n/\sin(k)| \leq M < \infty$ for any n and $k \in I$ (I as above). It is easy to show that then there are $0 < a < b < \infty$ such that

$$1 - \frac{a_n}{\sin(k)}\sin(2\theta) + \frac{a_n^2}{\sin^2(k)}\sin^2(\theta) \in (a,b)$$

for any n, θ and $k \in I$. Therefore in the previous argument we have to replace $1-\varepsilon$ and $1+\varepsilon$ with a and b. On the other hand, $x_n/x_{n+1} \to 0$ implies that for any γ there is n_0 such that $x_n/x_{n-1} > \gamma$ and $x_n > \gamma^n$ for $n > n_0$. Thus for any $\beta < 1$ choose γ large enough so that $\gamma^{1-\beta}a/b^\beta > 1$. The rest of the proof is identical with (1). ☐

This leaves us with non-decaying potentials and x_n growing geometrically. Therefore we will consider the following natural choice of potential. We take $v \neq 0$ and an integer $\gamma \geq 2$, and define

$$V_{v,\gamma}(x) \equiv \begin{cases} v & x = x_n \equiv \gamma^n \text{ for some } n \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

Notice that the increasing gaps in $V_{v,\gamma}$ ensure that the interval of energies $[-2,2]$, which is the essential spectrum of the free operator, is contained in the spectrum of each H_ϕ. Indeed, by the discrete version of a theorem of Klaus (*Theorem 3.13* in [1]), the essential spectrum of H_ϕ is

$$[-2,2] \cup \left\{ \operatorname{sgn}(v)\sqrt{4+v^2} \right\}.$$

In what follows, we will show that for suitable v and γ, the spectral measures μ_ϕ are α-continuous and $(1-\alpha)$-singular (for some $\alpha > 0$) in some sub-interval of $[-2,2]$, and so have fractional Hausdorff dimension there. It turns out that we will need small $|v|$ and large γ.

More precisely, we will show, that for certain v and γ, we have that for "most" k (in a sense to be specified later) within a given closed interval $I \subset (0,\pi)$ the function R_k has a suitable power growth with some power $\beta \in (0,\frac{1}{2})$. Then *Propositions 1.2, 1.3*, together with *Lemma 2.1* below, can be used to compute (bounds on) the dimension of the spectral measures. Notice that since by (1.5) R_k is constant on $[x_{n-1}+1, x_n]$, we only need to look at the growth of $\{R_k(x_n+1)\}_{n=1}^\infty$.

In what follows, we will not only prove fractional Hausdorff dimension, but also give bounds on it, and therefore we will need to enumerate those constants appearing in our argument, which affect the power of growth of R_k. We will denote these C_i.

In the present paper, we will consider spectral measures w.r.t. k, rather than w.r.t. $E = 2\cos(k)$. This will not affect the validity of the results because on any closed interval $I \subset (0,\pi)$ of k's (and we consider only such) the function $2\cos(k)$ is C^1 with bounded non-zero derivative. Therefore the dimensional properties of μ_ϕ w.r.t. $k \in I$ and w.r.t. $E \in J \equiv 2\cos(I)$ are identical. Nevertheless, our results will be stated in terms of E.

Finally, we mention that in (1.6) γ does not need to be integral. If one only requires $\gamma > 1$ and sets, for instance, $x_n = \lfloor \gamma^n \rfloor$, all our results continue to hold.

The rest of the paper is organized as follows. In Section 2 we present the main ideas of our proofs and results. Section 3 contains the abovementioned estimates on $\frac{\partial}{\partial k}\overline{\theta}_k(x_n)$. In Section 4 we prove fractional Hausdorff dimension of the spectral measures for almost all boundary conditions, along with bounds on this dimension (*Theorems 4.1, 4.2*). Section 5 contains the same results for all boundary conditions (*Theorems 5.1, 5.2*). We distinguish these two cases because in the case of a.e. ϕ the bounds we provide are considerably better than those for all ϕ. In Section 6 we consider certain randomization of the potential $V_{v,\gamma}$ given by (6.2) and prove exact fractional Hausdorff dimension for a.e. realization of the potential and a.e. boundary condition (*Theorem 6.3*). Finally, Section 7 contains the corresponding whole-line results (*Theorem 7.1*).

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2. Growth of Eigenfunctions

In this section, we will present a short tour of the proof of our main result. Technical details are left for later.

Let $J \subset (-2,2)$ be a given closed interval of energies. We let $I \subset (0,\pi)$ be such that $2\cos(I) = J$. We define $v_k \equiv -\frac{v}{\sin(k)}$ for $k \in I$, $w_1 \equiv \min_{k\in I}\{|v_k|\} > 0$, $w_2 \equiv \max_{k\in I}\{|v_k|\} < \infty$, $x_n \equiv \gamma^n$ for $n \geq 1$ and $x_0 \equiv 0$. Here $v \neq 0$ and $\gamma \geq 2$ are to be determined later.

We consider the half-line discrete Schrödinger operator H_ϕ given by (1.1), (1.2) with the potential (1.6), and we denote its spectral measure μ_ϕ. We will prove fractional Hausdorff dimension of $\mu_\phi(I\cap\cdot)$ for suitable v, γ and ϕ.

As mentioned earlier, we need to estimate the growth of $R_k(x_n+1)$. As in [5], using (1.5) and the Taylor series of $\ln(1+x)$, one obtains for $n\geq 1$

$$\begin{aligned}
\ln\big(R_k(x_n+1)\big)-\ln\big(R_k(x_{n-1}+1)\big) &= \frac{1}{2}\ln\Big(1+v_k\sin(2\overline{\theta}_k(x_n))+v_k^2\sin^2(\overline{\theta}_k(x_n))\Big)\\
&= \frac{1}{2}v_k\sin(2\overline{\theta}_k(x_n))+\frac{1}{4}v_k^2\Big(2\sin^2(\overline{\theta}_k(x_n))-\sin^2(2\overline{\theta}_k(x_n))\Big)+g_n(k)\\
&= \frac{v_k^2}{8}+\frac{v_k}{2}\sin(2\overline{\theta}_k(x_n))+\frac{v_k^2}{8}\Big(\cos(4\overline{\theta}_k(x_n))-2\cos(2\overline{\theta}_k(x_n))\Big)+g_n(k) \qquad (2.1)
\end{aligned}$$

where $|g_n(k)|<C_0|v_k|^3$ for some $C_0>0$. Here $g_n(k)$ is the sum of all third and higher order terms in v_k, and the last equality comes from

$$2\sin^2(\theta)-\sin^2(2\theta)=\tfrac{1}{2}-\cos(2\theta)+\tfrac{1}{2}\cos(4\theta).$$

Now take $v\neq 0$ small enough so that $w_2+w_2^2<1$ and $C_0w_2^3<w_1^2/8$, and define $d_1\equiv w_1^2/8-C_0w_2^3$ and $d_2\equiv w_2^2/8+C_0w_2^3$ If we let $0<C_1<d_1$ and $d_2<C_2<\infty$, then we have

$$\frac{v_k^2}{8}+g_n(k)\in[d_1,d_2]\subset(C_1,C_2) \qquad (2.2)$$

for any n and $k\in I$. From now on, v and C_0 (and thus also C_1 and C_2) will be fixed.

Here is our main idea. It follows from (2.2) that the contribution of the terms $v_k^2/8+g_n(k)$ to the size of $\ln(R_k(x_n+1))$ is within the interval $[d_1n,d_2n]$. If we could show that for large n the contribution of the remaining three (oscillating) terms in (2.1) is small compared to this, we would obtain estimates proving positive power of the growth of R_k. It is reasonable to hope for this because the oscillating terms change sign when n is varied, and so we can expect cancellations. This is the central idea of [5].

So let us assume for a while that for some $A\subseteq I$ with $|A|=0$ ($|A|$ being the Lebesgue measure of A)

$$\sum_{n=1}^{N}\sin(2\overline{\theta}_k(x_n))=o(N)\qquad\forall k\in I\backslash A \qquad (2.3)$$

and that the same holds for the other two oscillating terms. For any $k\in I\backslash A$ we have for large n (by (2.2))

$$R_k(x_n+1)\in(e^{C_1n},e^{C_2n})=(x_n^{\beta_1},x_n^{\beta_2}) \qquad (2.4)$$

where $\beta_i\equiv C_i/\ln(\gamma)$. If we choose γ to be large enough so that $0<\beta_1<\beta_2<\frac{1}{2}$, we get

$$c_1L^{1+2\beta_1}\leq\|R_k\|_L^2\leq c_2L^{1+2\beta_2} \qquad (2.5)$$

for large L and $c_i=c_i(k)$. Then *Proposition 1.2*, along with the theory of rank one perturbations [10], proves for a.e. boundary condition ϕ that the dimension of μ_ϕ is at least $1-2\beta_2$ (see the last paragraph of the proof of *Theorem 4.1*).

To obtain a good upper bound for the dimension, we need to prove an appropriate decay of the corresponding subordinate solutions. We will use the following result, the proof of which we postpone until the end of this section.

Lemma 2.1. *Let $x_1 < x_2 < \cdots$ be such that $\|T_E(x_n, x_{n-1})\| \leq B$ for some $E \in (-2,2)$ and $B < \infty$. Let us assume that $u \sim (R, \theta)$ is a generalized eigenfunction for energy E such that*

$$R(x_n + 1) = e^{\alpha_n}$$

where $\alpha_n = \sum_1^n (Z_j + X_j)$ with $Z_j \in [d_1, d_2]$ (for some $0 < d_1 \leq d_2 < \infty$) and $\sum_1^n X_j = o(n)$. Then there exists a subordinate solution $v \sim (P, \psi)$ for energy E such that for any $0 < d < d_1$ and for all sufficiently large n we have

$$P(x_n + 1) \leq e^{-dn}.$$

Remark. In principle, the lemma shows that the power of decay of the subordinate solution is the same as the power of growth of all other solutions for the same energy. Hence, in a sense, the result is optimal. Notice that information on only one growing solution is needed, but this has to satisfy a "steady growth" condition. Compare with Lemma 8.7 in [5], where two growing solutions are involved.

The lemma can be applied here because we have (2.2) and (2.3). Notice also that all the powers of the free transfer matrix

$$\begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}$$

for energy $E \in (-2,2)$ are uniformly norm-bounded. Hence, we also have a uniform (in n) bound on

$$\|T_E(x_n, x_{n-1})\| \leq \left\| \begin{pmatrix} E - v & -1 \\ 1 & 0 \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}^{x_n - x_{n-1} - 1} \right\|.$$

So for $k \in I \backslash A$ we let

$$Z_j = \frac{v_k^2}{8} + g_j(k)$$

and

$$X_j = \frac{v_k}{2} \sin(2\overline{\theta}_k(x_j)) + \frac{v_k^2}{8} \Big(\cos(4\overline{\theta}_k(x_j)) - 2\cos(2\overline{\theta}_k(x_j)) \Big).$$

If we now take d such that $C_1 < d < d_1$ (say $d = C_1 + \varepsilon$), we obtain the existence of a vector $\overline{u}_k^{\text{sub}} \in \mathbb{R}^2$ which generates the subordinate solution u_k^{sub} with $\|u_k^{\text{sub}}\|_L^2 \leq L^{1-2\beta_1-2\varepsilon/\ln(\gamma)}$. Hence by *Proposition 1.3* and the Gilbert-Pearson subordinacy theory [3], the dimension of μ_ϕ is at most $1 - 2\beta_1$ for a.e. ϕ (see the proof of *Theorem 4.1*).

Moreover, by the proof of *Lemma 2.1*, $\|T_E(x_n)\| \to \infty$ as $n \to \infty$ whenever $E \in 2\cos(I \backslash A)$. *Theorem 1.2* from [7] then implies absence of a.c. spectrum in I for all ϕ. The next paragraph proves absence of p.p. spectrum if $w_2 + w_2^2 < \ln(\gamma)$. Thus for γ large enough, we obtain purely s.c. spectrum in I for all ϕ.

But we can do even better. It turns out that with a slightly stronger assumption than (2.3), we can show fractional dimension *for all* boundary conditions! First notice that

$$\ln\big(R_k(x_n + 1)\big) - \ln\big(R_k(x_{n-1} + 1)\big) < \frac{w_2 + w_2^2}{2}$$

and so for all $k \in I$ we have $R_k(x_n+1) < x_n^{(w_2+w_2^2)/2\ln(\gamma)}$. Since R_k is constant on $[x_{n-1}+1, x_n]$, *Proposition 1.2* implies that the dimension is at least $1-(w_2+w_2^2)/\ln(\gamma)$ on all of I. Since $w_2+w_2^2<1$, this is strictly positive if $\gamma \geq 3$.

Now replace (2.3) with the (stronger) assumption that for any $\varepsilon > 0$ there exists $A^\varepsilon \subseteq I$, a set with dimension smaller than 1 such that

$$\varlimsup_{N\to\infty} \frac{1}{N}\left|\sum_{n=1}^{N} \sin(2\overline{\theta}_k(x_n))\right| \leq \varepsilon \qquad \forall k \in I\backslash A^\varepsilon, \tag{2.6}$$

and the same is true for the other two oscillating terms. Let us consider the "Dirichlet" measure μ_0. Since it has no a.c. part, it is supported on the set of k for which u_k is the subordinate solution. It follows that $\mu_0(I\backslash A^\varepsilon)=0$. Indeed, if $k \in I\backslash A^\varepsilon$, then for large n we have

$$R_k(x_n+1) > \exp\left\{\left(C_1 - \varepsilon\left(\frac{w_2}{2}+\frac{3w_2^2}{8}\right)\right)n\right\}.$$

If ε is small enough so that the exponent is positive, *Lemma 2.1* again shows the existence of a subordinate solution u_k^{sub}, and this must be different from u_k (by which we mean that u_k^{sub} is not a multiple of u_k). This implies that μ_0 is supported on A^ε.

Hence the dimension of μ_0 on I (which we know is positive) is at most $\dim(A^\varepsilon) < 1$, and therefore fractional. Considering instead of u_k the generalized eigenfunction for energy $2\cos(k)$ satisfying boundary condition $\phi \neq 0$, and assuming (2.6) for the corresponding $\overline{\theta}_{\phi,k}(x_n)$, we can obtain the same result for any ϕ. For details see the proof of *Theorem 5.1.*

Our considerations have, however, a pitfall. Neither (2.3) nor (2.6) need hold for such a large set of k's as we want. We cannot hope for this because we have only limited control of the argument of the sin term. Fortunately, we do not really need (2.3) and (2.6) in the presented form. This is because the non-oscillatory term $v_k^2/8$ gives us some space. We can "sacrifice" part of it, just as we did when we joined it with the $g_n(k)$ term, and still keep a power growth of R_k. More precisely, we will divide the sin term into two, one of which will be small with respect to the non-oscillatory term, whereas the other one will have enough "regularity" for (2.3) and (2.6) to hold. The two cos terms can be treated similarly.

Proof of Lemma 2.1. Let $d_0 = \frac{1}{2}(d+d_1)$ and let $v \sim (P,\psi)$ be any solution for energy E different from u. Let $p_n \equiv P(x_n+1)$, $r_n \equiv R(x_n+1)$ and $t_n \equiv \|T_E(x_n)\|$. The first ingredient in the proof is *Theorem 2.3* from [5] which states that there are $c_1, c_2 > 0$ such that

$$c_1 \max\{p_n, r_n\} \leq t_n \leq c_2 \max\{p_n, r_n\}.$$

Since by hypothesis $r_n \geq e^{d_0 n}/c_1$ for large n, it follows that for large enough n

$$t_n \geq e^{d_0 n}. \tag{2.7}$$

On the other hand, if $d_3 > \max\{d_2, \ln(B)\}$, then for large n

$$t_n \leq e^{d_3 n}.$$

Hence if

$$\delta \equiv \varliminf_{n\to\infty} \frac{\ln(t_n)}{n},$$

then $d_0 \leq \delta \leq d_3$.

From (2.7) we know that

$$\sum_{n=1}^{\infty} \frac{\|T_E(x_n, x_{n-1})\|^2}{\|T_E(x_n)\|^2} < \infty.$$

This is the assumption of our second ingredient, *Theorem 8.1* from [7], the proof of which (namely inequalities (8.5), (8.7)) yields the following. There is a vector $\overline{v} \in \mathbb{R}^2$ and $c_0 > 0$ such that

$$\|T_E(x_n)\overline{v}\|^2 \le c_0 t_n^2 s_n^2 + t_n^{-2}$$

with $s_n = \sum_{m=0}^{\infty} t_{n+m}^{-2}$. Our aim is to show that for large enough n this is smaller than e^{-2dn}.

The abovementioned *Theorem 8.1* also asserts that $\overline{v}$ generates the subordinate solution for energy E. One expects that this is different from u, generated by $\overline{u} \equiv (u(0), u(1))$, which is a growing solution. We will now prove this claim.

Let $\delta' < \delta < \delta''$ be such that $\delta'' < 2\delta'$. From the definition of δ we know that for large enough n

$$s_n \le e^{-2\delta' n}.$$

Also there are $\{n_j\}_{j=1}^{\infty}$ such that $t_{n_j} < e^{\delta'' n_j}$ and so

$$\|T_E(x_{n_j})\overline{v}\|^2 \le c_0 e^{2\delta'' n_j} e^{-4\delta' n_j} + e^{-2\delta'' n_j} \to 0$$

as $j \to \infty$. Since

$$\|T_E(x_{n_j})\overline{u}\|^2 = u(x_{n_j}+1)^2 + u(x_{n_j})^2 \ge \frac{R(x_{n_j}+1)^2}{2} \to \infty,$$

the subordinate solution is indeed different from u. Let us take it for v, and change P, p_n, c_1 and c_2 accordingly. Since *Theorem 8.1* from [7] also states that

$$\frac{\|T_E(x_n)\overline{v}\|}{\|T_E(x_n)\overline{u}\|} \to 0,$$

it then follows that $p_n < r_n$ for large n. Hence $c_1 r_n < t_n < c_2 r_n$ for large n.

Let $a_n = \alpha_n/n$. If n is large enough, then $a_n \in (d_0, d_3)$. Let $b \in \mathbb{N}$ be such that $d_0(b+1) > d_3$. Pick $\varepsilon > 0$ small enough so that $\varepsilon' \equiv \varepsilon(b+2) < (d_0 - d)/2$.

Since $a_n n = \sum_1^n (Z_j + X_j)$, for large n we have $\sum_1^n Z_j \ge (a_n - \varepsilon)n$ and $|\sum_1^n X_j| \le \varepsilon n$. Since

$$\ln(r_{n+m}) = a_{n+m}(n+m) = \sum_1^{n+m} (Z_j + X_j) \ge (a_n - \varepsilon)n + d_1 m - \varepsilon(n+m),$$

for $m \le bn$ we have

$$r_{n+m} \ge e^{(a_n - \varepsilon')n + d_1 m}.$$

Hence for large n

$$s_n \le c_1^{-2} \sum_{m=0}^{bn} e^{-2(a_n-\varepsilon')n - 2d_1 m} + c_1^{-2} \sum_{m=bn}^{\infty} e^{-2d_0(n+m)} \le c\left(e^{-2(a_n-\varepsilon')n} + e^{-2d_0(b+1)n}\right)$$

for some $c > 0$. It follows that

$$\begin{aligned}\|T_E(x_n)\overline{v}\|^2 &\leq 2cc_0c_2^2\left(e^{(2a_n-4(a_n-\varepsilon'))n} + e^{(2a_n-4d_0(b+1))n}\right) + c_1^{-2}e^{-2a_nn} \\ &\leq c'\left(e^{-2dn} + e^{-2d_3n} + e^{-2d_0n}\right) \leq 3c'e^{-2dn}.\end{aligned}$$

The rest is an easy computation. From (1.4) we have

$$P^2(x_n+1) \leq 2\left(v(x_n+1)^2 + v(x_n)^2\right) = 2\|T_E(x_n)\overline{v}\|^2 \leq 6c'e^{-2dn}$$

for large n. Since this holds for any $d < d_1$, the result follows. □

3. Estimates on Growth of $\overline{\theta}$

From now on let us write $\overline{\theta}(x_n, k)$ instead of $\overline{\theta}_k(x_n)$. As mentioned before, the key to our results are estimates on $\frac{\partial}{\partial k}\overline{\theta}(x_n,k)$ which we denote $\overline{\theta}'(x_n,k)$. Differentiating the first equation of (1.5) with respect to k, and using $\sin^2(\theta) = (1+\cot^2(\theta))^{-1}$, we get (as in [6])

$$\frac{\partial}{\partial k}\theta(x+1) = \frac{\frac{\partial}{\partial k}\overline{\theta}(x) - V(x)\frac{\cos(k)}{\sin^2(k)}\sin^2(\overline{\theta}(x))}{\sin^2(\overline{\theta}(x)) + \left[\cos(\overline{\theta}(x)) - \frac{V(x)}{\sin(k)}\sin(\overline{\theta}(x))\right]^2}.$$

For our potential $V_{v,\gamma}$ and for $x = x_n$, the denominator is

$$a_{n,k} = 1 + v_k\sin(2\overline{\theta}(x_n,k)) + v_k^2\sin^2(\overline{\theta}(x_n,k))$$

and so

$$\begin{aligned}\overline{\theta}'(x_{n+1},k) &= x_{n+1} - x_n + \theta'(x_n+1,k) \\ &= x_{n+1} - x_n + \frac{\overline{\theta}'(x_n,k)}{a_{n,k}} + \frac{v_k\cot(k)\sin^2(\overline{\theta}(x_n,k))}{a_{n,k}}.\end{aligned} \tag{3.1}$$

The denominator $a_{n,k}$ is in the interval $[C_3, C_4] \subset (1-(w_2+w_2^2), 1+(w_2+w_2^2))$ with

$$C_i \equiv 1 + \frac{w_2^2}{2} + (-1)^i w_2\sqrt{1+\frac{w_2^2}{4}}.$$

Notice that $C_3C_4 = 1$. The last fraction in (3.1) is in some interval $(-M, M)$ for any n and any $k \in I$. From (3.1) one can show that as $n \to \infty$, $\overline{\theta}'(x_n,k)$ gets close to x_n. Indeed, if for some n we have $\overline{\theta}'(x_n,k)/x_n \in (\Delta_1, \Delta_2)$ for all $k \in I$ (and some $\Delta_i \in \mathbb{R}$), then

$$\frac{\overline{\theta}'(x_{n+1},k)}{x_{n+1}} \in \left(1 + \frac{1}{\gamma}\left(\frac{\Delta_1}{C_4} - 1\right) - \frac{M}{\gamma^{n+1}},\ 1 + \frac{1}{\gamma}\left(\frac{\Delta_2}{C_3} - 1\right) + \frac{M}{\gamma^{n+1}}\right).$$

Iterating this, one proves that for any $D_1 < C_5 \equiv (\gamma-1)/(\gamma-C_3)$ and $D_2 > C_6 \equiv (\gamma-1)/(\gamma-C_4)$ there is n_0 such that for $n > n_0$ and all $k \in I$

$$\frac{\overline{\theta}'(x_n,k)}{x_n} \in (D_1, D_2). \tag{3.2}$$

This is because $(\gamma-1)/(\gamma-C_i^{-1})$ are the fixed points of the map $\Delta \mapsto 1 + (\Delta/C_i - 1)\gamma^{-1}$, and $C_3^{-1} = C_4$. Thus (D_1, D_2) is an interval containing 1, which can be made as small as we need by taking γ large enough. This will play an important role in our considerations.

For $n > n_0$ let $K_{n,1}$ and $K_{n,2}$ be the smallest and largest numbers in I such that $\overline{\theta}(x_n, K_{n,i})$ is an integral multiple of π. Notice that by (3.2) the distance of these numbers from the corresponding edges of I is smaller than $\pi D_1^{-1}\gamma^{-n}$. Let $k_{n,1} = K_{n,1} < k_{n,2} < \cdots < k_{n,j_n} = K_{n,2}$ be all numbers in I such that $\overline{\theta}(x_n, k_j)$ is an integral multiple of π. Let $I_{n,j} = [k_{n,j}, k_{n,j+1}]$. Then (3.2) implies that

$$|I_{n,j}| \in (\pi D_2^{-1}\gamma^{-n}, \pi D_1^{-1}\gamma^{-n}). \tag{3.3}$$

We will slightly alter the oscillating sin term on each $I_{n,j}$. This way we will obtain the (previously mentioned) more regular term for which we can prove (2.3) (and later (2.6)). This is the content of Sections 4 and 5. But before that, we present an additional argument.

It turns out that by considering $\overline{\theta}''(x_n, k)$ one can improve (3.2) and (3.3) on small scale (i.e., the scale of $I_{n,j}$). This improvement is not essential for our results; it only yields better numerical estimates for large γ. Differentiating (3.1) with respect to k one obtains

$$\overline{\theta}''(x_{n+1}, k) = \frac{\overline{\theta}''(x_n, k)}{a_{n,k}} - \left[\overline{\theta}'(x_n, k)\right]^2 \frac{2v_k \cos(2\overline{\theta}(x_n, k)) + v_k^2 \sin(2\overline{\theta}(x_n, k))}{a_{n,k}^2} + b_{n,k}\overline{\theta}'(x_n, k) + \tilde{b}_{n,k}$$

where $b_{n,k}, \tilde{b}_{n,k} \in (-M, M)$ for some M and all n, k. Then for $n > n_0$ (since $a_{n,k}^{-1} \le C_4$),

$$\frac{|\overline{\theta}''(x_{n+1}, k)|}{x_{n+1}^2} \le \frac{C_4}{\gamma^2}\frac{|\overline{\theta}''(x_n, k)|}{x_n^2} + \frac{D_2^2 C_4^2 (2w_2 + w_2^2)}{\gamma^2} + \frac{2MD_2}{\gamma^{n+1}}.$$

Similarly as above, by iterating this we obtain that for large enough n (say $n > n_0$ for a new n_0) and all $k \in I$

$$\frac{|\overline{\theta}''(x_n, k)|}{x_n^2} \le \frac{D_2^2 C_4^2 (2w_2 + w_2^2)}{\gamma^2 - 2}.$$

This is because the fixed point of the mapping $\Delta \mapsto C_4\Delta/\gamma^2 + D_2^2C_4^2(2w_2 + w_2^2)/\gamma^2$ is $D_2^2C_4^2(2w_2 + w_2^2)/(\gamma^2 - C_4)$, and because $C_4 < 2$.

If $n > n_0$, let $D_1^{n,j}$ and $D_2^{n,j}$ be such that for any $k \in I_{n,j}$

$$\frac{\overline{\theta}'(x_n, k)}{x_n} \in [D_1^{n,j}, D_2^{n,j}], \tag{3.2$'$}$$

and the interval $[D_1^{n,j}, D_2^{n,j}]$ is smallest possible. From (3.3) and the obtained estimate on $|\overline{\theta}''(x_n, k)|$ we have that

$$D_2^{n,j} - D_1^{n,j} \le \frac{\pi}{D_1\gamma^n}\frac{D_2^2C_4^2(2w_2 + w_2^2)}{\gamma^2 - 2}\gamma^{2n}\gamma^{-n} = \frac{\pi D_2^2C_4^2(2w_2 + w_2^2)}{D_1(\gamma^2 - 2)}.$$

Notice that $D_2 - D_1 \approx c/\gamma$, and so as $\gamma \to \infty$, the estimate (3.2$'$) is better than (3.2). We also have the obvious improvement of (3.3), namely

$$|I_{n,j}| \in \left[\pi(D_2^{n,j})^{-1}\gamma^{-n}, \pi(D_1^{n,j})^{-1}\gamma^{-n}\right]. \tag{3.3$'$}$$

Now let

$$C_7 \equiv \min\left\{\frac{\pi D_2^2C_4^2(2w_2 + w_2^2)}{D_1^2(\gamma^2 - 2)}, \frac{D_2 - D_1}{D_1}\right\}.$$

Since $D_1 \le D_1^{n,j} \le D_2^{n,j} \le D_2$, we have

$$\frac{D_2^{n,j} - D_1^{n,j}}{D_1^{n,j}} \le C_7 \le \frac{D_2 - D_1}{D_1}$$

for all j and $n > n_0$.

4. Fractional Dimension for A.E. Boundary Condition

Let us now turn to the announced division of the sin term in two. Since $(C_6 - C_5)/C_5 = (C_4 - C_3)/(\gamma - C_4)$, we can pick D_1, D_2 close to C_5, C_6 so that $(D_2 - D_1)/D_1$ is arbitrarily close to $(C_4 - C_3)/(\gamma - C_4)$. Since C_3 and C_4 (as well as C_1 and C_2) are independent of γ, and $C_7 \le (D_2 - D_1)/D_1$, we can make C_7 arbitrarily small by taking γ large enough. Let us do this so that

$$\frac{C_7}{\pi}\left(\frac{w_2}{2} + \frac{3w_2^2}{8}\right) < C_1. \tag{4.1}$$

For $k \in I$ and $n > n_0$ we let

$$Q_n(k) \equiv \begin{cases} |I_{n,j}|^{-1} \int_{I_{n,j}} \sin(2\overline{\theta}(x_n, \kappa))\, d\kappa & k \in I_{n,j} \\ 0 & k \in I \backslash [K_{n,1}, K_{n,2}] \end{cases}$$

and define

$$X_n(k) \equiv \sin(2\overline{\theta}(x_n, k)) - Q_n(k).$$

Notice that

$$\int_{I_{n,j}} X_n(k)\, dk = 0 \tag{4.2}$$

for any j. Also, by (3.2′)

$$|\sin(2\overline{\theta}(x_n, k)) - X_n(k)| = |Q_n(k)| \le \frac{2}{\pi}\frac{D_1^{n,j} - D_2^{n,j}}{D_1^{n,j} + D_2^{n,j}} \le \frac{C_7}{\pi}$$

for any $k \in I$ and $n > n_0$. This is because $|Q_n|$ is maximal possible on $I_{n,j}$ if $\overline{\theta}'$ equals $D_1^{n,j}\gamma^n$ when $\sin(2\overline{\theta})$ is positive and $D_2^{n,j}\gamma^n$ when $\sin(2\overline{\theta})$ is negative (or vice-versa). And in that case we have equality in the first inequality above.

Therefore we have

$$\left|\sum_{n=n_0+1}^{N} \frac{v_k}{2}\sin(2\overline{\theta}(x_n, k))\right| \le \frac{w_2}{2}\frac{C_7}{\pi}N + \left|\frac{v_k}{2}\sum_{n=n_0+1}^{N} X_n(k)\right|.$$

If we do the same with the other two oscillating terms (the corresponding $I_{n,j}$'s and X_n's will be slightly different), the three terms containing C_7 will add up to

$$\frac{C_7}{\pi}\left(\frac{w_2}{2} + \frac{3w_2^2}{8}\right)N < C_1 N.$$

So if we prove (2.3) for X_n in place of sin (and similarly for the two cos terms), we will still keep a power growth of R_k. We will be able to do this using (4.2).

To this end we need estimates on the covariances of the X_n's, so take $n > m > n_0$. Then for all j we have (4.2), whereas from (3.2) it follows that X_m takes on $I_{n,j}$ values within an interval of length at most $D_2\gamma^m|I_{n,j}| \le \pi D_2 D_1^{-1}\gamma^{m-n}$. Therefore

$$\left|\int_{I_{n,j}} X_m X_n \, dk\right| \le \frac{\pi D_2}{D_1}\gamma^{m-n}|I_{n,j}|.$$

Notice that this might not be true if $I_{n,j}$ contains some $k_{m,l}$, points where X_m may be discontinuous. But this is not a problem because the total length of such $I_{n,j}$'s is of order γ^{m-n}. Since also $|I\backslash[K_{n,1}, K_{n,2}]| \sim \gamma^{-n}$ and the X_n's are bounded, we get for $n > m > n_0$

$$\left|\int_I X_m X_n \, dk\right| \le c\gamma^{m-n}.$$

By Cauchy-Schwarz inequality

$$\begin{aligned}\int_I \left|\sum_{n=n_0+1}^{N} X_n\right| dk &\le c\left[\int_I \left(\sum_{n=n_0+1}^{N} X_n\right)^2 dk\right]^{1/2} \\ &= c\left[\int_I \sum_{n=n_0+1}^{N} X_n^2 + \sum_{n_0<m<n\le N} 2X_m X_n \, dk\right]^{1/2} \le c\,(N+cN)^{1/2} \le cN^{1/2}\end{aligned}$$

(the value of c changes in each inequality, but is independent of N). Thus

$$\int_I \left|\frac{\sum_{n=n_0+1}^{N^4} X_n}{N^4}\right| dk \le \frac{c}{N^2}$$

which means that for a.e. k

$$\frac{\sum_{n=n_0+1}^{N^4} X_n}{N^4} \to 0,$$

because $\sum_1^\infty N^{-2} < \infty$. Since the X_n's are bounded and $[(N+1)^4 - N^4]/N^4 \to 0$, it follows that for these k

$$\frac{\sum_{n=n_0+1}^{N} X_n}{N} \to 0$$

and so

$$\frac{\sum_{n=1}^{N} X_n}{N} \to 0.$$

Theorem 4.1. *Let H_ϕ be the discrete Schrödinger operator on $\mathbb{Z}^+$ with potential $V_{v,\gamma}$ given by* (1.6)*, and boundary condition ϕ. Let μ_ϕ be its spectral measure. For any closed interval of energies $J \subset (-2, 2)$ there is $v_0 > 0$ and $\gamma_0 \in \mathbb{N}$ such that if $0 < |v| < v_0$ and $\gamma \ge \gamma_0$ is an integer, then each μ_ϕ is purely singular continuous in J, and for a.e. ϕ the measure μ_ϕ has fractional Hausdorff dimension in J.*

Remark. A priori, γ_0 depends on v. However, since in (4.1) we have $C_1 = O(v^2)$, $C_7 = O(v\gamma^{-2})$ and $w_2 = O(v)$ as $v \to 0$, $\gamma \to \infty$, we can choose γ_0 uniformly for all small v.

Proof. Let $I \subset (0,\pi)$ be such that $2\cos(I) = J$. The above discussion and (2.2) show that there is $A \subseteq I$ with $|A| = 0$, such that for any $k \in I \backslash A$ there is $n_1(k)$ such that for $n > n_1(k)$

$$R_k(x_n+1) \in (e^{c_1 n}, e^{c_2 n}) = (x_n^{\beta_1}, x_n^{\beta_2}),$$

where

$$c_i \equiv C_i + (-1)^i \frac{C_7}{\pi}\left(\frac{w_2}{2} + \frac{3w_2^2}{8}\right) > 0 \qquad (i = 1,2)$$

and $\beta_i \equiv c_i/\ln(\gamma)$. Take γ large enough so that $0 < \beta_1 < \beta_2 < \frac{1}{2}$. It follows from the constancy of R_k on $[x_n+1, x_{n+1}]$ that (2.5) holds for $k \in I \backslash A$.

Using *Lemma 2.1* with

$$d_i + (-1)^i \frac{C_7}{\pi}\left(\frac{w_2}{2} + \frac{3w_2^2}{8}\right) \qquad (i = 1,2)$$

in place of d_1, d_2, one gets for these k the existence of a vector $\overline{u}_k^{\rm sub} \in \mathbb{R}^2$ such that the following holds. If $u_k^{\rm sub} \sim (P_k, \psi_k)$ is the generalized eigenfunction for energy $2\cos(k)$ generated by $\overline{u}_k^{\rm sub}$, then for some small ε

$$P_k(x_n+1) \leq e^{-(c_1+\varepsilon)n} = x_n^{-\beta_1 - \varepsilon/\ln(\gamma)}.$$

Since P_k is constant on $[x_{n-1}+1, x_n]$, we have

$$\|P_k\|_L^2 \leq c' L^{1-2\beta_1 - 2\varepsilon/\ln(\gamma)}. \tag{4.3}$$

Since $u_k^{\rm sub}$ is the subordinate solution for energy $2\cos(k)$, all other solutions for this energy grow (in power) no faster than R_k. Also, μ_ϕ restricted to I is supported on the set of those k, for which $u_k^{\rm sub}$ satisfies the boundary condition ϕ (this is because μ_ϕ has no a.c. part; see [3]), and so $P_k = R_{\phi,k}$. Then we have by *Propositions 1.2*, *1.3* and by (2.5), (4.3) that for any ϕ the restriction $\mu_\phi((I\backslash A)\cap\cdot)$ is $(1-2\beta_2)$-continuous and $(1-2\beta_1)$-singular. By the theory of rank one perturbations (*Theorem 1.8* in [10]) we know that $\mu_\phi(A) = 0$ for a.e. ϕ, and so for a.e. ϕ we have the same continuity/singularity of $\mu_\phi(I\cap\cdot)$. □

To get numerical bounds on the dimensions, one needs to evaluate constants C_i. These depend on w_i and therefore it is best to consider (for given v and large γ) a small interval I around each k, so that $w_i \approx |v_k|$. One then obtains bounds on local dimension of μ_ϕ which will be k-dependent. We will, however, first present an additional argument which will considerably facilitate this (by eliminating constants C_0, C_1 and C_2) as well as improve the obtained bounds.

Let us push the above ideas a little bit further. At the beginning of this argument we introduced the term $g_n(k)$ as the sum of all third and higher order terms in v_k. One can, however, write down all these terms explicitly, using the Taylor series of $\ln(1+x)$.

$$\begin{aligned}\ln\big(R_k(x_n+1)\big) - \ln\big(R_k(x_{n-1}+1)\big) &= \tfrac{1}{2}\ln\Big(1 + v_k\sin(2\overline{\theta}(x_n,k)) + v_k^2\sin^2(\overline{\theta}(x_n,k))\Big)\\ &= \sum_{\substack{a,b\geq 0\\ a+b\geq 1}} \frac{(-1)^{a+b+1}}{2a+2b}\binom{a+b}{a} v_k^{a+2b}\sin^a(2\overline{\theta}(x_n,k))\sin^{2b}(\overline{\theta}(x_n,k)).\end{aligned}$$

If $|v_k| + v_k^2 < 1$, then the sum of the amplitudes of the terms of this sum (with n fixed) converges absolutely.

Now notice that all terms with odd a are oscillating, whereas terms with even a do not change sign when varying n. This will allow us to obtain more precise bounds on the dimension, since, once again, the contribution of oscillating terms will be (for a.e. k) negligible in comparison with that of non-oscillating terms. This will be proved by the same methods as above. We will replace the oscillating term $\sin^a(2\overline{\theta}(x_n,k))\sin^{2b}(\overline{\theta}(x_n,k))$ by a more regular term $\sin^a(2\overline{\theta}(x_n,k))\sin^{2b}(\overline{\theta}(x_n,k)) - Q_{n,a,b}(k)$, with $Q_{n,a,b}(k)$ small. The fact that we now have an infinite number of terms will not cause problems because the sum of their amplitudes converges absolutely.

First we need to do what we have already done once. We will split each non-oscillating term in two, a constant and an oscillating term. The latter will be treated as the other oscillating terms. We define

$$F_{a,b} \equiv \frac{1}{\pi}\int_0^\pi \sin^a(2\theta)\sin^{2b}(\theta)\,d\theta.$$

If $2\nmid a$, then this is 0 and

$$W_{n,a,b}(k) \equiv \sin^a(2\overline{\theta}(x_n,k))\sin^{2b}(\overline{\theta}(x_n,k))$$

is oscillating. If $2|a$, the corresponding term is non-oscillating and we split it into $F_{a,b}$ and

$$W_{n,a,b}(k) \equiv \sin^a(2\overline{\theta}(x_n,k))\sin^{2b}(\overline{\theta}(x_n,k)) - F_{a,b}.$$

Let

$$F(v_k) \equiv \sum_{\substack{a,b\geq 0\\ a+b\geq 1}} \frac{(-1)^{a+b+1}}{2a+2b}\binom{a+b}{a} v_k^{a+2b} F_{a,b}$$

$$G(v_k) \equiv \sum_{\substack{a,b\geq 0\\ a+b\geq 1}} \frac{1}{2a+2b}\binom{a+b}{a} |v_k|^{a+2b}.$$

Then $F(v_k)$ is the sum of all the constant terms, and $G(v_k)$ is strictly larger than the sum of the amplitudes of the oscillating terms (for fixed n). Notice that

$$G(v_k) = -\tfrac{1}{2}\ln(1-|v_k|-v_k^2)$$

and (see [8, 6])

$$F(v_k) = \frac{1}{\pi}\int_0^\pi \frac{1}{2}\ln\Big(1+v_k\sin(2\theta)+v_k^2\sin^2(\theta)\Big)\,d\theta = \frac{1}{2}\ln\left(1+\frac{v_k^2}{4}\right).$$

We take γ large so that $C_7 G(v_k)/\pi < F(v_k)$ for all $k\in I$.

Now fix any a,b and consider all the $W_{n,a,b}$'s. We can do everything as above. Take the intervals $I_{n,j}$ (these will be different for $2|a$ and $2\nmid a$, just as they were for the sin and cos terms). As before, the integral of $W_{n,a,b}$ over any $I_{n,j}$ is close to 0 (it would be exactly 0 if $\overline{\theta}'(x_n,k)$ were constant on $I_{n,j}$). Thus from (3.2′) we know that there is $Q_{n,a,b}(k)$ such that $|Q_{n,a,b}(k)|\leq C_7/\pi$, and for $X_{n,a,b}(k)\equiv W_{n,a,b}(k)-Q_{n,a,b}(k)$ we have

$$\int_{I_{n,j}} X_{n,a,b}(k)\,dk = 0.$$

As before, one can use this to prove that

$$\frac{\sum_{n=1}^{N} X_{n,a,b}}{N} \to 0 \tag{4.4}$$

for a.e. k. This holds for any a, b. Since the number of these pairs is countable, we have that for a.e. k (4.4) holds for any a, b. Using the fact that the sum of the amplitudes of the oscillating terms is finite, we obtain that

$$\frac{1}{N}\sum_{n=1}^{N}\sum_{\substack{a,b\geq 0\\ a+b\geq 1}} \frac{(-1)^{a+b+1}}{2a+2b}\binom{a+b}{a} v_k^{a+2b} X_{n,a,b} \to 0$$

for a.e. k. Then for each such k and for large $n > n_1(k)$ we have

$$R_k(x_n+1) \in (e^{c_1 n}, e^{c_2 n}),$$

where $c_i = c_i(k) \equiv F(v_k) + (-1)^i C_7 G(v_k)/\pi$ (remember that $G(v_k)$ is strictly larger than the sum of the amplitudes). We know from [10] that for a.e. ϕ the spectral measure μ_ϕ is supported on the set of these k's because it is a set of full measure.

Let us now estimate C_7 for $|v_k| + v_k^2 < 1$. Let I be a small interval around k, so that w_2 is arbitrarily close to $|v_k|$. Then $C_4 - C_3 < 2(|v_k| + v_k^2)$ and $C_4 < 2$, and one can pick D_1, D_2 so that $(D_2 - D_1)/D_1 < 2(|v_k| + v_k^2)/(\gamma - 2)$ and $D_2/D_1 < \gamma/(\gamma-2)$. We trivially have $\pi(2w_2 + w_2^2)/(\gamma^2 - 2) < 7(|v_k| + v_k^2)/\gamma^2$ for small I and $\gamma \geq 5$. So we obtain

$$C_7 < \min\left\{\frac{28(|v_k| + v_k^2)}{(\gamma-2)^2}, \frac{2(|v_k| + v_k^2)}{\gamma - 2}\right\} \equiv C_7'. \tag{4.5}$$

Let I be small enough so that for any $k' \in I$

$$c_1(k'), c_2(k') \in \left(F(v_k) - \frac{C_7'}{\pi}G(v_k), F(v_k) + \frac{C_7'}{\pi}G(v_k)\right).$$

Repeating the proof of *Theorem 4.1* we obtain

Theorem 4.2. *With the notation of Theorem 4.1 let* $\gamma \geq 5$, $F(x) = \frac{1}{2}\ln(1 + \frac{x^2}{4})$ *and* $G(x) = -\frac{1}{2}\ln(1 - |x| - x^2)$. *For* $k \in (0, \pi)$ *let* $E(k) = 2\cos(k)$, $v_k = -\frac{v}{\sin(k)}$,

$$\alpha_1(k) \equiv 1 - 2\,\frac{F(v_k) + \frac{28(|v_k|+v_k^2)}{\pi(\gamma-2)^2}G(v_k)}{\ln(\gamma)},$$

$$\alpha_2(k) \equiv 1 - 2\,\frac{F(v_k) - \frac{28(|v_k|+v_k^2)}{\pi(\gamma-2)^2}G(v_k)}{\ln(\gamma)}.$$

Then for a.e. ϕ *we have for all* k *such that* $|v_k| + v_k^2 < 1$ *and* $\alpha_2(k) < 1$, *that on a small interval around* $E(k)$ *the measure* μ_ϕ *is* $\alpha_1(k)$*-continuous and* $\alpha_2(k)$*-singular.*

Remarks. 1. Notice that the hypotheses imply that $\alpha_1(k) > 0$ whenever $\alpha_2(k) < 1$.

2. So under the above conditions we know that for a.e. ϕ, the local dimension of the spectral measure is within an interval which is close to 1, and for large γ its size is small compared to its distance from 1.

5. Fractional Dimension for all Boundary Conditions

Now we turn to the proof for *all* boundary conditions. We return to considering the term $g_n(k)$ and three oscillating terms first. We again restrict our considerations to the sin term, the two cos terms are treated similarly.

It turns out that the X_n from Section 4 are not regular enough to prove (2.6) (with sin replaced by X_n). Therefore we need to make a new breakup of the oscillating term to obtain a more regular X_n. This is done in the Appendix in the proof of *Theorem A.1* (with $f_n(k) = 2\overline{\theta}(x_n, k)$, $\beta = 2$, $\delta_1 = 1 - D_1$ and $\delta_2 = D_2 - 1$). Using the reasoning from Section 2 we then obtain our main result:

Theorem 5.1. *Let H_ϕ be the discrete Schrödinger operator on $\mathbb{Z}^+$ with potential $V_{v,\gamma}$ given by* (1.6)*, and boundary condition ϕ. Let μ_ϕ be its spectral measure. For any closed interval of energies $J \subset (-2, 2)$ there is $v_0 > 0$ and $\gamma_0 \in \mathbb{N}$ such that if $0 < |v| < v_0$ and $\gamma \geq \gamma_0 v^{-2}$ is an integer, then for any ϕ, the measure μ_ϕ has fractional Hausdorff dimension in J.*

Remark. A priori, γ_0 depends on v. However, (5.1) below gives $\varepsilon = O(v)$ as $v \to 0$ (since $C_1 = O(v^2)$, $C_7 = O(v\gamma^{-2})$ and $w_2 = O(v)$). So if $\gamma = O(\varepsilon^{-2}) = O(v^{-2})$, then in (5.2) $\alpha(\varepsilon) < 1$ (because $D_1, D_2 \to 1$ as $v \to 0$). Compare with the remark after *Theorem 4.1.*

Proof. Let $I \subset (0, \pi)$ be such that $2\cos(I) = J$. Let $\alpha_1' \equiv 1 - (w_2 + w_2^2)/\ln(\gamma) > 0$ (with $w_2 + w_2^2 < 1$ and $\gamma \geq 3$). We already know that by *Proposition 1.2* μ_ϕ is α_1'-continuous on I.

By the Appendix, the absolute value of the "small" term $\sin(2\overline{\theta}(x_n, k)) - X_n(k)$ from the new breakup is at most $\frac{\pi}{2}(\delta_1 + \delta_2)(1 - \delta_1)^{-1} \leq \frac{\pi}{2}C_7$, and for all $\varepsilon > \varepsilon_0 = \varepsilon_0(D_1, D_2, \gamma) \equiv \big(2\ln(1 + \frac{10}{\gamma}D_2 D_1^{-1})\big)^{1/2}$ there is A^ε with $\dim(A^\varepsilon) < 1$ such that (2.6) with $X_n(k)$ in place of $\sin(2\overline{\theta}(x_n, k))$ is satisfied. Here $C_7, \varepsilon_0 \to 0$ as $D_1, D_2 \to 1$ and $\gamma \to \infty$, which is guaranteed when $\gamma \to \infty$. Hence we proceed as follows.

This time we let γ, D_1 and D_2 be such that

$$\frac{\pi}{2}C_7\left(\frac{w_2}{2} + \frac{3w_2^2}{8}\right) < C_1$$

(this is possible for all large γ). Then for $n > n_0$ we have

$$\left|\sum_{n=n_0+1}^{N} \frac{v_k}{2}\sin(2\overline{\theta}(x_n, k))\right| \leq \frac{w_2}{2}\frac{\pi}{2}C_7 N + \left|\frac{v_k}{2}\sum_{n=n_0+1}^{N} X_n(k)\right|.$$

We do the same with the other two oscillating terms, and the three terms containing C_7 add up to

$$\frac{\pi}{2}C_7\left(\frac{w_2}{2} + \frac{3w_2^2}{8}\right)N < C_1 N.$$

Next, we choose ε with

$$0 < \varepsilon\left(\frac{w_2}{2} + \frac{3w_2^2}{8}\right) < C_1 - \frac{\pi}{2}C_7\left(\frac{w_2}{2} + \frac{3w_2^2}{8}\right) \tag{5.1}$$

and we take γ large enough so that $\varepsilon_0 < \varepsilon$. By the Appendix there is a set A^ε such that

$$\dim(A^\varepsilon) \leq \alpha(\varepsilon) \equiv 1 - \frac{\varepsilon^2 - 2\ln\left(1 + \frac{10D_2}{\gamma D_1}\right)}{2\ln(\gamma)} \tag{5.2}$$

and (2.6) holds with sin replaced by X_n. Similarly for the other two oscillating terms, and we let A^ε be the union of the three sets.

From the above discussion it follows that for $k \in I \backslash A^\varepsilon$ and n large enough we have

$$R_k(x_n + 1) > e^{c_1 n} = x_n^{\beta_1}, \tag{5.3}$$

where

$$c_1 = C_1 - \left(\frac{\pi}{2} C_7 + \varepsilon\right)\left(\frac{w_2}{2} + \frac{3w_2^2}{8}\right) > 0$$

and $\beta_1 = c_1 / \ln(\gamma)$. It is known (see [4]) that for a.e. k w.r.t. μ_0 we have

$$\overline{\lim_{L\to\infty}} \frac{\|R_k\|_L}{L^{1/2}\ln(L)} < \infty.$$

Given (5.3), this can be true only if $\mu_0(I\backslash A^\varepsilon) = 0$. Thus $\mu_0(I \cap \cdot)$ is supported on A^ε.

Hence on I the measure μ_0 is $(\dim(A^\varepsilon) + \delta)$-singular for any $\delta > 0$. Since D_1, D_2 can be arbitrarily close to C_5, C_6, and since $C_6/C_5 < \gamma/(\gamma - 2)$ if $w_2 + w_2^2 < 1$, we have

$$\dim(A^\varepsilon) < \alpha_2' \equiv 1 - \frac{\varepsilon^2 - 2\ln\left(1 + \frac{10}{\gamma - 2}\right)}{2\ln(\gamma)}.$$

So μ_0 is α_2'-singular on I. If in this argument we replace u_k by the generalized eigenfunction for the same energy satisfying boundary condition ϕ, we obtain the same singularity for μ_ϕ. Now take γ large so that $\alpha_2' < 1$, and the result follows. □

At this point we can do the same as what we did after proving *Theorem 4.1*: consider an infinite sum of terms instead of $g_n(k)$.

Now we use a different type of regularization of the oscillating terms, but everything can be done as before, with one adjustment. We need to use *Theorem A.3* in place of *Theorem A.1*, which gives us a different bound for the difference of the oscillating term $\sin^a(2\overline{\theta}(x_n, k)) \sin^{2b}(\overline{\theta}(x_n, k))$ and its regularization $X_{n,a,b}(k)$, namely $\frac{\pi}{2}(a+b)C_7$. The derivative of the oscillating term enters here, and we use the (very crude) estimate

$$\left\|[\sin^a(2\theta)\sin^{2b}(\theta)]'\right\|_\infty \leq (a+b)\left\|[\sin(2\theta)]'\right\|_\infty.$$

So this time we need to take $F(v_k)$ and $G(v_k)$ as before (the latter will be the coefficient for ε), and also

$$\tilde{G}(v_k) = \sum_{\substack{a,b\geq 0\\ a+b\geq 1}} \frac{1}{2}\binom{a+b}{a} |v_k|^{a+2b} = \frac{1}{2}\,\frac{|v_k| + v_k^2}{1 - |v_k| - v_k^2}$$

which will be the coefficient for $\frac{\pi}{2}C_7$. We take γ large so that $\frac{\pi}{2}C_7\tilde{G}(v_k) < F(v_k)$, and take $\varepsilon > 0$ such that

$$\varepsilon G(v_k) < F(v_k) - \frac{\pi}{2}C_7\tilde{G}(v_k).$$

Then given any pair a, b we construct the set $A^{\varepsilon}_{a,b}$ for $X_{n,a,b}$. We show, as above, that its dimension is at most $\alpha(\varepsilon)$. Letting the (countable) union of these sets play the role of A^{ε} in the proof of *Theorem 5.1*, and considering (4.5), we obtain

Theorem 5.2. *With the notation of Theorem 5.1 let $\gamma \geq 5$. For $k \in (0, \pi)$ let $E(k) = 2\cos(k)$, $v_k = -\frac{v}{\sin(k)}$,*

$$\varepsilon_k \equiv \frac{\ln\left(1+\frac{v_k^2}{4}\right) - \frac{14\pi(|v_k|+v_k^2)^2}{(\gamma-2)^2(1-|v_k|-v_k^2)}}{-\ln(1-|v_k|-v_k^2)},$$

$$\alpha'_1(k) \equiv 1 - \frac{|v_k|+v_k^2}{\ln(\gamma)},$$

$$\alpha'_2(k) \equiv 1 - \frac{\varepsilon_k^2 - 2\ln\left(1+\frac{10}{\gamma-2}\right)}{2\ln(\gamma)}.$$

If k is such that $|v_k| + v_k^2 < 1$, $\varepsilon_k > 0$ and $\alpha'_2(k) < 1$, then on a small interval around $E(k)$ each μ_ϕ is $\alpha'_1(k)$-continuous and $\alpha'_2(k)$-singular.

Remarks. 1. Notice that the hypotheses imply that $\alpha'_1(k) > 0$.

2. So for large γ, the dimension of each spectral measure μ_ϕ is within an interval which is close to 1, and its size is comparable to its distance from 1. The estimate for a.e. ϕ in Section 4 is better by a factor of γ^{-2}.

To illustrate the obtained results we provide an

Example. Let us assume that v, γ and I are such that $\gamma \geq 10^4 v^{-2}$, and $|v_k| \leq \frac{1}{2}$ for any $k \in I$. Estimating the quantities in *Theorems 4.2* and *5.2*, one can obtain

$$[\alpha_1(k), \alpha_2(k)] \subseteq \left[1 - \frac{\ln\left(1+\frac{v_k^2}{4}\right)}{\ln(\gamma)} - \frac{10}{\gamma^2\ln(\gamma)},\ 1 - \frac{\ln\left(1+\frac{v_k^2}{4}\right)}{\ln(\gamma)} + \frac{10}{\gamma^2\ln(\gamma)}\right]$$

and

$$[\alpha'_1(k), \alpha'_2(k)] \subseteq \left[1 - \frac{|v_k|+v_k^2}{\ln(\gamma)},\ 1 - \frac{v_k^2}{500\ln(\gamma)}\right]$$

(notice that $v_k = -2v/\sqrt{4-E^2}$). For instance, take $v = \frac{1}{10}$, $\gamma = 10^6$ and $J = [-1.9, 1.9]$. Then for a.e. ϕ the local dimension of μ_ϕ at any $E \in J$ is in

$$\left[1 - \frac{\ln\left(1+\frac{1}{100(4-E^2)}\right)}{6\ln(10)} - \frac{1}{10^{12}},\ 1 - \frac{\ln\left(1+\frac{1}{100(4-E^2)}\right)}{6\ln(10)} + \frac{1}{10^{12}}\right]$$

and for all ϕ it is in

$$\left[1 - \frac{1}{20\ln(10)\sqrt{4-E^2}},\ 1 - \frac{1}{10^5\ln(10)(4-E^2)}\right] \subseteq \left[1 - \frac{1}{10},\ 1 - \frac{1}{10^6}\right].$$

6. Random Operators

In this section we consider potentials of the form (1.6), with certain randomness in the *position* of the special sites x_n. More precisely, x_n will be a random variable uniformly distributed over $\{\gamma^n - n, \dots, \gamma^n + n\}$. The fact that the size of these sets grows will yield certain "averaging" of $\ln(R_k(x_n + 1)) - \ln(R_k(x_{n-1} + 1))$ and thus a constant growth (in the limit $n \to \infty$) of $\ln(R_k(x_n + 1))$ for a.e. realization of the potential. As a consequence we will be able to compute the exact dimension of the spectral measures for these random potentials.

We begin with a standard result.

Lemma 6.1. *For a.e. $x \in \mathbb{R}$ there is q_0 such that for any integer $q > q_0$ we have*

$$\operatorname{dist}(qx, \mathbb{Z}) > \frac{1}{q^2}. \tag{6.1}$$

Proof. For any $n \in \mathbb{N}$ the measure of the set of $x \in [n, n+1)$, for which (6.1) fails, is $2q^{-2}$. This is summable in q, and so by the Borel-Cantelli Lemma the measure of those $x \in [n, n+1)$ for which (6.1) fails infinitely often is zero. $\square$

Let $\mathcal{X}$ be the set of all such x and let $\mathcal{K} = 2\pi\mathcal{X}$. Notice that $\mathcal{K}$ is a set of full measure, not intersecting $2\pi\mathbb{Q}$. The "averaging" result we need is also well-known.

Lemma 6.2. *Let $k \in \mathcal{K}$ and let $f \in C^4(\mathbb{R})$ have period 2π with $\int_0^{2\pi} f(t)\, dt = 0$. Then there is $C = C(k, f) < \infty$ such that for every $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$*

$$\left| \sum_{\ell=1}^{n} f(\theta + \ell k) \right| \le C.$$

Proof. Let $\sum_{q=-\infty}^{\infty} a_q e^{iqt}$ be the Fourier series of $f(t)$. Since f is C^4, we know that $|a_q| \le c q^{-4}$ for some c. Thanks to this fact all the sums appearing in this argument are pointwise absolutely convergent. Also, $a_0 = 0$ by the hypothesis. We have

$$\begin{aligned} \left| \sum_{\ell=1}^{n} f(\theta + \ell k) \right| \le \sum_q |a_q| \left| \sum_{\ell=1}^{n} e^{iq\ell k} \right| &= \sum_{q \neq 0} |a_q| \left| \frac{1 - e^{iqnk}}{1 - e^{iqk}} \right| \\ &\le \pi \sum_{q \neq 0} \frac{|a_q|}{\operatorname{dist}(qk, 2\pi\mathbb{Z})} \le c \sum_{q=1}^{\infty} \frac{1}{q^4 \operatorname{dist}(qx, \mathbb{Z})} \end{aligned}$$

where $x = k/2\pi \in \mathcal{X}$. By *Lemma 6.1* this is bounded by

$$c' + c \sum_{q=1}^{\infty} \frac{1}{q^2} \equiv C < \infty$$

where $c' < \infty$ depends on x and c. $\square$

Let ω_n be a random variable uniformly distributed over $\{-n, -n+1, \dots, n\}$ and let (Ω, ν) be the product probability space for these ω_n's ($n \in \mathbb{N}$). For $\omega = (\omega_1, \omega_2, \dots) \in \Omega$ let

$$V_{v,\gamma}^{(\omega)}(x) \equiv \begin{cases} v & x = x_n^{(\omega)} \equiv \gamma^n + \omega_n \text{ for some } n \ge 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

It is a consequence of the "smallness" of ω_n (compared to γ^n), that all previous results for $V_{v,\gamma}$ apply to each $V_{v,\gamma}^{(\omega)}$ as well. In this random case, however, we will prove and compute *exact* (*local*) *dimension* for the spectral measures for a.e. ω and a.e. boundary condition ϕ. Moreover, our results will apply to any v, k and γ, not only to small v_k and large γ.

Let us fix $k \in \mathcal{K}$ and define

$$X_n(\omega) \equiv \tfrac{1}{2}\ln\left(1 + v_k \sin\Big(2\overline{\theta}_k\left(x_n^{(\omega)}\right)\Big) + v_k^2 \sin^2\Big(\overline{\theta}_k\left(x_n^{(\omega)}\right)\Big)\right) - \tfrac{1}{2}\ln\left(1 + \frac{v_k^2}{4}\right).$$

Obviously $|X_n(\omega)| < M$ for some $M = M(k) < \infty$. We will prove that for a.e. ω

$$\frac{\sum_{n=1}^N X_n(\omega)}{N} \to 0, \tag{6.3}$$

which in turn implies (see below) that the exact dimension of $\mu_\phi^{(\omega)}$ for a.e. ω and a.e. ϕ is

$$1 - \frac{\ln\left(1 + \frac{v_k^2}{4}\right)}{\ln(\gamma)} = 1 - \frac{\ln\left(1 + \frac{v^2}{4-E^2}\right)}{\ln(\gamma)}.$$

We use the same methods as earlier. We exploit *Lemmas 6.1* and *6.2* to show that the expectations of the cross terms $X_m X_n$ for $m < n$ are small, which implies (6.3) for a.e. ω. This time, however, we consider expectations w.r.t. ω, rather than k.

Let

$$f(\theta) \equiv \tfrac{1}{2}\ln\Big(1 + v_k\sin(2\theta) + v_k^2\sin^2(\theta)\Big) - \tfrac{1}{2}\ln\left(1 + \frac{v_k^2}{4}\right).$$

Then $f(\theta) = f(\theta + 2\pi)$, $\int_0^{2\pi} f(\theta)\,d\theta = 0$, and f is C^4, so f is as in *Lemma 6.2*. Let $\delta, \omega \in \Omega$ be such that $\delta_j = \omega_j$ for $j = 1, \dots, n-1$, and $\delta_n = \omega_n + l$ for some l. Then the recursive relation for the EFGP transform of generalized eigenfunctions implies that if $X_n(\omega) = f(\theta)$ for some θ, then $X_n(\delta) = f(\theta + lk)$. This shows why *Lemma 6.2* enters in our argument.

For $m < n$ we have with $\mathbb{E} = \mathbb{E}_\omega$

$$\begin{aligned} |\mathbb{E}(X_m X_n)| &\le \max_{c_1, c_2, \dots, c_{n-1}} \left\{ \Big|\mathbb{E}\big(X_m X_n \,|\, \omega_1 = c_1, \dots, \omega_{n-1} = c_{n-1}\big)\Big| \right\} \\ &\le \max_{c_1, c_2, \dots, c_{n-1}} \Big\{ M\Big|\mathbb{E}\big(X_n \,|\, \omega_1 = c_1, \dots, \omega_{n-1} = c_{n-1}\big)\Big| \Big\}. \end{aligned}$$

By *Lemma 6.2* there exists $C < \infty$ such that the last expectation is at most $C/(2n+1)$. Hence for fixed $k \in \mathcal{K}$ there is $D = D(k) < \infty$ such that for $m < n$

$$|\mathbb{E}(X_m X_n)| < \frac{D}{n}.$$

Then

$$\mathbb{E}\left(\left|\sum_{n=1}^N X_n\right|\right) \le \mathbb{E}\left(\left(\sum_{n=1}^N X_n\right)^2\right)^{1/2} \le (M^2 N + 2DN)^{1/2} = cN^{1/2}.$$

As before, using summability of $\mathbb{E}\left(\left|N^{-4}\sum_{n=1}^{N^4} X_n\right|\right)$ and boundedness of X_n, one shows that (6.3) holds for a.e. ω.

We have proved (6.3) for a.e. k and a.e. ω. By the Fubini Theorem, (6.3) holds for a.e. ω and a.e. k. Thus for these (ω, k) we obtain an exact power of the growth for the solution u_k. Namely we have that for any

$$c_1 < \frac{1}{2} \ln\left(1 + \frac{v_k^2}{4}\right) < c_2$$

there is n_0 such that for $n > n_0$

$$R_k\left(x_n^{(\omega)} + 1\right) \in (e^{c_1 n}, e^{c_2 n}).$$

Also, *Lemma 2.1* gives us the existence of a subordinate solution $u_k^{\mathrm{sub}} \sim (P_k, \psi_k)$ for energy $2\cos(k)$, so that for large n

$$P_k\left(x_n^{(\omega)} + 1\right) \le e^{-c_1 n}.$$

Since for any such ω, for a.e. ϕ the spectral measure $\mu_\phi^{(\omega)}$ is supported on the set of the corresponding k's, one can conclude (by methods presented earlier) the following

Theorem 6.3. *Let (Ω, ν) be the above probability space. For $\omega \in \Omega$ let $H_\phi^{(\omega)}$ be the discrete Schrödinger operator on $\mathbb{Z}^+$ with potential $V_{v,\gamma}^{(\omega)}$, given by (6.2) with $v \neq 0$ and $\gamma > 1$, and boundary condition ϕ. Let $\mu_\phi^{(\omega)}$ be its spectral measure. Let*

$$J \equiv \left(-\sqrt{4 - \frac{v^2}{\gamma - 1}}, \sqrt{4 - \frac{v^2}{\gamma - 1}}\right)$$

if $v^2 < 4(\gamma - 1)$, and $J \equiv \emptyset$ otherwise. Then for a.e. ω and for a.e. ϕ the measure $\mu_\phi^{(\omega)}$ is purely singular continuous in J, with local dimension

$$1 - \frac{\ln\left(1 + \frac{v^2}{4 - E^2}\right)}{\ln(\gamma)},$$

and it is dense pure point in the rest of the interval $[-2, 2]$.

7. Whole Line Operators With Symmetric Potentials

It is readily seen that our results also apply to certain whole-line operators (satisfying (1.1) for $x \in \mathbb{Z}$). Let us consider the operator $\tilde{H}$ on $\ell^2(\mathbb{Z})$ with potential $\tilde{V}_{v,\gamma}$ given by (1.6) for $x \ge 1$ and by $\tilde{V}_{v,\gamma}(x) = \tilde{V}_{v,\gamma}(1 - x)$ for $x \le 0$. This potential is reflected about $\frac{1}{2}$. One easily sees that $\tilde{H} = \tilde{H}|_{\mathcal{E}} \oplus \tilde{H}|_{\mathcal{O}}$ where $\mathcal{E} = \{u \in \ell^2(\mathbb{Z}) \mid u(1 - x) = u(x) \text{ for all } x \in \mathbb{Z}\}$ and $\mathcal{O} = \{u \in \ell^2(\mathbb{Z}) \mid u(1 - x) = -u(x) \text{ for all } x \in \mathbb{Z}\}$ are, respectively, the even and odd subspaces of $\ell^2(\mathbb{Z})$.

On the other hand, if δ_1 is the delta function at $x = 1$, then

$$\begin{aligned} \tilde{H}|_{\mathcal{E}} &\cong H_0 + \delta_1 = H_{\frac{3\pi}{4}}, \\ \tilde{H}|_{\mathcal{O}} &\cong H_0 - \delta_1 = H_{\frac{\pi}{4}} \end{aligned} \tag{7.1}$$

by simply taking the restriction of u to $\mathbb{Z}^+$. Hence $\tilde{\mu} = \mu_{\pi/4} + \mu_{3\pi/4}$ is a spectral measure for $\tilde{H}$, and all we have proved about the measures for the half-line operators applies to $\tilde{\mu}$.

Notice that variation of boundary condition (i.e., of $V(1)$) in the half-line case correponds to varying $\tilde{V}(1) = \tilde{V}(0)$, as can be seen from (7.1). We conclude:

Theorem 7.1. *Let $\tilde{H}$ be the discrete Schrödinger operator on $\mathbb{Z}$, with potential $\tilde{V}_{v,\gamma}$ given by* (1.6) *for $x \geq 1$ and reflected about $\frac{1}{2}$. Let $\tilde{H}_\phi \equiv \tilde{H} - \tan(\phi)(\delta_0 + \delta_1)$ and let $\tilde{\mu}_\phi$ be the spectral measure of $\tilde{H}_\phi$. Then the statements of Theorems 4.1, 4.2, 5.1 and 5.2 hold for $\tilde{H}_\phi$.*

If instead of $\tilde{H}$ and $\tilde{V}_{v,\gamma}$ we consider $\tilde{H}^{(\omega)}$ and $\tilde{V}^{(\omega)}_{v,\gamma}$ (given by (6.2) *and reflected about $\frac{1}{2}$), and $\tilde{H}^{(\omega)}_\phi \equiv \tilde{H}^{(\omega)} - \tan(\phi)(\delta_0 + \delta_1)$, then the statement of Theorem 6.3 holds for $\tilde{H}^{(\omega)}_\phi$.*

Appendix. A Dimensional Estimate

Theorem A.1. *Let $\gamma > 1$, $\beta > 0$ and $\delta_1, \delta_2 \in [0,1)$. Let I be a finite interval and let $f_n \in C(I)$ be such that*

$$\frac{f_n'(k)}{\beta\gamma^n} \in [1-\delta_1, 1+\delta_2] \tag{A.1}$$

for all $n \in \mathbb{N}$ and a.e. $k \in I$. Let

$$F(k) \equiv \overline{\lim_{N\to\infty}} \frac{1}{N} \left| \sum_{n=1}^{N} \sin\big(f_n(k)\big) \right|$$

and $A^\varepsilon \equiv \{k \mid F(k) \geq \varepsilon\}$. Then there is $\varepsilon_0 = \varepsilon_0(\delta_1, \delta_2, \gamma)$ with $\varepsilon_0 \to 0$ as $\delta_1, \delta_2, \gamma^{-1} \to 0$ such that for $\varepsilon > \varepsilon_0$ the set A^ε has Hausdorff dimension less than 1.

Remarks. 1. In fact, we obtain

$$\dim(A^\varepsilon) \leq 1 - \frac{\left(\varepsilon - \frac{\pi}{2}\frac{\delta_1+\delta_2}{1-\delta_1}\right)^2 - 2\ln\left(1 + \frac{10}{\gamma}\frac{1+\delta_2}{1-\delta_1}\right)}{2\ln(\gamma)}$$

whenever $\varepsilon > \frac{\pi}{2}(\delta_1+\delta_2)(1-\delta_1)^{-1}$.

2. Similar questions have been studied using dynamical systems and Riesz measures (see, e.g., [2, 9]). The methods, however, seem to require $\delta_1 = \delta_2 = 0$ and $\gamma \in \mathbb{N}$.

3. This result is only useful if we can take ε smaller than 1. Hence even in the case $\delta_1 = \delta_2 = 0$ it applies only when γ is large (larger than $10/(\sqrt{e}-1)$).

4. See Section 5 for the application of this result in the present paper.

The rest of this appendix is devoted to the proof of the above estimate. First, we explain the presence of the term $\frac{\pi}{2}(\delta_1+\delta_2)(1-\delta_1)^{-1}$. The reason is the same as in Section 4. We need more regularity than the function $\sin(f_n(k))$ possesses, and so we will need to break it up in two terms. One of them will be small, with absolute value not more than $\frac{\pi}{2}(\delta_1+\delta_2)(1-\delta_1)^{-1}$, whereas the other one will be "regular" enough so that we will be able to prove for it the above theorem with $(\varepsilon - \dots)^2$ replaced by just ε^2. These two facts then yield the theorem as stated. We note that the regular term, denoted X_n, will be different from the X_n term from Section 4, which is not regular enough for the purposes of this argument.

Before we perform this breakup, notice that if we define intervals $I_{n,j} = [k_{n,j}, k_{n,j+1}]$ in the same way as in Section 3, but with $2\overline{\theta}(x_n, k)$ replaced by $f_n(k)$ (i.e., so that $f_n(k_{n,j})$ are multiples of 2π), then (A.1) implies

$$|I_{n,j}| \in \left[\frac{2\pi}{\beta D_2 \gamma^n}, \frac{2\pi}{\beta D_1 \gamma^n}\right] \tag{A.2}$$

with $D_1 \equiv 1 - \delta_1$ and $D_2 \equiv 1 + \delta_2$.

We now define for $k \in I$

$$\varphi_n(k) \equiv \begin{cases} f_n(k_{n,j}) + 2\pi \frac{k - k_{n,j}}{|I_{n,j}|} & k \in I_{n,j} \\ f_n(k) & k \in I \backslash [K_{n,1}, K_{n,2}]. \end{cases}$$

Notice that $\varphi_n(k_{n,j}) = f_n(k_{n,j})$ for all j, and $\varphi_n(k)$ is linear on each $I_{n,j}$. So if we let $X_n(k) \equiv \sin(\varphi_n(k))$, then X_n is a series of exact sin waves on intervals $I_{n,j}$. This is the type of regularity we need and X_n will be the regular term in our breakup.

Now we want to estimate the small term $\sin(f_n(k)) - X_n(k)$. To do that, we need an upper bound on $|f_n(k) - \varphi_n(k)|$. This will be maximal if $f_n'(k)$ equals $\beta D_1 \gamma^n$ on some interval $(k_{n,j}, k_{n,j} + c)$ and $\beta D_2 \gamma^n$ on $(k_{n,j} + c, k_{n,j+1})$ (or vice-versa), and the maximum will occur at $k_{n,j} + c$. Since in such case $\beta D_1 \gamma^n c + \beta D_2 \gamma^n (|I_{n,j}| - c) = 2\pi$ by the definition of $k_{n,j}$, we can compute $|I_{n,j}|$ and φ_n in terms of c, and then maximize for c. We obtain

$$|f_n(k) - \varphi_n(k)| \leq 2\pi \frac{\sqrt{D_2} - \sqrt{D_1}}{\sqrt{D_2} + \sqrt{D_1}}$$

which yields the above claimed estimate

$$|\sin(f_n(k)) - X_n(k)| \leq |f_n(k) - \varphi_n(k)| \leq 2\pi \frac{D_2 - D_1}{4D_1} \leq \frac{\pi}{2} \frac{\delta_1 + \delta_2}{1 - \delta_1}.$$

Now we only need to treat the term X_n. We start with a technical

Lemma A.2. *There is a constant c_0 such that for any $n \geq 1$ and any $0 \leq \varepsilon \leq \frac{1}{2}$ we have*

$$\sum_{j=0}^{\lfloor \frac{n}{2} - \varepsilon n \rfloor} \binom{n}{j} \leq c_0 n 2^n e^{-2\varepsilon^2 n}.$$

Remark. By the normal approximation to the binomial distribution, the left-hand side is roughly $2^n \Phi(-2\varepsilon\sqrt{n}) < 2^n e^{-2\varepsilon^2 n}$, where Φ is the standard normal distribution function. The extra factor n is added for convenience of proof and can be removed.

Proof. The sum is obviously smaller than $n\binom{n}{\lfloor \frac{n}{2} - \varepsilon n \rfloor}$, so we will estimate this. By Stirling's formula we have for $n \geq 1$

$$c_1 \sqrt{n} \left(\frac{n}{e}\right)^n < n! < c_2 \sqrt{n} \left(\frac{n}{e}\right)^n$$

for some $c_1, c_2 > 0$. Let $\lfloor \frac{n}{2} - \varepsilon n \rfloor = \left(\frac{1}{2} - \delta\right) n$ (hence $\varepsilon \leq \delta \leq \frac{1}{2}$). Then

$$\binom{n}{\left(\frac{1}{2} - \delta\right)n} \leq \frac{c_2 \sqrt{n} \left(\frac{n}{e}\right)^n}{c_1 \left[\frac{(\frac{1}{2} - \delta)n}{e}\right]^{(\frac{1}{2} - \delta)n} c_1 \sqrt{\frac{n}{2}} \left[\frac{(\frac{1}{2} + \delta)n}{e}\right]^{(\frac{1}{2} + \delta)n}} = c_0 \left[\left(\frac{1}{2} - \delta\right)^{\frac{1}{2} - \delta} \left(\frac{1}{2} + \delta\right)^{\frac{1}{2} + \delta}\right]^{-n}$$

Since $\varepsilon \le \delta$, it is sufficient to prove that

$$\left(\frac{1}{2}-\delta\right)^{\frac{1}{2}-\delta}\left(\frac{1}{2}+\delta\right)^{\frac{1}{2}+\delta} \ge 2^{-1}e^{2\delta^2}$$

for $0 \le \delta \le \frac{1}{2}$. This can be done by taking the logarithm of both sides of the inequality and observing that the resulting quantities have the same value and first derivative for $\delta = 0$, and the left hand side has a larger second derivative in $(0, \frac{1}{2})$. $\square$

Let us now study a simple example which will illustrate our strategy. Let us take $\tilde{X}_n(k) \equiv \operatorname{sgn}(\sin(2^n \pi k))$ for $k \in [0,1]$ with $\operatorname{sgn}(0) \equiv 1$. Thus $\tilde{X}_n$ takes only values 1 and -1, alternatively on intervals of lengths 2^{-n}. Let us denote

$$A^{\varepsilon}_{\tilde{X}} \equiv \left\{ k \,\middle|\, \overline{\lim_{N\to\infty}} \frac{\sum_{n=1}^{N} \tilde{X}_n(k)}{N} > \varepsilon \right\}.$$

We will show that the dimension of $A^{\varepsilon}_{\tilde{X}}$ is smaller than 1.

Let S_N be the union of those intervals $[j2^{-N}, (j+1)2^{-N}]$ (for $j = 0, \dots, 2^N - 1$), in which $\sum_{n=1}^{N} \tilde{X}_n(k) \ge \varepsilon N$. Their number equals the number of sequences of N symbols from the alphabet $\{-1, 1\}$ such that the number of occurences of -1 is at most $\lfloor (1-\varepsilon)\frac{N}{2} \rfloor$. For any N_1 the set $\bigcup_{N_1}^{\infty} S_N$ is obviously a (2^{-N_1})-cover of $A^{\varepsilon}_{\tilde{X}}$, and we have by *Lemma A.2*

$$\begin{aligned}
h^{\alpha}(A^{\varepsilon}_{\tilde{X}}) &\le \lim_{N_1\to\infty} \sum_{N=N_1}^{\infty} 2^{-\alpha N} \sum_{n=0}^{\lfloor \frac{(1-\varepsilon)}{2} N \rfloor} \binom{N}{n} \\
&\le \lim_{N_1\to\infty} \sum_{N=N_1}^{\infty} 2^{-\alpha N} c_0 N 2^N e^{-\frac{\varepsilon^2}{2}N} \\
&= \lim_{N_1\to\infty} \sum_{N=N_1}^{\infty} c_0 N \left(2^{1-\alpha} e^{-\frac{\varepsilon^2}{2}}\right)^N .
\end{aligned}$$

If $\alpha < 1$ is such that $2^{1-\alpha}e^{-\frac{\varepsilon^2}{2}} < 1$, we get $h^{\alpha}(A^{\varepsilon}_{\tilde{X}}) = 0$.

Since we can do the same for

$$B^{\varepsilon}_{\tilde{X}} \equiv \left\{ k \,\middle|\, \underline{\lim_{N\to\infty}} \frac{\sum_{n=1}^{N} \tilde{X}_n(k)}{N} < -\varepsilon \right\},$$

it follows, that the set

$$\left\{ k \,\middle|\, \overline{\lim_{N\to\infty}} \left| \frac{\sum_{n=1}^{N} \tilde{X}_n(k)}{N} \right| > \varepsilon \right\} = A^{\varepsilon}_{\tilde{X}} \cup B^{\varepsilon}_{\tilde{X}}$$

has dimension at most $\alpha < 1$.

We would like to prove now a similar result for our X_n's. There is, however, a problem. The technique used in the above example was applicable to finite-valued functions only. Thus we have to "discretize" X_n via another breakup into a "small" and a "nice" term. Pick

$p \in \mathbb{N}$ and define

$$Y_n(k) \equiv \frac{\lfloor pX_n(k) + \frac{1}{2} \rfloor}{p}. \tag{A.3}$$

Then Y_n takes values $\frac{j}{p}$ for $j = -p, \ldots, p$ and $|X_n(k) - Y_n(k)| \leq \frac{1}{2p}$. Later we will take $p \to \infty$, and then results which we prove for Y_n will apply to X_n as well.

Finally we will break up Y_n into p even simpler terms. Let

$$Y_{n,i}(k) \equiv \begin{cases} \operatorname{sgn}(Y_n(k)) & \text{if } |Y_n(k)| \geq \frac{i}{p}, \\ 0 & \text{otherwise}, \end{cases}$$

for $i = 1, \ldots, p$. Then $Y_{n,i}$ takes only values -1, 0 and 1, and $Y_n(k) = \frac{1}{p}\sum_{i=1}^{p} Y_{n,i}$.

It is obvious from the construction of $Y_{n,i}$ and the fact that X_n is a perfect sin wave on $I_{n,j}$, that on any $I_{n,j}$ we have

$$Y_{n,i}(k) = \begin{cases} 0 & k \in I^1_{n,j}(i) \equiv [k_{n,j}, k_{n,j} + a_i|I_{n,j}|), \\ 1 & k \in I^2_{n,j}(i) \equiv [k_{n,j} + a_i|I_{n,j}|, k_{n,j} + (\frac{1}{2} - a_i)|I_{n,j}|], \\ 0 & k \in I^3_{n,j}(i) \equiv (k_{n,j} + (\frac{1}{2} - a_i)|I_{n,j}|, k_{n,j} + (\frac{1}{2} + a_i)|I_{n,j}|], \\ -1 & k \in I^4_{n,j}(i) \equiv (k_{n,j} + (\frac{1}{2} + a_i)|I_{n,j}|, k_{n,j} + (1 - a_i)|I_{n,j}|), \\ 0 & k \in I^5_{n,j}(i) \equiv [k_{n,j} + (1 - a_i)|I_{n,j}|, k_{n,j+1}], \end{cases}$$

where $a_i = \frac{1}{2\pi} \arcsin\big((i - \frac{1}{2})/p\big)$. Hence we have

$$\begin{aligned} \frac{|I^1_{n,j}(i) \cup I^3_{n,j}(i) \cup I^5_{n,j}(i)|}{|I_{n,j}|} &= 4a_i, \\ \frac{|I^2_{n,j}(i)|}{|I_{n,j}|} = \frac{|I^4_{n,j}(i)|}{|I_{n,j}|} &= \frac{1}{2} - 2a_i. \end{aligned} \tag{A.4}$$

To complete our argument, we will prove that for $\varepsilon^2 > 2\ln(1 + \frac{10}{\gamma} D_2 D_1^{-1})$ the sets

$$A^\varepsilon_p \equiv \left\{ k \,\Big|\, \overline{\lim_{N\to\infty}} \frac{\sum_{n=1}^N Y_n(k)}{N} > \varepsilon \right\},$$

$$B^\varepsilon_p \equiv \left\{ k \,\Big|\, \underline{\lim}_{N\to\infty} \frac{\sum_{n=1}^N Y_n(k)}{N} < -\varepsilon \right\}$$

have dimension smaller than 1, and then deduce the same for X_n. To this end it is sufficient to show that for any $i = 1, \ldots, p$ the set

$$A^\varepsilon_p(i) \equiv \left\{ k \,\Big|\, \overline{\lim_{N\to\infty}} \frac{\sum_{n=1}^N Y_{n,i}(k)}{N} > \varepsilon \right\}$$

has dimension smaller than 1. This is because

$$A^\varepsilon_p \subseteq \bigcup_{i=1}^p A^\varepsilon_p(i),$$

and similarly for B^ε_p.

Fix some i and ε. Let I' be any closed sub-interval of I, not containing the endpoints of I. Let N_0 be such that for all $N > N_0$ we have $I' \subseteq (K_{N,1}, K_{N,2})$. Let us consider $\sum_{n=N_0+1}^{N+N_0} Y_{n,i}$ on I' and denote

$$A \equiv \left\{ k \in I' \middle| \ \overline{\lim_{N\to\infty}} \frac{\sum_{n=N_0+1}^{N+N_0} Y_{n,i}(k)}{N} > \varepsilon \right\}.$$

Obviously $A = A_p^\varepsilon(i) \cap I'$. Hence proving that $\dim(A) \le \alpha$ with $\alpha < 1$ independent of I' and i will be enough to show $\dim(A_p^\varepsilon) \le \alpha < 1$.

As in our simple example, we will cover A by a (recursively constructed) set of intervals. Let $N \ge 1$ and define $A_{N+N_0} \equiv \{k \in I' | \sum_{n=N_0+1}^{N+N_0} Y_{n,i}(k) > \varepsilon N\}$. Then for any $N_1 \ge 1$ we have $A \subseteq \bigcup_{N=N_1}^\infty A_{N+N_0}$. If $k \in A_{N+N_0}$ for some N, then the sequence $\{s_n\}_{N_0+1}^{N+N_0}$ with $s_n \equiv Y_{n,i}(k)$ can have at most $\lfloor (1-\varepsilon)N \rfloor$ zeros, and if it has l zeros, then it can have at most

$$\left\lfloor \frac{N-l}{2} - \frac{\varepsilon N}{2} \right\rfloor = \left\lfloor \frac{N-l}{2} - \frac{\varepsilon N}{2(N-l)}(N-l) \right\rfloor$$

occurences of -1. The number of such sequences (with l zeros) is at most

$$\binom{N}{l} \sum_{n=0}^{\lfloor \frac{N-l}{2} - \frac{\varepsilon N}{2(N-l)}(N-l) \rfloor} \binom{N-l}{n} \le \binom{N}{l} c_0 (N-l) 2^{N-l} e^{-\frac{\varepsilon^2 N^2}{2(N-l)}} \le c_0 N e^{-\frac{\varepsilon^2}{2}N} \binom{N}{l} 2^{N-l}$$

by *Lemma A.2.*

Let us pick one such sequence $\{s_n\}_{N_0+1}^{N+N_0}$ with l zeros. We will construct a covering of the set of those $k \in I'$ which generate this sequence, by intervals $I_{N+N_0+1,j}$. Let S_{N_0} be the union of those $I_{N_0+1,j}$ which have nonzero intersection with I'. Now construct inductively S_n from S_{n-1}, so that S_n is the union of those intervals $I_{n+1,j}$, which have nonzero intersection with the set

$$\tilde{S}_n \equiv \begin{cases} \bigcup_{I_{n,j} \subseteq S_{n-1}} I_{n,j}^2(i) & \text{if } s_n = 1, \\ \bigcup_{I_{n,j} \subseteq S_{n-1}} \left[I_{n,j}^1(i) \cup I_{n,j}^3(i) \cup I_{n,j}^5(i) \right] & \text{if } s_n = 0, \\ \bigcup_{I_{n,j} \subseteq S_{n-1}} I_{n,j}^4(i) & \text{if } s_n = -1. \end{cases}$$

We want to estimate $|S_{N+N_0}|$. This can be done recursively using (A.4) and these facts:

(1) $|S_{N_0}| \le |I|$
(2) If J is an interval and $n \in \mathbb{N}$, then the (Lebesgue) measure of the union of intervals $I_{n+1,j}$ which have nonzero intersection with J is at most $|J| + 2\left(\frac{2\pi}{\beta} D_1^{-1} \gamma^{-(n+1)}\right)$.
(3) For any j we have $\frac{2\pi}{\beta} D_1^{-1} \gamma^{-(n+1)} \le |I_{n,j}| D_2 (\gamma D_1)^{-1}$.

Here (2) and (3) follow from (A.2). The net result is

$$|S_{N+N_0}| \le |I| \left(4a_i + 6\frac{D_2}{\gamma D_1} \right)^l \left(\frac{1}{2} - 2a_i + 2\frac{D_2}{\gamma D_1} \right)^{N-l}.$$

Since S_{N+N_0} is a union of intervals $I_{N+N_0+1,j}$, their number is (by (A.2)) at most

$$|I| \left(4a_i + \frac{6D_2}{\gamma D_1} \right)^l \left(\frac{1}{2} - 2a_i + \frac{2D_2}{\gamma D_1} \right)^{N-l} \frac{\beta D_2}{2\pi} \gamma^{N+N_0+1}.$$

Therefore A_{N+N_0} can be covered by at most

$$\sum_{l=0}^{\lfloor(1-\varepsilon)N\rfloor} c_0 N e^{-\frac{\varepsilon^2}{2}N}\binom{N}{l}2^{N-l}|I|\left(4a_i+\frac{6D_2}{\gamma D_1}\right)^l\left(\frac{1}{2}-2a_i+\frac{2D_2}{\gamma D_1}\right)^{N-l}\frac{\beta D_2}{2\pi}\gamma^{N+N_0+1}$$
$$\leq cNe^{-\frac{\varepsilon^2}{2}N}\gamma^N\sum_{l=0}^{N}\binom{N}{l}\left(4a_i+\frac{6D_2}{\gamma D_1}\right)^l\left(\frac{1}{2}-2a_i+\frac{2D_2}{\gamma D_1}\right)^{N-l}2^{N-l}$$
$$= cN\left(\gamma e^{-\frac{\varepsilon^2}{2}}\left(1+\frac{10D_2}{\gamma D_1}\right)\right)^N$$

intervals $I_{N+N_0+1,j}$. Let us denote their union A'_{N+N_0}.

Now $\bigcup_{N=N_1}^{\infty} A'_{N+N_0}$ contains A, and by (A.2) it is a $\left(\frac{2\pi}{\beta}D_1^{-1}\gamma^{-N_1-N_0-1}\right)$-cover. This yields

$$h^\alpha(A) \leq \lim_{N_1\to\infty}\sum_{N=N_1}^{\infty} cN\left(\gamma e^{-\frac{\varepsilon^2}{2}}\left(1+\frac{10D_2}{\gamma D_1}\right)\right)^N\gamma^{-(N+N_0+1)\alpha}$$
$$= \lim_{N_1\to\infty}\sum_{N=N_1}^{\infty} cN\left(\gamma^{1-\alpha}e^{-\frac{\varepsilon^2}{2}}\left(1+\frac{10D_2}{\gamma D_1}\right)\right)^N.$$

Hence if

$$\gamma^{1-\alpha}e^{-\frac{\varepsilon^2}{2}}\left(1+\frac{10D_2}{\gamma D_1}\right)<1,$$

we get $h^\alpha(A)=0$. This is the case for any

$$\alpha > \alpha(\varepsilon) \equiv 1-\frac{\varepsilon^2-2\ln\left(1+\frac{10D_2}{\gamma D_1}\right)}{2\ln(\gamma)},$$

and so $\dim(A)\leq\alpha(\varepsilon)$. If $\varepsilon^2 > 2\ln(1+\frac{10}{\gamma}D_2D_1^{-1})$, then this is smaller than 1.

As mentioned earlier, it follows that $\dim(A_p^\varepsilon)\leq\alpha(\varepsilon)$. Notice that $\alpha(\varepsilon)$ does not depend on p. Thus if

$$A_X^\varepsilon \equiv \left\{k\,\middle|\,\overline{\lim_{N\to\infty}}\frac{\sum_{n=1}^N X_n(k)}{N}>\varepsilon\right\},$$

then obviously

$$A_X^\varepsilon \subseteq \bigcup_{p=1}^{\infty} A_p^\varepsilon,$$

and so $\dim(A_X^\varepsilon)\leq\alpha(\varepsilon)$. Since the same result holds for

$$B_X^\varepsilon \equiv \left\{k\,\middle|\,\underline{\lim_{N\to\infty}}\frac{\sum_{n=1}^N X_n(k)}{N}<-\varepsilon\right\},$$

we can take $A^\varepsilon \equiv A_X^\varepsilon\cup B_X^\varepsilon$. By the discussion at the beginning, this completes the proof.

We conclude with a generalization of *Theorem A.1.* We denote $x_\pm\equiv\max\{\pm x,0\}$.

Theorem A.3. *Theorem A.1 remains valid if we take*

$$F(k) \equiv \overline{\lim_{N\to\infty}} \frac{1}{N} \left| \sum_{n=1}^{N} G\big(f_n(k)\big) \right|$$

where $G : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:
(1) *G is continuous and piecewise C^1;*
(2) *$G(x) = G(x + 2\pi)$ and $\int_0^{2\pi} G(x)\, dx = 0$;*
(3) *There are $0 = x_1 < x_2 < \cdots < x_m = 2\pi$ such that G is monotone on $[x_j, x_{j+1}]$, and $G(x_{j+1}) = 0$ whenever $G(x_j) \neq 0$.*

Remarks. 1. The x_j's are the zeros and local maxima and minima of G in $[0, 2\pi]$. Between two zeros G has only one local extreme.

2. In fact, we obtain

$$\dim(A^{\varepsilon}) \le 1 - \frac{\left(\frac{\varepsilon}{\|G_+\|_\infty + \|G_-\|_\infty} - \frac{\pi}{2}\|G'\|_\infty \frac{\delta_1+\delta_2}{1-\delta_1}\right)^2 - 2\ln\left(1 + \frac{M}{\gamma}\frac{1+\delta_2}{1-\delta_1}\right)}{2\ln(\gamma)} \tag{A.5}$$

whenever $\varepsilon > \frac{\pi}{2}\|G'\|_\infty\big(\|G_+\|_\infty + \|G_-\|_\infty\big)(\delta_1 + \delta_2)(1 - \delta_1)^{-1}$, with M being the sum of twice the number of zeros and four times the larger of the numbers of local maxima and local minima of G. If in addition $G(\pi + x) = -G(\pi - x)$, then $\varepsilon(\|G_+\|_\infty + \|G_-\|_\infty)^{-1}$ in (A.5) is replaced by just $\varepsilon\|G\|_\infty^{-1}$.

3. So, for instance, for $G(x) = \sin(x)$ we have $M = 2 \cdot 3 + 4 \cdot 1 = 10$ as in *Theorem A.1.*

Proof outline. We first assume $G(\pi + x) = -G(\pi - x)$. In that case we proceed as before, but this time the functions $Y_{n,i}$ can take the value 0 on as many intervals as G has zeros, and the value 1 (resp. -1) on as many as G has maxima (resp. minima). That is why 10 is replaced by M.

If now G is not odd with respect to π, we notice that (A.3) does not guarantee Y_n to have zero average. However, using the fact that G has zero average, and that any upper/lower Riemann sum is larger/smaller than the Riemann integral, we can construct Y_n with $\int_0^{2\pi} Y_n(k)\, dk = 0$, $\|X_n - Y_n\|_\infty \le \frac{1}{p}$ and such that pY_n only takes integer values. The real price has to be paid when defining $Y_{n,i}$. If one wants them to have zero average, take only values 0 and $+1/-1$, and only on as many intervals as G has zeros and local maxima/minima, then one may need to take $\approx p(\|G_+\|_\infty + \|G_-\|_\infty)$ of them. Since their sum is still just pY_n, this time we obtain

$$A_p^{\varepsilon} \subseteq \bigcup_{i=1}^{p} A_p^{\varepsilon_1}(i)$$

with $\varepsilon_1 \equiv \varepsilon(\|G_+\|_\infty + \|G_-\|_\infty)^{-1}$. This finishes the proof. □

References

[1] H.L. Cycoon, R.G. Froese, W. Kirsch, B. Simon, *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry*, Springer-Verlag, Berlin Heidelberg, 1987.

[2] A.H. Fan and J. Schmeling, *On fast Birkhoff averaging*, preprint.

[3] D.J. Gilbert and D.B. Pearson, *On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators*, J. Math. Anal. Appl. **128** (1987), 30–56.

[4] S. Jitomirskaya and Y. Last, *Power-law subordinacy and singular spectra, I. Half-line operators*, Acta Math. **183** (1999), 171–189.

[5] A. Kiselev, Y. Last and B. Simon, *Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators*, Commun. Math. Phys. **194** (1998), 1–45.

[6] D. Krutikov and C. Remling, *Schrödinger operators with sparse potentials: asymptotics of the Fourier transform of the spectral measure*, Comm. Math. Phys. **223** (2001), 509–532

[7] Y. Last and B. Simon, *Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators*, Invent. Math. **135** (1999), 329–367.

[8] D.B. Pearson, *Singular continuous measures in scattering theory*, Commun. Math. Phys. **60** (1978), 13–36.

[9] J. Peyriere, *Études de quelques propriétés des produits de Riesz*, Ann. Inst. Fourier (Grenoble) **25** (1975), 127–169.

[10] B. Simon, *Spectral analysis of rank one perturbations and applications*, CRM Proceedings and Lecture Notes **8** (J. Feldman, R. Froese, L. Rosen, eds.), pp. 109–149, Amer. Math. Soc., Providence, RI, 1995.

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