Decoherence in a Two-Particle Model

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Abstract

We consider a simple one dimensional quantum system consisting of a heavy and a light particle interacting via a point interaction. The initial state is chosen to be a product state, with the heavy particle described by a coherent superposition of two spatially separated wave packets with opposite momentum and the light particle localized in the region between the two wave packets.
We characterize the asymptotic dynamics of the system in the limit of small mass ratio, with an explicit control of the error. We derive the corresponding reduced density matrix for the heavy particle and explicitly compute the (partial) decoherence effect for the heavy particle induced by the presence of the light one for a particular set up of the parameters.

1. Introduction

Decoherence has become the terminology for the irreversible suppression of interference of the wavefunction of a quantum system due to the interaction with an "environment" ([GJKKSZ], [BGJKS]).
The usual picture of decoherence, in the simple setting of a two particle system, goes as follows. Suppose one has a particle M with initial wave function $\varphi(x) = \varphi_l(x) + \varphi_r(x)$, representing the superposition of two wave packets φ_l, more or less supported "on the left of the origin" and heading to the right, and φ_r supported more or less on the "right of the origin" with an average velocity pointing to the left. Suppose that another particle m, described initially by the wave packet $\Phi(y)$, passes by and interacts with M. Assuming a small mass ratio between the second and the first particle, it is conceivable that the evolution of M will not be much affected by the interaction, while the scattering process undergone by the particle m will depend strongly on the position of the heavier particle. After interaction (which is assumed to be very fast) one then expects that the wave function describing the state of the system is of the type $\psi(x, y) = \varphi_l(x)\Phi_l(y) + \varphi_r(x)\Phi_r(y)$, where Φ_l and Φ_r will have spatial supports concentrated in distant regions for all later times. Therefore in the configuration space of the entire system the entangled state will appear as the sum of two disjoint components and the possibility of interference of the heavy particle wave packets will be reduced.
Notice that the reduced density matrix of the particle M has in this ideal case negligible off-diagonal elements. In this sense, interference has been reduced and the motion of the particle M has become more "classical". That is the way decoherence plays a role in the explanation of the emergence of classical behaviour from quantum mechanics.
For the relevance of the mechanism of decoherence in the classical limit of Quantum Mechanics in the language of Bohmian Mechanics see [ADGZ].
In this respect it is an interesting problem to separate "pure decoherence" from the other effects which an environment usually produces, which are dissipation and fluctuation. That is, one would like to have the motion of the system not much affected by the interaction with the environment, while the environment produces decoherence. It is unclear whether these desiderata, namely a good decoherence rate and a more or less unperturbed motion can be consistently fullfilled in realistic physical models ([GH]).
Explicit models where one can rigorously establish decoherence in this sense have been worked out in the last years. One such model has been studied in ([DS]) where the interaction of a particle with the radiation field has been considered. We shall now study another interaction which elaborates closer the idea of scattering of light particles (the environment) off a heavy particle (the system) (see [JZ],[GF],[T] for similar ideas).
We consider a very simple one dimensional model of a system (a heavy particle of mass M) plus environment (one light particle of mass m) interacting via a short range force ($\delta-$interaction). We consider this case as useful preparation for the treatment of a three dimensional gas of light particles interacting with the heavy particle, which we shall address in subsequent work.

We wish to stress some features that make the two-particle model with $\delta-$interaction (which has the advantage of being analytically easily accessible) particularly suitable as a model for decoherence.

1. there exists a simple dimensionless parameter in the problem, namely the fraction of the masses $\epsilon = \frac{m}{M}$.
2. Letting ϵ become small while keeping M fixed allows to approximate the solution of the Schrödinger equation of the two body problem by a scattering solution in which the heavy particle acts as a scattering center for the light one. The error is $\mathcal{O}(\epsilon)$. The time scale on which this approximation holds is of course given by the time the light particle needs to pass the heavy particle. This approximation is the starting point of the analysis in [JZ].
3. The decoherence effect (i.e. the amount by which the off diagonal elements of the reduced density matrix are reduced) can be explicitely computed (see (3.20) and the discussion follwing it) and is, in the relevant regime, of the order of $\alpha_0 m\hbar^{-2}\delta$, where δ is the initial spread of the light particle and α_0 is the strength of the potential ($(\alpha_0 m)^{-1}\hbar^2$ is the effective range of interaction) all of which can also be chosen ϵ-dependent.

We wish to warn the reader, that the point interaction we look at here in form of the δ-potential is for finite α_0 not a hard core interaction. The case $\alpha_0 \to \infty$ corresponds to hard core.

The paper is organised as follows.

In Section 2 we introduce the model and characterise the asymptotic dynamics of the two-particle system for small mass ratio and state the main approximation result.

In Section 3 we show the attenuation of the off-diagonal terms in the reduced density matrix for the heavy particle and we compute explicitly the probability distribution for the position of the heavy particle, showing reduction of the interference effects with respect to the non interacting case.

In Section 4 we give the proof of the main result of the paper.

In the appendix we recall the derivation of the explicit solution of the Schroedinger equation of the two-body system in interaction via a delta potential in dimension one.

2. Expression for small mass ratio

In this section we shall study the Schroedinger equation for the two-particle system in one dimension described by the hamiltonian

$$H = -\frac{\hbar^2}{2M}\Delta_R - \frac{\hbar^2}{2m}\Delta_r + \alpha_0\delta(r-R), \qquad \alpha_0 > 0 \tag{2.1}$$

In (2.1) we have denoted by R the position coordinate of the heavy particle with mass M and by r the position coordinate of the light particle with mass m. The interaction potential is chosen to be a repulsive point interaction of strenth α_0.
It is well known that (2.1) is a well defined positive and selfadjoint operator in $L^2(\mathbb{R}^2, drdR)$, which is also a solvable model ([AGH-KH]).
In fact, for an arbitrary initial state $\psi_0 = \psi_0(r, R)$, the solution of the Schroedinger equation can be explicitely written as (see [S] and the appendix)

$$\psi(t, r, R) = \int dr'dR'\psi_0(r', R')U_0^{\nu}\left(t, \frac{M}{\nu}(R-R') + \frac{\mu}{M}(r-r')\right)$$

$$\cdot\left[U_0^{\mu}(t, (r-R)-(r'-R')) - \frac{\mu\alpha_0}{\hbar^2}\int_0^{\infty} due^{-\frac{\mu\alpha_0}{\hbar^2}u}U_0^{\mu}(t, u+|r-R|+|r'-R'|)\right] \tag{2.2}$$

where we have introduced the reduced mass and the total mass of the system

$$\mu = \frac{mM}{m+M}, \quad \nu = m+M \tag{2.3}$$

and the integral kernel of the free unitary group $U_0^{\mathcal{M}}(t)$ corresponding to the mass $\mathcal{M} > 0$

$$U_0^{\mathcal{M}}(t, x-x') = e^{-i\frac{t}{\hbar}H_0^{\mathcal{M}}}(x-x') = \sqrt{\frac{\mathcal{M}}{2\pi i\hbar t}}e^{i\frac{\mathcal{M}}{2\hbar t}(x-x')^2}, \quad x, x' \in \mathbb{R} \tag{2.4}$$

We are interested in the case of an initial state in a product form. Then we fix two real valued smooth functions (for ease of formulation we assume that they are in Schwartz space $\mathcal{S}$)

$$f, g \in \mathcal{S}, \qquad \|f\| = \|g\| = 1 \tag{2.5}$$

where $\|\cdot\|$ denotes the norm in $L^2(\mathbb{R})$. For later use, it will be convenient to choose g even. Using f and g we define now the states in such a way that we can easily read of the relevant physical scales, i.e. we code the states by the physical parameters $R_0, P_0, \sigma, r_0, q_0, \delta$ as follows

$$f_{\sigma,R_0,P_0}(R) = \frac{1}{\sqrt{2}}\left[f^+_{\sigma,R_0,P_0}(R) + f^-_{\sigma,R_0,P_0}(R)\right] \tag{2.6}$$

$$f^{\pm}_{\sigma,R_0,P_0}(R) = \frac{1}{\sqrt{\sigma}}f\left(\frac{R \pm R_0}{\sigma}\right)e^{\pm i\frac{P_0}{\hbar}R} \tag{2.7}$$

$$g_{\delta,r_0,q_0}(r) = \frac{1}{\sqrt{\delta}}g\left(\frac{r-r_0}{\delta}\right)e^{i\frac{q_0}{\hbar}r} \tag{2.8}$$

$$\sigma, \delta, R_0, P_0, q_0 > 0, \quad r_0 \in \mathbb{R}, \quad R_0 > \sigma + \delta + |r_0| \tag{2.9}$$

The choice (2.9) is not essential for the most part of the paper, but it sets already a geometrical picture which puts the results in the right perspective (see below (2.10)). Later on we shall use this particular choice for computing effects. Note that the spread of the wave function of M is not given by σ but by R_0.
The initial state that we consider in the following is

$$\psi_0(r,R) = g_{\delta,r_0,q_0}(r) f_{\sigma,R_0,P_0}(R) \tag{2.10}$$

The initial state (2.10) is a (pure) product state for the whole system, i.e. no correlation is assumed between the two particles at time zero.
The heavy particle is assumed to be in a superposition of two spatially separated wave packets, one localized in $R = -R_0$ with mean value of the momentum P_0 and the other localized in $R = R_0$ with mean value of the momentum $-P_0$. The light particle is localized around r_0, in the region between the two wave packets, with positive mean momentum q_0.
To simplify the notation, in the rest of the paper we shall drop the dependence of the initial state on R_0, P_0, r_0, q_0. Moreover, for the convenience of the reader, we collect here some notation which will be used later on

$$\epsilon = \frac{m}{M}, \quad \mu = \frac{\epsilon}{1+\epsilon} M, \quad \nu = (1+\epsilon)M \tag{2.11}$$

$$\alpha = \frac{\alpha_0 m}{\hbar^2} \tag{2.12}$$

$$k_0 = \frac{q_0}{\hbar}, \qquad K = \frac{P_0}{\hbar} + k_0 \tag{2.13}$$

$$T \,:\, L^2(\mathbb{R}^2, dr dR) \to L^2(\mathbb{R}^2, dx_1 dx_2),$$

$$(Th)(x_1,x_2) \equiv h\left(x_2 + \frac{M}{m+M} x_1, x_2 - \frac{m}{m+M} x_1\right) \tag{2.14}$$

$$\Delta^\pm = (\pm R_0 - \sigma, \pm R_0 + \sigma) \tag{2.15}$$

and finally c will denote a positive numerical constant.
We shall now characterize the asymptotic behaviour of the wave function for small value of the mass ratio ϵ for the initial state (2.10). Letting m become small, keeping M fixed, the light particles spreads with speeds $v \sim \hbar/\delta m$ and the time by which the light particle passes M is of the order of R_0/v thus decreases with m, so that M does not change much its position during the passing of m.
The limit dynamics will hence describe a situation in which the light particle is scattered by the heavy one being in some fixed position, while the heavy particle moves freely. Nevertheless, we shall find that the free motion of the heavy particle is modified by the scattering event. In this heuristic argument we kept all the other physical parameters fixed except for the interaction strength. In fact, in order to keep the interaction effective on the light particle we need to scale α_0 in such a way that $\alpha_0 m \approx \mathcal{O}(1)$. There is of course no need to keep the other parameters

fixed, in fact one may well imagine δ and R_0 increasing with m, so that the kinetic energy of m stays finite and the spread of the M increases. We shall not discuss such choices here, but the estimates are detailed enough, so that other scalings can be easily discussed. This might become relevant in a model where the heavy particle is immersed in a gas of light particles.
In order to formulate the main result of this section, we define the integral operator

$$(W_+^{\gamma,x_0}h)(k) = \frac{1}{\sqrt{2\pi}}\int dx h(x)\left(e^{-ikx} + \mathcal{R}_\gamma(k)e^{-ix_0k}e^{i|k||x-x_0|}\right), \qquad \gamma > 0,\ x_0 \in \mathbb{R} \tag{2.16}$$

$$\mathcal{R}_\gamma(k) = -\frac{\gamma}{\gamma - i|k|} \tag{2.17}$$

where the integral kernel in (2.16) is the generalized eigenfunction of the hamiltonian

$$H_{\gamma,x_0} = -\frac{1}{2}\Delta + \gamma\delta(\cdot - x_0) \tag{2.18}$$

and $\mathcal{R}_\gamma(k)$ is the corresponding reflection coefficient (see e.g. [AGH-KH]). Moreover we introduce the wave operator Ω_+^{γ,x_0} associated to H_{γ,x_0}, explicitely given by

$$(\Omega_+^{\gamma,x_0}h)(x) = \left[(W_+^{\gamma,x_0})^{-1}\tilde{h}\right](x) \tag{2.19}$$

where $\tilde{h}$ denotes the Fourier transform of h.
With the above notation the asymptotic wave function, which will be denoted by $\psi^a(t)$, is explicitely characterized in the following theorem.

Theorem 1. *Given the initial state (2.10), then for any $t > 0$ the following estimate holds*

$$\|\psi(t) - \psi^a(t)\| < \left(\frac{A}{t} + B\right)\epsilon \tag{2.20}$$

where

$$\begin{aligned}\psi^a(t,r,R) &= \sqrt{\frac{m}{i\hbar t}}e^{i\frac{m}{2\hbar t}r^2}\int dy f_\sigma(y)U_0^M(t,R-y)\,(W_+^{\alpha,y}g_\delta)\left(\frac{mr}{\hbar t}\right)\\ &= \sqrt{\frac{m}{i\hbar t}}e^{i\frac{m}{2\hbar t}r^2}\int dy f_\sigma(y)U_0^M(t,R-y)\left[(\Omega_+^{\alpha,y})^{-1}g_\delta\right]^{\tilde{}}\left(\frac{mr}{\hbar t}\right)\end{aligned} \tag{2.21}$$

and A,B are positive, time-independent constants whose detailed dependence on the physical parameters characterising the interaction and the initial state will be given in section 4.

Remark 1. Note that $\psi^a(t,r,R)$ is close to what we described in the introduction. Think of f_σ as we do as consisting of two well concentrated wavepackets, then we have the light particle's scattered wavefunction correlated with the two mean positions of the heavy particles, i.e. read in $[(\Omega_+^{\alpha,y})^{-1}g_\delta]\tilde{}$ the y morally as the scattering center.

The result of theorem 1 can be rephrased in terms of reduced density matrix for the heavy particle, which is defined by the integral operator $\hat\rho(t)$ in $L^2(\mathbb{R})$ given by the kernel

$$\hat\rho(t,R,R') = \int dr \psi(t,r,R)\overline{\psi}(t,r,R') \tag{2.22}$$

We also introduce the integral operator $\hat\rho^a(t)$ defined by

$$\begin{aligned}\hat\rho^a(t,R,R') &= \int dr \psi^a(t,r,R)\overline{\psi^a}(t,r,R')\\ &= \int dy f_\sigma(y) U_0^M(t,R-y)\int dz \overline{f_\sigma}(z)\overline{U_0^M}(t,R'-z)\mathcal{I}(y,z)\end{aligned} \tag{2.23}$$

where

$$\mathcal{I}(y,z) \equiv \int dk (W_+^{\alpha,y}g_\delta)(k)\overline{(W_+^{\alpha,z}g_\delta)}(k) = \left((\Omega_+^{\alpha,z})^{-1}g_\delta,(\Omega_+^{\alpha,y})^{-1}g_\delta\right) \tag{2.24}$$

Formula (2.24), obtained through heuristic considerations, has been the main ingredient in the description of scattering induced decoherence in [JZ].
Observe that, from (2.23),(2.24) one has

$$\hat\rho^a(t) = U_0^M(t)\hat\rho_0^a U_0^M(-t) \tag{2.25}$$

where $\hat\rho_0^a$ is defined by the integral kernel

$$\hat\rho_0^a(y,z) = f_\sigma(y)\overline{f_\sigma}(z)\mathcal{I}(y,z) \tag{2.26}$$

It is easily seen that $\mathcal{I}(y,z) = \overline{\mathcal{I}}(z,y)$, $|\mathcal{I}(y,z)| \le 1$ and the equality holds only if $y=z$. Then $\hat\rho_0^a$ is a self-adjoint and trace-class operator, with $Tr(\hat\rho_0^a)=1$; it is also positive since

$$\begin{aligned}(h,\hat\rho_0^a h) &= \int dy \overline{h}(y)\int dz h(z) f_\sigma(y)\overline{f_\sigma}(z)\int dk (W^{\alpha,y}g_\delta)(k)\overline{(W^{\alpha,z}g_\delta)}(k)\\ &= \int dk \left|\int dy \overline{h}(y) f_\sigma(y)(W^{\alpha,y}g_\delta)(k)\right|^2\end{aligned} \tag{2.27}$$

Moreover we have

$$Tr((\hat{\rho}_0^a)^2) = \int dydz |f_\sigma(y)|^2 |f_\sigma(z)|^2 |\mathcal{I}(y,z)|^2 < 1 \tag{2.28}$$

We conclude that $\hat{\rho}_0^a$ and its free evolution $\hat{\rho}^a(t)$ are density matrices describing mixture states and by Theorem 1, for any $t > 0$, one has

$$Tr(|\hat{\rho}(t) - \hat{\rho}^a(t)|) < \left(\frac{A}{t} + B \right) \epsilon \tag{2.29}$$

This means that in our asymptotic regime the motion of the heavy particle is a free evolution. On the other hand the presence of the light particle has a relevant effect, since it produces a transition of the initial state of the heavy particle from $\hat{\rho}_0(y,z) = f_\sigma(y)\overline{f_\sigma}(z)$ to $\hat{\rho}_0^a(y,z)$.
We shall see in the next section that this is the origin of the decoherence effect on the heavy particle.
Finally, it is worth to mention that the dynamics of the system can be equivalently described by the Wigner function. From (2.23),(2.24) we see that the asymptotic form of the reduced Wigner function describing the motion of the heavy particle is the free evolution of

$$\hat{W}_0^a(R,P) = \frac{1}{2\pi} \int dx e^{iPx} f_\sigma(R - \frac{\hbar}{2}x) \overline{f_\sigma}(R + \frac{\hbar}{2}x) \mathcal{I}(R - \frac{\hbar}{2}x, R + \frac{\hbar}{2}x) \tag{2.30}$$

3. Size of Decoherence

Here we discuss an application of formula (2.23),(2.24) to a concrete example of quantum evolution and we give an explicit computation of the decoherence effect.
We shall consider the initial state (2.10) with the further assumptions (which are only done for ease of presentation)

$$f, g \in C_0^\infty(-1, +1), \tag{3.1}$$

and

$$\sigma \ll \frac{1}{\alpha} \ll R_0 - |r_0|, \qquad \delta \ll R_0 - |r_0| \tag{3.2}$$

i.e. the spreading in position of the wave packets (which are superposed in a gross superposition) of the heavy particle is much smaller than the effective range of the interaction and this, in

turn, is much smaller than the separation between the two particles. Moreover the light particle is well separated from each wave packet of the heavy one.
Notice that (3.2) obviously implies $\frac{\sigma}{R_0} \ll 1$.
Using assumptions (3.2) we can give an estimate of the basic object $\mathcal{I}(y,z)$ for $y, z \in \Delta^{\pm}$ and then we can find a more suitable expression for the reduced density matrix of the heavy particle. In order to formulate the result, we define the parameter

$$\Lambda = \int dk |\tilde{g}_\delta(k)|^2 \mathcal{T}_\alpha(k) = 1 + \int dk |\tilde{g}_\delta(k)|^2 \mathcal{R}_\alpha(k) \tag{3.3}$$

where

$$\mathcal{T}_\gamma(k) = -\frac{ik}{\gamma - ik} = 1 + \mathcal{R}_\gamma(k), \qquad \gamma > 0 \tag{3.4}$$

is the transmission coefficient associated to a point interaction of strength γ (see e.g. [AGH-KH]). Then we have

Proposition 2. *Assume (3.2). Then*

$$\sup_{y,z\in\Delta^{\pm}} |\mathcal{I}(y,z) - 1| < c\left(\alpha\sigma + \frac{1}{\alpha(R_0 - |r_0|)} + \frac{\delta}{R_0 - |r_0|}\right) \tag{3.5}$$

$$\sup_{y\in\Delta^{+}, z\in\Delta^{-}} |\mathcal{I}(y,z) - \Lambda| = \sup_{y\in\Delta^{-}, z\in\Delta^{+}} \left|\mathcal{I}(y,z) - \overline{\Lambda}\right| < c\left(\frac{1}{\alpha(R_0 - |r_0|)} + \frac{\delta}{R_0 - |r_0|}\right) \tag{3.6}$$

Proof. Using the shorthand notation $\beta = \alpha\delta$, we note that for $y \in \Delta^{-}$

$$\frac{1}{\sqrt{\delta}} \left(W_{+}^{\alpha,y} g_\delta\right)\left(\frac{k}{\delta}\right) = e^{i(k_0\delta - k)\frac{r_0}{\delta}} \tilde{g}(k - k_0\delta) + \mathcal{R}_\beta(k) e^{i(k_0\delta + |k|)\frac{r_0}{\delta} - i(k+|k|)\frac{y}{\delta}} \tilde{g}(|k| + k_0\delta) \tag{3.7}$$

and for $y \in \Delta^{+}$

$$\frac{1}{\sqrt{\delta}} \left(W_{+}^{\alpha,y} g_\delta\right)\left(\frac{k}{\delta}\right) = e^{i(k_0\delta - k)\frac{r_0}{\delta}} \tilde{g}(k - k_0\delta) + \mathcal{R}_\beta(k) e^{i(k_0\delta - |k|)\frac{r_0}{\delta} - i(k-|k|)\frac{y}{\delta}} \tilde{g}(|k| - k_0\delta) \tag{3.8}$$

where we have used the fact that

$$\tilde{g}_\delta(k) = \sqrt{\delta}\tilde{g}(k\delta - k_0\delta) e^{-i(k-k_0)r_0} \tag{3.9}$$

Then for $y, z \in \Delta^-$ we have

$$
\begin{aligned}
&\mathcal{I}(y,z) = 1 + \int dk |\tilde{g}(|k| + k_0\delta)|^2 |\mathcal{R}_\beta(k)|^2 e^{-i(k+|k|)\frac{y-z}{\delta}} \\
&+ \int dk \overline{\tilde{g}}(k - k_0\delta)\tilde{g}(|k| + k_0\delta)\mathcal{R}_\beta(k) e^{i(k+|k|)\frac{r_0-y}{\delta}} \\
&+ \int dk \overline{\tilde{g}}(|k| + k_0\delta)\tilde{g}(k - k_0\delta)\overline{\mathcal{R}_\beta}(k) e^{-i(k+|k|)\frac{r_0-z}{\delta}} \\
&= 1 + \int_0^\infty dk |\tilde{g}(k + k_0\delta)|^2 |\mathcal{R}_\beta(k)|^2 \left(e^{-2ik\frac{y-z}{\delta}} - 1\right) \\
&+ \int_0^\infty dk \overline{\tilde{g}}(k - k_0\delta)\tilde{g}(k + k_0\delta)\mathcal{R}_\beta(k) e^{2ik\frac{r_0-y}{\delta}} \\
&+ \int_0^\infty dk \overline{\tilde{g}}(k + k_0\delta)\tilde{g}(k - k_0\delta)\overline{\mathcal{R}_\beta}(k) e^{-2ik\frac{r_0-z}{\delta}} \\
&\equiv 1 + a_1 + a_2 + a_3
\end{aligned}
\tag{3.10}
$$

where we have used the identity $\mathcal{R}_\beta + \overline{\mathcal{R}_\beta} + 2|\mathcal{R}_\beta|^2 = 0$ and the fact that $\tilde{g}$ is even. Using (3.2) we easily estimate a_1

$$
\begin{aligned}
&|a_1| \le 2\frac{|y-z|}{\delta} \int_0^\infty dk |\tilde{g}(k + k_0\delta)|^2 k |\mathcal{R}_\beta(k)|^2 \le 4\frac{\sigma}{\delta} \int_0^\infty dk |\tilde{g}(k + k_0\delta)|^2 k |\mathcal{R}_\beta(k)|^2 \\
&\le 2\,\alpha\sigma
\end{aligned}
\tag{3.11}
$$

For the estimate of a_2 it is convenient to integrate by parts

$$
\begin{aligned}
|a_2| &= \left| \frac{1}{2i} \frac{\delta}{r_0 - y} \int_0^\infty dk \overline{\tilde{g}}(k - k_0\delta)\tilde{g}(k + k_0\delta)\mathcal{R}_\beta(k) \frac{d}{dk} e^{2ik\frac{r_0-y}{\delta}} \right| \\
&= \frac{\delta}{2|r_0 - y|} \left| \int_0^\infty dk \frac{d}{dk} \left(\overline{\tilde{g}}(k - k_0\delta)\tilde{g}(k + k_0\delta)\mathcal{R}_\beta(k) \right) e^{2ik\frac{r_0-y}{\delta}} - |\tilde{g}(k_0\delta)|^2 \right| \\
&\le \frac{\delta}{R_0 - |r_0|} \left[\frac{1}{2\pi} \left(\int dr |g(r)| \right)^2 + \frac{1}{\beta} \int_0^\infty dk \left| \overline{\tilde{g}}(k - k_0\delta)\tilde{g}(k + k_0\delta) \right| \right. \\
&\left. + \int_0^\infty dk \left| \overline{\tilde{g}'}(k - k_0\delta)\tilde{g}(k + k_0\delta) \right| + \int_0^\infty dk \left| \overline{\tilde{g}}(k - k_0\delta)\tilde{g}'(k + k_0\delta) \right| \right] \\
&\le \frac{\delta}{R_0 - |r_0|} \left[\frac{1}{2\pi} \left(\int dr |g(r)| \right)^2 + \frac{1}{\beta} \|g\|^2 + 2\|\tilde{g}'\| \right] \\
&\le \frac{\delta}{R_0 - |r_0|} \left(\frac{1}{\pi} + 2\|\tilde{g}'\| \right) + \frac{1}{\alpha(R_0 - |r_0|)}
\end{aligned}
\tag{3.12}
$$

The term a_3 is analysed exacly in the same way and then we get the estimate (3.5) for $y, z \in \Delta^-$. Since in the case $y, z \in \Delta^+$ the computation is similar we conclude that (3.5) holds.

In order to prove (3.6) we consider the case $y \in \Delta^+$ and $z \in \Delta^-$ (the case $y \in \Delta^-$ and $z \in \Delta^+$ can be treated exactly in the same way) and we obtain

$$
\begin{aligned}
&\mathcal{I}(y,z) = 1 + \int dk\overline{\tilde{g}}(|k|+k_0\delta)\tilde{g}(|k|-k_0\delta)|\mathcal{R}_\beta(k)|^2 e^{-2i|k|\frac{r_0}{\delta}+i(|k|-k)\frac{y}{\delta}+i(|k|+k)\frac{z}{\delta}} \\
&+ \int dk\overline{\tilde{g}}(k-k_0\delta)\tilde{g}(|k|-k_0\delta)\mathcal{R}_\beta(k)e^{-i(|k|-k)\frac{r_0-y}{\delta}} \\
&+ \int dk\overline{\tilde{g}}(|k|+k_0\delta)\tilde{g}(k-k_0\delta)\overline{\mathcal{R}_\beta}(k)e^{-i(|k|+k)\frac{r_0-z}{\delta}} \\
&= 1 + \int_0^\infty dk\left(|\tilde{g}(k-k_0\delta)|^2\mathcal{R}_\beta(k) + |\tilde{g}(k+k_0\delta)|^2\overline{\mathcal{R}_\beta}(k)\right) \\
&+ \int_0^\infty dk\overline{\tilde{g}}(k+k_0\delta)\tilde{g}(k-k_0\delta)|\mathcal{R}_\beta(k)|^2 e^{-2ik\frac{r_0-z}{\delta}} \\
&+ \int_0^\infty dk\overline{\tilde{g}}(k+k_0\delta)\tilde{g}(k-k_0\delta)\overline{\mathcal{R}_\beta}(k)e^{-2ik\frac{r_0-z}{\delta}} \\
&+ \int_0^\infty dk\overline{\tilde{g}}(k+k_0\delta)\tilde{g}(k-k_0\delta)|\mathcal{R}_\beta(k)|^2 e^{-2ik\frac{r_0-y}{\delta}} \\
&+ \int_0^\infty dk\overline{\tilde{g}}(k+k_0\delta)\tilde{g}(k-k_0\delta)\mathcal{R}_\beta(k)e^{-2ik\frac{r_0-y}{\delta}}
\end{aligned} \tag{3.13}
$$

The estimate of the last four terms of (3.13) proceeds exactly as the estimate of a_2 in (3.12). On the other hand

$$
\begin{aligned}
&1 + \int_0^\infty dk\left(|\tilde{g}(k-k_0\delta)|^2\mathcal{R}_\beta(k) + |\tilde{g}(k+k_0\delta)|^2\overline{\mathcal{R}_\beta}(k)\right) \\
&= \int dk|\tilde{g}(k-k_0\delta)|^2\left(\frac{-ik}{\beta-ik}\right)
\end{aligned} \tag{3.14}
$$

and this concludes the proof of the proposition. □

Proposition 2 allows us to find a further approximate form for the reduced density matrix.

Corollary 3. *Under the assumptions (3.2) and for any $t \geq 0$ we have*

$$
\left[Tr\left(\left(\hat{\rho}^a(t) - \hat{\rho}^f(t)\right)^2\right)\right]^{1/2} < c\left(\alpha\sigma + \frac{1}{\alpha(R_0-|r_0|)} + \frac{\delta}{R_0-|r_0|}\right) \tag{3.15}
$$

where

$$
\hat{\rho}^f(t) = U_0^M(t)\hat{\rho}_0^f U_0^M(-t) \tag{3.16}
$$

$$
\hat{\rho}_0^f(y,z) = \frac{1}{2}f_\sigma^+(y)\overline{f_\sigma^+}(z) + \frac{1}{2}f_\sigma^-(y)\overline{f_\sigma^-}(z) + \frac{\Lambda}{2}f_\sigma^+(y)\overline{f_\sigma^-}(z) + \frac{\overline{\Lambda}}{2}f_\sigma^-(y)\overline{f_\sigma^+}(z) \tag{3.17}
$$

Proof.

$$
\begin{aligned}
&Tr\left(\left(\hat{\rho}^a(t)-\hat{\rho}^f(t)\right)^2\right)=Tr\left(\left(\hat{\rho}^a_0-\hat{\rho}^f_0\right)^2\right)\\
&=\frac{1}{4}\int dydz\left|f^+_\sigma(y)\overline{f^+_\sigma}(z)(\mathcal{I}(y,z)-1)+f^-_\sigma(y)\overline{f^-_\sigma}(z)(\mathcal{I}(y,z)-1)\right.\\
&\left.+f^+_\sigma(y)\overline{f^-_\sigma}(z)(\mathcal{I}(y,z)-\Lambda)+f^-_\sigma(y)\overline{f^+_\sigma}(z)(\mathcal{I}(y,z)-\overline{\Lambda})\right|^2\\
&\leq \sup_{y,z\in\Delta^+}|\mathcal{I}(y,z)-1|^2+\sup_{y,z\in\Delta^-}|\mathcal{I}(y,z)-1|^2\\
&+\sup_{y\in\Delta^+,z\in\Delta^-}|\mathcal{I}(y,z)-\Lambda|^2+\sup_{y\in\Delta^-,z\in\Delta^+}|\mathcal{I}(y,z)-\overline{\Lambda}|^2
\end{aligned}
\tag{3.18}
$$

Using proposition 2 we conclude the proof. $\Box$

From corollary 3 and theorem 1 we conclude that the reduced density matrix for the heavy particle in the position representation can be approximated by the density matrix

$$
\begin{aligned}
\hat{\rho}^f(t,R,R')=&\frac{1}{2}(U^M_0(t)f^+_\sigma)(R)(U^M_0(-t)\overline{f^+_\sigma})(R')+\frac{1}{2}(U^M_0(t)f^-_\sigma)(R)(U^M_0(-t)\overline{f^-_\sigma})(R')\\
+&\frac{\Lambda}{2}(U^M_0(t)f^+_\sigma)(R)(U^M_0(-t)\overline{f^-_\sigma})(R')+\frac{\overline{\Lambda}}{2}(U^M_0(t)f^-_\sigma)(R)(U^M_0(-t)\overline{f^+_\sigma})(R')
\end{aligned}
\tag{3.19}
$$

with an explicit control of the error.
If the interaction with the light particle is switched off, i.e. for $\alpha=0$, we have $\Lambda=1$ and then (3.19) reduces to the pure state corresponding to the coherent superposition of the free evolution of the two wave packets $f^\pm_\sigma$.
On the other hand, if $\alpha>0$ one easily sees that $0<|\Lambda|<1$ and then (3.19) is a mixed state for which the interference terms are reduced by the factor Λ and this is the typical manifestation of the (partial) decoherence effect induced by the light particle on the heavy one.
The relevant parameter Λ (see (3.3)) is defined in terms of the probability distribution of the momentum of the light particle $|\tilde{g}_\delta(k)|^2$ and of the transmission coefficient $\mathcal{T}_\alpha(k)$.
Then the decoherence effect is emphasized if the fraction of transmitted wave for the light particle is small.
In particular, rescaling the integration variable in (3.3), one can also write

$$
\Lambda=1-\int dz|\tilde{g}(z)|^2\frac{\alpha\delta}{\alpha\delta-i(z+k_0\delta)}
\tag{3.20}
$$

Then one easily sees that for $k_0\gg\alpha$ the light particle is completely transmitted and $\Lambda\simeq 1$, i.e. the decoherence effect is negligible.

On the other hand, for $k_0 \ll \alpha$ the decoherence effect is nonzero and given by

$$\Lambda \simeq \int dz |\tilde{g}(z)|^2 |\mathcal{T}_{\alpha\delta}(z)|^2 = 1 - \int dz |\tilde{g}(z)|^2 \frac{\alpha^2\delta^2}{\alpha^2\delta^2 + z^2} \tag{3.21}$$

where we have used the fact that $\tilde{g}$ is even.
A further interesting question is the analysis of $\hat{\rho}^f(t)$ in the momentum representation.
Since momentum is a constant of the motion, the density matrix is simply given by

$$\begin{aligned}\frac{1}{\hbar}\tilde{\rho}_0^f\left(\frac{P}{\hbar},\frac{P'}{\hbar}\right) &= \frac{1}{2\hbar}\tilde{f}_\sigma^+\left(\frac{P}{\hbar}\right)\overline{\tilde{f}_\sigma^+}\left(\frac{P'}{\hbar}\right) + \frac{1}{2\hbar}\tilde{f}_\sigma^-\left(\frac{P}{\hbar}\right)\overline{\tilde{f}_\sigma^-}\left(\frac{P'}{\hbar}\right) \\ &+\frac{\Lambda}{2\hbar}\tilde{f}_\sigma^+\left(\frac{P}{\hbar}\right)\overline{\tilde{f}_\sigma^-}\left(\frac{P'}{\hbar}\right) + \frac{\overline{\Lambda}}{2\hbar}\tilde{f}_\sigma^-\left(\frac{P}{\hbar}\right)\overline{\tilde{f}_\sigma^+}\left(\frac{P'}{\hbar}\right)\end{aligned} \tag{3.22}$$

It is then clear that the decoherence effect is present also in the momentum representation and it is measured by the same parameter Λ.
Moreover, if $\tilde{f}_\sigma^+$ and $\tilde{f}_\sigma^-$ are well separated, one easily realizes that the probability distribution of the momentum remains essentially unchanged with respect to the unperturbed case $\Lambda = 1$, the error being of order ϵ.
We analyse now the evolution in the position representation of the heavy particle exploiting the approximate reduced density matrix $\hat{\rho}^f(t)$.
We shall explicitely show that the typical interference fringes produced by the superposition state when the interaction with the light particle is absent, i.e. for $\Lambda = 1$, are in fact reduced when the light particle is present, i.e. for $|\Lambda| < 1$.
In order to see the effect more clearly we assume

$$\frac{\sigma}{R_0} \ll \frac{\hbar}{\sigma P_0} \tag{3.23}$$

The effect of the interfernce terms becomes more relevant when the supports of the two wave pakets $U_0^M(t) f_\sigma^\pm$ have the maximal overlapping and this approximately happens at the time $t = \tau \equiv \frac{R_0 M}{P_0}$. Then, from (3.19), we consider

$$\begin{aligned}&n(\tau, R) \equiv \hat{\rho}^f(\tau, R, R) \\ &= \frac{1}{2}\left[|(U_0^M(\tau) f_\sigma^+)(R)|^2 + |(U_0^M(\tau) f_\sigma^-)(R)|^2 + 2\Re\left(\Lambda (U_0^M(\tau) f_\sigma^+)(R)\overline{(U_0^M(\tau) f_\sigma^-)(R)}\right)\right]\end{aligned} \tag{3.24}$$

Using (3.23) and a standard scattering estimate (see e.g. [RS]) we obtain

$$(U_0^M(\tau)f_\sigma^+)(R) = \sqrt{\frac{P_0\sigma}{2\pi\hbar R_0}} \int dx f\left(\frac{x+R_0}{\sigma}\right) e^{i\frac{P_0}{\hbar}x + i\frac{P_0}{2\hbar R_0}(R-x)^2}$$

$$= \sqrt{\frac{P_0\sigma}{i\hbar R_0}} e^{i\frac{P_0}{\hbar}\left(\frac{R^2}{2R_0}+R-\frac{R_0}{2}\right)} \tilde{f}\left(\frac{P_0\sigma}{\hbar R_0}R\right) + \mathcal{E}_0(R) \tag{3.25}$$

$$\|\mathcal{E}_0\| < \frac{P_0\sigma^2}{2\hbar R_0}\|\Delta\tilde{f}\| \tag{3.26}$$

Proceeding analogously for $(U_0^M(\tau)f_\sigma^-)(R)$ we find

$$n(\tau,R) = \frac{P_0\sigma}{\hbar R_0}\left|\tilde{f}\left(\frac{P_0\sigma}{\hbar R_0}R\right)\right|^2 \left(1+|\Lambda|\cos\left(\frac{2P_0}{\hbar}R+\varphi\right)\right) + \mathcal{E}_1(R) \tag{3.27}$$

$$\|\mathcal{E}_1\|_{L^1} < c\,\frac{P_0\sigma^2}{\hbar R_0} \tag{3.28}$$

where

$$\Lambda = |\Lambda|e^{i\varphi} \tag{3.29}$$

For $|\Lambda| < 1$, formula (3.27) shows that the presence of the light particle determines a reduction of the amplitude of the oscillations and a shift of the corresponding phases.
Notice that the shift is negligible if $k_0 \ll \alpha$.

4. Proof of theorem 1

The proof of theorem 1 will be obtained through the proof of three lemmas.

Lemma 4. *Given the initial state (2.10), for any $t \geq 0$ one has*

$$\|\psi(t) - \psi_1(t)\| < C_1\,\epsilon \tag{4.1}$$

where

$$\psi_1(t,r,R) = \int dy f_\sigma(y) U_0^\nu\left(t, \frac{M}{\nu}R + \frac{\mu}{M}r - y\right) \int dr' g_\delta(r'+y) U_{\alpha_0}^\mu(t, r-R, r') \tag{4.2}$$

and

$$C_1 = \left[\int dx x^2 \int dy \left| \frac{\partial}{\partial y} \left(f_\sigma(y) g_\delta(x+y) \right) \right|^2 \right]^{1/2} \tag{4.3}$$

Proof. Using the relative and the center of mass coordinates (see (2.14)), from (5.4) one has

$$(T\psi(t))(x_1, x_2) = \left(U_0^\nu(t) U_{\alpha_0}^\mu(t) T\psi_0 \right)(x_1, x_2) \tag{4.4}$$

where $U_{\alpha_0}^\mu(t)$ is defined in (5.10) of the appendix and

$$(T\psi_0)(x_1, x_2) = f_\sigma \left(x_2 - \frac{\mu}{M} x_1 \right) g_\delta \left(x_2 + \frac{M}{\nu} x_1 \right) \tag{4.5}$$

Moreover

$$\begin{aligned}
&(T\psi_1(t))(x_1, x_2) = \int dx_2' dx_1' f_\sigma(x_2') g_\delta(x_1' + x_2') U_0^\nu(t, x_2 - x_2') U_{\alpha_0}^\mu(t, x_1, x_1') \\
&\equiv \left(U_0^\nu(t) U_{\alpha_0}^\mu(t) T\psi_{01} \right)(x_1, x_2)
\end{aligned} \tag{4.6}$$

where

$$\psi_{01}(r, R) = f_\sigma \left(\frac{M}{\nu} R + \frac{\mu}{M} r \right) g_\delta \left(r - R + \frac{M}{\nu} R + \frac{\mu}{M} r \right) \tag{4.7}$$

Then we have with $\frac{\mu}{M} = \frac{\epsilon}{1+\epsilon}$, $\frac{M}{\nu} - 1 = -\frac{\epsilon}{1+\epsilon}$

$$\begin{aligned}
&\|\psi(t) - \psi_1(t)\|^2 = \|T\psi(t) - T\psi_1(t)\|^2 = \|T\psi_0 - T\psi_{01}\|^2 \\
&= \int dx_1 dx_2 \left| f_\sigma \left(x_2 - \frac{\mu}{M} x_1 \right) g_\delta \left(x_2 + \frac{M}{\nu} x_1 \right) - f_\sigma(x_2) g_\delta(x_1 + x_2) \right|^2 \\
&= \int dx_1 dx_2 \left| f_\sigma \left(x_2 - \frac{\epsilon}{1+\epsilon} x_1 \right) g_\delta \left(x_2 + x_1 - \frac{\epsilon}{1+\epsilon} x_1 \right) - f_\sigma(x_2) g_\delta(x_1 + x_2) \right|^2 \\
&= \int dx_1 dx_2 \left| F \left(x_1, x_2 - \frac{\epsilon}{1+\epsilon} x_1 \right) - F(x_1, x_2) \right|^2
\end{aligned} \tag{4.8}$$

where $F(x_1, x_2) = f_\sigma(x_2) g_\delta(x_1 + x_2)$. By a simple Plancherel argument, we have that

$$\begin{aligned}
&\int dx_1 dx_2 \left| F \left(x_1, x_2 - \frac{\epsilon}{1+\epsilon} x_1 \right) - F(x_1, x_2) \right|^2 \\
&= \int dx_1 \int dk \left| \tilde{F}(x_1, k) \left(e^{-i \frac{\epsilon}{1+\epsilon} x_1 k} - 1 \right) \right|^2
\end{aligned}$$

$$\leq \int dx_1 \int dk |\tilde{F}(x_1,k)|^2 \left(\frac{\epsilon}{1+\epsilon}x_1 k\right)^2$$
$$= \left(\frac{\epsilon}{1+\epsilon}\right)^2 \int dx_1 x_1^2 \int dx_2 \left|\frac{\partial}{\partial x_2}F(x_1,x_2)\right|^2 \tag{4.9}$$

from which the lemma follows. □

Since a small value of m in the interacting unitary group $U^{\mu}_{\alpha_0}(t)$ is equivalent to a large value of t, in the next lemma we use a typical scattering estimate to approximate $U^{\mu}_{\alpha_0}(t)$ in (4.2).

Lemma 5. *Given the initial state (2.10), for any $t>0$ one has*

$$\|\psi_1(t)-\psi_2(t)\| < \frac{C_2}{t}\,\epsilon \tag{4.10}$$

where

$$\psi_2(t,r,R) = \sqrt{\frac{m}{2\pi i\hbar t}}\sqrt{\frac{M}{2\pi i\hbar t}}e^{i\frac{m}{2\hbar t}r^2+i\frac{M}{2\hbar t}R^2}\int d\xi f_\sigma(\xi)e^{i\frac{M}{2\hbar t}\xi^2}e^{-i\left(\frac{M}{\hbar t}R+\frac{m}{\hbar t}r\right)\xi}$$
$$\cdot\int dr' g_\delta(r'+\xi)\left(e^{-i\frac{\mu}{\hbar t}(r-R)r'} - \frac{e^{i\frac{\mu}{\hbar t}|r-R||r'|}}{1-i\frac{\hbar}{\alpha_0 t}|r-R|}\right) \tag{4.11}$$

and

$$C_2 = c\frac{M}{\hbar}\left\{\int dx x^4|f_\sigma(x)|^2 + \int dx|f_\sigma(x)|^2\left[\int dy y^4|g_\delta(y+x)|^2 + \frac{1}{\alpha^3}\left(\int dy|g_\delta(y+x)|\right)^2\right.\right.$$
$$\left.\left.+\frac{1}{\alpha}\left(\int dy|y||g_\delta(y+x)|\right)^2 + \alpha\left(\int dy y^2|g_\delta(y+x)|\right)^2\right]\right\}^{1/2} \tag{4.12}$$

Proof. We shall first estimate the difference $\psi_1(t)-\hat{\psi}_2(t)$, where $\hat{\psi}_2(t)$ is explicitely given by

$$\hat{\psi}_2(t,r,R) = \sqrt{\frac{\mu}{2\pi i\hbar t}}e^{i\frac{\mu}{2\hbar t}(r-R)^2}\int dy f_\sigma(y)U^{\nu}_0\left(t,\frac{M}{\nu}R+\frac{\mu}{M}r-y\right)\int dr' g_\delta\left(r'+y\right)$$
$$\cdot\left(e^{-i\frac{\mu}{\hbar t}(r-R)r'} - \frac{e^{i\frac{\mu}{\hbar t}|r-R||r'|}}{1-i\frac{\hbar}{\alpha_0 t}|r-R|}\right) \tag{4.13}$$

From (4.2) we have

$$(T\psi_1)(t,x_1,x_2) = \int dy U_0^\nu(t,x_2-y)\varphi_1(t,x_1,y) \tag{4.14}$$

$$\varphi_1(t,x_1,x_2) = f_\sigma(x_2)\int dr' g_\delta(r'+x_2) U^\mu_{\alpha_0}(t,x_1,r') \tag{4.15}$$

and analogously for $\hat{\psi}_2(t)$ we write

$$(T\hat{\psi}_2(t))(x_1,x_2) = \int dy U_0^\nu(t,x_2-y)\varphi_2(t,x_1,y) \tag{4.16}$$

$$\varphi_2(t,x_1,x_2) = \sqrt{\frac{\mu}{2\pi i\hbar t}} e^{i\frac{\mu}{2\hbar t}x_1^2} f_\sigma(x_2)\int dr' g_\delta(r'+x_2)\left(e^{-i\frac{\mu}{\hbar t}x_1 r'} - \frac{e^{i\frac{\mu}{\hbar t}|x_1||r'|}}{1-i\frac{\hbar|x_1|}{\alpha_0 t}}\right) \tag{4.17}$$

Using the isometric character of the operators T and $U_0^\nu(t)$ and the explicit expression of $U^\mu_{\alpha_0}(t)$ (see (5.10)), we have

$$\begin{aligned}
&\|\psi_1(t)-\hat{\psi}_2(t)\|^2 = \|T\psi_1(t)-T\psi_2(t)\|^2 = \|\varphi_1(t)-\varphi_2(t)\|^2\\
&\leq 2\int dx_1 dx_2 \left| f_\sigma(x_2)\int dr' g_\delta(r'+x_2)\left(U_0^\mu(t,x_1-r') - \sqrt{\frac{\mu}{2\pi i\hbar t}} e^{i\frac{\mu}{2\hbar t}x_1^2 - i\frac{\mu}{\hbar t}x_1 r'}\right)\right|^2\\
&+2\int dx_1 dx_2 \left| f_\sigma(x_2)\int dr' g_\delta(r'+x_2)\left(\frac{\mu\alpha_0}{\hbar^2}\int_0^\infty du e^{-\frac{\mu\alpha_0}{\hbar^2}u} U_0^\mu(t,u+|x_1|+|r'|)\right.\right.\\
&\left.\left. - \sqrt{\frac{\mu}{2\pi i\hbar t}}\frac{1}{1-i\frac{\hbar|x_1|}{\alpha_0 t}} e^{i\frac{\mu}{2\hbar t}x_1^2 + i\frac{\mu}{\hbar t}|x_1||r'|}\right)\right|^2\\
&\equiv (I) + (II)
\end{aligned} \tag{4.18}$$

A standard estimate for the free unitary group (see e.g. [RS]) gives

$$\begin{aligned}
(I) &= \frac{\mu}{\pi\hbar t}\int dx_2 |f_\sigma(x_2)|^2 \int dx_1 \left|\int dr' g_\delta(r'+x_2)\left(e^{i\frac{\mu}{2\hbar t}r'^2}-1\right)e^{-i\frac{\mu}{\hbar t}x_1 r'}\right|^2\\
&= \frac{1}{\pi}\int dx_2 |f_\sigma(x_2)|^2 \int d\bar{x}_1 \left|\int dr' g_\delta(r'+x_2)\left(e^{i\frac{\mu}{2\hbar t}r'^2}-1\right)e^{-i\bar{x}_1 r'}\right|^2\\
&= 2\int dx_2 |f_\sigma(x_2)|^2 \int dr' |g_\delta(r'+x_2)|^2 \left|e^{i\frac{\mu}{2\hbar t}r'^2}-1\right|^2\\
&\leq \frac{1}{2}\left(\frac{\mu}{\hbar t}\right)^2 \int dx_2 |f_\sigma(x_2)|^2 \int dr' r'^4 |g_\delta(r'+x_2)|^2\\
&= \frac{\epsilon^2 M^2}{2(1+\epsilon)^2\hbar^2 t^2}\int dx_2 |f_\sigma(x_2)|^2 \int dr' r'^4 |g_\delta(r'+x_2)|^2
\end{aligned} \tag{4.19}$$

where in the second line of (4.19) we used Plancherel theorem.
Concerning (II), we introduce the change of variables

$$v = \frac{\mu\alpha_0}{\hbar^2}u, \quad y_1 = \frac{\mu}{\hbar t}x_1 \tag{4.20}$$

and use the identity

$$\int_0^\infty dv e^{-v+i\frac{\hbar|x_1|}{\alpha_0 t}v} = \frac{1}{1-i\frac{\hbar|x_1|}{\alpha_0 t}} \tag{4.21}$$

Then

$$\begin{aligned}
&(II) = \frac{1}{\pi}\int dx_2|f_\sigma(x_2)|^2\int dy_1\left|\int dr' g_\delta\left(r'+x_2\right)e^{i|y_1||r'|}\right. \\
&\left.\cdot\left(e^{i\frac{\mu}{2\hbar t}r'^2}\int_0^\infty dv e^{-v+i\frac{1+\epsilon}{\alpha}|y_1|v+i\frac{(1+\epsilon)m}{2\hbar t\alpha^2}v^2+i\frac{m}{\hbar t\alpha}|r'|v} - \int_0^\infty dv e^{-v+i\frac{1+\epsilon}{\alpha}|y_1|v}\right)\right|^2 \\
&\le \frac{2}{\pi}\int dx_2|f_\sigma(x_2)|^2\int dy_1\left|\int dr' g_\delta\left(r'+x_2\right)e^{i|y_1||r'|}e^{i\frac{\mu}{2\hbar t}r'^2}\int_0^\infty dv e^{-v+i\frac{1+\epsilon}{\alpha}|y_1|v}\right. \\
&\left.\cdot\left(e^{i\frac{(1+\epsilon)m}{2\hbar t\alpha^2}v^2+i\frac{m}{\hbar t\alpha}|r'|v}-1\right)\right|^2 \\
&+\frac{2}{\pi}\int dx_2|f_\sigma(x_2)|^2\int dy_1\left|\int dr' g_\delta\left(r'+x_2\right)e^{i|y_1||r'|}\left(e^{i\frac{\mu}{2\hbar t}r'^2}-1\right)\frac{1}{1-i\frac{1+\epsilon}{\alpha}|y_1|}\right|^2 \\
&\equiv (III)+(IV)
\end{aligned} \tag{4.22}$$

The estimate of (IV) is trivial

$$\begin{aligned}
&(IV) \le \frac{1}{2\pi}\left(\frac{\mu}{\hbar t}\right)^2\int dx_2|f_\sigma(x_2)|^2\int dy_1\frac{1}{1+\left(\frac{1+\epsilon}{\alpha}\right)^2y_1^2}\left(\int dr' r'^2\left|g_\delta\left(r'+x_2\right)\right|\right)^2 \\
&\le \frac{\epsilon^2M^2\alpha}{2(1+\epsilon)^3\hbar^2t^2}\int dx_2|f_\sigma(x_2)|^2\left(\int dr' r'^2|g_\delta(r'+x_2)|\right)^2
\end{aligned} \tag{4.23}$$

For the estimate of (III) it is convenient to integrate by parts the integral in the variable v

$$\begin{aligned}
&(III) = \frac{2}{\pi}\int dx_2|f_\sigma(x_2)|^2\int dy_1\frac{1}{1+\left(\frac{1+\epsilon}{\alpha}\right)^2y_1^2}\left|\int dr' g_\delta(r'+x_2)e^{i|y_1||r'|}e^{i\frac{\mu}{2\hbar t}r'^2}\right. \\
&\left.\cdot\int_0^\infty dv e^{-v+i\frac{1+\epsilon}{\alpha}|y_1|v+i\frac{(1+\epsilon)m}{2\hbar t\alpha^2}v^2+i\frac{m}{\hbar t\alpha}|r'|v}\frac{(1+\epsilon)\mu}{\hbar t\alpha}\left(\frac{1+\epsilon}{\alpha}v+|r'|\right)\right|^2
\end{aligned}$$

$$\le \frac{2(1+\epsilon)\mu^2}{\pi\hbar^2 t^2\alpha}\int dx_2|f_\sigma(x_2)|^2\int dy_1\frac{1}{1+y_1^2}\left[\int dr'|g_\delta(r'+x_2)|\int_0^\infty dv e^{-v}\left(\frac{1+\epsilon}{\alpha}v+|r'|\right)\right]^2$$

$$\le \frac{2\epsilon^2 M^2}{(1+\epsilon)\hbar^2 t^2\alpha}\int dx_2|f_\sigma(x_2)|^2\left[\int dr'|g_\delta(r'+x_2)|\left(\frac{1+\epsilon}{\alpha}+|r'|\right)\right]^2 \tag{4.24}$$

Finally, it remains to analise the difference $\psi_2(t)-\hat{\psi}_2(t)$. Using the explicit expression of $U_0^\nu(t)$ we have

$$\begin{aligned}&\psi_2(t,r,R)-\hat{\psi}_2(t,r,R)\\&=\sqrt{\frac{m}{2\pi i\hbar t}}\sqrt{\frac{M}{2\pi i\hbar t}}e^{i\frac{m}{2\hbar t}r^2+i\frac{M}{2\hbar t}R^2}\int d\xi f_\sigma(\xi)\left(e^{i\frac{M}{2\hbar t}\xi^2}-e^{i\frac{\nu}{2\hbar t}\xi^2}\right)e^{-i\left(\frac{M}{\hbar t}R+\frac{m}{\hbar t}r\right)\xi}\\&\cdot\int dr' g_\delta(r'+\xi)\left(e^{-i\frac{\mu}{\hbar t}(r-R)r'}-\frac{e^{i\frac{\mu}{\hbar t}|r-R||r'|}}{1-i\frac{\hbar}{\alpha_0 t}|r-R|}\right)\end{aligned} \tag{4.25}$$

Exploiting Plancherel theorem and the fact that the operator (2.16) is unitary one easily sees that

$$\|\psi_2(t)-\hat{\psi}_2(t)\|^2\le\left(\frac{M}{2\hbar t}\right)^2\epsilon^2\int d\xi\xi^4|f_\sigma(\xi)|^2 \tag{4.26}$$

From (4.18),(4.19),(4.22),(4.23),(4.24), (4.26) we conclude the proof of the lemma. □

In the last step we approximate (4.13) using the fact that the coordinates of the heavy particle are slowly varying with respect to the coordinates of the light one.

Lemma 6. *Given the initial state (2.10), for any $t>0$ we have*

$$\|\psi_2(t)-\psi^a(t)\|<C_3(t)\,\epsilon \tag{4.27}$$

where

$$\begin{aligned}C_3(t)=&\left[\int dz z^2\int dx\left|\frac{\partial}{\partial x}\left(\tilde{\hat{f}}_\sigma(z-x)\tilde{g}(x)\right)\right|^2\right]^{1/2}\\&+\frac{1}{\sqrt{2}\pi}\left\{\int dxdz\left[\frac{\alpha^2+z^2}{\alpha^2+x^2}\left(\int dr'|\zeta(z,r')|\right)^2+\frac{\alpha^2 z^2}{\alpha^2+x^2}\left(\int dr'|r'||\zeta(z,r')|\right)^2\right]\right\}^{1/2}\end{aligned} \tag{4.28}$$

with

$$C_3(t)<\frac{C_4}{t}+C_5 \tag{4.29}$$

and

$$\hat{f}_\sigma(\xi) = f_\sigma(\xi) e^{i\frac{M}{2\hbar t}\xi^2}, \qquad \zeta(z, r') = \int d\xi \hat{f}_\sigma(\xi) g_\delta(r' + \xi) e^{-iz\xi} \tag{4.30}$$

Proof. From (2.21) and (4.11) we have

$$\begin{aligned}
&\psi_2(t, r, R) - \psi^a(t, r, R) \\
&= \sqrt{\frac{m}{2\pi i\hbar t}} \sqrt{\frac{M}{2\pi i\hbar t}} e^{i\frac{m}{2\hbar t}r^2 + i\frac{M}{2\hbar t}R^2} \int d\xi f_\sigma(\xi) e^{i\frac{M}{2\hbar t}\xi^2} e^{-i\left(\frac{M}{\hbar t}R + \frac{m}{\hbar t}r\right)\xi} \\
&\cdot \left[\int dr' g_\delta(r' + \xi) \left(e^{-i\frac{\mu}{\hbar t}(r-R)r'} - e^{-i\frac{m}{\hbar t}rr'} \right) + \int dr' g_\delta(r' + \xi) \left(\frac{e^{i\frac{\mu}{\hbar t}|r-R||r'|}}{1 - i\frac{\hbar}{\alpha_0 t}|r - R|} - \frac{e^{i\frac{m}{\hbar t}|r||r'|}}{1 - i\frac{\hbar}{\alpha_0 t}|r|} \right) \right] \\
&\equiv (\psi_2 - \psi^a)_{fr}(t, r, R) + (\psi_2 - \psi^a)_{in}(t, r, R)
\end{aligned} \tag{4.31}$$

We will estimate separatly the two terms $(\psi_2 - \psi^a)_{fr}(t)$ and $(\psi_2 - \psi^a)_{in}(t)$.
Introducing the new integration variables

$$x = \frac{m}{\hbar t} r, \qquad z = \frac{MR + mr}{\hbar t} \tag{4.32}$$

and the function $\hat{f}_\sigma(\xi)$ defined in (4.30), we have

$$\begin{aligned}
&\| (\psi_2 - \psi^a)_{fr}(t) \|^2 \\
&= \frac{1}{(2\pi)^2} \int dx dz \left| \int d\xi \hat{f}_\sigma(\xi) e^{-iz\xi} \int dr' g_\delta(r' + \xi) \left(e^{-i\left(x - \frac{\epsilon}{1+\epsilon}z\right)r'} - e^{-ixr'} \right) \right|^2 \\
&= \frac{1}{2\pi} \int dx dz \left| \int d\xi \hat{f}_\sigma(\xi) e^{-iz\xi} \left(\tilde{g}_\delta\left(x - \frac{\epsilon}{1+\epsilon}z\right) e^{i\left(x - \frac{\epsilon}{1+\epsilon}z\right)\xi} - \tilde{g}_\delta(x) e^{ix\xi} \right) \right|^2 \\
&= \int dx dz \left| \tilde{\hat{f}}_\sigma\left(z - x + \frac{\epsilon}{1+\epsilon}z\right) \tilde{g}_\delta\left(x - \frac{\epsilon}{1+\epsilon}z\right) - \tilde{\hat{f}}_\sigma(z - x)\tilde{g}(x) \right|^2 \\
&= \int dx dz \left| G\left(x - \frac{\epsilon}{1+\epsilon}z, z\right) - G(x, z) \right|^2
\end{aligned} \tag{4.33}$$

where we introduced the function $G(x, z) = \tilde{\hat{f}}_\sigma(z - x)\tilde{g}(x)$.
Proceeding as in (4.9) we find

$$\| (\psi_2 - \psi^a)_{fr}(t) \|^2 \leq \left(\frac{\epsilon}{1+\epsilon} \right)^2 \int dz z^2 \int dx \left| \frac{\partial}{\partial x} \left(\tilde{\hat{f}}_\sigma(z - x)\tilde{g}(x) \right) \right|^2 \tag{4.34}$$

For the estimate of $(\psi_2 - \psi^a)_{in}(t)$ we use again the change of variables (4.32) and we introduce the function $\zeta(z, r')$ defined in (4.30)
Then we have

$$\begin{aligned}
&\|(\psi_2 - \psi^a)_{in}\|^2 \leq \frac{1}{(2\pi)^2} \int dx dz \left| \int dr' \zeta(z, r') \left(\frac{e^{i\left|x - \frac{\epsilon}{1+\epsilon} z\right| |r'|}}{1 - \frac{i}{\alpha}|(1+\epsilon)x - \epsilon z|} - \frac{e^{i|x||r'|}}{1 - \frac{i}{\alpha}|x|} \right) \right|^2 \\
&\leq \frac{1}{2\pi^2} \int dx dz \left| \frac{1}{1 - \frac{i}{\alpha}|(1+\epsilon)x - \epsilon z|} - \frac{1}{1 - \frac{i}{\alpha}|x|} \right|^2 \left(\int dr' |\zeta(z, r')| \right)^2 \\
&+ \frac{1}{2\pi^2} \int dx dz \frac{\alpha^2}{\alpha^2 + x^2} \left| \int dr' \zeta(z, r') \left(e^{i\left|x - \frac{\epsilon}{1+\epsilon} z\right| |r'|} - e^{i|x||r'|} \right) \right|^2 \\
&\leq \frac{\epsilon^2}{2\pi^2} \int dx dz \left[\frac{\alpha^2 + z^2}{\alpha^2 + x^2} \left(\int dr' |\zeta(z, r')| \right)^2 + \frac{\alpha^2 z^2}{\alpha^2 + x^2} \left(\int dr' |r'| |\zeta(z, r')| \right)^2 \right]
\end{aligned} \tag{4.35}$$

where we have used the estimates

$$\begin{aligned}
&\left| \frac{1}{1 - \frac{i}{\alpha}|(1+\epsilon)x - \epsilon z|} - \frac{1}{1 - \frac{i}{\alpha}|x|} \right|^2 = \frac{\alpha^2}{\alpha^2 + x^2} \frac{(|x| - |(1+\epsilon)x - \epsilon z|)^2}{\alpha^2 + ((1+\epsilon)x - \epsilon z)^2} \\
&\leq \epsilon^2 \frac{\alpha^2}{\alpha^2 + x^2} \frac{(x - z)^2}{\alpha^2 + ((1+\epsilon)x - \epsilon z)^2} \leq \left(\frac{\epsilon}{1+\epsilon} \right)^2 \frac{\alpha^2 + z^2}{\alpha^2 + x^2}
\end{aligned} \tag{4.36}$$

and

$$\left| e^{i\left|x - \frac{\epsilon}{1+\epsilon} z\right| |r'|} - e^{i|x||r'|} \right| \leq \frac{\epsilon}{1+\epsilon} |z||r'| \tag{4.37}$$

The function $\zeta(z, r')$ is smooth and, using repeated integration by parts, one easily sees that it is rapidly decreasing when its first argument goes to infinity. The computation is long but straightforward and we omit the details. The conclusion is that the integral in the last line of (4.35) is finite. Along the same line one can verify that (4.29) holds, where the constants C_4, C_5 are independent of time and then the proof of the lemma follows. □

Proof of theorem 1. It is an obvious consequence of lemmas 4, 5, 6. □

5. Appendix: explicit solution of the two-body problem

We recall here the solution of the Schrödinger equation

$$i\hbar \frac{\partial \psi(t)}{\partial t} = H\psi(t), \quad \psi(0) = \psi_0 \tag{5.1}$$

where H is the self-adjoint hamiltonian in $L^2(\mathbb{R}^2, drdR)$ given by (2.1). Using the unitary operator T (see (2.14) one obviously has

$$THT^{-1} = H_0^\nu + H_{\alpha_0}^\mu, \quad H_0^\nu = -\frac{\hbar^2}{2\nu}\Delta_{x_2}, \quad H_{\alpha_0}^\mu = -\frac{\hbar^2}{2\mu}\Delta_{x_1} + \alpha_0\delta(x_1) \tag{5.2}$$

where (x_1, x_2) are the relative and the center of mass coordinates

$$x_1 = r - R, \quad x_2 = \frac{mr + MR}{m + M} \tag{5.3}$$

Then the solution of (5.1) can be written as

$$\psi(t, r, R) = \left(T^{-1}U_0^\nu(t)U_{\alpha_0}^\mu(t)T\psi_0\right)(r, R)$$

$$= \int dr'dR'\psi_0(r', R')U_0^\nu\left(t, \frac{M}{\nu}(R - R') + \frac{\mu}{M}(r - r')\right)U_{\alpha_0}^\mu\left(t, r - R, r' - R'\right) \tag{5.4}$$

where the interacting unitary group $U_{\alpha_0}^\mu(t)$ is given by

$$U_{\alpha_0}^\mu(t, x, x') = e^{-i\frac{t}{\hbar}H_{\alpha_0}^\mu}(x, x'), \quad x, x' \in \mathbb{R} \tag{5.5}$$

We remark that the evolution (5.4) factorizes into a product of a free evolution in the center of mass coordinate and a one-body interacting evolution in the relative coordinate only if the initial state is of the form $\psi(r, R) = \psi_1(\frac{M}{\nu}R + \frac{\mu}{M}r)\psi_2(r - R)$.

In order to compute $(U_{\alpha_0}^\mu(t)\varphi_0)(x) \equiv \varphi(t, x)$ one has to solve the one-body Schrödinger equation

$$i\hbar\frac{\partial\varphi(t)}{\partial t} = -\frac{\hbar^2}{2\mu}\Delta_x\varphi(t) + \alpha_0\delta(x)\varphi(t), \quad \varphi(0) = \varphi_0 \tag{5.6}$$

Defining the rescaled wave function

$$\theta(s, z) = \varphi\left(\hbar s, \frac{\hbar}{\sqrt{\mu}}z\right) \tag{5.7}$$

one finds that $\theta(s)$ satisfies the corresponding equation with $\mu = \hbar = 1$

$$i\frac{\partial\theta(s)}{\partial s} = -\frac{1}{2}\Delta_z\theta(s) + \alpha_0\frac{\sqrt{\mu}}{\hbar}\delta(z)\theta(s), \quad \theta(0) = \theta_0, \;\; \theta_0(z) = \varphi_0\left(\frac{\hbar}{\sqrt{\mu}}z\right) \tag{5.8}$$

The solution of (5.8) can be found in [S]

$$\theta(s,z) = \left(\hat{U}_0^1(s)\theta_0\right)(z) - \alpha_0 \frac{\sqrt{\mu}}{\hbar} \int_0^\infty dv e^{-\alpha_0 \frac{\sqrt{\mu}}{\hbar} v} \int dz' \hat{U}_0^1(s, v + |z| + |z'|)\theta_0(z') \tag{5.9}$$

where $\hat{U}_0^1(s)$ is the free propagator with $\hbar = 1$. Noticing that $(U^\mu_{\alpha_0}(t)\varphi_0)(x) = \varphi(t,x) = \theta(\frac{t}{\hbar}, \frac{\sqrt{\mu}}{\hbar}x)$ one has

$$(U^\mu_{\alpha_0}(t)\varphi_0)(x) = (U^\mu_0(t)\varphi_0)(x) - \frac{\mu\alpha_0}{\hbar^2} \int_0^\infty du e^{-\frac{\mu\alpha_0}{\hbar^2} u} \int dx' U^\mu_0(t, u + |x| + |x'|)\varphi_0(x') \tag{5.10}$$

Using (5.4) and (5.10), we finally obtain the complete solution (2.2) of the Schroedinger equation (5.1).

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