SPECTRAL ANALYSIS OF DARBOUX TRANSFORMATIONS FOR THE FOCUSING NLS HIERARCHY

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Abstract. We study Darboux-type transformations associated with the focusing nonlinear Schrödinger equation (NLS_-) and their effect on spectral properties of the underlying Lax operator. The latter is a formally $\mathcal{J}$-self-adjoint (but non-self-adjoint) Dirac-type differential expression of the form

$$M(q) = i \begin{pmatrix} \frac{d}{dx} & -q \\ -\overline{q} & -\frac{d}{dx} \end{pmatrix},$$

satisfying $\mathcal{J} M(q) \mathcal{J} = M(q)^*$, where $\mathcal{J}$ is defined by $\mathcal{J} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mathcal{C}$, and $\mathcal{C}$ denotes the antilinear conjugation map in $\mathbb{C}^2$, $\mathcal{C}(a,b)^\top = (\overline{a},\overline{b})^\top$, $a,b \in \mathbb{C}$. As one of our principal results we prove that under the most general hypothesis $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ on q, the maximally defined operator $D(q)$ generated by $M(q)$ is actually $\mathcal{J}$-self-adjoint in $L^2(\mathbb{R})^2$. Moreover, we establish the existence of Weyl–Titchmarsh-type solutions $\Psi_+(z,\cdot) \in L^2([R,\infty))^2$ and $\Psi_-(z,\cdot) \in L^2((-\infty,R])$ for all $R \in \mathbb{R}$ of $M(q)\Psi_\pm(z) = z\Psi_\pm(z)$ for z in the resolvent set of D.

The Darboux transformations considered in this paper are the analog of the double commutation procedure familiar in the KdV and Schrödinger operator contexts. As in the corresponding case of Schrödinger operators, the Darboux transformations in question guarantee that the resulting potentials q are locally nonsingular. Moreover, we prove that the construction of N-soliton NLS_- potentials $q^{(N)}$ with respect to a general NLS_- background potential $q \in L^1_{\mathrm{loc}}(\mathbb{R})$, associated with the Dirac-type operators $D\big(q^{(N)}\big)$ and $D(q)$, respectively, amounts to the insertion of N complex conjugate pairs of $L^2(\mathbb{R})^2$-eigenvalues $\{z_1,\overline{z}_1,\dots,z_N,\overline{z}_N\}$ into the spectrum $\sigma(D(q))$ of $D(q)$, leaving the rest of the spectrum (especially, the essential spectrum $\sigma_{\mathrm{e}}(D(q))$) invariant, that is,

$$\sigma\big(D\big(q^{(N)}\big)\big) = \sigma(D(q)) \cup \{z_1,\overline{z}_1,\dots,z_N,\overline{z}_N\},$$
$$\sigma_{\mathrm{e}}\big(D\big(q^{(N)}\big)\big) = \sigma_{\mathrm{e}}(D(q)).$$

These results are obtained by establishing the existence of bounded transformation operators which intertwine the background Dirac operator $D(q)$ and the Dirac operator $D\big(q^{(N)}\big)$ obtained after N Darboux-type transformations.

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1. Introduction

Various methods of inserting eigenvalues in spectral gaps of one-dimensional Schrödinger operators $H(q)$ associated with differential expressions of the type

$$L(q) = -\frac{d^2}{dx^2} + q \qquad (1.\text{arabic@equation})$$

in $L^2(\mathbb{R})$ (or in $L^2((a,\infty))$, $a \geq -\infty$), with q real-valued and locally integrable, have attracted an enormous amount of attention. This is due to their prominent role in diverse fields such as the inverse scattering approach, supersymmetric quantum mechanics, level comparison theorems, as a tool to construct soliton solutions of the Korteweg-de Vries (KdV) hierarchy relative to (general) KdV background solutions, and in connection with Bäcklund transformations for the KdV hierarchy. The literature on this subject is too extensive to go into details here, but we refer to the detailed accounts given in [14], [15], [16, App. G], [17], [18] and the references cited therein. Historically, these methods of inserting eigenvalues go back to Jacobi [24] and Darboux [9] with decisive later contributions by Crum [8], Schmincke [40], and, especially, Deift [10].

Two particular methods turned out to be of special importance: The single commutation method, also called the Crum-Darboux method [8], [9] (actually going back at least to Jacobi [24]) and the double commutation method, to be found, for instance, in the seminal work of Gel'fand and Levitan [13]. (The latter can be obtained by a composition of two separate single commutation steps, explaining the name double commutation.)

The single commutation method, although very simply implemented, has the distinct disadvantage of relying on positivity properties of certain solutions ψ of $H(q)\psi = \lambda\psi$, which confines its applicability to the insertion of eigenvalues below the spectrum of $H(q)$ (assuming $H(q)$ to be bounded from below). A complete spectral characterization of this method has been provided by Deift [10] (see also [40]) on the basis of unitary equivalence of $A^*A|_{\ker(A)^\perp}$ and $AA^*|_{\ker(A^*)^\perp}$ for a densely defined closed linear operator A in a (complex, separable) Hilbert space.

The double commutation method on the other hand, allows one to insert eigenvalues into *any* spectral gap of $H(q)$. Although relatively simply implemented also, a complete spectral characterization of the double commutation method for Schrödinger-type operators was more recently achieved in [14] on the basis of Weyl–Titchmarsh m-function techniques and subsequently in [18] (for general Sturm–Liouville operators on arbitrary intervals) using a functional analytic approach based on the notion of (intertwining) transformation operators.

In this paper we concentrate on the analog of the double commutation method for Dirac-type operators associated with the Lax operator for the focusing nonlinear Schrödinger (NLS_-) hierarchy. Assuming q to be locally integrable, the Dirac-type operator corresponding to the Lax differential expression in the NLS_- case is associated with the 2×2 matrix-valued differential expression

$$M(q) = i\begin{pmatrix} \frac{d}{dx} & -q \\ -\overline{q} & -\frac{d}{dx} \end{pmatrix} \qquad (1.\text{arabic@equation})$$

for $x \in \mathbb{R}$ (cf., e.g., [12, Part I, Sect. I.2], [46], and [47]). The maximally defined Dirac-type operator associated with $M(q)$ in the (two-component) Hilbert space $L^2(\mathbb{R})^2$ will then be denoted by $D(q)$. By way of contrast, the corresponding (formally self-adjoint) Lax differential expression for the defocusing NLS_+ case is given by

$$i\begin{pmatrix} \frac{d}{dx} & -q \\ \overline{q} & -\frac{d}{dx} \end{pmatrix}. \qquad (1.\text{arabic@equation})$$

As it turns out there is no natural analog of the single commutation method for the Dirac operators associated with the focusing and defocusing nonlinear Schrödinger hierarchies (NLS$_\pm$). However, the complexified version of the NLS$_\pm$ hierarchies, the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy, supports two natural analogs of the single commutation method. In order to briefly describe them, we recall that the Dirac-type Lax differential expression associated with the AKNS hierarchy is given by

$$M(p,q) = i\begin{pmatrix} \frac{d}{dx} & -q \\ p & -\frac{d}{dx} \end{pmatrix} \qquad (1.\text{arabic@equation})$$

(cf. e.g., [1] and [16, Ch. 3]) in terms of two locally integrable coefficients p, q on $\mathbb{R}$. The focusing (NLS$_-$) and defocusing (NLS$_+$) nonlinear Schrödinger hierarchies are then associated with the constraints

$$\text{NLS}_\pm\colon\ p(x) = \pm\overline{q(x)}, \qquad (1.\text{arabic@equation})$$

respectively. In this paper we will concentrate on the focusing NLS$_-$ case only.

The two analogs of the single commutation method for the AKNS case, which are usually called elementary Darboux transformations, can then be described as follows. Suppose

$$M(p,q)\Psi(z_1,x) = z_1\Psi(z_1,x), \quad \Psi(z_1,x) = (\psi_1(z_1,x),\psi_2(z_1,x))^\top,\ (z_1,x)\in\mathbb{C}\times\mathbb{R}, \qquad (1.\text{arabic@equation})$$

and

$$M(p,q)\widetilde\Psi(\tilde z_1,x) = \tilde z_1\widetilde\Psi(\tilde z_1,x), \quad \widetilde\Psi(\tilde z_1,x) = (\tilde\psi_1(\tilde z_1,x),\tilde\psi_2(\tilde z_1,x))^\top,\ (\tilde z_1,x)\in\mathbb{C}\times\mathbb{R}. \qquad (1.\text{arabic@equation})$$

Then the two elementary Darboux transformations in the AKNS context are given by (cf. [27], [28])

$$(p,q)\mapsto(\hat p_{z_1},\hat q_{z_1}), \qquad (1.\text{arabic@equation})$$

where

$$\begin{aligned} &\hat p_{z_1}(x) = -\psi_2(z_1,x)/\psi_1(z_1,x), \\ &\hat q_{z_1}(x) = q'(x) - \psi_2(z_1,x)/\psi_1(z_1,x)q(x)^2 + 2iz_1q(x), \end{aligned} \qquad (1.\text{arabic@equation})$$

and

$$(p,q)\mapsto(\check p_{\tilde z_1},\check q_{\tilde z_1}), \qquad (1.\text{arabic@equation})$$

where

$$\begin{aligned} &\check p_{\tilde z_1}(x) = -p'(x) + \tilde\psi_1(\tilde z_1,x)/\tilde\psi_2(\tilde z_1,x)p(x)^2 + 2i\tilde z_1p(x), \\ &\check q_{\tilde z_1}(x) = \tilde\psi_1(\tilde z_1,x)/\tilde\psi_2(\tilde z_1,x). \end{aligned} \qquad (1.\text{arabic@equation})$$

Similar to the case of Schrödinger operators, the analog of the double commutation method for Dirac-type operators associated with (1.arabic@equation) is then obtained by an appropriate composition of the two elementary Darboux transformations (1.arabic@equation) and (1.arabic@equation). This two-step procedure is denoted by

$$(p,q) \mapsto (p^{(1)}_{z_1,\tilde z_1}, q^{(1)}_{z_1,\tilde z_1}) \qquad \text{(1.arabic@equation)}$$

and leads to (cf., e.g., [27], [28], [34, Sect. 4.2], [39])

$$\begin{aligned} p^{(1)}_{z_1,\tilde z_1}(x) &= p(x) - 2i(\tilde z_1 - z_1)\psi_2(z_1,x)\tilde\psi_2(\tilde z_1,x)/W(\Psi(z_1,x),\tilde\Psi(\tilde z_1,x)),\\ q^{(1)}_{z_1,\tilde z_1}(x) &= q(x) - 2i(\tilde z_1 - z_1)\psi_1(z_1,x)\tilde\psi_1(\tilde z_1,x)/W(\Psi(z_1,x),\tilde\Psi(\tilde z_1,x)). \end{aligned} \qquad \text{(1.arabic@equation)}$$

(Here $W(F,G)$ denotes the Wronskian of $F,G\in\mathbb{C}^2$). In contrast to (1.arabic@equation), (1.arabic@equation) and (1.arabic@equation), (1.arabic@equation), the two-step procedure (1.arabic@equation), (1.arabic@equation) with $\tilde z_1 = \overline{z_1}$, is compatible with the NLS$_\pm$ cases and one explicitly obtains the following Darboux-type transformation in the NLS$_-$ case,

$$q(x) \mapsto q^{(1)}_{z_1}(x) = q(x) + 4\,\mathrm{Im}(z_1)\frac{\psi_1(z_1,x)\overline{\psi_2(z_1,x)}}{|\psi_1(z_1,x)|^2 + |\psi_2(z_1,x)|^2}. \qquad \text{(1.arabic@equation)}$$

The transformation (1.arabic@equation) for Dirac-type operators assoiated with (1.arabic@equation) in the NLS$_-$ context represents the analog of the double commutation method for Schrödinger operators and leads to locally nonsingular NLS$_-$ potentials $q^{(1)}_{z_1}$, assuming q to be free of local singularities.

By analogy to the KdV and Schrödinger operator case, one expects the NLS$_-$ potential $q^{(1)}_{z_1}(x)$ to produce an eigenvalue at the spectral point z_1 for the associated Dirac operator $D\big(q^{(1)}_{z_1}\big)$, assuming z_1 to be a point in the resolvent set of the "background" operator $D(q)$. Actually, by a simple symmetry consideration, one expects a pair of eigenvalues $(z_1,\overline{z_1})$ in the point spectrum of $D\big(q^{(1)}_{z_1}\big)$. To prove this fact and to show that the remaining spectral characteristics (especially, the essential spectrum of $D(q)$) remain invariant under the Darboux-type transformation (1.arabic@equation), is the principal purpose of this paper. More precisely, if we denote by $q^{(N)}_{z_1,\dots,z_N}$ the NLS$_-$ potential obtained after an N-fold iteration of the Darboux-type transformation and by $D\big(q^{(N)}_{z_1,\dots,z_N}\big)$ the resulting Dirac operator, we will prove that

$$\sigma\big(D\big(q^{(N)}_{z_1,\dots,z_N}\big)\big) = \sigma(D(q)) \cup \{z_1,\overline{z}_1,\dots,z_N,\overline{z}_N\}, \qquad \text{(1.arabic@equation)}$$

$$\sigma_{\mathrm{p}}\big(D\big(q^{(N)}_{z_1,\dots,z_N}\big)\big) = \sigma_{\mathrm{p}}(D(q)) \cup \{z_1,\overline{z}_1,\dots,z_N,\overline{z}_N\}, \qquad \text{(1.arabic@equation)}$$

$$\sigma_{\mathrm{e}}\big(D\big(q^{(N)}_{z_1,\dots,z_N}\big)\big) = \sigma_{\mathrm{e}}(D(q)). \qquad \text{(1.arabic@equation)}$$

(Here $\sigma(S)$, $\sigma_{\mathrm{p}}(S)$, and $\sigma_{\mathrm{e}}(S)$ denote the spectrum, point spectrum, and essential spectrum of a densely defined closed operator S in a complex separable Hilbert space $\mathcal{H}$, cf. Section 5 for more details on spectra, etc.) Actually, we will go a step beyond (1.arabic@equation)–(1.arabic@equation) and establish the existence of bounded transformation operators which intertwine $D\big(q^{(N)}_{z_1,\dots,z_N}\big)$ and $D(q)$.

When trying to embark on proving results of the type (1.arabic@equation)–(1.arabic@equation) for Dirac-type operators associated with the differential expression (1.arabic@equation) in the NLS$_-$ context, one finds oneself at a distinct disadvantage when compared to the case of Schrödinger operators with real-valued potentials: While L in (1.arabic@equation) is formally self-adjoint for q real-valued,

$M(q)$ in (1.arabic@equation) is never self-adjoint (except, in the trivial case $q = 0$). As a consequence, the original approach to a complete spectral characterization of the double commutation method for Schrödinger operators in [14], based on Weyl–Titchmarsh theory and hence on spectral theory, is doomed from the start as there simply is no spectral theory for non-self-adjoint Dirac-type operators associated with (1.arabic@equation) under our general hypothesis $q \in L^1_{\mathrm{loc}(\mathbb{R})}$. That leaves one with only one possible line of attack, the analog of the transformation operator approach developed in the general Sturm–Liouville context in [18]. As it will turn out in Section 6, this approach is indeed successful although it requires more sophisticated and elaborate arguments compared to those in [18].

While the differential expression $M(q)$ in (1.arabic@equation) is never formally self-adjoint (if $q \neq 0$), it is, however, formally $\mathcal{J}$-self-adjoint, that is,

$$\mathcal{J} M(q) \mathcal{J} = M(q)^*. \qquad (1.\text{arabic@equation})$$

Here $\mathcal{J}$ is defined by

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{C}, \qquad (1.\text{arabic@equation})$$

and $\mathcal{C}$ denotes the antilinear conjugation map in $\mathbb{C}^2$,

$$\mathcal{C}(a,b)^\top = (\overline{a}, \overline{b})^\top, \quad a, b \in \mathbb{C}. \qquad (1.\text{arabic@equation})$$

As one of our principal results in this paper we will prove in Section 3 that under the most general hypothesis $q \in L^1_{\mathrm{loc}}(\mathbb{R})$, the maximally defined Dirac operator $D(q)$ associated with $M(q)$ is in fact $\mathcal{J}$-self-adjoint,

$$\mathcal{J} D(q) \mathcal{J} = D(q)^* \ (= D(-q)). \qquad (1.\text{arabic@equation})$$

As an aside we should mention that the corresponding maximally defined Lax operator associated with the defocusing NLS$_+$ differential expression (1.arabic@equation) is in fact self-adjoint assuming $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ only (this is proved in the references mentioned in Section 3). This should be contrasted with the case of one-dimensional Schrödinger differential expressions $L(q)$ in (1.arabic@equation) (the Lax differential expression associated with the KdV hierarchy). If $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ in (1.arabic@equation) is real-valued, then $L(q)$ is formally self-adjoint but the maximally defined operator $H(q)$ in $L^2(\mathbb{R})$ associated with $L(q)$ may not be self-adjoint. The latter situation occurs precisely when $L(q)$ is in the limit circle case (as opposed to the limit point case) at $+\infty$ and/or $-\infty$ (cf. [7, Ch. 9]). This is in sharp contrast to the focusing (respectively, defocusing) NLS case where $D(q)$ is always $\mathcal{J}$-self-adjoint (respectively, self-adjoint).

Summarizing, we derive the following principal new results in this paper, assuming the optimal condition $q \in L^1_{\mathrm{loc}}(\mathbb{R})$:

– $\mathcal{J}$-self-adjointness of $D(q)$.
– The existence of Weyl–Titchmarsh-type solutions of $M(q)\Psi = z\Psi$ for all $z \in \rho(D(q))$.
– The existence and boundedness of transformation operators intertwining the operators $D\big(q^{(N)}\big)$ and $D(q)$.
– A spectral analysis of NLS$_-$ Darboux transformations (cf. (1.arabic@equation)–(1.arabic@equation)).

Finally we briefly describe the content of each section. Section 2 first introduces our main notation and then proceeds to a review of Darboux transformations for AKNS and NLS$_-$ systems. Section 3 is devoted to a proof of the $\mathcal{J}$-self-adjointness property (1.arabic@equation) of $D(q)$, assuming $q \in L^1_{\text{loc}(\mathbb{R})}$ only. Section 4 constructs eigenvalues of $D\big(q^{(1)}_{z_1}\big)$ at pairs $z_1, \overline{z_1}$, $z_1 \in \rho(D(q))$ and associated $L^2(\mathbb{R})^2$-eigenfunctions. Section 5 derives some basic spectral properties of general Dirac-type operators $D(q)$ and establishes the existence of Weyl–Titchmarsh-type solutions associated with $M(q)$. This shows a remarkable similarity to self-adjoint systems and appears to be without precedent in this non-self-adjoint context. Our final Section 6 establishes the existence of bounded transformation operators intertwining $D\big(q^{(1)}_{z_1}\big)$ and $D(q)$ and then employs these transformation operators to prove the spectral properties (1.arabic@equation)–(1.arabic@equation).

All results in the principal part of this paper, Sections 3–6, are proved under the optimal condition $q \in L^1_{\text{loc}(\mathbb{R})}$. Moreover, practically all results in Sections 3–6 are new as long as one goes beyond bounded or periodic potentials q. In particular, Theorem 6.14 (characterizing transformation operators) and Theorem 6.15 (proving (1.arabic@equation)–(1.arabic@equation)) appear to be the first of their kind under any assumptions on q.

In this paper we confine ourselves to a stationary (i.e., time-independent) approach only. Applications to the time-dependent focusing NLS$_-$ equation and to nonlinear optics will be made in a subsequent paper [3].

2. Darboux-type Transformations for AKNS and NLS$_-$ Systems

In this section we take a close look at Darboux-type transformations for non-self-adjoint Dirac-type differential expressions $M(q)$ (cf. (2.arabic@equation)) applicable to AKNS systems, with special emphasis on the case of the focusing nonlinear Schrödinger equation NLS$_-$ (cf. (2.arabic@equation)).

Throughout this paper we use the following notation: $' = d/dx$; for a matrix A with complex-valued entries, $A^\top$ denotes the transposed matrix, $\overline{A}$ the matrix with complex conjugate entries, and $A^* = \overline{A}^\top = \overline{A^\top}$ the adjoint matrix. Occasionally, we use the following 2×2 matrices,

$$\begin{aligned} I_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \sigma_4 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_5 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_6 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \qquad \text{(2.arabic@equation)}$$

If $A = (a_1, a_2)^\top$ and $B = (b_1, b_2)^\top$ are 2×1 column-vectors, then $(A, B)_{\mathbb{C}^2} = B^* A = a_1\overline{b_1} + a_2\overline{b_2}$ denotes the usual scalar product in $\mathbb{C}^2$, $\|A\|_{\mathbb{C}^2} = (|a_1|^2 + |a_2|^2)^{1/2}$ the associated norm, and $A^\perp = (a_2, -a_1) = A^\top \sigma_4$ the 1×2 row-vector perpendicular to A (in the sense that $A^\perp A = 0$). We also use the notation

$$W(A, B) = B^\perp A = -A^\perp B = a_1 b_2 - a_2 b_1 = \big(A, \overline{B^\perp}\big)_{\mathbb{C}^2} \qquad \text{(2.arabic@equation)}$$

for the Wronskian of A and B. The space of 2×2 matrices with entries in $\mathbb{C}$ is denoted by $\mathbb{C}^{2\times 2}$ and the operator norm of a 2×2 matrix A induced by the usual norm in $\mathbb{C}^2$ is denoted by $\|A\|_{\mathbb{C}^{2\times 2}}$. In the following, $\Omega \subseteq \mathbb{R}$ denotes an open subset of $\mathbb{R}$ and $\text{AC}_{\text{loc}(\Omega)^{2\times m}}$, $m = 1, 2$, denotes the set of $2 \times m$ matrices with locally absolutely continuous entries on Ω (for 2×1 columns we use the corresponding notation $\mathrm{AC}_{\mathrm{loc}(\Omega)^2}$). We define[1] $L^p(\Omega)^{2\times m}$ and $L^p_{\mathrm{loc}}(\Omega)^{2\times m}$, $m = 1, 2$, to consist of $2\times m$ matrices with entries in $L^p(\Omega)$ and $L^p_{\mathrm{loc}}(\Omega)$, respectively. In the special case $\Omega = \mathbb{R}$ and $F, G \in L^2(\mathbb{R})^2$, the scalar product of F and G is denoted by $(F,G)_{L^2} = \int_{\mathbb{R}} dx\, (F(x), G(x))_{\mathbb{C}^2}$ with associated norm of F given by $\|F\|_{L^2} = \big(\int_{\mathbb{R}} dx\, \|F(x)\|^2_{\mathbb{C}^2}\big)^{1/2}$. Finally, the open complex upper (respectively, lower) half-plane is denoted by $\mathbb{C}_+$ (respectively, $\mathbb{C}_-$); the domain, range, and kernel (null space) of a linear operator T are denoted by $\mathrm{dom}(T)$, $\mathrm{ran}(T)$, and $\ker(T)$, respectively.

Hypothesis 2.1. *Let $\Omega \subseteq \mathbb{R}$ open and assume $p, q \in L^1_{\mathrm{loc}(\Omega)}$.*

Assuming Hypothesis 2.1 and $z \in \mathbb{C}$, we introduce the 2×2 matrix $U(z,p,q)$ and the 2×2 matrix-valued differential expression $M(p,q)$ by

$$U(z,p,q) = \begin{pmatrix} -iz & q \\ p & iz \end{pmatrix}, \quad M(p,q) = i\begin{pmatrix} \frac{d}{dx} & -q \\ p & -\frac{d}{dx} \end{pmatrix}. \qquad (2.\text{arabic@equation})$$

The functions p and q in (2.arabic@equation) are referred to as AKNS *potentials* due to the fact that $M(p,q)$ is the Lax differential expression associated with the AKNS hierarchy (see, e.g., [1] and [16, Ch. 3]). The particularly important special case $p = -\overline{q}$ will be referred to as the NLS$_-$ case (due to the obvious connection of (2.arabic@equation) with the zero curvature representation and the Lax operator for the focusing nonlinear Schrödinger equation, see, e.g., [12, Part 1, Sect. I.2], [46], and [47]), and then q is called an NLS$_-$ *potential*. For given $z \in \mathbb{C}$, $\Omega \subseteq \mathbb{R}$, and AKNS potentials (p,q), a function $\Psi(z,\cdot) \in \mathrm{AC}_{\mathrm{loc}(\Omega)^2}$ is called a *z-wave function associated with (p,q) on Ω* if $\Psi'(z,x) = U(z,p,q)\Psi(z,x)$ holds for a.e. $x \in \Omega$, that is, if $\Psi = (\psi_1,\psi_2)^\top$ satisfies the following first-order system of differential equations

$$\psi_1'(z,x) = -iz\psi_1(z,x) + q(x)\psi_2(z,x), \quad \psi_2'(z,x) = iz\psi_2(z,x) + p(x)\psi_1(z,x) \qquad (2.\text{arabic@equation})$$

for a.e. $x \in \Omega$. Equivalently, $\Psi(z) = (\psi_1(z),\psi_2(z))^\top \in \mathrm{AC}_{\mathrm{loc}(\Omega)^2}$ is a z-wave function associated with (p,q) on Ω if and only if $M(p,q)\Psi(z) = z\Psi(z)$ on Ω in the distributional sense. If for some $z \in \mathbb{C}$, $\Psi(z)$ and $\Phi(z)$ are z-wave functions associated with (p,q) on Ω, their Wronskian is well-known to be constant with respect to $x \in \Omega$,

$$\frac{d}{dx}W(\Psi(z,x),\Phi(z,x)) = 0, \quad x \in \Omega. \qquad (2.\text{arabic@equation})$$

More generally, if $\Psi(z_1)$ and $\Phi(z_2)$ are z_1- and z_2-wave functions associated with (p,q) on Ω, then

$$\begin{aligned} \frac{d}{dx}W(\Psi(z_1,x),\Phi(z_2,x)) &= i(z_2 - z_1)[\psi_1(z_1,x)\phi_2(z_2,x) + \psi_2(z_1,x)\phi_1(z_2,x)] \\ &= i(z_2 - z_1)\Psi(z_1,x)^\top\sigma_1\Phi(z_2,x), \qquad (2.\text{arabic@equation}) \end{aligned}$$

$$\Psi(z_1) = (\psi_1(z_1),\psi_2(z_1))^\top,\ \Phi(z_2) = (\phi_1(z_2),\phi_2(z_2))^\top,\ z_1,z_2 \in \mathbb{C}, \quad x \in \Omega.$$

In the NLS$_-$ case $p = -\overline{q}$ we use the notation

$$U(z,q) = \begin{pmatrix} -iz & q(x) \\ -\overline{q(x)} & iz \end{pmatrix}, \quad M(q) = i\begin{pmatrix} \frac{d}{dx} & -q \\ -\overline{q} & -\frac{d}{dx} \end{pmatrix} \qquad (2.\text{arabic@equation})$$

[1] For brevity we write $L^p(\Omega)$ for $L^p(\Omega; dx)$ and suppress the Lebesgue measure dx whenever possible.

instead of (2.arabic@equation), and we then call any distributional solution $\Psi(z)$ of $M(q)\Psi(z) = z\Psi(z)$ an NLS$_-$ z-wave function associated with q.

If $\Psi(z)$ is a z-wave function associated with (p,q), and $\Psi(z,x_0) = 0$ for some $x_0 \in \Omega$, then $\Psi(z,x)$ vanishes identically for all x in an open neighborhood of x_0 in Ω by the unique solvability of the Cauchy problem for (2.arabic@equation). Therefore, we will always assume that $\Psi(z,x) \neq 0$ for all $x \in \Omega$.

If $q = 0$ (and analogously if $p = 0$) a.e. on Ω, then the system (2.arabic@equation) decomposes and yields

$$\begin{aligned} \psi_1(z,x) &= C_1 \exp(-izx), \\ \psi_2(z,x) &= C_1 \bigg(\int_{x_0}^{x} dx'\, p(x') \exp(-2izx') \bigg) \exp(izx) + C_2 \exp(izx) \end{aligned} \tag{2.arabic@equation}$$

for some constants $C_1, C_2 \in \mathbb{C}$, $x_0 \in \Omega$.

For further use we collect some simple consequences of (2.arabic@equation). First we introduce the antilinear (i.e., conjugate linear) operator $\mathcal{J}$ defined by

$$\mathcal{J} = \sigma_1 \mathcal{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{C}, \quad \mathcal{J}^2 = I_2, \tag{2.arabic@equation}$$

with $\mathcal{C}$ the antilinear conjugation map

$$\mathcal{C}(a,b)^\top = (\overline{a}, \overline{b})^\top, \quad a, b \in \mathbb{C}. \tag{2.arabic@equation}$$

Moreover, we introduce the antilinear involution $\mathcal{K}$ defined by

$$\mathcal{K} = \sigma_4 \mathcal{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{C}, \quad \mathcal{K}^2 = -I_2. \tag{2.arabic@equation}$$

We also note that

$$\|\mathcal{J}F\|_{\mathbb{C}^2} = \|F\|_{\mathbb{C}^2}, \quad \|\mathcal{K}F\|_{\mathbb{C}^2} = \|F\|_{\mathbb{C}^2}, \quad F \in \mathbb{C}^2. \tag{2.arabic@equation}$$

Lemma 2.2. *Assume Hypothesis* 2.1, $z \in \mathbb{C}$, *and suppose* $\Psi(z) = (\psi_1(z), \psi_2(z))^\top$ *is a z-wave function associated with* (p,q) *on* Ω. *Then the following assertions hold.*
(*i*) $\phi(z,x) = -\psi_2(z,x)/\psi_1(z,x)$ *satisfies the Riccati-type equation*

$$-\phi'(z,x) + q(x)\phi(z,x)^2 + 2iz\phi(z,x) - p(x) = 0 \tag{2.arabic@equation}$$

on the set $\{x \in \Omega \,|\, \psi_1(z,x) \neq 0\}$.
(*ii*) $\varphi(z,x) = \psi_1(z,x)/\psi_2(z,x)$ *satisfies the Riccati-type equation*

$$-\varphi'(z,x) - p(x)\varphi(z,x)^2 - 2iz\varphi(z,x) + q(x) = 0 \tag{2.arabic@equation}$$

on the set $\{x \in \Omega \,|\, \psi_2(z,x) \neq 0\}$.
Assume in addition the NLS$_-$ *case* $p = -\overline{q}$. *Then the following assertions hold.*
(*iii*) *If* $\Psi(z) = (\psi_1(z), \psi_2(z))^\top$ *is a z-wave function associated with q then*

$$\mathcal{K}\Psi(z) = \sigma_4 \mathcal{C}\Psi(z) = \big(\overline{\psi_2(z)}, -\overline{\psi_1(z)}\big)^\top = (\Psi(z)^\perp)^* \tag{2.arabic@equation}$$

is a $\overline{z}$-wave function associated with q and

$$M(q) = -\mathcal{K} M(q) \mathcal{K}. \tag{2.arabic@equation}$$

(*iv*) *The following identity holds*

$$\big(\|\Psi(z,x)\|_{\mathbb{C}^2}^2\big)' = 2\mathrm{Im}(z)\big[|\psi_1(z,x)|^2 - |\psi_2(z,x)|^2\big]. \tag{2.arabic@equation}$$

(v) $M(q)$ and $M(q)^*$ are formally unitarily equivalent in the sense that

$$M(q)^* = \sigma_3 M(q)\sigma_3 = M(-q). \qquad (2.\text{arabic@equation})$$

In addition, $M(q)$ is formally $\mathcal{J}$-self-adjoint in the following sense

$$\mathcal{J}M(q)\mathcal{J} = M(q)^*. \qquad (2.\text{arabic@equation})$$

Consider AKNS potentials $p, q \in L^1_{\text{loc}(\Omega)}$. Fix $z, z_1, \tilde z_1 \in \mathbb{C}$, a z_1-wave function $\Psi(z_1)$, and a $\tilde z_1$-wave function $\tilde\Psi(\tilde z_1)$ associated with (p,q). Our objective is to construct new potentials $p^{(1)}, q^{(1)} \in L^1_{\text{loc}(\Omega^{(1)})}$ for some open set $\Omega^{(1)} \subseteq \Omega$, and the corresponding z-wave functions associated with $(p^{(1)}, q^{(1)})$ on $\Omega^{(1)}$. In the NLS$_-$ case $p = -\overline{q}$ we choose $\tilde z_1 = \overline{z_1}$ and $\tilde\Psi(\overline{z_1}) = \mathcal{K}\Psi(z_1)$, see Lemma 2.2 (iii).

Remark 2.3. *Let $\Gamma \in \mathrm{AC}_{\text{loc}(\Omega)^{2\times 2}}$, $A, B \in L^1(\Omega)^{2\times 2}$ for some open subset $\Omega \subseteq \mathbb{R}$, suppose that the following identity holds*

$$\Gamma'(x) + \Gamma(x)A(x) - B(x)\Gamma(x) = 0 \ \text{ for a.e. } x \in \Omega, \qquad (2.\text{arabic@equation})$$

and assume that $\Phi \in \mathrm{AC}_{\text{loc}(\Omega)^2}$ satisfies the first-order system $\Phi' = A\Phi$ on Ω. Then the function $\Phi^{(1)}$, defined by $\Phi^{(1)} = \Gamma\Phi$, satisfies $\Phi^{(1)} \in \mathrm{AC}_{\text{loc}(\Omega)^2}$ and the first-order system $(\Phi^{(1)})' = B\Phi^{(1)}$ on Ω.

Lemma 2.4. *Assume Hypothesis 2.1 and $z, z_1 \in \mathbb{C}$. In addition, suppose $\Psi(z_1) = (\psi_1(z_1), \psi_2(z_1))^\top$ is a z_1-wave function associated with (p,q) on Ω and introduce*

$$\hat\Omega_{z_1} = \{x \in \Omega \,|\, \psi_1(z_1, x) \neq 0\}. \qquad (2.\text{arabic@equation})$$

Define $\hat p_{z_1}$, $\hat q_{z_1}$, $\hat\Gamma_0(q, \hat p_{z_1})$, and $\hat\Gamma(z, q, \hat p_{z_1})$ on $\hat\Omega_{z_1}$ by

$$\hat p_{z_1}(x) = -\psi_2(z_1,x)/\psi_1(z_1,x), \quad \hat q_{z_1}(x) = q'(x) + \hat p_{z_1}(x)q(x)^2 + 2iz_1 q(x), \qquad (2.\text{arabic@equation})$$

$$\hat\Gamma_0(x, q, \hat p_{z_1}) = -\frac{1}{2}\begin{pmatrix} q(x)\hat p_{z_1}(x) & q(x) \\ \hat p_{z_1}(x) & 1 \end{pmatrix}, \qquad (2.\text{arabic@equation})$$

$$\hat\Gamma(z, x, q, \hat p_{z_1}) = i(z - z_1)\sigma_5 + \hat\Gamma_0(x, q, \hat p_{z_1}). \qquad (2.\text{arabic@equation})$$

Then $\Gamma = \hat\Gamma$ satisfies (2.arabic@equation) *on $\hat\Omega_{z_1}$ with A and B given by*

$$A(z,x) = U(z,p,q), \quad B(z,x,z_1) = U(z, \hat p_{z_1}, \hat q_{z_1}), \quad x \in \hat\Omega_{z_1}. \qquad (2.\text{arabic@equation})$$

Proof. Using Lemma 2.2 (i) one verifies that $\Gamma = \hat\Gamma_0$ satisfies (2.arabic@equation) with $A = U(z_1, p, q)$ and $B = U(z_1, \hat p_{z_1}, \hat q_{z_1})$. Since $U(z) = U(z_1) - i(z - z_1)\sigma_3$, the conclusion follows. ☐

Similarly, by Lemma 2.2 (ii), one has the following analogous result.

Lemma 2.5. *Assume Hypothesis 2.1 and let $z, \tilde z_1 \in \mathbb{C}$. In addition, suppose that $\tilde\Psi(\tilde z_1) = (\tilde\psi_1(\tilde z_1), \tilde\psi_2(\tilde z_1))^\top$ is a $\tilde z_1$-wave function associated with (p,q) on Ω and introduce*

$$\check\Omega_{\tilde z_1} = \{x \in \Omega \,|\, \tilde\psi_2(\tilde z_1, x) \neq 0\}. \qquad (2.\text{arabic@equation})$$

Define $\check p_{\tilde z_1}$, $\check q_{\tilde z_1}$, $\check\Gamma_0(p, \check q_{\tilde z_1})$, *and* $\check\Gamma(z, p, \check q_{\tilde z_1})$ *on* $\check\Omega_{\tilde z_1}$ *by*

$$\check p_{\tilde z_1}(x) = -p'(x) + \check q_{\tilde z_1}(x)p(x)^2 + 2i\tilde z_1 p(x), \quad \check q_{\tilde z_1}(x) = \tilde\psi_1(\tilde z_1, x)/\tilde\psi_2(\tilde z_1, x), \tag{2.arabic@equation}$$

$$\check\Gamma_0(x, p, \check q_{\tilde z_1}) = \frac{1}{2}\begin{pmatrix} -1 & \check q_{\tilde z_1}(x) \\ p(x) & -p(x)\check q_{\tilde z_1}(x)\end{pmatrix}, \tag{2.arabic@equation}$$

$$\check\Gamma(z, x, p, \check q_{\tilde z_1}) = i(z - \tilde z_1)\sigma_6 + \check\Gamma_0(x, p, \check q_{\tilde z_1}). \tag{2.arabic@equation}$$

Then $\Gamma = \check\Gamma$ *satisfies* (2.arabic@equation) *on* $\check\Omega_{\tilde z_1}$ *with* A *and* B *given by*

$$A(z, x) = U(z, p, q), \quad B(z, x, \tilde z_1) = U(z, \check p_{\tilde z_1}, \check q_{\tilde z_1}), \quad x \in \check\Omega_{\tilde z_1}. \tag{2.arabic@equation}$$

The Darboux-type transformations characterized by (2.arabic@equation) and (2.arabic@equation) are also called elementary Darboux transformations. They have been discussed, for instance, in [27] and [28]. In the special context of algebro-geometric AKNS solutions, the effect of elementary Darboux transformations on the underlying compact hyperelliptic curve (in connection with the insertion and deletion of eigenvalues as well as the isospectral case) was studied in detail in [15], [16, App. G] (see also [19], [20]).

Next, we will construct the transformation matrix $\Gamma(z, \Psi(z_1), \tilde\Psi(\tilde z_1))$ that satisfies equation (2.arabic@equation) with $A(z, x) = U(z, p, q)$ and $B(z, x) = U\big(z, p^{(1)}_{z_1,\tilde z_1}, q^{(1)}_{z_1,\tilde z_1}\big)$ as the product of $\hat\Gamma(z, \hat p_{z_1}, \check q_{\tilde z_1})$ and $\check\Gamma(z, q, \hat p_{z_1})$. Since we will choose $\tilde z_1 = \overline{z_1}$ in the NLS$_-$ context, we omit the $\tilde z_1$-dependence in $p^{(1)}$, $q^{(1)}$, $\Omega^{(1)}$, $\Phi^{(1)}$, etc., in the NLS$_-$ case in the following.

Theorem 2.6. *Assume Hypothesis* 2.1.
(i) Suppose $z, z_1, \tilde z_1 \in \mathbb{C}$, *and assume that* $\Psi(z_1) = (\psi_1(z_1), \psi_2(z_1))^\top$ *and* $\tilde\Psi(\tilde z_1) = (\tilde\psi_1(\tilde z_1), \tilde\psi_2(\tilde z_1))^\top$ *are* z_1- *and* $\tilde z_1$-*wave functions, respectively, associated with* (p, q) *on* Ω. *In addition, introduce*

$$\Omega^{(1)}_{z_1,\tilde z_1} = \{x \in \Omega \,|\, W(\Psi(z_1, x), \tilde\Psi(\tilde z_1, x)) \neq 0\}. \tag{2.arabic@equation}$$

Define $p^{(1)}_{z_1,\tilde z_1}$, $q^{(1)}_{z_1,\tilde z_1}$, *and* $\Gamma(z, \Psi(z_1), \tilde\Psi(\tilde z_1))$ *on* $\Omega^{(1)}_{z_1,\tilde z_1}$ *by*

$$p^{(1)}_{z_1,\tilde z_1}(x) = p(x) - 2i(\tilde z_1 - z_1)\psi_2(z_1, x)\tilde\psi_2(\tilde z_1, x)/W(\Psi(z_1, x), \tilde\Psi(\tilde z_1, x)), \tag{2.arabic@equation}$$

$$q^{(1)}_{z_1,\tilde z_1}(x) = q(x) - 2i(\tilde z_1 - z_1)\psi_1(z_1, x)\tilde\psi_1(\tilde z_1, x)/W(\Psi(z_1, x), \tilde\Psi(\tilde z_1, x)), \tag{2.arabic@equation}$$

$$\begin{aligned}\Gamma(z, x, \Psi(z_1), \tilde\Psi(\tilde z_1)) = {}& -\frac{i}{2}zI_2 - \frac{i}{2}W(\Psi(z_1, x), \tilde\Psi(\tilde z_1, x))^{-1} \\ & \times \big(\tilde z_1\tilde\Psi(\tilde z_1, x)\Psi(z_1, x)^\perp - z_1\Psi(z_1, x)\tilde\Psi(\tilde z_1, x)^\perp\big).\end{aligned} \tag{2.arabic@equation}$$

Then Γ *satisfies the first-order system*

$$\begin{aligned}&\Gamma'(z, x, \Psi(z_1), \tilde\Psi(\tilde z_1)) + \Gamma(z, x, \Psi(z_1), \tilde\Psi(\tilde z_1))U(z, p, q) \\ &\quad - U\big(z, p^{(1)}_{z_1,\tilde z_1}, q^{(1)}_{z_1,\tilde z_1}\big)\Gamma(z, x, \Psi(z_1), \tilde\Psi(\tilde z_1)) = 0\end{aligned} \tag{2.arabic@equation}$$

a.e. on $\Omega^{(1)}_{z_1,\tilde z_1}$. *Thus, if* $\Upsilon(z)$ *is a* z*-wave function associated with* (p,q) *on* Ω*, then* $\Upsilon^{(1)}_{z_1,\tilde z_1}(z)$*, defined by*

$$\Upsilon^{(1)}_{z_1,\tilde z_1}(z,x) = \Gamma(z,x,\Psi(z_1),\tilde\Psi(\tilde z_1))\Upsilon(z,x), \qquad \text{(2.arabic@equation)}$$

is a z*-wave function associated with* $(p^{(1)}_{z_1,\tilde z_1}, q^{(1)}_{z_1,\tilde z_1})$ *on* $\Omega^{(1)}_{z_1,\tilde z_1}$.

(*ii*) *Assume the* NLS$_-$ *case* $p=-\overline{q}$ *and* $z,z_1\in\mathbb{C}$. *Then, for* $\tilde z_1=\overline{z_1}$, $\tilde\Psi(\tilde z_1) = \mathcal{K}\Psi(z_1) = \big(\overline{\psi_2(z_1)}, -\overline{\psi_1(z_1)}\big)^\top$, *and*

$$\Omega^{(1)}_{z_1} = \{x\in\Omega \,|\, W(\Psi(z_1,x),\mathcal{K}\Psi(z_1,x))\neq 0\}, \qquad \text{(2.arabic@equation)}$$

formulas (2.arabic@equation)–(2.arabic@equation) *simplify to*

$$q^{(1)}_{z_1}(x) = q(x) + 4\mathrm{Im}(z_1)\psi_1(z_1,x)\overline{\psi_2(z_1,x)}\|\Psi(z_1,x)\|^{-2}_{\mathbb{C}^2}, \qquad \text{(2.arabic@equation)}$$

$$\begin{aligned} p^{(1)}_{z_1}(x) &= p(x) - 4\mathrm{Im}(z_1)\psi_2(z_1,x)\overline{\psi_1(z_1,x)}\|\Psi(z_1,x)\|^{-2}_{\mathbb{C}^2} \\ &= -\overline{q^{(1)}_{z_1}(x)}, \end{aligned} \qquad \text{(2.arabic@equation)}$$

$$\begin{aligned} \Gamma(z,x,\Psi(z_1),\mathcal{K}\Psi(z_1)) &= -(i/2)(z-z_1)I_2 \\ &\quad + \mathrm{Im}(z_1)\|\Psi(z_1,x)\|^{-2}_{\mathbb{C}^2}\mathcal{K}\Psi(z_1,x)\Psi(z_1,x)^\perp \end{aligned} \qquad \text{(2.arabic@equation)}$$

for $x\in\Omega^{(1)}_{z_1}$. *In particular, if* $\Upsilon(z)$ *is a* z*-wave function associated with* q *on* Ω*, then* $\Upsilon^{(1)}_{z_1}(z)$*, defined by*

$$\Upsilon^{(1)}_{z_1}(z,x) = \Gamma(z,x,\Psi(z_1),\mathcal{K}\Psi(z_1))\Upsilon(z,x), \qquad \text{(2.arabic@equation)}$$

is a z*-wave function associated with* $q^{(1)}_{z_1}$ *on* $\Omega^{(1)}_{z_1}$ (*which may vanish identically w.r.t.* $x\in\Omega^{(1)}_{z_1}$, *cf. Remark* 2.8).

Proof. First, we use Lemma 2.4 for $z=\tilde z_1$ and $\Psi(z_1)$ to construct $\hat p_{z_1}$, $\hat q_{z_1}$, $\hat\Gamma(\tilde z_1, q, \hat p_{z_1})$ as in (2.arabic@equation)–(2.arabic@equation). Define $\hat\Psi_{z_1}(\tilde z_1) = \hat\Gamma(\tilde z_1, q, \hat p_{z_1})\Psi(\tilde z_1)$. By Lemma 2.4 and Remark 2.3 we conclude that $\hat\Psi_{z_1}(\tilde z_1)$ is a $\tilde z_1$-wave function associated with $(\hat p_{z_1}, \hat q_{z_1})$. Moreover,

$$\begin{aligned} \hat\Psi_{z_1}(\tilde z_1,x) &= i(\tilde z_1 - z_1)\psi_1(\tilde z_1,x)\begin{pmatrix}1\\0\end{pmatrix} \\ &\quad - \frac{1}{2}\begin{pmatrix} q(x) & 0 \\ 0 & 1\end{pmatrix}\big(\hat p_{z_1}(x)\psi_1(\tilde z_1,x) + \psi_2(\tilde z_1,x)\big)\begin{pmatrix}1\\1\end{pmatrix}, \end{aligned} \qquad \text{(2.arabic@equation)}$$

where $\hat p_{z_1}$ is defined in (2.arabic@equation). We now apply Lemma 2.5 replacing $\tilde\Psi(\tilde z_1)$ by $\hat\Psi_{z_1}(\tilde z_1)$ and (p,q) by $(\hat p_{z_1}, \hat q_{z_1})$. Then $\check q_{z_1,\tilde z_1} = \hat\psi_{1,z_1}(\tilde z_1)/\hat\psi_{2,z_1}(\tilde z_1)$, as required by (2.arabic@equation), coincides with $q^{(1)}_{z_1,\tilde z_1}$, as defined in (2.arabic@equation),

$$\check q_{z_1,\tilde z_1}(x) = q^{(1)}_{z_1,\tilde z_1}(x). \qquad \text{(2.arabic@equation)}$$

By formula (2.arabic@equation) for $\check p_{\tilde z_1,z_1}$ and Lemma 2.2 (i) for $\hat p_{z_1} = -\psi_2(z_1)/\psi_1(z_1)$ one infers

$$\begin{aligned} \check p_{\tilde z_1,z_1}(x) &= -\hat p'_{z_1}(x) + q^{(1)}_{z_1,\tilde z_1}(x)\hat p_{z_1}(x)^2 + 2i\tilde z_1\hat p_{z_1}(x) \\ &= -\big(q(x)\hat p_{z_1}(x)^2 + 2iz_1\hat p_{z_1}(x) - p(x)\big) + q^{(1)}_{z_1,\tilde z_1}(x)\hat p_{z_1}(x)^2 + 2i\tilde z_1\hat p_{z_1}(x) \\ &= p^{(1)}_{z_1,\tilde z_1}(x). \end{aligned} \qquad \text{(2.arabic@equation)}$$

Using (2.arabic@equation) and (2.arabic@equation)–(2.arabic@equation) one computes

$$
\begin{aligned}
&\check\Gamma(z,\hat p_{z_1},\check q_{z_1,\tilde z_1})\hat\Gamma(z,q,\hat p_{z_1})\\
&= -\left((z-\tilde z_1)\sigma_6 + \frac{i}{2}\begin{pmatrix}1 & 0\\ 0 & -\hat p_{z_1}\end{pmatrix}\begin{pmatrix}1 & -\check q_{z_1,\tilde z_1}\\ 1 & -\check q_{z_1,\tilde z_1}\end{pmatrix}\right)\\
&\quad\times\left((z-z_1)\sigma_5 + \frac{i}{2}\begin{pmatrix}q & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\hat p_{z_1} & 1\\ \hat p_{z_1} & 1\end{pmatrix}\right)\\
&= -(i/2)zI_2 - (i/2)W(\Psi(z_1),\tilde\Psi(\tilde z_1))^{-1}\\
&\quad\times\begin{pmatrix}\tilde z_1\psi_2(z_1)\tilde\psi_1(\tilde z_1) - z_1\psi_1(z_1)\tilde\psi_2(\tilde z_1) & -(\tilde z_1 - z_1)\psi_1(z_1)\tilde\psi_1(\tilde z_1)\\ (\tilde z_1 - z_1)\psi_2(z_1)\tilde\psi_2(\tilde z_1) & -\tilde z_1\psi_1(z_1)\tilde\psi_2(\tilde z_1) + z_1\psi_2(z_1)\tilde\psi_1(\tilde z_1)\end{pmatrix}\\
&= \Gamma(z,\Psi(z_1),\tilde\Psi(\tilde z_1)). \qquad (2.arabic@equation)
\end{aligned}
$$

To check (2.arabic@equation) one uses Lemmas 2.4 and 2.5,

$$
\begin{aligned}
&\Gamma'(z,\Psi(z_1),\tilde\Psi(\tilde z_1))\\
&= \check\Gamma'(z,\hat p_{z_1},\check q_{z_1,\tilde z_1})\hat\Gamma(z,q,\hat p_{z_1}) + \check\Gamma(z,\hat p_{z_1},\check q_{z_1,\tilde z_1})\hat\Gamma'(z,q,\hat p_{z_1})\\
&= [-\check\Gamma(z,\hat p_{z_1},\check q_{z_1,\tilde z_1})U(z,\hat p_{z_1},\hat q_{z_1})\\
&\qquad + U(z,\check p_{z_1,\tilde z_1},\check q_{z_1,\tilde z_1})\check\Gamma(z,\hat p_{z_1},\check q_{z_1,\tilde z_1})]\hat\Gamma(z,q,\hat p_{z_1})\\
&\quad + \check\Gamma(z,\hat p_{z_1},\check q_{z_1,\tilde z_1})[-\hat\Gamma(z,q,\hat p_{z_1})U(z,p,q) + U(z,\hat p_{z_1},\hat q_{z_1})\hat\Gamma(z,q,\hat p_{z_1})]\\
&= U(z,\check p_{z_1,\tilde z_1},\check q_{z_1,\tilde z_1})\Gamma(z,\Psi(z_1),\tilde\Psi(\tilde z_1)) - \Gamma(z,\Psi(z_1),\tilde\Psi(\tilde z_1))U(z,p,q). \qquad (2.arabic@equation)
\end{aligned}
$$

Formulas (2.arabic@equation)–(2.arabic@equation) follow from (2.arabic@equation)–(2.arabic@equation) since by Lemma 2.2 (iii),

$$W(\Psi(z_1),\tilde\Psi(\overline{z_1})) = W(\Psi(z_1),\mathcal{K}\Psi(z_1)) = -\|\Psi(z_1)\|^2_{\mathbb{C}^2}. \qquad (2.arabic@equation)$$

$\square$

For a general treatment of Bäcklund (Darboux) and gauge transformations and their interrelations we refer, for instance, to [34, Sect. 4.1] and [39].

Finally we add a few more facts valid in the NLS_- case.

Remark 2.7. *Assume the* NLS_- *case* $p=-\overline{q}$. *If* $\Upsilon(z)=(\upsilon_1(z),\upsilon_2(z))^\top$ *is a* z*-wave function associated with* q*, then* $\mathcal{K}\Upsilon(z,x)$ *is a* $\overline{z}$*-wave function associated with* q *and*

$$
\begin{aligned}
\Upsilon^{(1)}_{z_1}(\overline{z},x) &= \Gamma(\overline{z},x,\Psi(z_1),\mathcal{K}\Psi(z_1))\Upsilon(\overline{z},x)\\
&= \Gamma(\overline{z},x,\Psi(z_1),\mathcal{K}\Psi(z_1))\mathcal{K}\Upsilon(z,x)\\
&= \left(\overline{\upsilon^{(1)}_{2,z_1}(z,x)}, -\overline{\upsilon^{(1)}_{1,z_1}(z,x)}\right)^\top\\
&= \mathcal{K}\Upsilon^{(1)}_{z_1}(z,x) \qquad (2.arabic@equation)
\end{aligned}
$$

is a $\overline{z}$*-wave function associated with* $q^{(1)}_{z_1}$ *(cf. Lemma 2.2 (iii)).*

Remark 2.8. *Assume the* NLS_- *case* $p=-\overline{q}$.
(i) Take $z=z_1$ *and* $\Upsilon(z_1)=\Psi(z_1)$ *in Theorem 2.6. Then*

$$\Psi^{(1)}_{z_1}(z_1,x) = \Gamma(z_1,x,\Psi(z_1),\mathcal{K}\Psi(z_1))\Psi(z_1,x) = 0 \qquad (2.arabic@equation)$$

since $\Psi(z_1)^\perp \Psi(z_1) = 0$.
(ii) Take $z = \overline{z_1}$ *and* $\mathcal{K}\Upsilon(z_1) = \mathcal{K}\Psi(z_1)$ *in Theorem* 2.6. *Then*

$$\mathcal{K}\Psi^{(1)}_{z_1}(z_1, x) = \Gamma(\overline{z_1}, x, \Psi(z_1), \mathcal{K}\Psi(z_1))\mathcal{K}\Psi(z_1, x) = 0 \qquad (2.\text{arabic@equation})$$

since $\Psi(z_1)^\perp \mathcal{K}\Psi(z_1) = \|\Psi(z_1)\|^2_{\mathbb{C}^2}$ *by Lemma* 2.2 *(iii)*.

In the NLS$_-$ case, Theorem 2.6 also shows that $q^{(1)}_{z_1}$ is locally nonsingular whenever q is. More precisely, one has the following result.

Corollary 2.9. *Assume the* NLS$_-$ *case* $p = -\overline{q}$ *and suppose* $z_1 \in \mathbb{C}$. *Then, if* $q \in L^p_{\text{loc}(\mathbb{R})}$ *for some* $p \in [1, \infty) \cup \{\infty\}$ *(respectively, if* $q \in C^k(\mathbb{R})$ *for some* $k \in \mathbb{N}_0$*), the* NLS$_-$ *potential* $q^{(1)}_{z_1}$ *given by* (2.arabic@equation) *also satisfies* $q^{(1)}_{z_1} \in L^p_{\text{loc}(\mathbb{R})}$ *(respectively,* $q^{(1)}_{z_1} \in C^k(\mathbb{R})$*).*

Proof. Since $\big|\psi_1(z_1, x)\overline{\psi_2(z_1, x)}\big| \|\Psi(z_1, x)\|^{-2}_{\mathbb{C}^2} \leq 1/2$, one concludes from (2.arabic@equation) that $(q^{(1)}_{z_1} - q) \in L^\infty(\mathbb{R})$. Again by (2.arabic@equation), q and $q^{(1)}_{z_1}$ share the same L^p_{loc} and C^k properties since by (2.arabic@equation) (with $p = -\overline{q}$) one has $\partial^m_x \psi_j(z_1, \cdot) \in \text{AC}_{\text{loc}}(\mathbb{R})$, $j = 1, 2$, whenever $\partial^m_x q \in L^1_{\text{loc}(\mathbb{R})}$. □

3. $\mathcal{J}$-Self-Adjointness of NLS$_-$ Dirac-Type Operators

It is a known fact that the Dirac-type Lax differential expression in the defocusing NLS$_+$ case is always in the limit point case at $\pm\infty$. Put differently, the maximally defined Dirac-type operator corresponding to the defocusing NLS$_+$ case (cf. (1.arabic@equation)) is always self-adjoint. Classical references in this context are [32, Sect. 8.6], [45], which use some additional conditions (such as real-valuedness and/or regularity) of the coefficient q. A simple proof of this fact under most general conditions on q was recently communicated to us by Hinton [23] (cf. also [5], [6] and [29] for matrix-valued extensions of this result). In this section we show that the analogous result holds for Dirac-type differential expressions $M(q)$ in (2.arabic@equation) in the focusing NLS$_-$ case, when self-adjointness is replaced by $\mathcal{J}$-self-adjointness.

First, we recall some basic facts about $\mathcal{J}$-symmetric and $\mathcal{J}$-self-adjoint operators in a complex Hilbert space $\mathcal{H}$ (see, e.g., [11, Sect. III.5] and [21, p. 76]) with scalar product denoted by $(\cdot, \cdot)_\mathcal{H}$ (linear in the first and antilinear in the second place) and corresponding norm denoted by $\|\cdot\|_\mathcal{H}$. Let $\mathcal{J}$ be a conjugation operator in $\mathcal{H}$, that is, $\mathcal{J}$ is an antilinear involution satisfying

$$(\mathcal{J}u, v)_\mathcal{H} = (\mathcal{J}v, u)_\mathcal{H} \text{ for all } u, v \in \mathcal{H}, \quad \mathcal{J}^2 = I. \qquad (3.\text{arabic@equation})$$

In particular,

$$(\mathcal{J}u, \mathcal{J}v)_\mathcal{H} = (v, u)_\mathcal{H}, \quad u, v \in \mathcal{H}. \qquad (3.\text{arabic@equation})$$

A densely defined linear operator S in $\mathcal{H}$ is called $\mathcal{J}$-symmetric if

$$S \subseteq \mathcal{J}S^*\mathcal{J} \text{ (equivalently, if } \mathcal{J}S\mathcal{J} \subseteq S^*). \qquad (3.\text{arabic@equation})$$

Clearly, (3.arabic@equation) is equivalent to

$$(\mathcal{J}u, Sv)_\mathcal{H} = (\mathcal{J}Su, v)_\mathcal{H}, \quad u, v \in \text{dom}(S). \qquad (3.\text{arabic@equation})$$

Here S^* denotes the adjoint operator of S in $\mathcal{H}$. If S is $\mathcal{J}$-symmetric, so is its closure $\overline{S}$. The operator S is called $\mathcal{J}$-self-adjoint if

$$S = \mathcal{J}S^*\mathcal{J} \ \text{ (equivalently, if } \mathcal{J}S\mathcal{J} = S^*). \qquad (3.\text{arabic@equation})$$

Finally, a densely defined, closable operator T is called essentially $\mathcal{J}$-self-adjoint if its closure $\overline{T}$ is $\mathcal{J}$-self adjoint, that is, if

$$\overline{T} = \mathcal{J}T^*\mathcal{J}. \qquad (3.\text{arabic@equation})$$

Next, assuming S to be $\mathcal{J}$-symmetric, one introduces the following inner product $(\cdot,\cdot)_*$ on $\operatorname{dom}(\mathcal{J}S^*\mathcal{J}) = \operatorname{dom}(S^*\mathcal{J})$ according to [26] (see also [37]),

$$(u,v)_* = (\mathcal{J}u,\mathcal{J}v)_{\mathcal{H}} + (S^*\mathcal{J}u, S^*\mathcal{J}v)_{\mathcal{H}}, \quad u,v \in \operatorname{dom}(\mathcal{J}S^*\mathcal{J}), \qquad (3.\text{arabic@equation})$$

which renders $\operatorname{dom}(\mathcal{J}S^*\mathcal{J})$ a Hilbert space. Then the following theorem holds (I denotes the identity operator in $\mathcal{H}$).

Theorem 3.1 (Race [37])**.** *Let S be a densely defined closed $\mathcal{J}$-symmetric operator. Then*

$$\operatorname{dom}(\mathcal{J}S^*\mathcal{J}) = \operatorname{dom}(S) \oplus_* \ker((S^*\mathcal{J})^2 + I), \qquad (3.\text{arabic@equation})$$

where $\oplus_*$ means the orthogonal direct sum with respect to the inner product $(\cdot,\cdot)_*$. In particular, a densely defined closed $\mathcal{J}$-symmetric operator S is $\mathcal{J}$-self-adjoint if and only if

$$\ker((S^*\mathcal{J})^2 + I) = \{0\}. \qquad (3.\text{arabic@equation})$$

We will apply (3.arabic@equation) to (maximally defined) Dirac-type operators associated with the differential expression $M(q)$ in (2.arabic@equation) relevant to the focusing NLS_- hierarchy and prove the fundamental fact that such Dirac operators are always $\mathcal{J}$-self-adjoint under most general conditions on the coefficient q (cf. Theorem 3.5).

It will be convenient to make the following NLS_- assumption throughout the remainder of this section.

Hypothesis 3.2. *Suppose $q \in L^1_{\text{loc}(\mathbb{R})}$ and assume the* NLS_- *case $p = -\overline{q}$.*

Given Hypothesis 3.2, we now introduce the following maximal and minimal Dirac-type operators in $L^2(\mathbb{R})^2$ associated with the differential expression $M(q)$,

$$D_{\max}(q)F = M(q)F, \qquad (3.\text{arabic@equation})$$
$$F \in \operatorname{dom}(D_{\max}(q)) = \big\{G \in L^2(\mathbb{R})^2 \,\big|\, G \in AC_{\text{loc}(\mathbb{R})^2;\, M(q)G \in L^2(\mathbb{R})^2}\big\},$$
$$D_{\min}(q)F = M(q)F, \qquad (3.\text{arabic@equation})$$
$$F \in \operatorname{dom}(D_{\min}(q)) = \{G \in \operatorname{dom}(D_{\max}(q)) \,|\, \operatorname{supp}(G) \text{ is compact}\}.$$

It follows by standard techniques (see, e.g., [32, Ch. 8] and [45]) that under Hypothesis 3.2, $D_{\min}(q)$ is densely defined and closable in $L^2(\mathbb{R})^2$ and $D_{\max}(q)$ is a densely defined closed operator in $L^2(\mathbb{R})^2$. Moreover (cf. (2.arabic@equation)), one infers (see, e.g., [32, Lemma 8.6.2] and [45] in the analogous case of symmetric Dirac operators)

$$\overline{D_{\min}(q)} = D_{\max}(-q)^*, \ \text{ or equivalently, } \ D_{\min}(q)^* = D_{\max}(-q). \qquad (3.\text{arabic@equation})$$

The following result will be the crucial ingredient in the proof of Theorem 3.5, the principal result of this section.

Theorem 3.3. *Assume Hypothesis* 3.2. *Let $N(q)$ be the following* (*formally self-adjoint*) *differential expression*

$$N(q) = i \begin{pmatrix} \frac{d}{dx} & -q \\ \overline{q} & \frac{d}{dx} \end{pmatrix} \qquad (3.\text{arabic@equation})$$

and denote by $\widetilde{D}_{\max}(q)$ the maximally defined Dirac-type operator in $L^2(\mathbb{R})^2$ associated with $N(q)$,

$$\begin{aligned} &\widetilde{D}_{\max}(q)F = N(q)F, \qquad (3.\text{arabic@equation}) \\ &F \in \operatorname{dom}(\widetilde{D}_{\max}(q)) = \big\{G \in L^2(\mathbb{R})^2 \,\big|\, G \in AC_{\operatorname{loc}(\mathbb{R})^2;\; N(q)G \in L^2(\mathbb{R})^2}\big\}. \end{aligned}$$

Then one infers:
(i) The following identity holds

$$M(-q)M(q) = N(q)^2. \qquad (3.\text{arabic@equation})$$

(ii) Let $U_q = U_q(x)$ satisfy the initial value problem

$$U_q' = \begin{pmatrix} 0 & q \\ -\overline{q} & 0 \end{pmatrix} U_q, \quad U_q(0) = I_2. \qquad (3.\text{arabic@equation})$$

Then $\{U_q(x)\}_{x\in\mathbb{R}}$ is a family of unitary matrices in $\mathbb{C}^2$ with entries in $AC_{\operatorname{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying

$$U_q^{-1} N(q) U_q = i\frac{d}{dx} I_2. \qquad (3.\text{arabic@equation})$$

(iii) Let $\mathcal{U}_q$ denote the multiplication operator with $U_q(\cdot)$ on $L^2(\mathbb{R})^2$. Then $\widetilde{D}_{\max}(q)$ is unitarily equivalent to the maximally defined operator in $L^2(\mathbb{R})^2$ associated with the differential expression $i\frac{d}{dx}I_2$,

$$\begin{aligned} &\mathcal{U}_q^{-1}\widetilde{D}_{\max}(q)\mathcal{U}_q = \Big(i\frac{d}{dx}I_2\Big)_{\max}, \qquad (3.\text{arabic@equation}) \\ &\operatorname{dom}\Big(\Big(i\frac{d}{dx}I_2\Big)_{\max}\Big) = H^{1,2}(\mathbb{R})^2 = \big\{F \in L^2(\mathbb{R})^2 \,\big|\, F \in AC_{\operatorname{loc}}(\mathbb{R})^2;\; F' \in L^2(\mathbb{R})^2\big\}. \end{aligned}$$

Moreover,

$$\begin{aligned} &\mathcal{U}_q^{-1} D_{\max}(-q) D_{\max}(q)\mathcal{U}_q = \Big(-\frac{d^2}{dx^2}I_2\Big)_{\max}, \qquad (3.\text{arabic@equation}) \\ &\operatorname{dom}\Big(\Big(-\frac{d^2}{dx^2}I_2\Big)_{\max}\Big) = H^{2,2}(\mathbb{R})^2 \\ &\qquad = \big\{F \in L^2(\mathbb{R})^2 \,\big|\, F, F' \in AC_{\operatorname{loc}}(\mathbb{R})^2;\; F', F'' \in L^2(\mathbb{R})^2\big\}. \end{aligned}$$

Proof. That $N(q)$ is formally self-adjoint and $M(-q)M(q) = N(q)^2$ as stated in (i) is an elementary matrix calculation.
To prove (ii), we note that the initial value problem (3.arabic@equation) is well-posed in the sense of Carathéodory since $q \in L^1_{\operatorname{loc}}(\mathbb{R})$ (cf., e.g., [22, p. 45–46]) with a solution matrix U_q with entries in $AC_{\operatorname{loc}(\mathbb{R})}$. Moreover, for each $x \in \mathbb{R}$, $U_q(x)$ is

a unitary matrix in $\mathbb{C}^2$, since $U_q' = -B(q)U_q$, with $B(q) = \left(\begin{smallmatrix} 0 & -q \\ \overline{q} & 0 \end{smallmatrix}\right)$ skew-adjoint. Thus, the entries $U_{q,j,k}$, $1 \le j,k \le 2$ of U_q (as well as those of U_q^{-1}) actually satisfy

$$U_{q,j,k} \in AC_{\text{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad 1 \le j,k \le 2. \qquad (3.\text{arabic@equation})$$

Next, fix $F \in AC_{\text{loc}}(\mathbb{R})^2$, such that $\mathcal{U}_q^{-1}F \in H^{1,2}(\mathbb{R})^2$. Then

$$\begin{aligned} U_q\left(i\frac{d}{dx}I_2\right)U_q^{-1}F &= i\frac{d}{dx}F + iU_q\frac{d}{dx}(U_q^{-1})F \\ &= i\frac{d}{dx}F + iU_q(U_q^{-1}B(q)^*)F \\ &= N(q)F, \qquad (3.\text{arabic@equation}) \end{aligned}$$

where we used the fact that $(U_q^{-1})' = U_q^{-1}B(q)^*$. Thus, (ii) follows.
Moreover, by (3.arabic@equation) one concludes $\text{dom}(\widetilde{D}_{\max}(q)) = \mathcal{U}_q H^{1,2}(\mathbb{R})^2$ by (3.arabic@equation) and the fact that U_q is unitary in $\mathbb{C}^2$. This proves (3.arabic@equation).
Clearly (i) and (ii) yield the relation

$$U_q^{-1}M(-q)M(q)U_q = -\frac{d^2}{dx^2}I_2. \qquad (3.\text{arabic@equation})$$

Thus, (3.arabic@equation) will follow once we prove the following facts:

(i) $U_qF \in L^2(\mathbb{R})^2$ if and only if $F \in L^2(\mathbb{R})^2$, (3.arabic@equation)

(ii) $U_qF \in AC_{\text{loc}(\mathbb{R})^2}$ if and only if $F \in AC_{\text{loc}(\mathbb{R})^2}$, (3.arabic@equation)

(iii) $M(q)U_qF \in L^2(\mathbb{R})^2$ if and only if $F' \in L^2(\mathbb{R})^2$, (3.arabic@equation)

(iv) $M(q)U_qF \in AC_{\text{loc}(\mathbb{R})^2}$ if and only if $F' \in AC_{\text{loc}(\mathbb{R})^2}$, (3.arabic@equation)

(v) $M(-q)M(q)U_qF \in L^2(\mathbb{R})^2$ if and only if $F'' \in L^2(\mathbb{R})^2$. (3.arabic@equation)

Clearly (3.arabic@equation) and (3.arabic@equation) hold since U_q is unitary in $\mathbb{C}^2$. (3.arabic@equation) is valid since $U_{q,j,k}, U_{q,j,k}^{-1} \in AC_{\text{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $j,k = 1,2$. Next, an explicit computation yields

$$M(q)U_qF = i\begin{pmatrix} U_{q,1,1}f_1' + U_{q,1,2}f_2' \\ -U_{q,2,1}f_1' - U_{q,2,2}f_2' \end{pmatrix}, \quad F = (f_1, f_2)^\top. \qquad (3.\text{arabic@equation})$$

Introducing

$$V_q = \sigma_3 U_q \sigma_3 = \begin{pmatrix} U_{q,1,1} & -U_{q,1,2} \\ -U_{q,2,1} & U_{q,2,2} \end{pmatrix}, \qquad (3.\text{arabic@equation})$$

one infers $V_{q,j,k}, V_{q,j,k}^{-1} \in AC_{\text{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $j,k = 1,2$ and

$$V_q^{-1}M(q)U_qF = i(f_1', -f_2')^\top, \qquad (3.\text{arabic@equation})$$

and hence (3.arabic@equation) and (3.arabic@equation) hold. This proves (3.arabic@equation).

□

Remark 3.4. *We note that by* (3.arabic@equation), $\widetilde{D}_{\max}(q)$, *the maximally defined, self-adjoint operator in* $L^2(\mathbb{R})^2$ *associated with the* 2×2 *matrix-valued differential expression* $N(q)$ *has a purely absolutely continuous spectrum that equals* $\mathbb{R}$,

$$\sigma(\widetilde{D}_{\max}(q)) = \sigma_{\text{ac}}(\widetilde{D}_{\max}(q)) = \mathbb{R}. \qquad (3.\text{arabic@equation})$$

(*We refer to Section* 5 *for a discussion of various spectral decompositions. In the present context we just note that* $\sigma(T)$ *and* $\sigma_{\rm ac}(T)$ *denote the spectrum and absolutely continuous spectrum of a self-adjoint operator* T *in a separable complex Hilbert space* $\mathcal{H}$.)

The principal result of this section then reads as follows.

Theorem 3.5. *Assume Hypothesis* 3.2. *Then the minimally defined Dirac-type operator* $D_{\min}(q)$ *associated with the Lax differential expression*

$$M(q) = i\begin{pmatrix} \frac{d}{dx} & -q \\ -\overline{q} & -\frac{d}{dx} \end{pmatrix} \qquad \text{(3.arabic@equation)}$$

introduced in (3.arabic@equation) *is essentially* $\mathcal{J}$*-self-adjoint in* $L^2(\mathbb{R})^2$, *that is,*

$$\overline{D_{\min}(q)} = \mathcal{J} D_{\min}(q)^* \mathcal{J}, \qquad \text{(3.arabic@equation)}$$

where $\mathcal{J}$ *is the conjugation defined in* (2.arabic@equation). *Moreover,*

$$\overline{D_{\min}(q)} = D_{\max}(q) \qquad \text{(3.arabic@equation)}$$

and hence $D_{\max}(q)$ *is* $\mathcal{J}$*-self-adjoint.*

Proof. We first recall (cf. (3.arabic@equation))

$$D_{\min}(q)^* = D_{\max}(-q) \qquad \text{(3.arabic@equation)}$$

and also note

$$\mathcal{J} D_{\max}(-q)\mathcal{J} = D_{\max}(q). \qquad \text{(3.arabic@equation)}$$

Since $\overline{D_{\min}(q)}$ is closed and $\mathcal{J}$-symmetric (cf. (2.arabic@equation)), its $\mathcal{J}$-self-adjointness is equivalent to showing that (cf. (3.arabic@equation))

$$\ker\big(D_{\min}(q)^*\mathcal{J}D_{\min}(q)^*\mathcal{J} + I\big) = \ker(D_{\max}(-q)D_{\max}(q) + I) = \{0\}. \qquad \text{(3.arabic@equation)}$$

Since by (3.arabic@equation) $D_{\max}(-q)D_{\max}(q)$ is unitarily equivalent to $(-d^2/dx^2 I_2)_{\max} \geq 0$, one concludes $D_{\max}(-q)D_{\max}(q) \geq 0$ and hence (3.arabic@equation) obviously holds. The fact (3.arabic@equation) then follows from (3.arabic@equation) and (3.arabic@equation) since

$$\overline{D_{\min}(q)} = \mathcal{J} D_{\min}(q)^*\mathcal{J} = \mathcal{J} D_{\max}(-q)\mathcal{J} = D_{\max}(q). \qquad \text{(3.arabic@equation)}$$

□

As mentioned in the introductory paragraph to this section, Theorem 3.5 in the $\mathcal{J}$-self-adjoint context can be viewed as an analog of the result of the corresponding (self-adjoint) Dirac operator relevant in the defocusing NLS$_+$ case of the nonlinear Schrödinger equation.

4. Constructing $L^2(\mathbb{R})^2$-Wave Functions for $\mathcal{J}$-Self-Adjoint Dirac-Type Operators

In this section we discuss how to construct $L^2(\mathbb{R})^2$-wave functions for non-self-adjoint (but $\mathcal{J}$-self-adjoint) Dirac-type operators associated with the Lax differential expression for the NLS$_-$ system.

By Remark 2.8, in order to obtain a nonzero z_1-wave function associated with $q^{(1)}_{z_1}$, we have to apply the transformation matrix $\Gamma(z_1, \Psi(z_1), \mathcal{K}\Psi(z_1))$ to a z_1-wave function $\Phi(z_1)$ associated with q that is linearly independent with the original z_1-wave function $\Psi(z_1)$ associated with q. Similarly, in order to obtain a nonzero $\overline{z_1}$-wave function associated with $q^{(1)}_{z_1}$, we have to apply the transformation matrix $\Gamma(\overline{z_1}, \Psi(z_1), \mathcal{K}\Psi(z_1))$ to a $\overline{z_1}$-wave function $\mathcal{K}\Phi(z_1)$ associated with q that is linearly independent with the original $\overline{z_1}$-wave function $\mathcal{K}\Psi(z_1)$ associated with q. The function $\Phi(z_1)$ is constructed as follows.

Let $\Psi(z) = (\psi_1(z), \psi_2(z))^\top$, $z \in \mathbb{C}$, be a z-wave function associated with q on $\mathbb{R}$ and introduce

$$\Omega_z = \{x \in \mathbb{R} \,|\, \psi_1(z,x)\psi_2(z,x) \neq 0\}. \qquad (4.\text{arabic@equation})$$

Next, consider

$$\Psi^{\#}(z,x) = (1/2)(\psi_2(z,x)^{-1}, -\psi_1(z,x)^{-1})^\top, \quad x \in \Omega_z, \qquad (4.\text{arabic@equation})$$

such that

$$W(\Psi^{\#}(z,x), \Psi(z,x)) = 1. \qquad (4.\text{arabic@equation})$$

Let $x_0, x \in \Omega_z$ such that $[x_0, x] \subseteq \Omega_z$ and define

$$R(z,x,x_0) = -\frac{1}{2}\int_{x_0}^{x} dx' \bigg(\frac{\overline{q(x')}}{\psi_2(z,x')^2} + \frac{q(x')}{\psi_1(z,x')^2} \bigg), \qquad (4.\text{arabic@equation})$$

$$\Phi(z,x) = \Psi^{\#}(z,x) + R(z,x,x_0)\Psi(z,x), \quad [x_0,x] \subseteq \Omega_z. \qquad (4.\text{arabic@equation})$$

Using (2.arabic@equation), we have $(\Psi^{\#})' = U(p,q)\Psi^{\#} - R'\Psi$. Thus, $W(\Phi, \Psi) = 1$ and $\Phi(z)$ is a z-wave function associated with q, since

$$\Phi' = (\Psi^{\#})' + R'\Psi + R\Psi' = U\Psi^{\#} - R'\Psi + R'\Psi + RU\Psi = U\Phi. \qquad (4.\text{arabic@equation})$$

We note that $\Psi^\perp \Psi^{\#} = 1$ implies $\mathcal{K}\Psi(z)\Psi(z)^\perp\Phi(z) = \mathcal{K}\Psi(z)$.

Thus, if $\Gamma(z, \Psi(z_1), \mathcal{K}\Psi(z_1))$ is defined using $\Psi(z_1)$ as in (2.arabic@equation), then the z_1-wave function $\Phi^{(1)}_{z_1}(z_1)$ associated with $q^{(1)}_{z_1}$, as prescribed in Theorem 2.6, is computed as follows

$$\begin{aligned}\Phi^{(1)}_{z_1}(z_1,x) &= \Gamma(z_1, x, \Psi(z_1), \mathcal{K}\Psi(z_1))\Phi(z_1,x) \\ &= \operatorname{Im}(z_1)\|\Psi(z_1,x)\|^{-2}_{\mathbb{C}^2}\mathcal{K}\Psi(z_1,x). \end{aligned} \qquad (4.\text{arabic@equation})$$

Moreover, by Remark 2.7, $\mathcal{K}\Phi^{(1)}_{z_1}(z_1)$ is computed as

$$\begin{aligned}\mathcal{K}\Phi^{(1)}_{z_1}(z_1,x) &= \Gamma(\overline{z_1}, x, \Psi(z_1), \mathcal{K}\Psi(z_1))\mathcal{K}\Phi(z_1,x) \\ &= -\operatorname{Im}(z_1)\|\Psi(z_1,x)\|^{-2}_{\mathbb{C}^2}\Psi(z_1,x). \end{aligned} \qquad (4.\text{arabic@equation})$$

By (4.arabic@equation) and (2.arabic@equation), for each $z \in \mathbb{C}$, the z-wave function $\Phi^{(1)}_{z_1}(z)$ associated with $q^{(1)}_{z_1}$ (constructed using the z-wave function $\Phi(z)$ associated with q) is computed by

$$\begin{aligned}\Phi^{(1)}_{z_1}(z,x) &= \Gamma(z, x, \Psi(z_1), \mathcal{K}\Psi(z_1))\Phi(z,x) \\ &= -(i/2)(z-z_1)\Phi(z,x) + \Phi^{(1)}_{z_1}(z_1,x)\Psi(z_1,x)^\perp\Phi(z,x) \\ &= -(i/2)(z-z_1)\Phi(z,x) - W(\Psi(z_1,x), \Phi(z,x))\Phi^{(1)}_{z_1}(z_1,x). \end{aligned} \qquad (4.\text{arabic@equation})$$

Formulas (2.arabic@equation) and (4.arabic@equation) now imply

$$q^{(1)}_{z_1}(x) = q(x) + 4\phi^{(1)}_{1,z_1}(z_1,x)\psi_1(z_1,x), \qquad (4.\text{arabic@equation})$$

where $\Phi^{(1)}_{z_1}(z,x) = (\phi^{(1)}_{1,z_1}(z,x), \phi^{(1)}_{2,z_1}(z,x))^\top$, $\Psi(z,x) = (\psi_1(z,x), \psi_2(z,x))^\top$.

Remark 4.1. *We emphasize that while $R(z,x,x_0)$ in* (4.arabic@equation)*, and hence $\Phi(z,x)$ in* (4.arabic@equation)*, in general, will have singularities on $\mathbb{R}$, the formulas* (4.arabic@equation)–(4.arabic@equation) *are well-defined for all $x \in \mathbb{R}$.*

The next hypothesis will be crucial in our attempt to construct z_1- and $\overline{z}_1$-wave functions in $L^2(\mathbb{R})^2$ associated with the Dirac-type differential expression $M\big(q^{(1)}_{z_1}\big)$.

Hypothesis 4.2. *Suppose $q \in L^1_{\text{loc}}(\mathbb{R})$, assume the* NLS$_-$ *case $p = -\overline{q}$, and let $z_0 \in \mathbb{C}$. Suppose $\Psi(z_0)$ to be a z_0-wave function associated with q that satisfies the condition $\|\Psi(z_0,\cdot)\|^{-1}_{\mathbb{C}^2} \in L^2(\mathbb{R})$, that is,*

$$\int_{-\infty}^{\infty} dx\, \|\Psi(z_0,x)\|^{-2}_{\mathbb{C}^2} < \infty. \qquad (4.\text{arabic@equation})$$

If a z_0-wave function $\Psi(z_0)$ associated with q satisfies condition (4.arabic@equation), we will henceforth say that $\Psi(z_0)$ satisfies Hypothesis 4.2 at z_0.

Remark 4.3. *Assume Hypothesis* 3.2 *and suppose that $\Psi(z)$ satisfies Hypothesis* 4.2 *at z. Then,*
(i) by Hölder's inequality, $\Psi(z) \notin L^2((-\infty,R])^2 \cup L^2([R,\infty))^2$ for all $R \in \mathbb{R}$.
(ii) $\mathcal{K}\Psi(z)$ satisfies Hypothesis 4.2 *at $\overline{z}$ by Lemma* 2.2 *(iii).*

Remark 4.4. *Assume Hypothesis* 3.2 *and let $\lambda \in \mathbb{R}$. Then all λ-wave functions $\Psi(\lambda)$ associated with q satisfy $\|\Psi(\lambda,x)\|_{\mathbb{C}^2} = c(\lambda)$ (independently of $x \in \mathbb{R}$). In particular, $\|\Psi(\lambda,\cdot)\|_{\mathbb{C}^2}, \|\Psi(\lambda,\cdot)\|^{-1}_{\mathbb{C}^2} \notin L^2([R,\pm\infty))$, $R \in \mathbb{R}$, and hence there exists no λ-wave function associated with q that satisfies Hypothesis* 4.2 *at $\lambda \in \mathbb{R}$.*

The principal result of this section then reads as follows.

Theorem 4.5. *Assume Hypothesis* 3.2*. Let $z_1 \in \mathbb{C}\backslash\mathbb{R}$ and suppose that the z_1-wave function $\Psi(z_1)$ associated with q satisfies Hypothesis* 4.2 *at z_1. Let $q^{(1)}_{z_1}$ be given by* (2.arabic@equation)*. Then z_1 and $\overline{z_1}$ are eigenvalues of the maximal Dirac-type operator $D_{\max}\big(q^{(1)}_{z_1}\big)$ associated with $M\big(q^{(1)}_{z_1}\big)$ of geometric multiplicity equal to one. The corresponding eigenfunctions $\Phi^{(1)}_{z_1}(z_1)$ and $\mathcal{K}\Phi^{(1)}_{z_1}(z_1)$ are given by* (4.arabic@equation) *and* (4.arabic@equation)*, respectively, that is, one has*

$$\Phi^{(1)}_{z_1}(z_1), \mathcal{K}\Phi^{(1)}_{z_1}(z_1) \in \operatorname{dom}\big(D_{\max}\big(q^{(1)}_{z_1}\big)\big), \qquad (4.\text{arabic@equation})$$

$$D_{\max}\big(q^{(1)}_{z_1}\big)\Phi^{(1)}_{z_1}(z_1) = z_1\Phi^{(1)}_{z_1}(z_1), \qquad (4.\text{arabic@equation})$$

$$D_{\max}\big(q^{(1)}_{z_1}\big)\mathcal{K}\Phi^{(1)}_{z_1}(z_1) = \overline{z_1}\mathcal{K}\Phi^{(1)}_{z_1}(z_1). \qquad (4.\text{arabic@equation})$$

Proof. Indeed, using (4.arabic@equation) at z_1 one obtains

$$0 < \big\|\Phi^{(1)}_{z_1}(z_1)\big\|^2_{L^2} = |\text{Im}(z_1)|^2 \int_{-\infty}^{\infty} dx\, \|\Psi(z_1,x)\|^{-4}_{\mathbb{C}^2} \|\Psi(z_1,x)\|^2_{\mathbb{C}^2} < \infty. \qquad (4.\text{arabic@equation})$$

In order to show that z_1 has geometric multiplicity equal to one as an eigenvalue of $D_{\max}\big(q^{(1)}_{z_1}\big)$, we next assume that $\Phi^{(1)}_{z_1}(z_1,\cdot) \in L^2(\mathbb{R})^2$ and $\tilde{\Phi}^{(1)}_{z_1}(z_1,\cdot) \in L^2(\mathbb{R})^2$

are linearly independent z_1-wave functions associated with $q_{z_1}^{(1)}$. Then clearly $\Phi_{z_1}^{(1)}(z_1,\cdot), \tilde\Phi_{z_1}^{(1)}(z_1,\cdot) \in \operatorname{dom}\big(D_{\max}\big(q_{z_1}^{(1)}\big)\big)$ and

$$W\big(\Phi_{z_1}^{(1)}(z_1,\cdot), \tilde\Phi_{z_1}^{(1)}(z_1,\cdot)\big) \in L^1(\mathbb{R}). \qquad (4.\text{arabic@equation})$$

However, since by (2.arabic@equation), $W(\Phi_{z_1}^{(1)}(z_1,x), \tilde\Phi_{z_1}^{(1)}(z_1,x))$ is constant with respect to $x\in\mathbb{R}$, (4.arabic@equation) represents a contradiction and hence z_1 has geometric multiplicity equal to one. The analogous arguments apply to $\overline{z_1}$. □

The argument that $D_{\max}\big(q_{z_1}^{(1)}\big)$ has only eigenvalues of geometric multiplicity equal to one applies of course in complete generality to any $D(q)$ with $q \in L^1_{\text{loc}(\mathbb{R})}$.

Next we show that condition (4.arabic@equation) is preserved under iterations, a fact of great relevance in connection with the multi-soliton solutions relative to arbitrary backgrounds discussed at the end of Section 6.

Lemma 4.6. *Assume Hypothesis* 3.2 *and let* $z_1, z_2 \in \mathbb{C}$. *Fix a* z_1*-wave function* $\Psi(z_1)$ *and a* z_2*-wave function* $\Phi(z_2)$ *associated with* q. *Using* $\Psi(z_1)$*, construct the* NLS$_-$ *potential* $q_{z_1}^{(1)}$ *by formula* (2.arabic@equation) *and consider the transformation matrix* $\Gamma(z_2,\Psi(z_1),\mathcal{K}\Psi(z_1))$ *given by formula* (2.arabic@equation) *for* $z=z_2$. *Let* $\Phi_{z_1}^{(1)}(z_2)$ *be the* z_2*-wave function associated with* $q_{z_1}^{(1)}$ *as in* (4.arabic@equation). *If* $\Phi(z_2)$ *satisfies Hypothesis* 4.2 *at* z_2*, then* $\Phi_{z_1}^{(1)}(z_2)$ *satisfies Hypothesis* 4.2 *at* z_2.

Proof. Without loss of generality we may assume $\operatorname{Im}(z_1)\operatorname{Im}(z_2) \geq 0$. Formulas (2.arabic@equation), (4.arabic@equation), and (4.arabic@equation) imply

$$\begin{aligned}
&\big\|\Phi_{z_1}^{(1)}(z_2)\big\|_{\mathbb{C}^2}^2\\
&= \frac14|z_2-z_1|^2\|\Phi(z_2)\|_{\mathbb{C}^2}^2 + \frac14|\overline{z_1}-z_1|^2\|\Psi(z_1)\|_{\mathbb{C}^2}^{-4}\|\mathcal{K}\Psi(z_1))\Psi(z_1)^{\perp}\Phi(z_2)\|_{\mathbb{C}^2}^2\\
&\quad - \frac12\operatorname{Re}\big(((z_2-z_1)\Phi(z_2),(\overline{z_1}-z_1)\|\Psi(z_1)\|_{\mathbb{C}^2}^{-2}\mathcal{K}\Psi(z_1)\Psi(z_1)^{\perp}\Phi(z_2))_{\mathbb{C}^2}\big)\\
&= \frac14|z_2-z_1|^2\|\Phi(z_2)\|_{\mathbb{C}^2}^2 + \frac14|\overline{z_1}-z_1|^2\|\Psi(z_1)\|_{\mathbb{C}^2}^{-2}|\Psi(z_1)^{\perp}\Phi(z_2)|^2\\
&\quad + \frac12\|\Psi(z_1)\|_{\mathbb{C}^2}^{-2}|\Psi(z_1)^{\perp}\Phi(z_2)|^2\operatorname{Re}((z_2-z_1)(\overline{z_1}-z_1))\\
&= \frac14|z_2-z_1|^2\|\Phi(z_2)\|_{\mathbb{C}^2}^2 + \|\Psi(z_1)\|_{\mathbb{C}^2}^{-2}|\Psi(z_1)^{\perp}\Phi(z_2)|^2\operatorname{Im}(z_1)\operatorname{Im}(z_2)\\
&\geq \frac14|z_2-z_1|^2\|\Phi(z_2)\|_{\mathbb{C}^2}^2. \qquad (4.\text{arabic@equation})
\end{aligned}$$

Thus, if $\int_{-\infty}^{\infty} dx\, \|\Phi(z_2,x)\|_{\mathbb{C}^2}^{-2} < \infty$, then $\int_{-\infty}^{\infty} dx\, \big\|\Phi_{z_1}^{(1)}(z_2,x)\big\|_{\mathbb{C}^2}^{-2} < \infty$. □

It will be shown in Remark 6.1 that for any $z \in \rho\big(D_{\max}\big(q_{z_1}^{(1)}\big)\big)$ all but two z-wave functions of $D_{\max}\big(q_{z_1}^{(1)}\big)$ satisfy Hypothesis 4.2 at z.

5. Some Spectral Properties and the Existence of Weyl–Titchmarsh-type Solutions for $\mathcal{J}$-Self-Adjoint Dirac-Type Operators

The principal purpose of this section is to establish the existence of Weyl–Titchmarsh-type solutions for formally $\mathcal{J}$-self-adjoint Dirac differetial expressions $M(q)$ associated with the focusing NLS$_-$ case. The latter are well-known to exist in the case of self-adjoint Dirac operators (in particular, they are well-known to exist in the context of the defocusing NLS$_+$ equation) and are known to be a fundamental ingredient in the spectral analysis in the self-adjoint context (cf., for instance, [31, Chs. 3, 4,]). As far as we know, no such result appears to be known in the general $\mathcal{J}$-self-adjoint case studied in this section. Along the way we also collect some results concerning spectral properties of $D_{\max}(q)$.

Thus, assuming the NLS$_-$ case and hence the basic Hypothesis 3.2 throughout this section, we adopt the simplified notation (cf. Section 3)

$$D(q) = \overline{D_{\min}(q)} = D_{\max}(q), \quad q \in L^1_{\text{loc}}(\mathbb{R}). \qquad (5.\text{arabic@equation})$$

We also denote by I_2 the identity operator in $L^2(\mathbb{R})^2$ (as well as in $\mathbb{C}^2$). Moreover, we find it convenient to introduce the following notations,

$$L^2_{\text{loc}}([-\infty,\infty)) = \big\{f\colon \mathbb{R} \to \mathbb{C} \text{ is measurable} \,\big|\, f \in L^2((-\infty,R]) \text{ for all } R \in \mathbb{R}\big\},$$
$$L^2_{\text{loc}}((-\infty,\infty]) = \big\{f\colon \mathbb{R} \to \mathbb{C} \text{ is measurable} \,\big|\, f \in L^2([R,\infty)) \text{ for all } R \in \mathbb{R}\big\}. \qquad (5.\text{arabic@equation})$$

We start with the following auxiliary result (the variation of parameters formula).

Lemma 5.1. *Assume Hypothesis* 3.2 *and let* $(z,x_0) \in \mathbb{C}\times\mathbb{R}$. *Let* $\Psi_1(z)$ *and* $\Psi_2(z)$ *be linearly independent* z*-wave functions for* $M(q)$ *defined on* $[x_0,\infty)$ *and denote by* $\Xi(z,x) = \big[\Psi_1(z,x), \Psi_2(z,x)\big]$ *the* 2×2 *fundamental matrix solution of* $M(q)\Xi(z) = z\Xi(z)$. *Assume the Wronskian of* Ψ_1 *and* Ψ_2 *satisfies* $W(\Psi_1(z,x),\Psi_2(z,x)) = \det(\Xi(z,x)) = 1$ *for some* (*and hence for all*) $x \in [x_0,\infty)$. *Moreover, suppose* $B \in L^1_{\text{loc}}([x_0,\infty))^{2\times 2}$. *Then* $\Phi(z,\cdot) \in AC_{\text{loc}}(\mathbb{R})^2$ *satisfies* $M(q)\Phi(z) = z\Phi(z) + B\Phi(z)$ *if and only if*

$$\Phi(z,x) = \Xi(z,x)C + \int_{x_0}^{x} dx'\, \Xi(z,x)\Xi(z,x')^{-1}B(x')\Phi(z,x'), \quad x \geq x_0 \qquad (5.\text{arabic@equation})$$

for some $C = (c_1,c_2)^\top \in \mathbb{C}^2$ *independent of* x. *Moreover,* $\Phi(z,x) = 0$ *for all* $x \geq x_0$ *if and only if* $C = 0$.

Proof. The computation

$$\begin{aligned} M(q)\big(\Phi(x) - \int_{x_0}^{x} dx'\, \Xi(x)\Xi(x')^{-1}B(x')\Phi(x')\big) \\ &= z\Phi(x) + B(x)\Phi(x) - M(q)\bigg(\Xi(x)\int_{x_0}^{x} dx'\, \Xi(x')^{-1}B(x')\Phi(x')\bigg) \\ &= z\bigg(\Phi(x) - \int_{x_0}^{x} dx'\, \Xi(x)\Xi(x')^{-1}B(x')\Phi(x')\bigg) \qquad (5.\text{arabic@equation}) \end{aligned}$$

shows that Φ satisfies $M(q)\Phi(z) = z\Phi(z) + B\Phi(z)$ if and only if Φ satisfies (5.arabic@equation) since Ξ is a fundamental matrix for the first-order linear differential system $M(q)\Psi(z) = z\Psi(z)$. That $\Phi(z,x) = 0$ for all $x \geq x_0$ if and only if $C = 0$ follows upon iterating the Volterra-type integral equation (5.arabic@equation) in a standard manner. □

Next, we find it convenient to recall a number of basic definitions and well-known facts in connection with the spectral theory of non-self-adjoint operators (we refer to [11, Chs. I, III, IX], [21, Sects. 1, 21–23], and [38, p. 178–179] for more details). Let S be a densely defined closed operator in a complex Hilbert space $\mathcal{H}$. Denote by $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear operators on $\mathcal{H}$. The spectrum, $\sigma(S)$, point spectrum (the set of eigenvalues), $\sigma_{\text{p}}(S)$, continuous spectrum, $\sigma_{\text{c}}(S)$, residual spectrum, $\sigma_{\mathrm{r}}(S)$, approximate point spectrum, $\sigma_{\mathrm{a}}(S)$, essential spectrum, $\sigma_{\mathrm{e}}(S)$, field of regularity, $\pi(S)$, resolvent set, $\rho(S)$, and $\Delta(S)$ are defined by

$$\sigma(S) = \{\lambda \in \mathbb{C} \,|\, (S-\lambda I)^{-1} \in \mathcal{B}(\mathcal{H})\}, \quad \text{(5.arabic@equation)}$$

$$\sigma_{\mathrm{p}}(S) = \{\lambda \in \mathbb{C} \,|\, \ker(S-\lambda I) \neq \{0\}\}, \quad \text{(5.arabic@equation)}$$

$$\sigma_{\mathrm{c}}(S) = \{\lambda \in \sigma(S) \,|\, \ker(S-\lambda I) = \{0\} \text{ and } \operatorname{ran}(S-\lambda I) \text{ is dense in } \mathcal{H} \text{ but not equal to } \mathcal{H}\}, \quad \text{(5.arabic@equation)}$$

$$\sigma_{\mathrm{r}}(S) = \{\lambda \in \mathbb{C} \,|\, \ker(S-\lambda I) = \{0\} \text{ and } \operatorname{ran}(S-\lambda I) \text{ is not dense in } \mathcal{H}\}, \quad \text{(5.arabic@equation)}$$

$$\sigma_{\mathrm{a}}(S) = \{\lambda \in \mathbb{C} \,|\, \text{there exists a sequence } \{f_n\}_{n\in\mathbb{N}} \text{ with } \|f_n\|_{\mathcal{H}} = 1,\ n \in \mathbb{N}, \text{ and } \lim_{n\to\infty} \|(S-\lambda I)f_n\|_{\mathcal{H}} = 0\}, \quad \text{(5.arabic@equation)}$$

$$\sigma_{\mathrm{e}}(S) = \{\lambda \in \mathbb{C} \,|\, \text{there exists a sequence } \{f_n\}_{n\in\mathbb{N}} \text{ s.t. } \{f_n\}_{n\in\mathbb{N}} \text{ contains no convergent subsequence, } \|f_n\|_{\mathcal{H}} = 1,\ n \in \mathbb{N}, \text{ and } \lim_{n\to\infty} \|(S-\lambda I)f_n\|_{\mathcal{H}} = 0\}, \quad \text{(5.arabic@equation)}$$

$$\pi(S) = \{z \in \mathbb{C} \,|\, \text{there exists } k_z > 0 \text{ s.t. } \|(S-zI)u\|_{\mathcal{H}} \geq k_z \|u\|_{\mathcal{H}} \text{ for all } u \in \operatorname{dom}(S)\}, \quad \text{(5.arabic@equation)}$$

$$\rho(S) = \mathbb{C}\backslash\sigma(S), \quad \text{(5.arabic@equation)}$$

$$\Delta(S) = \{z \in \mathbb{C} \,|\, \dim(\ker(S-z)) < \infty \text{ and } \operatorname{ran}(S-z) \text{ is closed}\}, \quad \text{(5.arabic@equation)}$$

respectively. One then has

$$\sigma(S) = \sigma_{\mathrm{p}}(S) \cup \sigma_{\mathrm{c}}(S) \cup \sigma_{\mathrm{r}}(S) \quad \text{(disjoint union)} \quad \text{(5.arabic@equation)}$$

$$= \sigma_{\mathrm{p}}(S) \cup \sigma_{\mathrm{e}}(S) \cup \sigma_{\mathrm{r}}(S), \quad \text{(5.arabic@equation)}$$

$$\sigma_{\mathrm{c}}(S) \subseteq \sigma_{\mathrm{e}}(S)\backslash(\sigma_{\mathrm{p}}(S) \cup \sigma_{\mathrm{r}}(S)), \quad \text{(5.arabic@equation)}$$

$$\sigma_{\mathrm{r}}(S) = \sigma_{\mathrm{p}}(S^*)^*\backslash\sigma_{\mathrm{p}}(S), \quad \text{(5.arabic@equation)}$$

$$\sigma_{\mathrm{a}}(S) = \{\lambda \in \mathbb{C} \,|\, \text{for all } \varepsilon > 0, \text{ there exists } 0 \neq f_\varepsilon \text{ s.t. } \|(S-\lambda)f_\varepsilon\|_{\mathcal{H}} \leq \varepsilon \|f_\varepsilon\|_{\mathcal{H}}\} \quad \text{(5.arabic@equation)}$$

$$= \mathbb{C}\backslash\pi(S), \quad \text{(5.arabic@equation)}$$

$$\sigma(S)\backslash\sigma_{\mathrm{a}}(S) \subseteq \sigma_{\mathrm{r}}(S), \quad \sigma(S)\backslash\sigma_{\mathrm{a}}(S) \text{ is open}, \quad \text{(5.arabic@equation)}$$

$$\sigma_{\mathrm{e}}(S) = \{\lambda \in \mathbb{C} \,|\, \text{there exists a sequence } \{g_n\}_{n\in\mathbb{N}} \text{ s.t. w-}\lim_{n\to\infty} g_n = 0, \|g_n\|_{\mathcal{H}} = 1,\ n \in \mathbb{N}, \text{ and } \lim_{n\to\infty} \|(S-\lambda I)g_n\|_{\mathcal{H}} = 0\} \quad \text{(5.arabic@equation)}$$

$$= \mathbb{C}\backslash\Delta(S), \quad \text{(5.arabic@equation)}$$

$$\sigma_{\mathrm{e}}(S) \subseteq \sigma_{\mathrm{a}}(S) \subseteq \sigma(S) \ \text{(all three sets are closed)}, \quad \text{(5.arabic@equation)}$$

$$\rho(S) \subseteq \pi(S) \subseteq \Delta(S) \ \text{(all three sets are open)}. \quad \text{(5.arabic@equation)}$$

Here ω^* in the context of (5.arabic@equation) denotes the complex conjugate of the set $\omega \subseteq \mathbb{C}$, that is,

$$\omega^* = \{\overline{\lambda} \in \mathbb{C} \,|\, \lambda \in \omega\}. \quad \text{(5.arabic@equation)}$$

For future reference we note that the sequence $\{f_n\}_{n\in\mathbb{N}}$ in (5.arabic@equation) (and the sequence $\{g_n\}_{n\in\mathbb{N}}$ in (5.arabic@equation)) is called a *singular* (*or Weyl*) *sequence* of S corresponding to λ. We also note that there are several versions of the concept of the essential spectrum in the non-self-adjoint context (cf. [11, Ch. IX]) but we will only use the one in (5.arabic@equation) (respectively, (5.arabic@equation)) in this paper.

In the special case where S is $\mathcal{J}$-self-adjoint one obtains the following simplifications (cf. [11, p. 118], [21, p. 76]):

$$\sigma(S) = \sigma_{\rm p}(S) \cup \sigma_{\rm c}(S) \quad (5.arabic@equation)$$
$$= \sigma_{\rm p}(S) \cup \sigma_{\rm e}(S), \quad (5.arabic@equation)$$
$$\sigma_{\rm r}(S) = \emptyset, \quad (5.arabic@equation)$$
$$\sigma_{\rm p}(S) = \sigma_{\rm p}(S^*)^*, \quad (5.arabic@equation)$$
$$\sigma_{\rm a}(S) = \sigma(S), \quad (5.arabic@equation)$$
$$\pi(S) = \rho(S), \quad (5.arabic@equation)$$

whenever S is $\mathcal{J}$-self-adjoint. Note that $\pi(S) = \emptyset$ may occur for $\mathcal{J}$-symmetric operators (see [37]) in sharp contrast to the case of densely defined, closed, symmetric operators T, where any nonreal number is in the field of regularity $\pi(T)$.

Returning to the NLS$_-$ case, we next recall an elementary but useful consequence of (2.arabic@equation) and (2.arabic@equation).

Lemma 5.2. *Assume Hypothesis* 3.2. *Then*

$$\mathcal{K}D(q)\mathcal{K} = -D(q), \quad (5.arabic@equation)$$
$$\sigma_3 D(q)\sigma_3 = D(q)^* = D(-q). \quad (5.arabic@equation)$$

Consequently,

$$\sigma(D(q))^* = \sigma(D(q)) = \sigma(D(-q)). \quad (5.arabic@equation)$$

Moreover,

$$\text{if } z_0 \in \sigma_{\rm p}(D(q)) \text{ and } D(q)F = z_0 F \text{ for some } F \in \operatorname{dom}(D(q)),$$
$$\text{then } D(q)\mathcal{K}F = \overline{z_0}\mathcal{K}F. \quad (5.arabic@equation)$$

Proof. Relations (5.arabic@equation) and (5.arabic@equation) are clear from (2.arabic@equation), (2.arabic@equation), (3.arabic@equation), (3.arabic@equation), and (5.arabic@equation). Since $(D(q)^* - zI_2)^{-1} = [(D(q) - \overline{z}I_2)^{-1}]^*$, $\sigma(D(q)^*) = \sigma(D(q))^*$ and hence (5.arabic@equation) follows from (5.arabic@equation). Finally, (5.arabic@equation) is clear from (5.arabic@equation) and $\mathcal{K}^2 = -I_2$. $\square$

Next we introduce the basic hypothesis to be assumed for the remainder of this section.

Hypothesis 5.3. *Suppose $q \in L^1_{\rm loc}(\mathbb{R})$, assume the* NLS$_-$ *case $p = -\overline{q}$, and suppose that the ($\mathcal{J}$-self-adjoint) operator $D(q)$ has nonempty resolvent set, $\rho(D(q)) \neq \emptyset$.*

Recalling the standard notation

$$\operatorname{nul}(T) = \dim(\ker(T)), \quad (5.arabic@equation)$$
$$\operatorname{def}(T) = \dim(\operatorname{ran}(T)^\perp) = \dim(\ker(T^*)) = \operatorname{nul}(T^*), \quad (5.arabic@equation)$$

where T denotes a densely defined closed operator in $\mathcal{H}$, we can state the following fundamental result, establishing the existence of Weyl–Titchmarsh-type solutions for $\mathcal{J}$-self-adjoint Dirac-type operators relevant to the NLS$_-$ case.

Theorem 5.4. *Assume Hypothesis* 5.3 *and pick $z \in \rho(D(q))$. Then there exist two unique* (*up to constant multiples*) *linearly independent z-wave functions $\Psi_-(z,\cdot)$ and $\Psi_+(z,\cdot)$ associated with q satisfying*

$$\Psi_-(z,\cdot) \in L^2_{\mathrm{loc}}([-\infty,\infty)), \quad \Psi_+(z,\cdot) \in L^2_{\mathrm{loc}}((-\infty,\infty]), \qquad (5.\text{arabic@equation})$$

$$\|\Psi_-(z,\cdot)\|_{\mathbb{C}^2}^{-1} \in L^2_{\mathrm{loc}}((-\infty,\infty]), \quad \|\Psi_+(z,\cdot)\|_{\mathbb{C}^2}^{-1} \in L^2_{\mathrm{loc}}([-\infty,\infty)), \qquad (5.\text{arabic@equation})$$

$$\lim_{x\to\pm\infty} \|\Psi_\pm(z,x)\|_{\mathbb{C}^2} = \lim_{x\to\mp\infty} \|\Psi_\pm(z,x)\|_{\mathbb{C}^2}^{-1} = 0, \qquad (5.\text{arabic@equation})$$

$$\sup_{r\in\mathbb{R}} \bigg[\bigg(\int_{-\infty}^r dx\, \|\Psi_-(z,x)\|_{\mathbb{C}^2}^2\bigg)\bigg(\int_r^\infty dx\, \|\Psi_+(z,x)\|_{\mathbb{C}^2}^2\bigg)\bigg] < \infty. \qquad (5.\text{arabic@equation})$$

The corresponding $\overline{z}$-wave functions $\mathcal{K}\Psi_\pm(z,x)$ associated with q satisfy (5.arabic@equation)–(5.arabic@equation) *with z replaced by $\overline{z}$.*

Proof. We prove the existence of the two z-wave functions following the lines of [11, Sect 10.4]. To this end we introduce the operators

$$\begin{aligned}
&D_{\max}(q;-\infty)F = M(q)F,\\
&F \in \mathrm{dom}(D_{\max}(q;-\infty)) = \big\{G \in L^2((-\infty,0]) \,|\, G \in AC_{\mathrm{loc}}((-\infty,0]),\\
&\qquad\qquad M(q)G \in L^2((-\infty,0])\big\},
\end{aligned} \qquad (5.\text{arabic@equation})$$

$$\begin{aligned}
&D_{\min}(q;-\infty)F = M(q)F,\\
&F \in \mathrm{dom}(D_{\min}(q;-\infty)) = \{G \in \mathrm{dom}(D_{\max}(q;-\infty)) \mid G(0) = 0;\\
&\qquad\qquad \mathrm{supp}(G) \subset (-\infty,0] \text{ is compact}\},
\end{aligned} \qquad (5.\text{arabic@equation})$$

$$\begin{aligned}
&D_{\max}(q;\infty)F = M(q)F,\\
&F \in \mathrm{dom}(D_{\max}(q;\infty)) = \big\{G \in L^2([0,\infty)) \,|\, G \in AC_{\mathrm{loc}}([0,\infty)),\\
&\qquad\qquad M(q)G \in L^2([0,\infty))\big\},
\end{aligned} \qquad (5.\text{arabic@equation})$$

$$\begin{aligned}
&D_{\min}(q;\infty)F = M(q)F,\\
&\mathrm{dom}(D_{\min}(q;\infty)) = \{G \in \mathrm{dom}(D_{\max}(q;\infty)) \,|\, G(0) = 0;\\
&\qquad\qquad \mathrm{supp}(G) \subset [0,\infty) \text{ is compact}\}.
\end{aligned} \qquad (5.\text{arabic@equation})$$

In close analogy to [11, Theorem III.10.20] one can prove that for all $z \in \rho(D(q))$,

$$\mathrm{def}\big(\overline{D_{\min}(q)} - zI_2\big) = \mathrm{def}\big(\overline{D_{\min}(q;-\infty)} - zI_2\big) + \mathrm{def}\big(\overline{D_{\min}(q;\infty)} - zI_2\big) - 2. \qquad (5.\text{arabic@equation})$$

Since $D(q) = \overline{D_{\min}(q)}$ is $\mathcal{J}$-self-adjoint and $z \in \rho(D(q))$, we necessarily have (see, e.g., [11, Theorem III.5.5]) that

$$\mathrm{def}\big(\overline{D_{\min}(q)} - zI_2\big) = \mathrm{def}(D(q) - zI_2) = 0. \qquad (5.\text{arabic@equation})$$

Thus,

$$\mathrm{def}\big(\overline{D_{\min}(q;-\infty)} - zI_2\big) + \mathrm{def}\big(\overline{D_{\min}(q;\infty)} - zI_2\big) = 2. \qquad (5.\text{arabic@equation})$$

We claim that the only possibility is

$$\operatorname{def}\big(\overline{D_{\min}(q;-\infty)} - zI_2\big) = \operatorname{def}\big(\overline{D_{\min}(q;\infty)} - zI_2\big) = 1. \qquad (5.\text{arabic@equation})$$

Indeed, arguing by contradiction, we assume, for instance, that

$$\operatorname{def}\big(\overline{D_{\min}(q;-\infty)} - z_0 I_2\big) = 0. \qquad (5.\text{arabic@equation})$$

This implies

$$\operatorname{nul}\big(D_{\max}(-q;\infty) - \overline{z_0} I_2\big) = 2 \qquad (5.\text{arabic@equation})$$

since $(D_{\min}(q);\infty)^* = D_{\max}(-q;\infty)$. Thus all $\overline{z_0}$-wave functions for the NLS$_-$ potential $-q$ are in $L^2_{\text{loc}}((-\infty,\infty])^2$. According to Remark 4.4, this is clearly impossible for $\overline{z_0} \in \mathbb{R} \cap \rho(D(-q))$. Next we show that this is impossible also for $\overline{z_0} \in (\mathbb{C}\backslash\mathbb{R}) \cap \rho(D(-q))$. Using Lemma 2.2 (iii), we can simplify notations and replace $\overline{z_0}$ by z_0 without loss of generality. Moreover, to avoid confusion with the change $q \to -q$ and the corresponding change for $q^{(1)}_{z_0}$, we simply use q instead of $-q$ in the proof of (5.arabic@equation) below.

To this end, we fix $z_0 \in \rho(D(q))$ with $\operatorname{Im}(z_0) \neq 0$ and $\Psi_1(z_0), \Psi_2(z_0)$ two linearly independent z_0-wave functions associated with the background potential q. The latter are in $L^2([0,\infty))^2$ by hypothesis (5.arabic@equation). Then for any z_0-wave function $\Psi(z_0) = (\psi_1(z_0), \psi_2(z_0))^\top$ associated with the NLS$_-$ potential q, one infers (cf. (4.arabic@equation) and (4.arabic@equation)) that $\Phi^{(1)}_{z_0}(z_0) = \operatorname{Im}(z_0)\mathcal{K}\Psi(z_0)\|\Psi(z_0)\|^{-2}_{\mathbb{C}^2}$ is a z_0-wave function associated with the NLS$_-$ potential

$$q^{(1)}_{z_0} = q + 4\operatorname{Im}(z_0)\psi_1(z_0)\overline{\psi_2(z_0)}\|\Psi(z_0)\|^{-2}_{\mathbb{C}^2}. \qquad (5.\text{arabic@equation})$$

Thus $\Phi^{(1)}_{z_0}(z_0)$ satisfies

$$z_0\Phi^{(1)}_{z_0}(z_0) = M(q^{(1)})\Phi^{(1)}_{z_0}(z_0) = M(q)\Phi^{(1)}_{z_0}(z_0) - B(z_0)\Phi^{(1)}_{z_0}(z_0), \qquad (5.\text{arabic@equation})$$

where

$$B(z_0,x) = 4i\operatorname{Im}(z_0)\|\Psi(z_0,x)\|^{-2}_{\mathbb{C}^2} \begin{pmatrix} 0 & \psi_1(z_0,x)\overline{\psi_2(z_0,x)} \\ \overline{\psi_1(z_0,x)}\psi_2(z_0,x) & 0 \end{pmatrix} \qquad (5.\text{arabic@equation})$$

belongs to $L^\infty(\mathbb{R})^{2\times 2}$. In particular,

$$\operatorname{ess\,sup}_{x\in\mathbb{R}}\|B(z_0,x)\|_{\mathbb{C}^{2\times 2}} \leq 2|\operatorname{Im}(z_0)|. \qquad (5.\text{arabic@equation})$$

Since no confusion can arise we occasionally suppress the explicit z_0-dependence in the calculations below. The variation of parameters formula (5.arabic@equation) then yields the following for the fundamental system of solutions $\Xi = [\Psi_1, \Psi_2]$, $W(\Psi_1,\Psi_2) = 1$, associated with q and z_0:

$$\begin{aligned} \Phi^{(1)}_{z_0}(x) &= \Xi(x)C + \int_a^x dx'\, \Xi(x)\Xi(x')^{-1}B(x')\Phi^{(1)}_{z_0}(x') \qquad (5.\text{arabic@equation}) \\ &= c_1\Psi_1(x) + c_2\Psi_2(x) + \Xi(x)\int_a^x dx'\, \Xi(x')^{-1}B(x')\Phi^{(1)}_{z_0}(x'), \quad x \geq a > 0, \end{aligned}$$

where $C = (c_1, c_2)^\top \neq 0$ depends on a and we assume $W(\Psi_1,\Psi_2) = 1$ according to Lemma 5.1. Hence one obtains

$$\|\Xi(x)\|_{\mathbb{C}^{2\times 2}} \leq 2^{1/2}\max\big(\|\Psi_1(x)\|_{\mathbb{C}^2}, \|\Psi_2(x)\|_{\mathbb{C}^2}\big). \qquad (5.\text{arabic@equation})$$

Moreover, writing $\Psi_j = (\psi_{1,j}, \psi_{2,j})^\top$, $j = 1, 2$, one infers

$$\|\psi_{k,j}(z_0,\cdot)\|_{L^2([a,\infty))} \leq K(a), \quad j,k = 1,2 \tag{5.arabic@equation}$$

for some constant $K(a) > 0$ with

$$\lim_{a\uparrow\infty} K(a) = 0. \tag{5.arabic@equation}$$

Hence,

$$\begin{aligned}\int_a^x dx'\, \|\Xi(x')^{-1}\|^2_{\mathbb{C}^{2\times 2}} &\leq 2\int_a^x dx'\, \max\left(\left\| \begin{pmatrix} \psi_{2,2}(x') \\ -\psi_{2,1}(x') \end{pmatrix} \right\|^2_{\mathbb{C}^2}, \left\| \begin{pmatrix} -\psi_{1,2}(x') \\ \psi_{1,1}(x') \end{pmatrix} \right\|^2_{\mathbb{C}^2}\right) \\ &\leq 4K(a)^2. \end{aligned} \tag{5.arabic@equation}$$

Thus, (5.arabic@equation)–(5.arabic@equation) yield for $x \geq a$,

$$\begin{aligned}\big\|\Phi^{(1)}_{z_0}(x)\big\|_{\mathbb{C}^2} &\leq |c_1|\|\Psi_1(x)\|_{\mathbb{C}^2} + |c_2|\|\Psi_2(x)\|_{\mathbb{C}^2} \\ &\quad + 2^{1/2}4|\mathrm{Im}(z_0)|K(a)\max\big(\|\Psi_1(x)\|_{\mathbb{C}^2}, \|\Psi_2(x)\|_{\mathbb{C}^2}\big)\bigg(\int_a^x dx'\, \big\|\Phi^{(1)}_{z_0}(x')\big\|^2_{\mathbb{C}^2}\bigg)^{1/2}. \end{aligned} \tag{5.arabic@equation}$$

Squaring (5.arabic@equation) and integrating the result from a to x, one estimates

$$\begin{aligned} &\int_a^x dx'\, \big\|\Phi^{(1)}_{z_0}(x')\big\|^2_{\mathbb{C}^2} \\ &\leq 3|c_1|^2 \int_a^x dx'\, \|\Psi_1(x')\|^2_{\mathbb{C}^2} + 3|c_2|^2\int_a^x dx'\, \|\Psi_2(x')\|^2_{\mathbb{C}^2} \\ &\quad + 96|\mathrm{Im}(z_0)|^2K(a)^2 \int_a^x dx' \int_a^{x'} dx''\, \max\big(\|\Psi_1(x')\|_{\mathbb{C}^2}, \|\Psi_2(x')\|_{\mathbb{C}^2}\big)\big\|\Phi^{(1)}_{z_0}(x'')\big\|^2_{\mathbb{C}^2} \\ &\leq 6K(a)^2\big(|c_1|^2 + |c_2|^2\big) \\ &\quad + 96|\mathrm{Im}(z_0)|^2K(a)^2 \int_a^x dx'' \int_{x''}^{x} dx'\, \max\big(\|\Psi_1(x')\|_{\mathbb{C}^2}, \|\Psi_2(x')\|_{\mathbb{C}^2}\big)\big\|\Phi^{(1)}_{z_0}(x'')\big\|^2_{\mathbb{C}^2} \\ &\leq 6K(a)^2\big(|c_1|^2 + |c_2|^2\big) \\ &\quad + 96|\mathrm{Im}(z_0)|^2K(a)^2 \int_a^x dx''\, \big\|\Phi^{(1)}_{z_0}(x'')\big\|^2_{\mathbb{C}^2} \int_a^x dx'\, \max\big(\|\Psi_1(x')\|_{\mathbb{C}^2}, \|\Psi_2(x')\|_{\mathbb{C}^2}\big) \\ &\leq 6K(a)^2\big(|c_1|^2 + |c_2|^2\big) + 192|\mathrm{Im}(z_0)|^2K(a)^4 \int_a^x dx''\, \big\|\Phi^{(1)}_{z_0}(x'')\big\|^2_{\mathbb{C}^2}. \end{aligned} \tag{5.arabic@equation}$$

Here we applied the Fubini–Tonelli theorem to the integrand

$$\max\big(\|\Psi_1(x')\|_{\mathbb{C}^2}, \|\Psi_2(x')\|_{\mathbb{C}^2}\big)\big\|\Phi^{(1)}_{z_0}(x'')\big\|^2_{\mathbb{C}^2}\chi_{[a,x']}(x'') \geq 0 \tag{5.arabic@equation}$$

(χ_Λ the characteristic function of the set $\Lambda \subset \mathbb{R}$) to prove equality of the iterated integrals $\int_a^x dx' \int_a^{x'} dx'' \cdots$ and $\int_a^x dx'' \int_{x''}^x dx' \cdots$ in (5.arabic@equation). Hence, if one chooses a large enough (such that $192|\mathrm{Im}(z_0)|^2K(a)^4 < 1$), then

$$\big[1 - 192|\mathrm{Im}(z_0)|^2K(a)^4\big]\int_a^x dx'\, \big\|\Phi^{(1)}_{z_0}(z_0,x')\big\|^2_{\mathbb{C}^2} \leq 6K(a)^2\big(|c_1|^2 + |c_2|^2\big), \tag{5.arabic@equation}$$

and thus, $\Phi^{(1)}_{z_0}(z_0,\cdot) \in L^2([a,\infty))^2$. Since $\big\|\Phi^{(1)}_{z_0}(z_0,\cdot)\big\|_{\mathbb{C}^2} = \|\Psi(z_0,\cdot)\|^{-1}_{\mathbb{C}^2}$ this contradicts the assumption $\Psi(z_0,\cdot) \in L^2([0,\infty))^2$. This proves (5.arabic@equation).

Finally, if $\Psi_-(z)$ and $\Psi_+(z)$ satisfying (5.arabic@equation) were linearly dependent then it would follow that $z \in \sigma_p(D(q))$ by (5.arabic@equation), contradicting the initial assumption $z \in \rho(D(q))$. Summing up, (5.arabic@equation) implies existence and uniqueness (up to constant multiples) of $\Psi_\pm(z)$ satisfying (5.arabic@equation).

To prove (5.arabic@equation) we assume without loss of generality that

$$W(\Psi_-(z,x),\Psi_+(z,x)) = 1, \quad x \in \mathbb{R}. \qquad \text{(5.arabic@equation)}$$

Then one computes

$$1 + |(\Psi_-(z,x),\Psi_+(z,x))_{\mathbb{C}^2}|^2 = \|\Psi_+(z,x)\|^2_{\mathbb{C}^2}\|\Psi_-(z,x)\|^2_{\mathbb{C}^2} \geq 1 \qquad \text{(5.arabic@equation)}$$

and hence,

$$\|\Psi_\mp(z,x)\|^{-1}_{\mathbb{C}^2} \leq \|\Psi_\pm(z,x)\|_{\mathbb{C}^2}, \quad x \in \mathbb{R}. \qquad \text{(5.arabic@equation)}$$

Thus, $\|\Psi_-(z,\cdot)\|^{-1}_{\mathbb{C}^2} \in L^2_{\text{loc}}((-\infty,\infty])$. The fact that $\|\Psi_+(z,\cdot)\|^{-1}_{\mathbb{C}^2} \in L^2_{\text{loc}}([-\infty,\infty))$ in (5.arabic@equation) is proved analogously.

By (5.arabic@equation) and (5.arabic@equation) one infers

$$\liminf_{x\to\pm\infty} \|\Psi_\pm(z,x)\|_{\mathbb{C}^2} = \liminf_{x\to\pm\infty} \|\Psi_\mp(z,x)\|^{-1}_{\mathbb{C}^2} = 0. \qquad \text{(5.arabic@equation)}$$

To prove (5.arabic@equation) one first integrates (2.arabic@equation) to obtain

$$\|\Psi_\pm(z,x_2)\|^2_{\mathbb{C}^2} - \|\Psi_\pm(z,x_1)\|^2_{\mathbb{C}^2} = 2\mathrm{Im}(z)\int_{x_1}^{x_2} dx\, \big[|\psi_{1,\pm}(z,x)|^2 - |\psi_{2,\pm}(z,x)|^2\big]. \qquad \text{(5.arabic@equation)}$$

Thus, $\lim_{x\to\pm\infty} \|\Psi_\pm(z,x)\|_{\mathbb{C}^2}$ exist and hence equal 0 by (5.arabic@equation). By (5.arabic@equation) one then infers

$$\lim_{x\to\pm\infty} \|\Psi_\mp(z,x)\|^{-1}_{\mathbb{C}^2} = 0. \qquad \text{(5.arabic@equation)}$$

What remains to be proved is (5.arabic@equation). To this end consider the Green's function for the Dirac-type operator $D(q)$. It can be written in terms of the z-wave functions

$$\Psi_-(z,x) = (\psi_{1,-}(z,x),\psi_{2,-}(z,x))^\top, \quad \Psi_+(z,x) = (\psi_{1,+}(z,x),\psi_{2,+}(z,x))^\top, \qquad \text{(5.arabic@equation)}$$

whose existence is guaranteed by (5.arabic@equation) and whose Wronskian is still assumed to satisfy (5.arabic@equation). More precisely, define the 2×2 matrix-valued function on $\mathbb{R}^2$

$$G(z,x,y) = -\begin{cases} \Psi_+(z,x)\Psi_-(z,y)^\top, & x \leq y, \\ \Psi_-(z,x)\Psi_+(z,y)^\top, & x \geq y, \end{cases} \qquad z \in \rho(D(q)). \qquad \text{(5.arabic@equation)}$$

Explicitly,

$$G(z,x,y) = -\begin{cases} \begin{pmatrix} \psi_{1,+}(z,x)\psi_{1,-}(z,y) & \psi_{1,+}(z,x)\psi_{2,-}(z,y) \\ \psi_{2,+}(z,x)\psi_{1,-}(z,y) & \psi_{2,+}(z,x)\psi_{2,-}(z,y) \end{pmatrix}, & x \le y, \\ \begin{pmatrix} \psi_{1,-}(z,x)\psi_{1,+}(z,y) & \psi_{1,-}(z,x)\psi_{2,+}(z,y) \\ \psi_{2,-}(z,x)\psi_{1,+}(z,y) & \psi_{2,-}(z,x)\psi_{2,+}(z,y) \end{pmatrix}, & x \ge y. \end{cases} \tag{5.arabic@equation}$$

To prove that (5.arabic@equation) indeed represents the Green's function associated with the $\mathcal{J}$-self-adjoint operator $D(q)$ one can argue as follows. One introduces a densely defined operator $R(z)$ in $L^2(\mathbb{R})^2$ by

$$(R(z)F)(x) = \int_{\mathbb{R}} dx'\, G(z,x,x')F(x'), \quad z \in \rho(D(q)),\ x \in \mathbb{R}, \tag{5.arabic@equation}$$

$$F \in \operatorname{dom}(R(z)) = \big\{G \in L^2(\mathbb{R}))^2 \,|\, \operatorname{supp}(G) \text{ is compact}\big\}. \tag{5.arabic@equation}$$

By inspection one infers that

$$R(z)F \in AC_{\mathrm{loc}}(\mathbb{R})^2 \cap L^2(\mathbb{R})^2 \text{ and } M(q)R(z)F \in L^2(\mathbb{R})^2. \tag{5.arabic@equation}$$

Thus, $R(z)$ maps $L^2(\mathbb{R})^2$-elements of compact support into the domain of $D(q)$. Moreover, an explicit computation shows that

$$\begin{aligned} &(M(q)-z)\big[R(z)F - (D(q)-z)^{-1}F\big] \\ &\quad = (D(q)-z)\big[R(z)F - (D(q)-z)^{-1}F\big] = 0, \quad F \in \operatorname{dom}(R(z)). \end{aligned} \tag{5.arabic@equation}$$

Since by hypothesis, $z \in \rho(D(q))$, (5.arabic@equation) implies

$$R(z)F = (D(q)-z)^{-1}F, \quad F \in \operatorname{dom}(R(z)). \tag{5.arabic@equation}$$

Hence $R(z)$ extends boundedly to all of $L^2(\mathbb{R})^2$ and its closure coincides with the resolvent $(D(q)-z)^{-1}$ of $D(q)$. Thus, $z \in \rho(D(q))$ is equivalent to the boundedness of the operator

$$\begin{aligned} L^2(\mathbb{R})^2 \ni F(x) \longmapsto &\int_{\mathbb{R}} G(z,x,y)F(y)\,dy \\ &= \Psi_+(z,x)\int_{-\infty}^{x} \Psi_-(z,y)^\top F(y)dy \\ &\quad + \Psi_-(z,x)\int_{x}^{\infty} \Psi_+(z,y)^\top F(y)dy \end{aligned} \tag{5.arabic@equation}$$

in $L^2(\mathbb{R})$. Taking $z \in \rho(D(q))$ and $F = (f,0)^\top$ and $F = (0,f)^\top$ it follows that the operators

$$\begin{aligned} L^2(\mathbb{R}) \ni f(x) \mapsto \psi_{j,+}(z,x) &\int_{-\infty}^{x} \psi_{k,-}(z,y)f(y)dy \\ + \psi_{j,-}(z,x) &\int_{x}^{\infty} \psi_{k,+}(z,y)f(y)dy, \quad j,k=1,2 \end{aligned} \tag{5.arabic@equation}$$

are bounded in $L^2(\mathbb{R})$. The last statement implies the relations (in fact, it is equivalent to them, cf. [35] and Lemma 6.2)

$$\sup_{r\in\mathbb{R}}\left[\left(\int_{-\infty}^{r}|\psi_{k,-}(x)|^2\,dx\right)\left(\int_{r}^{\infty}|\psi_{j,+}(x)|^2\,dx\right)\right]<+\infty,\quad j,k=1,2. \tag{5.arabic@equation}$$

Indeed, we will prove next that (5.arabic@equation) follows from (5.arabic@equation). For simplicity we consider the case $j=k=1$. (The proof for the remaining combinations of indices j,k proceeds analogously, cf. also [4]). Assuming (5.arabic@equation) for $j=k=1$, there exists a constant $C>0$ such that

$$\begin{aligned}&\int_{\mathbb{R}}\left|\psi_{1,+}(x)\int_{-\infty}^{x}\psi_{1,-}(y)f(y)\,dy+\psi_{1,-}(x)\int_{x}^{\infty}\psi_{1,+}(y)f(y)dy\right|^2dx\\&\leq C\int_{\mathbb{R}}|f(x)|^2\,dx.\end{aligned} \tag{5.arabic@equation}$$

For fixed $r\in\mathbb{R}$ and $f\in L^2(\mathbb{R})$ satisfying $f(x)=0$, for all $x>r$, the inequality (5.arabic@equation) implies (restricting the interval of integration)

$$\left(\int_{r}^{\infty}|\psi_{1,+}(x)|^2\,dx\right)\left|\int_{-\infty}^{r}\psi_{1,-}(y)f(y)\,dy\right|^2\leq C\int_{-\infty}^{r}|f(x)|^2\,dx. \tag{5.arabic@equation}$$

Thus, choosing $f(x)=\overline{\psi_{1,-}(x)}$, for $x\leq r$ and $f(x)=0$ otherwise (then clearly $f\in L^2(\mathbb{R})$) one obtains (5.arabic@equation) with $j=k=1$. Since $\|\Psi_\pm(x)\|^2_{\mathbb{C}^2}=|\psi_{1,\pm}(x)|^2+|\psi_{2,\pm}(x)|^2$, (5.arabic@equation) yields (5.arabic@equation).

Finally, $\|\mathcal{K}\Psi(x)\|_{\mathbb{C}^2}=\|\Psi(x)\|_{\mathbb{C}^2}$ proves the statement about the $\overline{z}$-wave functions associated with q. □

The solutions $\Psi_\pm(z,x)$ in (5.arabic@equation) are analogs of the familiar Weyl–Titchmarsh solutions in the context of self-adjoint Dirac-type operators (cf. [31, Ch. 3]).

The following is a consequence of (5.arabic@equation) and Remark 4.4.

Corollary 5.5. *Assume $q\in L^1_{\text{loc}}(\mathbb{R})$ and suppose the* NLS$_-$ *case $p=-\overline{q}$. Then*

$$\sigma_{\mathrm{c}}(D(q))\supseteq\mathbb{R}, \tag{5.arabic@equation}$$

$$\sigma_{\mathrm{e}}(D(q))\supseteq\mathbb{R}, \tag{5.arabic@equation}$$

$$\sigma_{\mathrm{p}}(D(q))\cap\mathbb{R}=\emptyset. \tag{5.arabic@equation}$$

Proof. Relation (5.arabic@equation) holds by Remark 4.4 which excludes the existence of an $L^2(\mathbb{R})^2$ solution F of $M(q)F=\lambda F$ near $\pm\infty$ for all $\lambda\in\mathbb{R}$. To prove (5.arabic@equation) we can restrict ourselves to the case in which $\rho(D(q))\neq\emptyset$. Pick $\lambda_0\in\mathbb{R}$. Then Remark 4.4, (5.arabic@equation), and (5.arabic@equation) imply that $\lambda_0\notin\rho(D(q))\cup\sigma_{\mathrm{p}}(D(q))$. Since $\sigma_{\mathrm{r}}(D(q))=\emptyset$ by (5.arabic@equation), it follows that $\lambda_0\in\sigma_{\mathrm{c}}(D(q))$ by (5.arabic@equation). Relation (5.arabic@equation) is then obvious from (5.arabic@equation). □

As a consequence of (5.arabic@equation), our frequent assumption $z\in\rho(D(q))$ (especially in the next Section 6) automatically implies $z\in\mathbb{C}\backslash\mathbb{R}$.

Interesting restrictions on the permissible location of eigenvalues of $\mathcal{J}$-self-adjoint Dirac-type operators $D(q)$ under strong additional constraints on q were recently derived in [25].

Remark 5.6. *Given normalized Weyl–Titchmarsh-type solutions $\Psi_\pm(z,x,x_0)$ of $M(q)\Psi(z,x)=z\Psi(z,x)$ for $z\in\rho(D(q))$ satisfying*

$$\psi_{1,\pm}(z,x_0,x_0)=1, \quad z\in\rho(D(q)) \qquad (5.\text{arabic@equation})$$

for some $x_0\in\mathbb{R}$, one can formally introduce associated Weyl–Titchmarsh m-functions as follows. Denote by $\Xi(z,x,x_0)$ a normalized 2×2 fundamental system of solutions of

$$M(q)\Psi(z,x)=z\Psi(z,x), \quad z\in\mathbb{C} \qquad (5.\text{arabic@equation})$$

at some $x_0\in\mathbb{R}$, that is, $\Xi(z,x,x_0)$ satisfies (5.arabic@equation) *for a.e. $x\in\mathbb{R}$ and*

$$\Xi(z,x_0,x_0)=I_2, \quad z\in\mathbb{C}. \qquad (5.\text{arabic@equation})$$

One then partitions $\Xi(z,x,x_0)$ as

$$\Xi(z,x,x_0)=(\Theta(z,x,x_0)\ \Phi(z,x,x_0))=\begin{pmatrix}\theta_1(z,x,x_0) & \phi_1(z,x,x_0)\\ \theta_2(z,x,x_0) & \phi_2(z,x,x_0)\end{pmatrix}, \qquad (5.\text{arabic@equation})$$

where $\theta_j(z,x,x_0)$ and $\phi_j(z,x,x_0)$, $j=1,2$, are entire with respect to $z\in\mathbb{C}$ and normalized according to (5.arabic@equation)*. Then the normalized Weyl–Titchmarsh solutions $\Psi_\pm(z,x,x_0)$ can be expressed in terms of the basis $(\Theta(z,x,x_0)\ \Phi(z,x,x_0))$ as*

$$\Psi_\pm(z,x,x_0)=\Theta(z,x,x_0)+m_\pm(z,x_0)\Phi(z,x,x_0), \quad z\in\rho(D(q)) \qquad (5.\text{arabic@equation})$$

for some coefficients $m_\pm(z,x_0)$. Clearly, $m_\pm(z,x_0)$ are analytic on $\rho(D(q))$ and they are the obvious analogs of the half-line Weyl–Titchmarsh coefficients, familiar from (*second-order scalar and first-order 2×2*) *self-adjoint differential and difference operators (cf., e.g.,* [2, § VII.1], [31, Chs. 2, 3]*). It is tempting to conjecture that appropriate boundary values of $m_\pm(z,x_0)$ as z approaches $\sigma(D(q))$ encode the spectral information on $D(q)$, but this is left for future investigations.*

6. Transformation Operators for $\mathcal{J}$-Self-Adjoint Dirac-Type Operators

The principal goal in this section is to construct transformation operators in $L^2(\mathbb{R})^2$ that intertwine the $\mathcal{J}$-self-adjoint operators $D(q)$ and $D\big(q_{z_1}^{(1)}\big)$ corresponding to the Lax differential expressions $M(q)$ and $M\big(q_{z_1}^{(1)}\big)$ in the focusing NLS$_-$-case and to use these transformation operators to relate the spectra of $D(q)$ and $D\big(q_{z_1}^{(1)}\big)$, the principal goal of this paper.

In the following we always assume Hypothesis 5.3 and freely use the notation established in Section 5 for $D(q)$ as the maximally defined $\mathcal{J}$-self-adjoint Dirac operator in the focusing NLS$_-$ case and the Weyl–Titchmarsh-type solutions $\Psi_\pm(z,x)$, $z\in\rho(D(q))$, established in Theorem 5.4.

We start with an elementary but important observation.

Remark 6.1. *Since the two z-wave functions $\Psi_-(z)$ and $\Psi_+(z)$ of $D(q)$ are linearly independent,*

$$W(\Psi_-(z,x),\Psi_+(z,x))=c(z)\neq 0, \qquad (6.\text{arabic@equation})$$

all other (nontrivial) z-wave functions Ψ *satisfy*

$$\Psi(z) = \alpha\Psi_-(z) + \beta\Psi_+(z) \text{ for some } \alpha, \beta \in \mathbb{C}\backslash\{0\}. \qquad \text{(6.arabic@equation)}$$

In addition, as we will prove next,

$$\|\Psi(z,\cdot)\|_{\mathbb{C}^2}^{-1} \in L^2(\mathbb{R}). \qquad \text{(6.arabic@equation)}$$

Indeed, $\Psi(z) = \alpha\Psi_-(z) + \beta\Psi_+(z)$ *and hence*

$$\|\Psi(z)\|_{\mathbb{C}^2}^{-1} \leq \big||\alpha|\|\Psi_+(z)\|_{\mathbb{C}^2} - |\beta|\|\Psi_-(z)\|_{\mathbb{C}^2}\big|^{-1}, \qquad \text{(6.arabic@equation)}$$

(5.arabic@equation)*, and the fact that* $\Psi_\pm(z,\cdot) \in AC_{\mathrm{loc}(\mathbb{R})^2}$*, yield the existence of constants* $C_\pm > 0$ *such that*

$$\begin{aligned}\|\Psi(z,x)\|_{\mathbb{C}^2}^{-1} &= \big(\|\Psi_\pm(z,x)\|_{\mathbb{C}^2}\|\Psi(z,x)\|_{\mathbb{C}^2}^{-1}\big)\|\Psi_\pm(z,x)\|_{\mathbb{C}^2}^{-1} \\ &\leq C_\pm\|\Psi_\pm(z,x)\|_{\mathbb{C}^2}^{-1}, \quad x \in \mathbb{R}. \qquad \text{(6.arabic@equation)}\end{aligned}$$

By (5.arabic@equation) *this implies that all z-wave functions* $\Psi(z)$ *associated with* q*, except* $\Psi_\pm(z)$*, satisfy* (4.arabic@equation)*. Hence, Hypothesis* 5.3 *guarantees the existence of z-wave functions* $\Psi(z)$ *satisfying Hypothesis* 4.2 *for all* $z \in \rho(D(q))$*. In particular, all but two z-wave functions of* $D(q)$ *(viz.,* $\Psi_\pm(z)$*) satisfy Hypothesis* 4.2 *at* $z \in \rho(D(q))$*.*

Without loss of generality we will restrict our considerations to the special case $\alpha = \beta = 1$ in (6.arabic@equation) for the remainder of this section up to (6.arabic@equation), that is, we choose

$$\Psi(z) = \Psi_-(z) + \Psi_+(z) \qquad \text{(6.arabic@equation)}$$

in the following.

Next, we pick some fixed $z_1 \in \rho(D(q))$. We take $\Psi(z_1)$ as in (6.arabic@equation), $\Psi(z_1) = \Psi_-(z_1)+\Psi_+(z_1)$, where $\Psi_\pm(z_1)$ satisfy (5.arabic@equation)–(5.arabic@equation) with z replaced by z_1, and let $\mathcal{K}\Psi(z_1)$ be the corresponding $\overline{z_1}$-wave function associated with q. By Theorem 5.4, $\mathcal{K}\Psi_\pm(z_1,\cdot)$ satisfy (5.arabic@equation)–(5.arabic@equation) with z replaced by $\overline{z_1}$. Moreover,

$$\mathcal{K}\Psi(z_1) = \mathcal{K}\Psi_-(z_1) + \mathcal{K}\Psi_+(z_1). \qquad \text{(6.arabic@equation)}$$

Define the NLS potential $q_{z_1}^{(1)}$ as in (2.arabic@equation) and consider the z_1-wave function $\Phi_{z_1}^{(1)}(z_1) = (\phi_1^{(1)}(z_1), \phi_2^{(1)}(z_1))^\top$ associated with $q_{z_1}^{(1)}$ as defined in (4.arabic@equation),

$$\Phi_{z_1}^{(1)}(z_1,x) = \mathrm{Im}(z_1)\|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-2}\mathcal{K}\Psi(z_1,x), \qquad \text{(6.arabic@equation)}$$

and the $\overline{z_1}$-wave function $\mathcal{K}\Phi_{z_1}^{(1)}(z_1)$ associated with $q_{z_1}^{(1)}$ as defined in (4.arabic@equation),

$$\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x) = -\mathrm{Im}(z_1)\|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-2}\Psi(z_1,x). \qquad \text{(6.arabic@equation)}$$

We recall that according to (4.arabic@equation) the new NLS$_-$ potential is then given by

$$q_{z_1}^{(1)}(x) = q(x) + 4\phi_1^{(1)}(z_1,x)\psi_1(z_1,x). \qquad \text{(6.arabic@equation)}$$

Of course, both $D(q)$ and $D\big(q_{z_1}^{(1)}\big)$ (associated with the differential expressions $M(q)$ and $M\big(q_{z_1}^{(1)}\big)$, respectively) are $\mathcal{J}$-self-adjoint by Theorem 3.5 and Corollary 2.9.

In order to motivate the introduction of transformation operators one can argue as follows. Since

$$\Gamma(z,\Psi(z_1),\mathcal{K}\Psi(z_1)) = -(i/2)(z-z_1)I_2 + \Phi_{z_1}^{(1)}(z_1)\Psi(z_1)^{\perp}, \quad (6.\text{arabic@equation})$$

one computes, for every z-wave function $\Upsilon(z,\cdot)$ associated with q,

$$\begin{aligned}
&\Upsilon_{z_1}^{(1)}(z,x) = \Gamma(z,x,\Psi(z_1),\mathcal{K}\Psi(z_1))\Upsilon(z,x) \\
&= -(i/2)(z-z_1)\Upsilon(z,x) + \Phi_{z_1}^{(1)}(z_1,x)\Psi(z_1,x)^{\perp}\Upsilon(z,x) \\
&= -(i/2)(z-z_1)\Upsilon(z,x) - \Phi_{z_1}^{(1)}(z_1,x)W(\Psi(z_1,x),\Upsilon(z,x)) \\
&= -(i/2)(z-z_1)\Upsilon(z,x) - \Phi_{z_1}^{(1)}(z_1,x)W(\Psi_-(z_1,x)+\Psi_+(z_1,x),\Upsilon(z,x)) \\
&= -(i/2)(z-z_1)\bigg\{\Upsilon(z,x) + 2\Phi_{z_1}^{(1)}(z_1,x) \\
&\quad \times \bigg[\int_{-a}^{x} dx'\, \Psi_-(z_1,x')^{\top}\sigma_1\Upsilon(z,x') - \int_{x}^{a} dx'\, \Psi_+(z_1,x')^{\top}\sigma_1\Upsilon(z,x')\bigg]\bigg\} \qquad (6.\text{arabic@equation}) \\
&\quad - \Phi_{z_1}^{(1)}(z_1,x)[W(\Psi_-(z_1,-a),\Upsilon(z,-a)) + W(\Psi_+(z_1,a),\Upsilon(z,a))], \quad a>0.
\end{aligned}$$

To arrive at (6.arabic@equation), we used the integrated form of (2.arabic@equation). Replacing $\Upsilon(z)$ by $F \in L^2(\mathbb{R})^2$ in (6.arabic@equation), noting that

$$\liminf_{a\uparrow\infty} |W(\Psi_\pm(z_1,\pm a),F(\pm a))| = 0, \quad F \in L^2(\mathbb{R})^2, \qquad (6.\text{arabic@equation})$$

and repeating the same argument with $\Upsilon(z)$ replaced by $\mathcal{K}\Upsilon(z)$, then leads to the introduction of the following transformation operators T_{z_1} and $\widetilde{T}_{\overline{z_1}}$ in $L^2(\mathbb{R})^2$,

$$\begin{aligned}
&L^2(\mathbb{R})^2 \ni F(x) \mapsto (T_{z_1}F)(x) = F(x) + 2\Phi_{z_1}^{(1)}(z_1,x) \qquad (6.\text{arabic@equation}) \\
&\qquad \times \bigg[\int_{-\infty}^{x} dx'\, \Psi_-(z_1,x')^{\top}\sigma_1 F(x') - \int_{x}^{\infty} dx'\, \Psi_+(z_1,x')^{\top}\sigma_1 F(x')\bigg], \\
&L^2(\mathbb{R})^2 \ni F(x) \mapsto (\widetilde{T}_{\overline{z_1}}F)(x) = F(x) + 2\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x) \qquad (6.\text{arabic@equation}) \\
&\qquad \times \bigg[\int_{-\infty}^{x} dx'\, (\mathcal{K}\Psi_-(z_1,x'))^{\top}\sigma_1 F(x') - \int_{x}^{\infty} dx'\, (\mathcal{K}\Psi_+(z_1,x'))^{\top}\sigma_1 F(x')\bigg], \\
&\qquad\qquad z_1 \in \rho(D(q)).
\end{aligned}$$

That T_{z_1} and $\widetilde{T}_{\overline{z_1}}$ are in fact bounded operators in $L^2(\mathbb{R})^2$ follows from Lemmas 6.2 and 6.3 below.

Lemma 6.2 (Talenti [42], Tomaselli [44] (see also [4], [35]).
Let $f \in L^2(\mathbb{R})$, $U \in L^2((-\infty,R])$, $V \in L^2([R,\infty))$ for all $R \in \mathbb{R}$. Then the following assertions (i)–(iii) are equivalent:
(i) There exists a finite constant $C > 0$ such that

$$\int_{\mathbb{R}} dx \left| U(x)\int_x^\infty dx'\, V(x')f(x')\right|^2 \le C\int_{\mathbb{R}} dx\, |f(x)|^2. \qquad (6.\text{arabic@equation})$$

(ii) There exists a finite constant $D > 0$ such that

$$\int_{\mathbb{R}} dx \left| V(x)\int_{-\infty}^x dx'\, U(x')f(x')\right|^2 \le D\int_{\mathbb{R}} dx\, |f(x)|^2. \qquad (6.\text{arabic@equation})$$

(iii)

$$\sup_{r\in\mathbb{R}}\bigg[\bigg(\int_{-\infty}^{r} dx\,|U(x)|^2\bigg)\bigg(\int_{r}^{\infty} dx\,|V(x)|^2\bigg)\bigg] < \infty. \qquad (6.\text{arabic@equation})$$

Lemma 6.3. *Assume Hypothesis* 5.3 *and* $z_1 \in \rho(D(q))$. *Then the operators*

$$L^2(\mathbb{R}) \ni f(x) \mapsto \frac{1}{\|\Psi_\pm(z_1,x)\|_{\mathbb{C}^2}}\int_{\pm\infty}^{x} dx'\,\|\Psi_\pm(z_1,x')\|_{\mathbb{C}^2} f(x'), \qquad (6.\text{arabic@equation})$$

$$L^2(\mathbb{R}) \ni f(x) \mapsto \|\Psi_\pm(z_1,x)\|_{\mathbb{C}^2}\int_{\mp\infty}^{x} dx'\,\frac{1}{\|\Psi_\pm(z_1,x')\|_{\mathbb{C}^2}} f(x') \quad (6.\text{arabic@equation})$$

are bounded in $L^2(\mathbb{R})$.

Proof. Using (5.arabic@equation), (5.arabic@equation), (5.arabic@equation), and (5.arabic@equation) one obtains

$$\sup_{r\in\mathbb{R}}\bigg(\int_{-\infty}^{r} dx\,\|\Psi_-(z_1,x)\|_{\mathbb{C}^2}^2\bigg)\bigg(\int_{r}^{\infty} dx\,\|\Psi_-(z_1,x)\|_{\mathbb{C}^2}^{-2}\bigg) < \infty \quad (6.\text{arabic@equation})$$

and

$$\sup_{r\in\mathbb{R}}\bigg(\int_{-\infty}^{r} dx\,\|\Psi_+(z_1,x)\|_{\mathbb{C}^2}^{-2}\bigg)\bigg(\int_{r}^{\infty} dx\,\|\Psi_+(z_1,x)\|_{\mathbb{C}^2}^{2}\bigg) < \infty. \qquad (6.\text{arabic@equation})$$

By Lemma 6.2, (6.arabic@equation) is a necessary and sufficient condition for the operators associated with $\Psi_-(z_1,\cdot)$ in (6.arabic@equation) and (6.arabic@equation) to be bounded in $L^2(\mathbb{R})$. Similarly, (6.arabic@equation) is a necessary and sufficient condition for the operators associated with $\Psi_+(z_1,\cdot)$ in (6.arabic@equation) and (6.arabic@equation) to be bounded in $L^2(\mathbb{R})$. ☐

Thus, (6.arabic@equation), (4.arabic@equation), and (4.arabic@equation) imply the following result.

Corollary 6.4. *Assume Hypothesis* 5.3 *and* $z_1 \in \rho(D(q))$. *Then the operators* T_{z_1} *and* $\widetilde{T}_{\overline{z_1}}$ *defined in* (6.arabic@equation) *and* (6.arabic@equation) *are bounded operators in* $L^2(\mathbb{R})^2$.

Proof. By (6.arabic@equation) one obtains $\big\|\Phi_{z_1}^{(1)}(z_1,x)\big\|_{\mathbb{C}^2} = |\mathrm{Im}(z_1)|\|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-1}$. Applying (6.arabic@equation) then yields the existence of constants $C_\pm > 0$ such that

$$\begin{aligned}\big\|\Phi_{z_1}^{(1)}(z_1,x)\big\|_{\mathbb{C}^2} &= |\mathrm{Im}(z_1)|\|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-1}\\ &\le |\mathrm{Im}(z_1)|C_\pm\|\Psi_\pm(z_1,x)\|_{\mathbb{C}^2}^{-1}, \quad x\in\mathbb{R}. \qquad (6.\text{arabic@equation})\end{aligned}$$

At this point an application of Lemma 6.3 proves the boundedness of T_{z_1} in $L^2(\mathbb{R})^2$. Since $\|\mathcal{K}G\|_{\mathbb{C}^2} = \|G\|_{\mathbb{C}^2}$ for all $G\in\mathbb{C}^2$, the same arguments prove boundedness of $\widetilde{T}_{\overline{z_1}}$ in $L^2(\mathbb{R})^2$. ☐

In order to motivate the introduction of the inverse transformation operators, one inverts the matrix $\Gamma(z,\Psi(z_1),\mathcal{K}\Psi(z_1))$ to obtain

$$\Gamma(z,\Psi(z_1),\mathcal{K}\Psi(z_1))^{-1} = c(z,z_1)\big[-(i/2)(z-\overline{z_1})I_2 + \mathcal{K}\Psi(z_1)\mathcal{K}\Phi_{z_1}^{(1)}(z_1)^{\perp}\big], \qquad (6.\text{arabic@equation})$$

where

$$c(z,z_1) = -4(z-z_1)^{-1}(z-\overline{z}_1)^{-1}. \tag{6.arabic@equation}$$

Thus, for z-wave functions $\Upsilon_{z_1}^{(1)}(z,x)$ associated with $q_{z_1}^{(1)}$,

$$\begin{aligned}
&\Upsilon(z,x) = \Gamma(z,x,\Psi(z_1),\mathcal{K}\Psi(z_1))^{-1}\Upsilon_{z_1}^{(1)}(z,x)\\
&= c(z,z_1)\big[-(i/2)(z-\overline{z_1})\Upsilon_{z_1}^{(1)}(z,x) + \mathcal{K}\Psi(z_1,x)\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x)^{\perp}\Upsilon_{z_1}^{(1)}(z,x)\big]\\
&= c(z,z_1)\big[-(i/2)(z-\overline{z_1})\Upsilon_{z_1}^{(1)}(z,x) - \mathcal{K}\Psi(z_1,x)W\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x),\Upsilon_{z_1}^{(1)}(z,x)\big)\big]\\
&= c(z,z_1)\big\{-(i/2)(z-\overline{z_1})\Upsilon_{z_1}^{(1)}(z,x)\\
&\qquad - [\mathcal{K}\Psi_-(z_1,x) + \mathcal{K}\Psi_+(z_1,x)]W\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x),\Upsilon_{z_1}^{(1)}(z,x)\big)\big\}\\
&= 2i(z-z_1)^{-1}\bigg\{\Upsilon_{z_1}^{(1)}(z,x) + 2\mathcal{K}\Psi_+(z_1,x)\int_{-a}^{x} dx'\,\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^{\top}\sigma_1\Upsilon_{z_1}^{(1)}(z,x')\\
&\qquad - 2\mathcal{K}\Psi_-(z_1,x)\int_{x}^{a} dx'\,\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^{\top}\sigma_1\Upsilon_{z_1}^{(1)}(z,x')\bigg\}\\
&\quad - c(z,z_1)\big[\mathcal{K}\Psi_-(z_1,x)W\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,-a),\Upsilon_{z_1}^{(1)}(z,-a)\big)\\
&\qquad + \mathcal{K}\Psi_+(z_1,x)W\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,a),\Upsilon_{z_1}^{(1)}(z,a)\big)\big], \quad a>0.
\end{aligned} \tag{6.arabic@equation}$$

Replacing $\Upsilon_{z_1}^{(1)}(z)$ by $F \in L^2(\mathbb{R})^2$ in (6.arabic@equation), and repeating the same argument for $\mathcal{K}\Upsilon_{z_1}^{(1)}(z)$ instead of $\Upsilon_{z_1}^{(1)}(z)$, then leads to the introduction of the following (inverse) transformation operators $\widehat{S}_{\overline{z_1}}$ and $\widetilde{S}_{z_1}$ in $L^2(\mathbb{R})^2$,

$$\begin{aligned}
L^2(\mathbb{R})^2 \ni F(x) \mapsto (\widehat{S}_{\overline{z_1}}F)(x) &= F(x)\\
&+ 2\mathcal{K}\Psi_+(z_1,x)\int_{-\infty}^{x} dx'\,\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^{\top}\sigma_1 F(x')\\
&- 2\mathcal{K}\Psi_-(z_1,x)\int_{x}^{\infty} dx'\,\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^{\top}\sigma_1 F(x'),
\end{aligned} \tag{6.arabic@equation}$$

$$\begin{aligned}
L^2(\mathbb{R})^2 \ni F(x) \mapsto (\widetilde{S}_{z_1}F)(x) &= F(x)\\
&+ 2\Psi_+(z_1,x)\int_{-\infty}^{x} dx'\,\Phi_{z_1}^{(1)}(z_1,x')^{\top}\sigma_1 F(x')
\end{aligned} \tag{6.arabic@equation}$$

$$- 2\Psi_-(z_1,x)\int_{x}^{\infty} dx'\,\Phi_{z_1}^{(1)}(z_1,x')^{\top}\sigma_1 F(x'), \quad z_1 \in \rho(D(q)).$$

Equations (6.arabic@equation), (4.arabic@equation), and (4.arabic@equation) then imply the following result.

Corollary 6.5. *Assume Hypothesis* 5.3 *and* $z_1 \in \rho(D(q))$. *Then the operators* $\widehat{S}_{\overline{z_1}}$ *and* $\widetilde{S}_{z_1}$ *defined in* (6.arabic@equation) *and* (6.arabic@equation) *are bounded operators in* $L^2(\mathbb{R})^2$.

Proof. Using again the estimate (6.arabic@equation), one can follow the arguments in the proof of Corollary 6.4. ∎

We define two vectors $F, G \in L^2(\mathbb{R})^2$ to be $\mathcal{J}$-orthogonal if

$$(F, \mathcal{J}G)_{L^2} = 0, \tag{6.arabic@equation}$$

and we then write

$$F \perp_{\mathcal{J}} G. \tag{6.arabic@equation}$$

Since

$$(F(x), \mathcal{J}G(x))_{\mathbb{C}^2} = (\mathcal{J}G(x))^* F(x) = G(x)^\top \sigma_1 F(x), \tag{6.arabic@equation}$$

$$F \perp_{\mathcal{J}} G \text{ is equivalent to } (F, \mathcal{J}G)_{L^2} = \int_{\mathbb{R}} dx\, G(x)^\top \sigma_1 F(x) = 0, \quad F, G \in L^2(\mathbb{R})^2.$$

We also introduce the following notation of the $\mathcal{J}$-orthogonal complement $\mathcal{V}^{\perp_{\mathcal{J}}}$ to a subset $\mathcal{V} \subset L^2(\mathbb{R})^2$,

$$\mathcal{V}^{\perp_{\mathcal{J}}} = \{F \in L^2(\mathbb{R})^2 \,|\, F \perp_{\mathcal{J}} G \text{ for all } G \in \mathcal{V}\}. \tag{6.arabic@equation}$$

In analogy to the orthogonality property of eigenvectors corresponding to different (necessarily real) eigenvalues of a symmetric operator in some complex Hilbert space $\mathcal{H}$, one infers the $\mathcal{J}$-orthogonality of eigenvectors corresponding to different eigenvalues of a $\mathcal{J}$-symmetric operator S in $\mathcal{H}$. Indeed, $Sf_j = z_j f_j$, $j = 1, 2$, with $z_1 \neq z_2$ implies

$$\overline{z_2}(\mathcal{J}f_1, f_2)_{\mathcal{H}} = (\mathcal{J}f_1, Sf_2)_{\mathcal{H}} = (\mathcal{J}Sf_1, f_2)_{\mathcal{H}} = (\mathcal{J}z_1 f_1, f_2)_{\mathcal{H}} = \overline{z_1}(\mathcal{J}f_1, f_2)_{\mathcal{H}} \tag{6.arabic@equation}$$

and hence

$$(z_1 - z_2)(f_1, \mathcal{J}f_2)_{\mathcal{H}} = 0, \text{ implying } (f_1, \mathcal{J}f_2)_{\mathcal{H}} = 0. \tag{6.arabic@equation}$$

Next, let σ_0 be an isolated subset of $\sigma\big(D\big(q_{z_1}^{(1)}\big)\big)$ in the sense that σ_0 can be surrounded by a positively oriented, rectifiable, simple, closed countour $\gamma_{\sigma_0} \subset \rho\big(D\big(q_{z_1}^{(1)}\big)\big)$ separating σ_0 from the remaining spectrum $\sigma\big(D\big(q_{z_1}^{(1)}\big)\big)\backslash\sigma_0$. Then the Riesz projection onto the spectral subspace corresponding to σ_0 is given by

$$P_{z_1}^{(1)}(\sigma_0) = -\frac{1}{2\pi i}\int_{\gamma_{\sigma_0}} dz\, \big(D\big(q_{z_1}^{(1)}\big) - zI_2\big)\big)^{-1}. \tag{6.arabic@equation}$$

The spectral subspace $\Sigma_{z_1}^{(1)}(\sigma_0)$ corresponding to σ_0 is then the range of the Riesz projection, $\Sigma_{z_1}^{(1)}(\sigma_0) = P_{z_1}^{(1)}(\sigma_0)L^2(\mathbb{R})^2$.

We record the following result.

Lemma 6.6. *Assume Hypothesis* 5.3.
(i) Let $z_0, \tilde{z}_0 \in \sigma_p\big(D\big(q_{z_1}^{(1)}\big)\big)$, $z_0 \neq \tilde{z}_0$. Then the $L^2(\mathbb{R})^2$-eigenfunctions $\Phi_{z_1}^{(1)}(z_0)$, $\Phi_{z_1}^{(1)}(\tilde{z}_0)$ corresponding to z_0 and $\tilde{z}_0$, respectively, are $\mathcal{J}$-orthogonal,

$$\Phi_{z_1}^{(1)}(z_0) \perp_{\mathcal{J}} \Phi_{z_1}^{(1)}(\tilde{z}_0). \tag{6.arabic@equation}$$

(ii) Let $z_0 \in \sigma_p\big(D\big(q_{z_1}^{(1)}\big)\big)$ and σ_0 an isolated subset of $\sigma\big(D\big(q_{z_1}^{(1)}\big)\big)$ which does not contain z_0. Then the $L^2(\mathbb{R})^2$-eigenfunction $\Phi_{z_1}^{(1)}(z_0)$ corresponding to z_0 is $\mathcal{J}$-orthogonal to the spectral subspace $\Sigma_{z_1}^{(1)}(\sigma_0)$ corresponding to σ_0,

$$\Phi_{z_1}^{(1)}(z_0) \perp_{\mathcal{J}} \Sigma_{z_1}^{(1)}(\sigma_0). \tag{6.arabic@equation}$$

Proof. Assertion (i) is clear from (6.arabic@equation) and (6.arabic@equation). To prove (ii), one chooses a positively oriented, rectifiable, simple, closed contour γ_{σ_0} which separates σ_0 and $\{z_0\}$. For $F \in L^2(\mathbb{R})^2$ one then computes

$$\begin{aligned}
\big(P_{z_1}^{(1)}(\sigma_0)F, \mathcal{J}\Phi_{z_1}^{(1)}(z_0)\big)_{L^2} &= -\frac{1}{2\pi i}\int_{\gamma_{\sigma_0}} dz\, \big(D\big(q_{z_1}^{(1)}\big) - zI_2\big)^{-1}F, \mathcal{J}\Phi_{z_1}^{(1)}(z_0)\big)_{L^2} \\
&= -\frac{1}{2\pi i}\int_{\gamma_{\sigma_0}} dz\, \big(F, \mathcal{J}\big(D\big(q_{z_1}^{(1)}\big) - zI_2\big)^{-1}\Phi_{z_1}^{(1)}(z_0)\big)_{L^2} \\
&= -\frac{1}{2\pi i}\int_{\gamma_{\sigma_0}} dz\, (z_0 - z)^{-1}\big(F, \mathcal{J}\Phi_{z_1}^{(1)}(z_0)\big)_{L^2} = 0.
\end{aligned}$$

Here we used the fact that since $D\big(q_{z_1}^{(1)}\big)$ is $\mathcal{J}$-self-adjoint, so is $\big(D\big(q_{z_1}^{(1)}\big) - zI_2\big)^{-1}$. □

Remark 6.7. *From Lemma* 6.6 *and Theorem* 4.5 *one concludes that*

$$\Phi_{z_1}^{(1)}(z_1) \perp_{\mathcal{J}} \mathcal{K}\Phi_{z_1}^{(1)}(z_1) \qquad (6.\text{arabic@equation})$$

since z_1 *is a nonreal eigenvalue. Thus,*

$$\Phi_{z_1}^{(1)}(z_1) \in \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}. \qquad (6.\text{arabic@equation})$$

Remark 6.8. *By Corollary* 6.5, *the operator* $\widehat{S}_{\overline{z_1}}$, $z_1 \in \rho(D(q))$, *is well-defined and bounded in* $L^2(\mathbb{R})^2$. *However, for future considerations it is more appropriate to restrict it to the closed subspace* $\big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}$. *Hence from now on, we will denote by* $S_{\overline{z_1}}$ *the restriction of* $\widehat{S}_{\overline{z_1}}$ *to* $\big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}$,

$$S_{\overline{z_1}} = \widehat{S}_{\overline{z_1}}\big|_{\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\}^{\perp_{\mathcal{J}}}}, \quad z_1 \in \rho(D(q)). \qquad (6.\text{arabic@equation})$$

An elementary computation (*based on* (6.arabic@equation)) *then reveals that*

$$(S_{\overline{z_1}}G)(x) = (\widehat{S}_{\overline{z_1}}G)(x) = G(x) + 2\mathcal{K}\Psi(z_1, x)\int_{-\infty}^{x} dx'\, \big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1, x')\big)^{\top}\sigma_1 G(x'),$$
$$G \in \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}. \qquad (6.\text{arabic@equation})$$

Next, we prove several results leading up to the principal theorems of this section.

Lemma 6.9. *Assume Hypothesis* 5.3 *and* $z_1 \in \rho(D(q))$. *Then*

$$\operatorname{ran}(T_{z_1}) \subseteq \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}, \quad \Phi_{z_1}^{(1)}(z_1) \in \ker(S_{\overline{z_1}}), \qquad (6.\text{arabic@equation})$$

$$\big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2} = \operatorname{Im}(z_1)(2W(\Psi_+(z_1), \Psi_-(z_1)))^{-1} \neq 0, \qquad (6.\text{arabic@equation})$$

$$T_{z_1}S_{\overline{z_1}}G = G - \frac{\big(G, \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}}{\big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}}\Phi_{z_1}^{(1)}(z_1), \quad G \in \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}, \qquad (6.\text{arabic@equation})$$

$$S_{\overline{z_1}}T_{z_1}F = F, \quad F \in L^2(\mathbb{R})^2. \qquad (6.\text{arabic@equation})$$

Proof. For brevity, we introduce

$$\xi(x;F) = \int_{-\infty}^{x} dx'\, \Psi_-(z_1,x')^\top \sigma_1 F(x') - \int_x^\infty dx'\, \Psi_+(z_1,x')^\top \sigma_1 F(x'), \tag{6.arabic@equation}$$

$$F \in L^2(\mathbb{R})^2,\ x \in \mathbb{R},$$

such that

$$(T_{z_1}F)(x) = F(x) + 2\Phi_{z_1}^{(1)}(z_1,x)\xi(x;F), \quad F \in L^2(\mathbb{R})^2. \tag{6.arabic@equation}$$

Using

$$\xi_x(x;F) = \Psi(z_1,x)^\top \sigma_1 F(x) \text{ for a.e. } x \in \mathbb{R}, \tag{6.arabic@equation}$$

one computes

$$\begin{aligned}
&\big(T_{z_1}F, \mathcal{J}\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2} = \big(F, \mathcal{J}\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2} + 2\big(\Phi_{z_1}^{(1)}(z_1)\xi(\cdot;F), \mathcal{J}\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2} \\
&= \big(F, \mathcal{J}\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2} - (\mathrm{Im}(z_1))^{-1} \int_{\mathbb{R}} dx\, \big(\big\|\Phi_{z_1}^{(1)}(z_1,x)\big\|_{\mathbb{C}^2}^2\big)_x \xi(x;F) \\
&= \big(F, \mathcal{J}\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2} + (\mathrm{Im} z_1)^{-1} \int_{\mathbb{R}} dx\, \big\|\Phi_{z_1}^{(1)}(z_1,x)\big\|_{\mathbb{C}^2}^2 \Psi(z_1,x)^\top \sigma_1 F(x) \\
&= \big(F, \mathcal{J}\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2} + \overline{\mathrm{Im}(z_1)} \int_{\mathbb{R}} dx\, \|\Psi(z_1,x)\|_{\mathbb{C}}^{-2} \Psi(z_1,x)^\top \sigma_1 F(x) \\
&= 0, \quad F \in L^2(\mathbb{R})^2.
\end{aligned} \tag{6.arabic@equation}$$

To justify the integration by parts step in (6.arabic@equation) one can argue as follows. By (6.arabic@equation), the estimate

$$\begin{aligned}
&\big\|\Phi_{z_1}^{(1)}(z_1,x)\big\|_{\mathbb{C}^2}^2 |\xi(x;F)| \\
&\le |\mathrm{Im}(z_1)|^2 \|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-2} \bigg[\int_{-\infty}^x dx'\, \|\Psi_-(z_1,x')\|_{\mathbb{C}^2} \|F(x')\|_{\mathbb{C}^2} \\
&\qquad + \int_x^\infty dx'\, \|\Psi_+(z_1,x')\|_{\mathbb{C}^2} \|F(x')\|_{\mathbb{C}^2}\bigg] \\
&\le |\mathrm{Im}(z_1)|^2 \|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-1} \bigg[C_- \|\Psi_-(z_1,x)\|_{\mathbb{C}^2}^{-1} \int_{-\infty}^x dx'\, \|\Psi_-(z_1,x')\|_{\mathbb{C}^2} \|F(x')\|_{\mathbb{C}^2} \\
&\qquad + C_+ \|\Psi_+(z_1,x)\|_{\mathbb{C}^2}^{-1} \int_x^\infty dx'\, \|\Psi_+(z_1,x')\|_{\mathbb{C}^2} \|F(x')\|_{\mathbb{C}^2}\bigg]
\end{aligned} \tag{6.arabic@equation}$$

yields

$$\big\|\Phi_{z_1}^{(1)}(z_1,\cdot)\big\|_{\mathbb{C}^2}^2 |\xi(\cdot;F)| \in L^1(\mathbb{R}), \tag{6.arabic@equation}$$

using (6.arabic@equation) and (6.arabic@equation). Thus,

$$\liminf_{x\to\pm\infty} \big\|\Phi_{z_1}^{(1)}(z_1,x)\big\|_{\mathbb{C}^2}^2 |\xi(x;F)| = 0 \tag{6.arabic@equation}$$

(actually, $\lim_{x\to\pm\infty} |\cdots| = 0$ in (6.arabic@equation) since all Lebesgue integrals involved in (6.arabic@equation) are finite), which was to be proven. Hence, one concludes $\mathcal{K}\Phi_{z_1}^{(1)}(z_1) \perp_{\mathcal{J}} \mathrm{ran}(T_{z_1})$.

Next, using (2.arabic@equation), Lemma 2.2 (iii), (6.arabic@equation), (6.arabic@equation), and (6.arabic@equation), one computes

$$\begin{aligned}
&\big(S_{\overline{z_1}}\Phi_{z_1}^{(1)}(z_1)\big)(x) = \Phi_{z_1}^{(1)}(z_1,x) + 2\mathcal{K}\Psi(z_1,x)\int_{-\infty}^{x} dx'\,\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^\top \sigma_1 \Phi^{(1)}(z_1,x') \\
&= \Phi_{z_1}^{(1)}(z_1,x) - 2\mathcal{K}\Psi(z_1,x)(2\mathrm{Im}(z_1))^{-1} W\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x), \Phi_{z_1}^{(1)}(z_1,x)\big) \\
&= \Phi_{z_1}^{(1)}(z_1,x) - \mathcal{K}\Psi(z_1,x)(\mathrm{Im}(z_1))^{-1}\big\|\Phi_{z_1}^{(1)}(z_1,x)\big\|_{\mathbb{C}^2}^2 \\
&= \Phi_{z_1}^{(1)}(z_1,x) - \mathrm{Im}(z_1)\|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-2}\mathcal{K}\Psi(z_1,x) \\
&= 0 \qquad\qquad (6.\text{arabic@equation})
\end{aligned}$$

and hence $\Phi_{z_1}^{(1)}(z_1) \in \ker(S_{\overline{z_1}})$.

For the proof of (6.arabic@equation) we next assume $G \in \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}$. Using

$$2\mathrm{Im}(z)\Psi_\mp(z_1,x)^\top\sigma_1\mathcal{K}\Psi(z_1,x) = W(\Psi_\mp(z_1,x),\mathcal{K}\Psi(z,x))_x \qquad (6.\text{arabic@equation})$$

(cf. (2.arabic@equation)), one then computes

$$\begin{aligned}
&(T_{z_1}S_{\overline{z_1}}G)(x) = (S_{\overline{z_1}}G)(x) + 2\Phi_{z_1}^{(1)}(z_1,x)\xi(x;S_{\overline{z_1}}G) \\
&= G(x) + 2\mathcal{K}\Psi(z_1,x)\int_{-\infty}^{x} dx'\,\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^\top\sigma_1 G(x') + 2\Phi_{z_1}^{(1)}(z_1,x)\xi(x;G) \\
&\quad + 2(\mathrm{Im}(z_1))^{-1}\Phi_{z_1}^{(1)}(z_1,x) \qquad\qquad (6.\text{arabic@equation}) \\
&\qquad \times\bigg[\int_{-\infty}^{x} dx'\,W(\Psi_-(z_1,x'),\mathcal{K}\Psi(z_1,x'))_{x'}\int_{-\infty}^{x'} dx''\,\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x'')\big)^\top\sigma_1 G(x'') \\
&\qquad\quad - \int_{x}^{\infty} dx'\,W(\Psi_+(z_1,x'),\mathcal{K}\Psi(z_1,x'))_{x'}\int_{-\infty}^{x'} dx''\,\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x'')\big)^\top\sigma_1 G(x'')\bigg].
\end{aligned}$$

By (6.arabic@equation) one estimates

$$\begin{aligned}
&\bigg|W(\Psi_\mp(z_1,x),\mathcal{K}\Psi(z_1,x))\int_{\mp\infty}^{x} dx'\,\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^\top\sigma_1 G(x')\bigg| \\
&\le \|\Psi_\mp(z_1,x)\|_{\mathbb{C}^2}\|\Psi(z_1,x)\|_{\mathbb{C}^2}\int_{\mp\infty}^{x} dx'\,\big\|\Phi_{z_1}^{(1)}(z_1,x')\big\|_{\mathbb{C}^2}\|G(x')\|_{\mathbb{C}^2} \\
&\le \|\Psi_\mp(z_1,x)\|_{\mathbb{C}^2}\|\Psi(z_1,x)\|_{\mathbb{C}^2}|\mathrm{Im}(z_1)|\int_{\mp\infty}^{x} dx'\,\big\|\Psi(z_1,x')\big\|_{\mathbb{C}^2}^{-1}\|G(x')\|_{\mathbb{C}^2} \\
&\le |\mathrm{Im}(z_1)|C_\pm\|\Psi_\mp(z_1,x)\|_{\mathbb{C}^2}[\|\Psi_-(z_1,x)\|_{\mathbb{C}^2} + \|\Psi_+(z_1,x)\|_{\mathbb{C}^2}] \\
&\quad \times\int_{\mp\infty}^{x} dx'\,\big\|\Psi_\pm(z_1,x')\big\|_{\mathbb{C}^2}^{-1}\|G(x')\|_{\mathbb{C}^2}, \qquad\qquad (6.\text{arabic@equation})
\end{aligned}$$

and hence the left-hand side of (6.arabic@equation) is in $L^1((\mp\infty,R])$ for all $R\in\mathbb{R}$ by Lemma 6.3. Thus,

$$\liminf_{x\to\mp\infty}\bigg|W(\Psi_\mp(z_1,x),\mathcal{K}\Psi(z_1,x))\int_{\mp\infty}^{x} dx'\,\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^\top\sigma_1 G(x')\bigg| = 0. \qquad (6.\text{arabic@equation})$$

An integration by parts, using (6.arabic@equation) and

$$(\mathrm{Im}(z_1))^{-1}\Phi_{z_1}^{(1)}(z_1,x)W(\Psi(z_1,x),\mathcal{K}\Psi(z_1,x)) = -\mathcal{K}\Psi(z_1,x), \qquad (6.\text{arabic@equation})$$

then yields

$$\begin{aligned}(T_{z_1}S_{\overline{z_1}}G)(x) &= G(x) + 2\Phi_{z_1}^{(1)}(z_1,x)\xi(x;G) - 2(\mathrm{Im}(z_1))^{-1}\Phi_{z_1}^{(1)}(z_1,x)\\ &\quad\times\bigg[\int_{-\infty}^{x} dx'\, W(\Psi_-(z_1,x'),\mathcal{K}\Psi(z_1,x'))\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^\top\sigma_1 G(x')\\ &\qquad - \int_x^\infty dx'\, W(\Psi_+(z_1,x'),\mathcal{K}\Psi(z_1,x'))\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^\top\sigma_1 G(x')\bigg]\\ &= G(x) + 2\Phi_{z_1}^{(1)}(z_1,x)\\ &\quad\times\bigg[\int_{-\infty}^{x} dx'\,\Theta_-(z_1,x')^\top\sigma_1 G(x') - \int_x^\infty dx'\,\Theta_+(z_1,x')^\top\sigma_1 G(x')\bigg].\end{aligned} \tag{6.arabic@equation}$$

(Again, $\lim_{x\to\pm\infty}|\cdots|=0$ in (6.arabic@equation) since all Lebesgue integrals involved are finite). Here we introduced the abbreviation

$$\Theta_\pm(z_1,x) = \Psi_\pm(z_1,x) + \Psi(z_1,x)\|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-2} W(\Psi_\pm(z_1,x),\mathcal{K}\Psi(z_1,x)). \tag{6.arabic@equation}$$

Actually, using the Jacobi identity

$$AW(B,C) + BW(C,A) + CW(A,B) = 0, \quad A,B,C\in\mathbb{C}^2, \tag{6.arabic@equation}$$

one infers

$$\Psi_\pm(z_1)W(\mathcal{K}\Psi(z_1),\Psi(z_1)) + \Psi(z_1)W(\Psi_\pm(z_1),\mathcal{K}\Psi(z_1)) = \mathcal{K}\Psi(z_1)W(\Psi_\pm(z_1),\Psi(z_1)), \tag{6.arabic@equation}$$

which in turn implies

$$\Theta_\pm(z_1,x) = (\mathrm{Im}(z_1))^{-1}W(\Psi_\pm(z_1,x),\Psi_\mp(z_1,x))\Phi_{z_1}^{(1)}(z_1,x). \tag{6.arabic@equation}$$

Combining (6.arabic@equation) and (6.arabic@equation) results in

$$\begin{aligned}(T_{z_1}S_{\overline{z_1}}G)(x) &= G(x) + 2(\mathrm{Im}(z_1))^{-1}W(\Psi_-(z_1,x),\Psi_+(z_1,x))\Phi_{z_1}^{(1)}(z_1,x)\\ &\quad\times\int_{\mathbb{R}} dx'\,\Phi_{z_1}^{(1)}(z_1,x')^\top\sigma_1 G(x').\end{aligned} \tag{6.arabic@equation}$$

Thus,

$$\begin{aligned}(T_{z_1}S_{\overline{z_1}}G)(x) &= G(x)\\ &\quad + 2(\mathrm{Im}(z_1))^{-1}W(\Psi_-(z_1,x),\Psi_+(z_1,x))\big(G,\mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}\Phi_{z_1}^{(1)}(z_1,x).\end{aligned} \tag{6.arabic@equation}$$

Applying (6.arabic@equation) to $G=\Phi_{z_1}^{(1)}(z_1)\in\big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}$, one obtains the relation

$$0 = 1 + 2(\mathrm{Im}(z_1))^{-1}W(\Psi_-(z_1,x),\Psi_+(z_1,x))\big(\Phi_{z_1}^{(1)}(z_1),\mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}, \tag{6.arabic@equation}$$

which proves (6.arabic@equation). Insertion of (6.arabic@equation) into (6.arabic@equation) proves (6.arabic@equation).

Finally, let $F \in L^2(\mathbb{R})^2$. Then $T_{z_1}F \in \big\{\mathcal{K}\Phi^{(1)}_{z_1}(z_1)\big\}^{\perp_{\mathcal{J}}}$ by (6.arabic@equation) and hence (6.arabic@equation) yields

$$\begin{aligned}
&(S_{\overline{z_1}}T_{z_1}F)(x) = F(x) + 2\Phi^{(1)}_{z_1}(z_1,x)\xi(x;F) \\
&\qquad + 2\mathcal{K}\Psi(z_1,x)\int_{-\infty}^{x} dx'\,\big(\mathcal{K}\Phi^{(1)}_{z_1}(z_1,x')\big)^{\top}\sigma_1\big[F(x') + 2\Phi^{(1)}(z_1,x')\xi(x';F)\big] \\
&= F(x) + 2\Phi^{(1)}_{z_1}(z_1,x)\xi(x;F) + 2\mathcal{K}\Psi(z_1,x)\int_{-\infty}^{x} dx'\,\big(\mathcal{K}\Phi^{(1)}_{z_1}(z_1,x')\big)^{\top}\sigma_1 F(x') \\
&\quad - 2(\mathrm{Im}(z_1))^{-1}\mathcal{K}\Psi(z_1,x)\int_{-\infty}^{x} dx'\,\big(W\big(\mathcal{K}\Phi^{(1)}_{z_1}(z_1,x'),\Phi^{(1)}(z_1,x')\big)\big)_{x'}\xi(x',F),
\end{aligned} \tag{6.arabic@equation}$$

using

$$-2\mathrm{Im}(z_1)\big(\mathcal{K}\Phi^{(1)}_{z_1}(z_1,x)\big)^{\top}\sigma\Phi^{(1)}_{z_1}(z_1,x) = W\big(\mathcal{K}\Phi^{(1)}_{z_1}(z_1,x),\Phi^{(1)}_{z_1}(z_1,x)\big)_x. \tag{6.arabic@equation}$$

The estimate

$$\begin{aligned}
&\big\|\mathcal{K}\Psi(z_1,x)W\big(\mathcal{K}\Phi^{(1)}_{z_1}(z_1,x),\Phi^{(1)}_{z_1}(z_1,x)\big)\xi(x;F)\big\|_{\mathbb{C}^2} \\
&\le \|\Psi(z_1,x)\|_{\mathbb{C}^2}2|\mathrm{Im}(z_1)|\big\|\Phi^{(1)}_{z_1}(z_1,x)\big)\|^2_{\mathbb{C}^2}|\xi(x;F)| \\
&\le 2|\mathrm{Im}(z_1)|^3\|\Psi(z_1,x)\|^{-1}_{\mathbb{C}^2}|\xi(x;F)| \\
&= 2|\mathrm{Im}(z_1)|^3\|\Psi(z_1,x)\|^{-1}_{\mathbb{C}^2}\bigg[\int_{-\infty}^{x} dx'\,\|\Psi_-(z_1,x')\|_{\mathbb{C}^2}\|F(x')\|_{\mathbb{C}^2} \\
&\qquad\qquad + \int_{x}^{\infty} dx'\,\|\Psi_+(z_1,x')\|_{\mathbb{C}^2}\|F(x')\|_{\mathbb{C}^2}\bigg] \\
&\le 2|\mathrm{Im}(z_1)|^3\bigg[C_-\|\Psi_-(z_1,x)\|^{-1}_{\mathbb{C}^2}\int_{-\infty}^{x} dx'\,\|\Psi_-(z_1,x')\|_{\mathbb{C}^2}\|F(x')\|_{\mathbb{C}^2} \\
&\qquad\qquad + C_+\|\Psi_+(z_1,x)\|^{-1}_{\mathbb{C}^2}\int_{x}^{\infty} dx'\,\|\Psi_+(z_1,x')\|_{\mathbb{C}^2}\|F(x')\|_{\mathbb{C}^2}\bigg]
\end{aligned} \tag{6.arabic@equation}$$

then proves

$$\big\|\mathcal{K}\Psi(z_1)W\big(\mathcal{K}\Phi^{(1)}_{z_1}(z_1),\Phi^{(1)}_{z_1}(z_1)\big)\xi(\cdot;F)\big\|_{\mathbb{C}^2} \in L^2(\mathbb{R}) \tag{6.arabic@equation}$$

by (6.arabic@equation). An integration by parts in the last term of (6.arabic@equation), using (6.arabic@equation), (6.arabic@equation), and

$$\begin{aligned}
&\liminf_{x\downarrow-\infty}\big\|\mathcal{K}\Psi(z_1,x)W\big(\mathcal{K}\Phi^{(1)}_{z_1}(z_1,x),\Phi^{(1)}_{z_1}(z_1,x)\big)\xi(x;F)\big\|_{\mathbb{C}^2} \\
&\le |2\mathrm{Im}(z_1)|\liminf_{x\downarrow-\infty}\|\Psi(z_1,x)\|_{\mathbb{C}^2}\big\|\Phi^{(1)}_{z_1}(z_1,x)\big\|^2_{\mathbb{C}^2}|\xi(x;F)| = 0
\end{aligned} \tag{6.arabic@equation}$$

by (6.arabic@equation) (in fact, $\lim_{x\downarrow-\infty}|\cdots| = 0$ in (6.arabic@equation) since all Lebesgue integrals involved are finite), then proves $S_{\overline{z_1}}T_{z_1}F = F$ and hence (6.arabic@equation). ∎

Next, we further restrict $S_{\overline{z_1}}$ and define the operator $S_{z_1,\overline{z_1}}$ by

$$S_{z_1,\overline{z_1}} = S_{\overline{z_1}}\big|_{\{\Phi^{(1)}_{z_1}(z_1),\mathcal{K}\Phi^{(1)}_{z_1}(z_1)\}^{\perp_{\mathcal{J}}}} = \widehat{S}_{\overline{z_1}}\big|_{\{\Phi^{(1)}_{z_1}(z_1),\mathcal{K}\Phi^{(1)}_{z_1}(z_1)\}^{\perp_{\mathcal{J}}}}. \tag{6.arabic@equation}$$

Lemma 6.10. *Assume Hypothesis* 5.3 *and* $z_1 \in \rho(D(q))$. *Then*

$$\ker(T_{z_1}) = \{0\}, \quad \operatorname{ran}(T_{z_1}) = \big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}, \qquad (6.\text{arabic@equation})$$

$$\ker(S_{\overline{z_1}}) = \operatorname{span}\{\Phi_{z_1}^{(1)}(z_1)\}, \quad \operatorname{ran}(S_{\overline{z_1}}) = L^2(\mathbb{R})^2. \qquad (6.\text{arabic@equation})$$

Moreover, $S_{z_1,\overline{z_1}}$ *is the inverse of* T_{z_1}, *that is,*

$$\ker(S_{z_1,\overline{z_1}}) = \{0\}, \quad \operatorname{ran}(S_{z_1,\overline{z_1}}) = L^2(\mathbb{R})^2, \qquad (6.\text{arabic@equation})$$

$$S_{z_1,\overline{z_1}} T_{z_1} = I_2 \ \text{ on } \ L^2(\mathbb{R})^2, \qquad (6.\text{arabic@equation})$$

$$T_{z_1} S_{z_1,\overline{z_1}} = I_2 \ \text{ on } \ \big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}. \qquad (6.\text{arabic@equation})$$

Proof. Suppose $T_{z_1}F = 0$ for some $F \in L^2(\mathbb{R})^2$. Then (6.arabic@equation) yields $0 = S_{\overline{z_1}} T_{z_1} F = F$ and hence $\ker(T_{z_1}) = \{0\}$. The assertion $\operatorname{ran}(S_{\overline{z_1}}) = L^2(\mathbb{R})^2$ in (6.arabic@equation) is also clear from (6.arabic@equation). Next, assume $S_{\overline{z_1}}G = 0$ for some $G \in \operatorname{dom}(S_{\overline{z_1}}) = \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}$ (cf. (6.arabic@equation)). Then (6.arabic@equation) implies

$$0 = T_{z_1} S_{\overline{z_1}} G = G - \frac{\big(G, \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}}{\big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}} \Phi_{z_1}^{(1)}(z_1), \quad G \in \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}, \qquad (6.\text{arabic@equation})$$

and hence $G = c\Phi_{z_1}^{(1)}(z_1)$ for some $c \in \mathbb{C}$. This proves (6.arabic@equation).

Next, suppose $G \in \big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}$. Then $\big(G, \mathcal{J}\Phi_{z_1}^{(1)}\big)_{L^2} = 0$ and (6.arabic@equation) imply $T_{z_1} S_{\overline{z_1}} G = G$ and hence

$$\big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}} \subseteq \operatorname{ran}(T_{z_1}) \subseteq \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}, \qquad (6.\text{arabic@equation})$$

taking $\operatorname{ran}(T_{z_1}) \subseteq \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}$ in (6.arabic@equation) into account. The computation

$$\big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J} T_{z_1} S_{\overline{z_1}} G\big)_{L^2} = \big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}G\big)_{L^2} - \big(G, \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2} = 0 \qquad (6.\text{arabic@equation})$$

then proves $\Phi_{z_1}^{(1)}(z_1) \perp_{\mathcal{J}} \operatorname{ran}(T_{z_1})$ and

$$\operatorname{ran}(T_{z_1}) = \big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}} \qquad (6.\text{arabic@equation})$$

since $\operatorname{ran}(S_{\overline{z_1}}) = L^2(\mathbb{R})^2$.

Finally, (6.arabic@equation) and (6.arabic@equation) are clear from (6.arabic@equation), (6.arabic@equation), (6.arabic@equation), and (6.arabic@equation). ☐

Next we state an auxiliary result.

Lemma 6.11. *Assume Hypothesis* 3.2 *and let* $\xi \in AC_{\mathrm{loc}}(\mathbb{R})$, $\Phi = (\phi_1, \phi_2)^\top \in AC_{\mathrm{loc}}(\mathbb{R})^2$, $F = (f_1, f_2)^\top \in \operatorname{dom}(D(q))$, *and*

$$A_- \in \big\{G \in AC_{\mathrm{loc}}(\mathbb{R})^2 \,\big|\, G, M(q)G \in L^2_{\mathrm{loc}}([-\infty, \infty))^2\big\}, \qquad (6.\text{arabic@equation})$$

$$A_+ \in \big\{G \in AC_{\mathrm{loc}}(\mathbb{R})^2 \,\big|\, G, M(q)G \in L^2_{\mathrm{loc}}((-\infty, \infty])^2\big\}. \qquad (6.\text{arabic@equation})$$

Then,

$$(M(q)\xi\Phi)(x) = i\xi'(x)(\phi_1(x), -\phi_2(x))^\top + \xi(x)(M(q)\Phi)(x) \ \text{for a.e. } x \in \mathbb{R} \qquad (6.\text{arabic@equation})$$

and

$$\int_{-\infty}^{x} dx'\, A_-(x')^\top \sigma_1 (M(q)F)(x') = \int_{-\infty}^{x} dx'\, (M(q)A_-)(x')^\top \sigma_1 F(x') + iA_-(x)^\perp F(x), \qquad (6.\text{arabic@equation})$$

$$\int_{x}^{\infty} dx'\, A_+(x')^\top \sigma_1 (M(q)F)(x') = \int_{x}^{\infty} dx'\, (M(q)A_+)(x')^\top \sigma_1 F(x') - iA_+(x)^\perp F(x) \qquad (6.\text{arabic@equation})$$

for all $x \in \mathbb{R}$.

Proof. Assertion (6.arabic@equation) follows directly from the definition of $M(q)$. To prove assertion (6.arabic@equation) one integrates by parts and obtains

$$\begin{aligned}
&\int_{-\infty}^{x} dx'\, A_-(x')^\top \sigma_1 (M(q)F)(x') = i[a_{2,-}(x)f_1(x) - a_{1,-}(x)f_2(x)] \\
&\quad + i\int_{-\infty}^{x} dx'\, (-a_{1,-}\overline{q}f_1 - a_{2,-}qf_{2,-} + a'_{1,-}f_2 - a'_{2,-}f_1)(x') \\
&= \int_{-\infty}^{x} dx'\, ((M(q)A_-)(x'))^\top \sigma_1 F(x') + iA_-(x)^\perp F(x), \qquad (6.\text{arabic@equation})
\end{aligned}$$

using $\liminf_{x\downarrow -\infty} |A_-(x)^\perp F(x)| = 0$ (actually, $\lim_{x\downarrow-\infty} |\cdots| = 0$). Relation (6.arabic@equation) is proved analogously. ☐

The following result shows that T_{z_1} and $S_{\overline{z_1}}$ intertwine $D\big(q^{(1)}_{z_1}\big)$ and $D(q)$.

Lemma 6.12. *Assume Hypothesis* 5.3 *and* $z_1 \in \rho(D(q))$. *Then,*

$$D\big(q^{(1)}_{z_1}\big)T_{z_1} = T_{z_1}D(q), \qquad (6.\text{arabic@equation})$$

$$S_{\overline{z_1}}D\big(q^{(1)}_{z_1}\big) = D(q)S_{\overline{z_1}}. \qquad (6.\text{arabic@equation})$$

Proof. Using formulas (4.arabic@equation) one infers

$$M\big(q^{(1)}_{z_1}\big) - M(q) = 4i\begin{pmatrix} 0 & -\phi^{(1)}_{1,z_1}(z_1)\psi_1(z_1) \\ \phi^{(1)}_{2,z_1}(z_1)\psi_2(z_1) & 0 \end{pmatrix}. \qquad (6.\text{arabic@equation})$$

Relation (6.arabic@equation) applied to $M(q^{(1)})$ with $\Phi = \Phi^{(1)}_{z_1}(z_1)$ and (cf. (6.arabic@equation))

$$\xi(x) = \xi(x;F) = \int_{-\infty}^{x} dx'\, \Psi_-(z_1,x')^\top \sigma_1 F(x') - \int_{x}^{\infty} dx'\, \Psi_+(z_1,x')^\top \sigma_1 F(x'), \quad F \in \operatorname{dom}(D(q)) \qquad (6.\text{arabic@equation})$$

then yields

$$\big(M\big(q^{(1)}_{z_1}\big)\big(\xi(\cdot;F)\,\Phi^{(1)}_{z_1}(z_1)\big)\big)(x) = i\Psi(z_1,x)^\top \sigma_1 F(x)(\phi^{(1)}_{1,z_1}(z_1,x), -\phi^{(1)}_{2,z_1}(z_1,x))^\top + z_1\Phi^{(1)}_{z_1}(z_1,x)\xi(x;F) \qquad (6.\text{arabic@equation})$$

since $M\big(q_{z_1}^{(1)}\big)\Phi_{z_1}^{(1)}(z_1) = z_1\Phi_{z_1}^{(1)}(z_1)$. By (6.arabic@equation), with $A_- = \Psi_-(z_1)$ and $A_+ = \Psi_+(z_1)$, one infers

$$\begin{aligned}\int_{\pm\infty}^{x} dx'\,\Psi_\pm(z_1,x')^\top\sigma_1(M(q)F)(x') = z_1\int_{\pm\infty}^{x} dx'\,\Psi_\pm(z_1,x')^\top\sigma_1F(x')\\ + i\Psi_\pm(z_1,x)^\perp F(x), \quad F\in\operatorname{dom}(D(q))\end{aligned} \tag{6.arabic@equation}$$

since $(M(q)\Psi_\pm(z_1))^\top = z_1\Psi_\pm(z_1)^\top$. Thus,

$$\begin{aligned}&\big(M\big(q_{z_1}^{(1)}\big)T_{z_1} - T_{z_1}M(q)\big)F = \big(M\big(q_{z_1}^{(1)}\big) - M(q)\big)F\\ &\quad + 2M\big(q_{z_1}^{(1)}\big)\big(\Phi^{(1)}(z_1)\xi(\cdot;F)\big) - 2\Phi_{z_1}^{(1)}(z_1)\xi(\cdot;M(q)F)\\ &= 4i\big(-\phi_{1,z_1}^{(1)}(z_1)\psi_1(z_1)f_2, \phi_{2,z_1}^{(1)}(z_1)\psi_2(z_1)f_1\big)^\top\\ &\quad + 2i\big[\Psi(z_1)^\top\sigma_1F\big]\big(\phi_1^{(1)}(z_1), -\phi_2^{(1)}(z_1)\big)^\top + 2z_1\Phi_{z_1}^{(1)}(z_1)\xi(\cdot;F)\\ &\quad - 2z_1\Phi_{z_1}^{(1)}(z_1)\xi(\cdot;F) - 2i\big[\Psi(z_1)^\perp F\big]\big(\phi_{1,z_1}^{(1)}(z_1), \phi_{2,z_1}^{(1)}(z_1)\big)^\top\\ &= 0, \quad F\in\operatorname{dom}(D(q)).\end{aligned} \tag{6.arabic@equation}$$

This computation also proves that

$$T_{z_1}\operatorname{dom}(D(q)) \subseteq \operatorname{dom}\big(D(q_{z_1}^{(1)})\big) \text{ and hence } T_{z_1}D(q) \subseteq D\big(q_{z_1}^{(1)}\big)T_{z_1}. \tag{6.arabic@equation}$$

Next, choosing $G\in\operatorname{dom}(D(q)S_{\overline{z_1}})$ (i.e., $G\in L^2(\mathbb{R})^2$ such that $S_{\overline{z_1}}G\in\operatorname{dom}(D(q))$), (6.arabic@equation) implies

$$\begin{aligned}&S_{\overline{z_1}}D(q_{z_1}^{(1)})T_{z_1}S_{\overline{z_1}}G = S_{\overline{z_1}}D(q_{z_1}^{(1)})\big[G - \big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\Phi_{z_1}^{(1)}\big)_{L^2}^{-1}\big(G, \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}\big]\\ &= S_{\overline{z_1}}D(q_{z_1}^{(1)})G = S_{\overline{z_1}}T_{z_1}D(q)S_{\overline{z_1}}G = D(q)S_{\overline{z_1}}G, \quad G\in\operatorname{dom}(D(q)S_{\overline{z_1}}).\end{aligned} \tag{6.arabic@equation}$$

Here we successively used (6.arabic@equation), $S_{\overline{z_1}}\Phi_{z_1}^{(1)}(z_1) = 0$ (cf. (6.arabic@equation)), and (6.arabic@equation). Thus, one concludes

$$\operatorname{dom}\big(D(q_{z_1}^{(1)})\big) \subseteq \operatorname{dom}(D(q)S_{\overline{z_1}}) \text{ and hence } S_{\overline{z_1}}D\big(q_{z_1}^{(1)}\big) \subseteq D(q)S_{\overline{z_1}}. \tag{6.arabic@equation}$$

Hence,

$$\operatorname{dom}\big(D(q_{z_1}^{(1)})T_{z_1}\big) = \big\{F\in L^2(\mathbb{R})^2 \,\big|\, T_{z_1}F\in\operatorname{dom}\big(D(q_{z_1}^{(1)})\big)\big\} \subseteq \operatorname{dom}(D(q)) \tag{6.arabic@equation}$$

since $S_{\overline{z_1}}(T_{z_1}F) = F\in\operatorname{dom}(D(q))$ for $F\in\operatorname{dom}\big(D\big(q_{z_1}^{(1)}\big)T_{z_1}\big)$ by (6.arabic@equation). Combining (6.arabic@equation) and (6.arabic@equation) then proves (6.arabic@equation). Equations (6.arabic@equation), (6.arabic@equation), and (6.arabic@equation) in turn imply

$$\begin{aligned}&S_{\overline{z_1}}D(q_{z_1}^{(1)})T_{z_1}S_{\overline{z_1}} = S_{\overline{z_1}}D(q_{z_1}^{(1)})\big[I - \big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\Phi_{z_1}^{(1)}\big)_{L^2}^{-1}\big(\cdot, \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}\big]\\ &= S_{\overline{z_1}}D(q_{z_1}^{(1)}) = S_{\overline{z_1}}T_{z_1}D(q)S_{\overline{z_1}} = D(q)S_{\overline{z_1}}\end{aligned} \tag{6.arabic@equation}$$

and hence (6.arabic@equation). $\square$

Remark 6.13. *One can prove, similarly to Lemma* 6.12*, that*

$$
\begin{aligned}
&D\big(q_{z_1}^{(1)}\big)\widetilde{T}_{\overline{z_1}} = \widetilde{T}_{\overline{z_1}}D(q) \ \text{ and } \ \ker(\widetilde{T}_{\overline{z_1}}) = \{0\}, \\
&\widetilde{S}_{z_1}D\big(q_{z_1}^{(1)}\big) = D(q)\widetilde{S}_{z_1} \ \text{ and } \ \ker(\widetilde{S}_{z_1}) = \operatorname{span}\big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\} \qquad (6.\text{arabic@equation})
\end{aligned}
$$

and that $\widetilde{T}_{\overline{z_1}}$ *and* $\widetilde{S}_{z_1}\big|_{\{\Phi_{z_1}^{(1)}(z_1),\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\}^{\perp_{\mathcal{J}}}}$ *are transformation operators* (*inverse to each other*)*.*

Summarizing the results of Corollaries 6.4 and 6.5 and Lemmas 6.9–6.12 obtained thus far, we are now in position to state one of the principal results of this section.

Theorem 6.14. *Assume Hypothesis* 5.3 *and* $z_1 \in \rho(D(q))$*. Then* T_{z_1} *and* $S_{\overline{z_1}}$ *are bounded linear operators in* $L^2(\mathbb{R})^2$ *which intertwine the operators* $D(q)$ *and* $D\big(q_{z_1}^{(1)}\big)$*,*

$$D\big(q_{z_1}^{(1)}\big)T_{z_1} = T_{z_1}D(q), \qquad (6.\text{arabic@equation})$$

$$S_{\overline{z_1}}D\big(q_{z_1}^{(1)}\big) = D(q)S_{\overline{z_1}}. \qquad (6.\text{arabic@equation})$$

Moreover,

$$\ker(T_{z_1}) = \{0\}, \quad \operatorname{ran}(T_{z_1}) = \big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}, \qquad (6.\text{arabic@equation})$$

$$\ker(S_{\overline{z_1}}) = \operatorname{span}\big\{\Phi_{z_1}^{(1)}(z_1)\big\}, \quad \operatorname{ran}(S_{\overline{z_1}}) = L^2(\mathbb{R})^2, \qquad (6.\text{arabic@equation})$$

and $S_{z_1,\overline{z_1}}$ (*cf.* (6.arabic@equation)) *is the inverse of* T_{z_1}*, that is,*

$$S_{z_1,\overline{z_1}}T_{z_1} = I_2 \ \text{ on } \ L^2(\mathbb{R})^2, \qquad (6.\text{arabic@equation})$$

$$T_{z_1}S_{z_1,\overline{z_1}} = I_2 \ \text{ on } \ \big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}. \qquad (6.\text{arabic@equation})$$

An analogous statement can be formulated when z_1 is replaced by $\overline{z_1}$, that is, for the operators $T_{\overline{z_1}}$ and $\widetilde{S}_{z_1}$, but we chose not to dwell on it here due to the symmetry of the arguments.

We conclude with the principal spectral theoretic result of this paper.

Theorem 6.15. *Assume Hypothesis* 5.3 *and*[2] $z_1 \in \rho(D(q))$*. Then*

$$\sigma(D\big(q_{z_1}^{(1)}\big)) = \sigma(D(q)) \cup \{z_1, \overline{z_1}\}, \qquad (6.\text{arabic@equation})$$

$$\sigma_{\mathrm{p}}\big(D\big(q_{z_1}^{(1)}\big)\big) = \sigma_{\mathrm{p}}(D(q)) \cup \{z_1, \overline{z_1}\}, \qquad (6.\text{arabic@equation})$$

$$\sigma_{\mathrm{e}}\big(D\big(q_{z_1}^{(1)}\big)\big) = \sigma_{\mathrm{e}}(D(q)). \qquad (6.\text{arabic@equation})$$

In other words, constructing the new NLS_- *potential* $q_{z_1}^{(1)}$ *amounts to inserting a pair of complex conjugate* (*nonreal*) $L^2(\mathbb{R})^2$*-eigenvalues,* z_1 *and* $\overline{z_1}$*, into the spectrum of the background operator* $D(q)$*, leaving the rest of its spectrum invariant.*

Proof. We will prove (6.arabic@equation) and (6.arabic@equation) from which (6.arabic@equation) follows by (5.arabic@equation). By Theorem 4.5 one has $z_1, \overline{z_1} \in \sigma_{\mathrm{p}}\big(D\big(q_{z_1}^{(1)}\big)\big)$. Thus, (6.arabic@equation) is equivalent to

$$\sigma_{\mathrm{p}}\big(D\big(q_{z_1}^{(1)}\big)\big)\backslash\{z_1, \overline{z_1}\} = \sigma_{\mathrm{p}}(D(q)). \qquad (6.\text{arabic@equation})$$

Next we denote

$$X^{(1)} = \big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}. \qquad (6.\text{arabic@equation})$$

[2]We recall that $z \in \rho(D(q))$ implies $z \in \mathbb{C}\backslash\mathbb{R}$ by Corollary 5.5.

Since $D(q_{z_1}^{(1)})$ is $\mathcal{J}$-self-adjoint, $X^{(1)}$ is a closed, $D\big(q_{z_1}^{(1)}\big)$-invariant subspace. In addition, we denote by $D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}$ the part of $D\big(q_{z_1}^{(1)}\big)$ in $X^{(1)}$ with

$$\operatorname{dom}\big(D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}\big) = X^{(1)} \cap \operatorname{dom}\big(D\big(q_{z_1}^{(1)}\big)\big). \qquad (6.\text{arabic@equation})$$

From (6.arabic@equation) and Theorem 6.14 it follows that

$$D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}} = TD(q)T^{-1} \text{ on } X^{(1)} \qquad (6.\text{arabic@equation})$$

and

$$D(q) = T^{-1}D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}T \text{ on } L^2(\mathbb{R})^2, \qquad (6.\text{arabic@equation})$$

with $T = T_{z_1}$ and $T^{-1} = \widehat{S}_{\overline{z_1}}\big|_{\big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}}$ (cf. Lemma 6.10). Assuming $\mu \in \sigma_{\mathrm{p}}(D(q))$ and

$$D(q)F(\mu) = \mu F(\mu), \quad 0 \neq F(\mu) \in \operatorname{dom}(D(q)), \qquad (6.\text{arabic@equation})$$

then (6.arabic@equation) implies

$$D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}TF(\mu) = \mu TF(\mu). \qquad (6.\text{arabic@equation})$$

Since $\ker(T) = \{0\}$ (cf. (6.arabic@equation)), this implies

$$\sigma_{\mathrm{p}}(D(q)) \subseteq \sigma_{\mathrm{p}}\big(D\big(q_{z_1}^{(1)}\big)\big) \text{ and hence } \sigma_{\mathrm{p}}(D(q)) \subseteq \sigma_{\mathrm{p}}\big(D\big(q_{z_1}^{(1)}\big)\big)\backslash\{z_1, \overline{z_1}\}. \qquad (6.\text{arabic@equation})$$

Conversely, assuming $\nu \in \sigma_{\mathrm{p}}\big(D\big(q_{z_1}^{(1)}\big)\big)\backslash\{z_1, \overline{z_1}\}$ and

$$D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}F^{(1)}(\nu) = \nu F^{(1)}(\nu), \quad 0 \neq F^{(1)}(\nu) \in \operatorname{dom}\big(D\big(q_{z_1}^{(1)}\big)\big), \qquad (6.\text{arabic@equation})$$

one infers from (6.arabic@equation) that

$$D(q)T^{-1}F^{(1)}(\nu) = \nu T^{-1}F^{(1)}(\nu). \qquad (6.\text{arabic@equation})$$

Since $\ker(T^{-1}) = \{0\}$ (cf. (6.arabic@equation)), one concludes $T^{-1}F^{(1)}(\nu) \neq 0$ and hence

$$\sigma_{\mathrm{p}}(D(q)) \supseteq \sigma_{\mathrm{p}}\big(D\big(q_{z_1}^{(1)}\big)\big)\backslash\{z_1, \overline{z_1}\}, \qquad (6.\text{arabic@equation})$$

implying (6.arabic@equation).

Next, one observes that (6.arabic@equation) also implies

$$\sigma_{\mathrm{e}}(D(q)) = \sigma_{\mathrm{e}}\big(D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}\big). \qquad (6.\text{arabic@equation})$$

To prove (6.arabic@equation) we note that (6.arabic@equation) implies

$$\sigma_{\mathrm{e}}(D(q)) = \sigma_{\mathrm{e}}\big(D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}\big) \subseteq \sigma_{\mathrm{e}}\big(D\big(q_{z_1}^{(1)}\big)\big). \qquad (6.\text{arabic@equation})$$

Conversely, let $\lambda \in \sigma_{\mathrm{e}}\big(D\big(q_{z_1}^{(1)}\big)\big)$ and $\{G_n\}_{n\in\mathbb{N}}$ be a singular sequence of $D\big(q_{z_1}^{(1)}\big)$ corresponding to λ, that is, a bounded sequence in $\operatorname{dom}\big(D\big(q_{z_1}^{(1)}\big)\big)$ without any convergent subsequence such that $\lim_{n\to\infty}\big(D\big(q_{z_1}^{(1)}\big) - \lambda I\big)G_n = 0$. Since $L^2(\mathbb{R})^2$ permits the direct sum decomposition (not to be confused with an orthogonal direct sum decomposition)

$$L^2(\mathbb{R})^2 = X^{(1)} \dot{+} \operatorname{span}\big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\} \qquad (6.\text{arabic@equation})$$

(here one uses the fact that $\Phi^{(1)}_{z_1}(z_1) \perp_{\mathcal{J}} \mathcal{K}\Phi^{(1)}_{z_1}(z_1)$, cf. (6.arabic@equation)), one can write

$$G_n = F_n + \frac{\big(G_n, \mathcal{J}\Phi^{(1)}_{z_1}(z_1)\big)_{L^2}}{\big(\Phi^{(1)}_{z_1}(z_1), \mathcal{J}\Phi^{(1)}_{z_1}(z_1)\big)_{L^2}} \Phi^{(1)}_{z_1}(z_1) + \frac{\big(G_n, \mathcal{J}\mathcal{K}\Phi^{(1)}_{z_1}(z_1)\big)_{L^2}}{\big(\mathcal{K}\Phi^{(1)}_{z_1}(z_1), \mathcal{J}\mathcal{K}\Phi^{(1)}_{z_1}(z_1)\big)_{L^2}} \mathcal{K}\Phi^{(1)}_{z_1}(z_1), \tag{6.arabic@equation}$$

where $F_n \in X^{(1)} \cap \mathrm{dom}\big(D\big(q^{(1)}_{z_1}\big)\big)$ is the projection of G_n onto the space $X^{(1)}$, the projection being parallel to the subspace span$\big\{\Phi^{(1)}_{z_1}(z_1), \mathcal{K}\Phi^{(1)}_{z_1}(z_1)\big\}$. Since the coefficients of $\Phi^{(1)}_{z_1}(z_1)$ and $\mathcal{K}\Phi^{(1)}_{z_1}(z_1)$ in (6.arabic@equation) are bounded with respect to $n \in \mathbb{N}$, one can assume that they are convergent by restricting to a subsequence. Thus, for some real numbers c_1 and c_2, one has

$$\big(D\big(q^{(1)}_{z_1}\big) - \lambda I\big)F_n \underset{n\to\infty}{\longrightarrow} c_1\Phi^{(1)}_{z_1}(z_1) + c_2\mathcal{K}\Phi^{(1)}_{z_1}(z_1). \tag{6.arabic@equation}$$

Since $X^{(1)}$ is $D\big(q^{(1)}_{z_1}\big)$-invariant, $F_n \in X^{(1)}$ implies $\big(D\big(q^{(1)}_{z_1}\big) - \lambda I\big)F_n \in X^{(1)}$, and since $X^{(1)}$ is closed, the limit $c_1\Phi^{(1)}_{z_1}(z_1) + c_2\mathcal{K}\Phi^{(1)}_{z_1}(z_1)$ also belongs to $X^{(1)}$. But this implies $c_1 = c_2 = 0$, because $\big(\Phi^{(1)}_{z_1}(z_1), \mathcal{J}\mathcal{K}\Phi^{(1)}_{z_1}(z_1)\big)_{L^2} = 0$ (cf. (6.arabic@equation)) and $\big(\Phi^{(1)}_{z_1}(z_1), \mathcal{J}\Phi^{(1)}_{z_1}(z_1)\big)_{L^2} \neq 0$, $\big(\mathcal{K}\Phi^{(1)}_{z_1}(z_1), \mathcal{J}\mathcal{K}\Phi^{(1)}_{z_1}(z_1)\big)_{L^2} \neq 0$ (cf. (6.arabic@equation)). Thus, one concludes that $\{F_n\}_{n\in\mathbb{N}} \subset X^{(1)}$ is a singular sequence of $D\big(q^{(1)}_{z_1}\big)\big|_{X^{(1)}}$ corresponding to λ since it is bounded and has no convergent subsequence. (Otherwise, by (6.arabic@equation), $\{G_n\}_{n\in\mathbb{N}}$ would have a convergent subsequence, contradicting the assumption that it is a singular sequence). It follows that $\lambda \in \sigma_{\mathrm{e}}\big(D\big(q^{(1)}_{z_1}\big)\big|_{X^{(1)}}\big) = \sigma_{\mathrm{e}}(D(q))$ and hence $\sigma_{\mathrm{e}}(D(q)) = \sigma_{\mathrm{e}}\big(D\big(q^{(1)}_{z_1}\big)\big|_{X^{(1)}}\big) \supseteq \sigma_{\mathrm{e}}\big(D\big(q^{(1)}_{z_1}\big)\big)$, proving (6.arabic@equation). ☐

Theorems 6.14 and 6.15 are new. We note that they are proven under the optimal assumption $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ (but they seem to be new under virtually any assumptions on q).

In the special periodic case where the machinery of Floquet theory can be applied, the issue of isospectral Darboux transformations is briefly mentioned in [33, Theorem 3]. This excludes the insertion of eigenvalues as in Theorem 6.15. Inserting eigenvalues into the spectrum of a self-adjoint one-dimensional Dirac operator (not applicable in the present NLS$_-$ context) was investigated by means of transformation operators (along the lines of [18]) in [43].

We conclude this section with a few facts on N-soliton NLS$_-$ potentials. As shown in Lemma 4.6, the insertion of pairs of complex conjugate eigenvalues into the spectrum of the background operator $D(q)$ can be iterated. To fix the proper notation, we now slightly extend our approach of Section 6 and we consider a more general linear combination $\Psi_{\gamma_k}(z,x)$ of $\Psi_\pm(z,x)$: Assuming $z_k \in \mathbb{C}\backslash\mathbb{R}$, $k = 1,\dots,N$, one defines

$$\Psi_{\gamma_k}(z_k,x) = \Psi_-(z_k,x) + \gamma_k\Psi_+(z_k,x), \quad \gamma_k \in \mathbb{C}\backslash\{0\},\ k = 1\dots,N \tag{6.arabic@equation}$$

(as opposed to to our choice $\gamma = 1$ in (6.arabic@equation)). In obvious notation one then denotes the corresponding Nth iteration of the construction of $q^{(1)}_{z_1}(x)$ presented in Sections 2 and 4 (cf. (2.arabic@equation), (4.arabic@equation)), identifying $\Psi(z_1,x)$ and $\Psi_{\gamma_1}(z_1,x)$, by $q^{(N)}_{z_1,\dots z_N,\gamma_1,\dots,\gamma_N}(x)$.

In order to describe a well-known explicit formula for $q^{(N)}_{z_1,\dots z_N,\gamma_1,\dots,\gamma_N}(x)$ (cf., e.g., [34, Sect. 4.2], [36], [41]) one introduces the quantities

$$\varphi_k(x) = \frac{\psi_{2,-}(z_k,x) + \gamma_k \psi_{2,+}(z_k,x)}{\psi_{1,-}(z_k,x) + \gamma_k \psi_{1,+}(z_k,x)}, \qquad (6.\text{arabic@equation})$$

and the $2N \times 2N$ Vandermonde-type matrices

$$V_{2N}(x) = \begin{pmatrix} 1 & z_1 & \dots & z_1^{N-1} & \varphi_1 & z_1\varphi_1 & \dots & z_1^{N-1}\varphi_1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_N & \dots & z_N^{N-1} & \varphi_N & z_N\varphi_N & \dots & z_N^{N-1}\varphi_N \\ 1 & z_{N+1} & \dots & z_{N+1}^{N-1} & \varphi_{N+1} & z_{N+1}\varphi_{N+1} & \dots & z_{N+1}^{N-1}\varphi_{N+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_{2N} & \dots & z_{2N}^{N-1} & \varphi_{2N} & z_{2N}\varphi_{2N} & \dots & z_{2N}^{N-1}\varphi_{2N} \end{pmatrix} \qquad (6.\text{arabic@equation})$$

and

$$\widetilde{V}_{2N}(x) = \begin{pmatrix} 1 & z_1 & \dots & z_1^{N-1} & \varphi_1 & z_1\varphi_1 & \dots & z_1^{N} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_N & \dots & z_N^{N-1} & \varphi_N & z_N\varphi_N & \dots & z_N^{N} \\ 1 & z_{N+1} & \dots & z_{N+1}^{N-1} & \varphi_{N+1} & z_{N+1}\varphi_{N+1} & \dots & z_{N+1}^{N} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_{2N} & \dots & z_{2N}^{N-1} & \varphi_{2N} & z_{2N}\varphi_{2N} & \dots & z_{2N}^{N} \end{pmatrix}. \qquad (6.\text{arabic@equation})$$

The Nth iteration $q^{(N)}_{z_1,\dots z_N,\gamma_1,\dots,\gamma_N}(x)$ is then explicitly given by

$$q^{(N)}_{z_1,\dots z_N,\gamma_1,\dots,\gamma_N}(x) = q(x) - 2i\frac{\det(\widetilde{V}_{2N}(x))}{\det(V_{2N}(x))}. \qquad (6.\text{arabic@equation})$$

Denoting by $D(q^{(N)}_{z_1,\dots z_N,\gamma_1,\dots,\gamma_N})$ the associated $\mathcal{J}$-self-adjoint operator in $L^2(\mathbb{R})^2$, repeated application of Theorem 6.15 then yields

$$\sigma(D(q^{(N)}_{z_1,\dots z_N,\gamma_1,\dots,\gamma_N})) = \sigma(D(q)) \cup \{z_1, \overline{z_1}, \dots, z_N, \overline{z_N}\}, \qquad (6.\text{arabic@equation})$$

$$\sigma_{\mathrm{p}}(D(q^{(N)}_{z_1,\dots z_N,\gamma_1,\dots,\gamma_N})) = \sigma_{\mathrm{p}}(D(q)) \cup \{z_1, \overline{z_1}, \dots, z_N, \overline{z_N}\}, \qquad (6.\text{arabic@equation})$$

$$\sigma_{\mathrm{e}}(D(q^{(N)}_{z_1,\dots z_N,\gamma_1,\dots,\gamma_N})) = \sigma_{\mathrm{e}}(D(q)). \qquad (6.\text{arabic@equation})$$

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