The Infinite Volume Limit of Dissipative Abelian Sandpiles

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Abstract: We construct the thermodynamic limit of the stationary measures of the Bak-Tang-Wiesenfeld sandpile model with a dissipative toppling matrix (sand grains may disappear at each toppling). We prove uniqueness and mixing properties of this measure and we obtain an infinite volume ergodic Markov process leaving it invariant. We show how to extend the Dhar formalism of the 'abelian group of toppling operators' to infinite volume in order to obtain a compact abelian group with a unique Haar measure representing the uniform distribution over the recurrent configurations that create finite avalanches[1].

1 Introduction

The abelian sandpile is a model on a lattice in which a discrete height-variable (e.g. representing the slope of a sandpile at that site) is associated to each lattice site. A configuration of height variables is called stable if the height at any site is less than a critical value γ, otherwise it is called unstable. The mechanism of stabilization of an unstable configuration is by a sequence of so-called topplings. In such a toppling, the site gives one (sand) grain to each of its neighboring sites which in their turn can become "unstable" and "topple" etc., until every site has again a subcritical height-value. An unstable site thus creates an "avalanche" involving possibly the toppling of many sites around it. The dynamics on stable configurations consists in adding at

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randomly chosen sites and stabilizing the configuration. Since the avalanche of topplings can involve many sites (depending on the configuration) this dynamics is highly non-local. Therefore, defining infinite volume limits poses a non-trivial problem.

Since its appearance in [1], the abelian sandpile model has been studied intensively, see for instance [4], [5], [8], [14], and also [12] for a mathematical review of the main properties of the model in finite volume. The main technical tool in the analysis of the model is the "abelian group" of toppling operators which can be identified with the set of recurrent configurations.

Our aim is to define the model on infinite graphs. The main problem to overcome is the non-locality or extreme sensitivity to boundary-conditions of the dynamics. We have constructed in [9] and [10] the infinite volume standard sandpile process on the one-dimensional lattice and on homogeneous trees. In the present paper we focus on the thermodynamic limit for *dissipative* models. There, the infinite graph S is a subgraph of the regular lattice $\mathbb{Z}^d$ and on each site the height has a critical value γ, an integer not less than the maximal number $\mathcal{N}$ of neighbors of a site in S. The finite volume rule now starts as follows: choose a site x at random from the volume V and add one grain to it. Suppose that x has $\mathcal{N}_x$ nearest neighbors and that the new height at x is $\gamma+1$. Then, it topples by giving to each of its nearest neighbors one grain and dissipating $\gamma-\mathcal{N}_x$ grains to a sink associated to the volume. We say that the site x is dissipative when $\gamma > \mathcal{N}_x$ and the model is dissipative when this happens for a considerable fraction of sites. This condition can be rephrased in terms of the simple random walk on S with a sink associated to the dissipative sites: the model is dissipative when the Green's function decays fast enough in the lattice distance, see (2.12) below for a precise formulation. Dissipative abelian sandpile models have appeared in the physics literature in [15] and [3], where it was argued that dissipation removes criticality, that is, correlation functions decay exponentially fast uniformly in the volume. From the point of view of defining the thermodynamic limit, the main simplification of dissipative models is that there is a stronger control of the non-locality: more precisely, the probability that a site y is influenced by addition on x decays exponentially fast (or at least in a summable way) in the distance between the sites. Hence avalanche clusters are almost surely finite.

1.1 Results

Our three main results are:

1. The extension of the Dhar formalism to infinite volume sandpile dynamics. That includes the construction of a compact abelian group of recurrent configurations on which we can define addition (of sand) operations.

2. The construction of the thermodynamic limit of the finite volume stationary measure with exponential decay of correlations in the case of "strong dissipativity".

3. The construction of an infinite volume sandpile process which converges exponentially fast to its unique stationary measure.

1.2 Plan of the paper

The paper is organized as follows: in Section 2 we repeat some of the basic results on the abelian sandpile model in finite volume and we introduce the definition of dissipativity, with examples. In Section 3 we show how to extend the dynamics on infinite volume recurrent configurations and we recover the group structure of "addition of recurrent configurations." In Section 4 we prove existence and ergodic properties of the infinite volume dynamics. Section 5 is devoted to the proof of exponential decay of correlations.

2 Finite volume model

In this section we recall some definitions and properties of abelian sandpiles in finite volume. In [4], [5], [8], [14] and [12], the reader will find more details.

The infinite graphs S on which we construct the dissipative abelian sandpile dynamics are $S = \mathbb{Z}^d$, and "strips", that is, $S = \mathbb{Z} \times \{1, \ldots, \ell\}$, for some integer $\ell > 1$ (notice that $\ell = 1$ corresponds to $S = \mathbb{Z}^d$ with $d = 1$). Finite subsets of S will be denoted by V, W; we write $\mathcal{S} = \{W \subset S : W \text{ finite}\}$. We denote by ∂V the external boundary of V: all the sites in $S \setminus V$ that have a nearest neighbor in V. Let $\mathcal{N}$ be the maximal number of neighbors of a site in S, e.g., $\mathcal{N} = 2d$ for $S = \mathbb{Z}^d$ and $\mathcal{N} = 4$ for $S = \mathbb{Z} \times \{1, \ldots, \ell\}, \ell \geq 3$. The state space of the process in infinite volume is $\Omega = \{1, \ldots, \gamma\}^S$, with some integer $\gamma \geq \mathcal{N}$.

We fix $V \in \mathcal{S}$, a nearest neighbor connected subset of S. Then $\Omega_V = \{1, \ldots, \gamma\}^V$ is the state space of the process in the finite volume V. We denote by $\mathcal{N}_V(x)$ the number of nearest neighbors of x in V.

A *(infinite volume) height configuration* η is a mapping from S to $\mathbb{N} = \{1, 2, \ldots\}$ assigning to each site x a "number of sand grains" $\eta(x) \geq 1$. If $\eta \in \Omega$, it is called a *stable* configuration. Otherwise η is *unstable*. For $\eta \in \Omega$, η_V is its restriction to V, and for $\eta, \zeta \in \Omega$, $\eta_V \zeta_{V^c}$ denotes the configuration whose restriction to V (resp. V^c) coincides with η_V (resp. ζ_{V^c}).

The configuration space Ω is endowed with the product topology, making it into a compact metric space. A function $f : \Omega \to \mathbb{R}$ is *local* if there is a finite $W \subset S$ such that $\eta_W = \zeta_W$ implies $f(\eta) = f(\zeta)$. The minimal (in the sense of set ordering) such W is called the *dependence set* of f, and is denoted by D_f. A local function can be seen as a function on Ω_W for all $W \supset D_f$, and every function on Ω_W can be seen as a local function on Ω. The set $\mathcal{L}$ of all local functions is uniformly dense in the set $\mathcal{C}(\Omega)$ of all continuous functions on Ω.

2.1 The dynamics in finite volume

The toppling matrix Δ on S is defined by, for $x, y \in S$,

$$\begin{aligned} \Delta_{xx} &= \gamma, \\ \Delta_{xy} &= -1 \quad \text{if } x \text{ and } y \text{ are nearest neighbors}, \\ \Delta_{xy} &= 0 \quad \text{otherwise} \end{aligned} \tag{2.1}$$

We denote by Δ^V the restriction of Δ to $V \times V$.

A site $x \in V$ is called a *dissipative site in the volume* V if

$$\sum_{y \in V} \Delta_{xy} > 0.$$

Thus if $\gamma > \mathcal{N}$, every site is dissipative. If $\gamma = \mathcal{N}$, the internal boundary sites of V (that is all the sites in V that have a nearest neighbor in $S \setminus V$), are the only dissipative sites in V.

To define the sandpile dynamics, we first introduce the *toppling of a site* x as the mapping $T_x : \mathbb{N}^V \to \mathbb{N}^V$ defined by

$$\begin{aligned} T_x(\eta)(y) &= \eta(y) - \Delta^V_{xy} \text{ if } \eta(x) > \Delta^V_{xx}, \\ &= \eta(y) \text{ otherwise.} \end{aligned} \tag{2.2}$$

In words, site x topples if and only if its height is strictly larger than $\Delta^V_{xx} = \gamma$, by transferring $-\Delta^V_{xy} \in \{0, 1\}$ grains to site $y \neq x$ and losing itself in total $\Delta^V_{xx} = \gamma$ grains. As a consequence, if the site is dissipative, then, upon toppling, some grains are lost. Toppling rules commute on unstable configurations, that is, for $x, y \in V$ such that $\eta(x) > \gamma = \Delta^V_{xx}$ and $\eta(y) > \gamma = \Delta^V_{yy}$:

$$T_x\left(T_y(\eta)\right) = T_y\left(T_x(\eta)\right)$$

For $\eta \in \mathbb{N}^V$, we say that $\zeta \in \Omega_V$ *arises from* η *by toppling* if there exists a k-tuple $(x_1, \ldots, x_k)$ of sites in V such that

$$\zeta = (\prod_{i=1}^{k} T_{x_i})(\eta)$$

The *toppling transformation* is the mapping $\mathcal{T} : \mathbb{N}^V \to \Omega_V$ defined by the requirement that $\mathcal{T}(\eta)$ arises from η by toppling. The fact that stabilization of an unstable configuration is always possible follows from the existence of dissipative sites. The fact that $\mathcal{T}$ is *well-defined*, that is, that the same final stable configuration is obtained irrespective of the order of the topplings, is a consequence of the commutation property, see [12] for a complete proof.

For $\eta \in \mathbb{N}^V$ and $x \in V$, let η^x denote the configuration obtained from η by adding one grain to site x, that is $\eta^x(y) = \eta(y) + \delta_{x,y}$. The *addition operator* defined by

$$a_{x,V} : \Omega_V \to \Omega_V; \eta \mapsto a_{x,V}\eta = \mathcal{T}(\eta^x) \tag{2.3}$$

represents the effect of adding a grain to the stable configuration η and letting a stable configuration arise by toppling. Because $\mathcal{T}$ is well-defined, the composition of addition operators is commutative. We can now define a discrete time Markov chain $\{\eta_n : n \geq 0\}$ on Ω_V by picking a point $x \in V$ randomly at each discrete time step and applying the addition operator $a_{x,V}$ to the configuration. We define also a continuous time Markov process $\{\eta_t : t \geq 0\}$ with infinitesimal generator

$$L_V^{0,\varphi} f(\eta) = \sum_{x \in V} \varphi(x)[f(a_{x,V}\eta) - f(\eta)]; \tag{2.4}$$

this is a pure jump process on Ω_V, where $\varphi : S \to (0, \infty)$ is the *addition rate function*.

2.2 Recurrent configurations, invariant measure

The Markov chain $\{\eta_n, n \geq 0\}$ (or its continuous time version $\{\eta_t\}$) has a unique recurrent class $\mathcal{R}_V$, and its stationary measure μ_V is the uniform measure on that class, that is,

$$\mu_V = \frac{1}{|\mathcal{R}_V|} \sum_{\eta \in \mathcal{R}_V} \delta_\eta. \tag{2.5}$$

A configuration $\eta \in \Omega_V$ belongs to $\mathcal{R}_V$ if it passes the *burning algorithm* (see [4]), which is described as follows. Pick $\eta \in \Omega_V$ and erase the set E_1 of all sites $x \in V$ with a height strictly larger than the number of neighbors of that site in V, that is, satisfying the inequality

$$\eta(x) > \mathcal{N}_V(x)$$

Iterate this procedure for the new volume $V \setminus E_1$, and so on. If at the end some non-empty subset V_f is left, η satisfies, for all $x \in V_f$,

$$\eta(x) \leq \mathcal{N}_{V_f}(x)$$

The restriction η_{V_f} is then called a *forbidden subconfiguration* (fsc). If V_f is empty, the configuration is called *allowed*. The set $\mathcal{A}_V$ of allowed configurations coincides with the set of recurrent configurations, $\mathcal{A}_V = \mathcal{R}_V$ (see [8], [12], [14]).

A recurrent configuration is thus nothing but a configuration without forbidden subconfigurations. This extends to infinite volume:

Definition 2.6 *A configuration $\eta \in \Omega$ is called recurrent if for any $V \in \mathcal{S}$, $\eta_V \in \mathcal{R}_V$.*

The set $\mathcal{R}$ of all recurrent configurations forms a perfect (hence uncountable) subset of Ω. This means that $\mathcal{R}$ is closed (hence compact) and every element $\eta \in \mathcal{R}$ is the limit of a sequence $\eta_n \in \mathcal{R}, \eta_n \neq \eta$.

On the set $\mathcal{R}_V$, the finite volume addition operators $a_{x,V}$ can be inverted and they generate a finite abelian group. This group is characterized by the closure relation

$$\prod_{y \in V} a_{y,V}^{\Delta_{xy}^V} = \mathrm{Id} \tag{2.7}$$

By the group property, the uniform measure μ_V is invariant under the action of $a_{x,V}$ and of $a_{x,V}^{-1}$.

2.3 Toppling numbers

For $x, y \in V$ and $\eta \in \Omega_V$, let $n_V(x, y, \eta)$ denote the *number of topplings* at site y by adding a grain at x, that is, the number of times we have to apply the operator T_y to stabilize η^x in the volume V. We have the relation

$$\eta(y) + \delta_{x,y} = a_{x,V}\eta(y) + \sum_{z \in V} \Delta_{yz}^V n_V(x, z, \eta) \tag{2.8}$$

Defining

$$G_V(x,y) = \int \mu_V(d\eta)\; n_V(x,y,\eta) \tag{2.9}$$

one obtains, by integrating (2.8) over μ_V:

$$G_V(x,y) = (\Delta^V)^{-1}_{xy}. \tag{2.10}$$

In the limit $V \uparrow S$, G_V converges to the Green's function G of the simple random walk on S with a sink associated to the dissipative sites (that is every site x is linked with $\gamma - \mathcal{N}_S(x)$ edges to a sink and the walk stops when it reaches the sink). By (2.9), the probability that a site y topples by addition at x in volume V is bounded by $G_V(x,y)$.

Definition 2.11 *We say that the sandpile model is dissipative if*

$$\sup_{x\in S} \sum_{y\in S} G(x,y) < +\infty \tag{2.12}$$

In our examples, if $\gamma > 2d$ for $\mathbb{Z}^d$ or $\gamma \geq 4$ for strips, the Green's function $G(x,y)$ decays exponentially in the lattice distance between x and y and hence (2.1) defines a dissipative model. From now on, we restrict ourselves to these cases.

Definition 2.13 *For any integer n, let ν_{W_n} be a probability measure on Ω_{W_n}, with $W_n \in \mathcal{S}$, $W_n \uparrow S$. Then ν_{W_n} converges to a probability measure ν on Ω if for any $f \in \mathcal{L}$,*

$$\lim_{n\to\infty} \int f d\nu_{W_n} = \int f d\nu.$$

We denote by $\mathcal{I}$ the set of all limit points of $\{\mu_V : V \in \mathcal{S}\}$ in the sense of Definition 2.13. By compactness of Ω, $\mathcal{I}$ is a non-empty compact convex set. Moreover, by (2.5) and Definition 2.6, any $\mu \in \mathcal{I}$ concentrates on $\mathcal{R}$ (see [10]).

2.4 Untoppling numbers

On the set $\mathcal{R}_V$ the addition operators $a_{x,V}$ are invertible. The action of the inverse operator on a recurrent configuration can be defined recursively as follows, see [8]. Consider $\eta \in \mathcal{R}_V$ and $x \in V$. Remove one grain from η at site x. If the resulting configuration is recurrent, it is $a^{-1}_{x,V}\eta$, otherwise it contains a forbidden subconfiguration (fsc) in $V_1 \subset V$. In that case "untopple" the sites in V_1. By untoppling of a site z we mean that the sites are updated according to the rule $\eta(y) \to \eta(y) + \Delta_{zy}$. Iterate this procedure until a recurrent configuration is obtained: the latter coincides with $a^{-1}_{x,V}\eta$.
As an example, consider a graph with just three sites $a \sim b \sim c$ for $\gamma = 2$. The configuration 212 is recurrent. After removal of one grain at site c, we get 211, which contains the fsc 11. Untoppling site b gives 130, and untoppling site c gives 122, which is recurrent. Conversely, one verifies that addition at site c on 122 gives back the original configuration 212.

Call $n_V^-(x,y,\eta)$ the number of untopplings at site y by removing one grain from x and from untoppling sites until a recurrent configuration is obtained. As in the previous section, one easily proves the relation

$$\int n_V^-(x,y,\eta)\mu_V(d\eta) = G_V(x,y) \tag{2.14}$$

3 The group of addition operators in infinite volume

In this section we show how to obtain the group of addition operators in the infinite volume limit. The assumption of dissipativity is crucial in order to obtain a *compact* abelian group in the thermodynamic limit.

3.1 Addition operator

The finite volume addition operators $a_{x,V}$ (cf. (2.3)) are defined on Ω via

$$a_{x,V} : \Omega \to \Omega : \eta \mapsto a_{x,V}\eta = (a_{x,V}\eta_V)_V \eta_{V^c}. \tag{3.1}$$

(with some slight abuse of notation). Similarly, the inverses are defined on $\mathcal{R}$ via

$$a_{x,V}^{-1} : \mathcal{R} \to \Omega : \eta \mapsto (a_{x,V}^{-1}\eta_V)_V \eta_{V^c} \tag{3.2}$$

Remark that if $\eta \in \mathcal{R}$, then $(a_{x,V}\eta)_W \in \mathcal{R}_W$ for all $W \subset V$ but $a_{x,V}\eta$ is not necessarily an element of $\mathcal{R}$.

Definition 3.3 *For $\eta \in \Omega$, we say that the limit of the finite volume addition operators is defined on η if for every $x \in S$, there exists $\Lambda_0 \in \mathcal{S}$ such that for any $\Lambda \in \mathcal{S}, \Lambda \supset \Lambda_0$, $a_{x,\Lambda}\eta = a_{x,\Lambda_0}\eta$; in that case, we write*

$$a_x\eta = a_{x,\Lambda_0}\eta$$

Similarly, for $\eta \in \mathcal{R}$, we say that the limit of the finite volume inverse addition operators is defined on η if for every $x \in S$, there exists $\Lambda_0 \in \mathcal{S}$ such that for any $\Lambda \in \mathcal{S}, \Lambda \supset \Lambda_0$, $a_{x,\Lambda}^{-1}\eta = a_{x,\Lambda_0}^{-1}\eta$; we write

$$a_x^{-1}\eta = a_{x,\Lambda_0}^{-1}\eta$$

Remark that if $\eta \in \mathcal{R}$ and a_x is defined on η, then $a_x\eta \in \mathcal{R}$.

Lemma 3.4 *Assume* (2.12). *For any $\mu \in \mathcal{I}$ there exists a tail measurable subset $\overline{\Omega} \subset \Omega$ such that:*

1. *$\mu(\overline{\Omega}) = 1$;*
2. *The limit of the finite volume addition operators and their inverses is defined on every $\eta \in \overline{\Omega}$.*

Moreover, every $\mu \in \mathcal{I}$ is invariant under the action of a_x and a_x^{-1} , that is, for all $x \in S$ and $f \in \mathcal{L}$

$$\int f(a_x\eta)\mu(d\eta) = \int f(a_x^{-1}\eta)\mu(d\eta) = \int f(\eta)\mu(d\eta) \tag{3.5}$$

and $a_x a_x^{-1} = a_x^{-1} a_x = id$ on $\overline{\Omega}$.

Proof. We prove the result for the addition operators, the analogue for the inverses is proved along the same lines by replacing "number of topplings" by "number of untopplings".

Pick $W_k \in \mathcal{S}, W_k \uparrow S$ such that $\mu_{W_k} \to \mu$ and $x \in S$. We have to prove that

$$\mu\left[\forall \Lambda_0 \in \mathcal{S}, \exists V \supset \Lambda_0 : a_{x,V}\eta \neq a_{x,\Lambda_0}\eta\right] = 0 \tag{3.6}$$

We enumerate $S = \{x_n : n \in \mathbb{N}\}$, with $V_n = \{x_1, \ldots, x_n\}$ such that $V_n \uparrow S$, $x_n \in \partial V_{n-1}$. If $a_{x,V}\eta \neq\ a_{x,V_n}\eta$, then some boundary site of V_n has toppled under addition at x in volume V. This implies that for every m such that $V_m \supset V$ some external boundary site of V_n topples upon addition at x in V_m. Therefore, the left hand side of (3.6) is bounded by

$$\mu\left[\forall n \in \mathbb{N}, \exists p \geq n, \exists y \in \partial V_n : n_{V_p}(x,y,\eta) \geq 1\right]$$

and we have to estimate

$$\mu\left[\exists p \geq n, \exists y \in \partial V_n : n_{V_p}(x,y,\eta) \geq 1\right] \tag{3.7}$$

Since $n_{V_p}(x,y,\eta) \leq n_{V_{p+1}}(x,y,\eta)$,

$$\begin{aligned}
\mu\left(\exists p \geq n, \exists y \in \partial V_n : n_{V_p}(x,y,\eta) \geq 1\right) &\leq \lim_{k\to\infty} \mu\left(\exists y \in \partial V_n : n_{V_k}(x,y,\eta) \geq 1\right) \\
&\leq \lim_{k\to\infty} \sum_{y\in\partial V_n} \int n_{V_k}(x,y,\eta)\mu(d\eta) \\
&\leq \lim_{k\to\infty} \sum_{y\in\partial V_n} \int n_{W_k}(x,y,\eta)\mu_{W_k}(d\eta) \\
&= \sum_{y\in\partial V_n} G(x,y)
\end{aligned}$$

which implies that (3.7) converges to zero as n tends to infinity, by condition (2.12). Finally, (3.5) follows easily from Definition 3.3, $\mu(\overline{\Omega}) = 1$, $f \in \mathcal{L}$, and the invariance of μ_V under the finite volume addition operators $a_{x,V}$ and $a_{x,V}^{-1}$. ∎

Notice that for $\eta \in \overline{\Omega}$, we can take the limit $V \uparrow S$ in (2.8) and write

$$\eta(y) + \delta_{x,y} = a_x\eta(y) + \sum_{z\in S} \Delta_{yz} n_S(x,z,\eta) \tag{3.8}$$

for any $x, y \in S$, where $n_S(x,z,\eta)$, the number of topplings at site $z \in S$ by adding a grain at x, satisfies $\sum_{z\in S} n_S(x,z,\eta) < +\infty$.

Lemma 3.9 *Assume* (2.12). *For any $\mu \in \mathcal{I}$ there exists a tail measurable subset $\Omega^o \subset \overline{\Omega}$ with $\mu(\Omega^o) = 1$ such that for any $V \in \mathcal{S}$ and $n_x, x \in V$ integers, the product $\prod_{x\in V} a_x^{n_x}$ is well-defined, as the limit of $\prod_{x\in V} a_{x,\Lambda}^{n_x}$ as $\Lambda \to S$, on every $\eta \in \Omega^o$.*

Proof. We fix $V \in \mathcal{S}, x \in V, n_x$ a positive integer, and we prove that $a_x^{n_x}$ is well-defined on $\overline{\Omega}$ (the case of negative n_x is similar and the extension to finite products it straightforward). Following the same lines as in the preceding proof, we have to replace (3.6) by

$$\mu\left[\forall \Lambda_0 \in \mathcal{S}, \exists \Lambda \supset \Lambda_0 : a_{x,\Lambda}^{n_x}\eta \neq a_{x,\Lambda_0}^{n_x}\eta\right] = 0$$

We denote by $E_{V_p}(n_x, x, z, \eta)$ the event that addition in V_p of n_x grains at x causes at least one toppling at z. As these events are increasing in p, we estimate

$$\begin{aligned}\mu\left(\exists p \geq n, \exists y \in \partial V_n : E_{V_p}(n_x, x, y, \eta) \geq 1\right) &\leq \lim_{k\to\infty} \sum_{y\in\partial V_n} \mu_{W_k}(E_{W_k}(n_x, x, y, \eta)) \\ &\leq \sum_{y\in\partial V_n} n_x G(x, y)\end{aligned}$$

where the last inequality is a consequence of (2.10) and (3.5). From this we deduce that for any $V \in \mathcal{S}$, $n = (n_x, x \in V) \in \mathbb{Z}^V$, the product $\prod_{x\in V} a_x^{n_x}$ is well-defined on a tail measurable set $\Omega(V, n)$ of μ-measure one. The set Ω^o is then the countable intersection

$$\Omega^o = \cap_{V\in\mathcal{S}, n\in\mathbb{Z}^V} \Omega(V, n)$$

of tail measurable μ-measure one sets. ∎

The following proposition extends this to addition on infinite products.

Proposition 3.10 *Assume* (2.12). *If $n = (n_x, x \in S) \in \mathbb{Z}^S$ satisfies $\sum_{x\in S} |n_x| G(0, x) < +\infty$, the product $\prod_{x\in S} a_x^{n_x}$ is well-defined on a set $\Omega(n)$ of μ-measure 1, for every $\mu \in \mathcal{I}$.*

Proof. Take $n_x \geq 0$ for every $x \in S$; the case of negative n_x is treated again by replacing "topplings" with "untopplings". It suffices to show that for every $\Lambda_0 \in \mathcal{S}$

$$\mu\left(\exists V_0, \forall V \supset V_0, \forall y \in \Lambda_0 : \left(\prod_{x\in V} a_x^{n_x}\eta\right)(y) = \left(\prod_{x\in V_0} a_x^{n_x}\eta\right)(y)\right) = 1$$

or

$$\lim_{V_0\uparrow S} \mu\left(\exists V \supset V_0, \exists y \in \Lambda_0 : \left(\prod_{x\in V} a_x^{n_x}\eta\right)(y) \neq \left(\prod_{x\in V_0} a_x^{n_x}\eta\right)(y)\right) = 0 \qquad (3.11)$$

The left hand side of (3.11) is bounded by the sum

$$\sum_{y\in\Lambda_0} \mu\left(\exists V \supset V_0 : \left(\prod_{x\in V} a_x^{n_x}\eta\right)(y) \neq \left(\prod_{x\in V_0} a_x^{n_x}\eta\right)(y)\right) \qquad (3.12)$$

If none of the external boundary points of Λ_0 topples upon addition of n_z grains at $z \in V \setminus V_0$ to the configuration $\left(\prod_{x\in V_0} a_x^{n_x}\eta\right)$, we have that for all $y \in \Lambda_0$:

$$\left(\prod_{x\in V} a_x^{n_x}\eta\right)(y) = \left(\prod_{x\in V_0} a_x^{n_x}\eta\right)(y)$$

Since μ is invariant under the a_x, see (3.5), the sum (3.12) is bounded from above by

$$\sum_{y\in\Lambda_0}\sum_{|x-y|=1}\sum_{z\in V_0^c}\mu\left(E_S(n_z,z,x,\eta)\right) \le \sum_{y\in\Lambda_0}\sum_{|x-y|=1}\sum_{z\in V_0^c} n_z G(z,x)$$

which implies (3.11) by the hypothesis on n. ∎

3.2 Group structure

Here we show that the product $\prod_{x\in S} a_x^{n_x}$ can be defined on any recurrent configuration, provided we identify recurrent configurations which differ by a multiple of Δ.

Given $n \in \mathbb{Z}^S$ and $\eta \in \mathcal{R}$, we consider the set

$$A_n(\eta) = \{\xi \in \mathcal{R} : \exists m \in \mathbb{Z}^S, \eta + n = \xi + \Delta m\}$$

Similarly, for subtraction,

$$S_n(\eta) = \{\xi \in \mathcal{R} : \exists m \in \mathbb{Z}^S, \eta - n = \xi + \Delta m\}$$

Fix $n \in \mathbb{Z}^S$ so that

$$\sup_{y\in S}\sum_{x\in S}[|n_x| + 2\gamma]G(y,x) = B < +\infty \tag{3.13}$$

and let

$$\Omega_n = \{\eta \in \mathcal{R} : S_n(\eta) \neq \emptyset, A_n(\eta) \neq \emptyset\}$$

be the set of recurrent configurations for which both addition and subtraction with n gives rise to a new recurrent configuration, modulo the toppling matrix applied to an integer function.

Lemma 3.14 *$\Omega_n = \mathcal{R}$.*

Proof. We prove that Ω_n is closed. Let $(\eta_k)_{k\geq 0}$ be a sequence in Ω_n which converges to η as $k \to \infty$. For each k, there exist $\eta_k^{\pm} \in \mathcal{R}$ and $m_k^{\pm} \in [-B,B]^S$ such that

$$\eta_k \pm n = \eta_k^{\pm} + \Delta m_k^{\pm} \tag{3.15}$$

Since $\mathcal{R} \times [-B,B]^S$ is compact, there exists a subsequence $k_i \to \infty$ such that $\eta_{k_i}^{\pm} \to \eta^{\pm}$ and $m_{k_i}^{\pm} \to m^{\pm}$. Taking limits along this subsequence in (3.15) yields

$$\eta \pm n = \eta^{\pm} + \Delta m^{\pm},$$

that is, $\eta \in \Omega_n$. Looking back at Proposition 3.10, $\Omega(n) \cap \mathcal{R} \subset \Omega_n$ and $\Omega(n)$ is a μ-measure one (hence non-empty) tail set. Therefore it is dense and $\Omega_n = \mathcal{R}$. ∎

Definition 3.16 *Two recurrent configurations $\eta, \zeta \in \mathcal{R}$ are called equivalent, and we write $\eta \sim \zeta$, if there exists $m \in \mathbb{Z}^S$ such that*

$$\eta = \zeta + \Delta m \tag{3.17}$$

Remark 3.18
1. *For all $n \in \mathbb{Z}^S, \eta \in \mathcal{R}$, if $\zeta, \zeta' \in A_n(\eta)$ (or $\zeta, \zeta' \in S_n(\eta)$), then $\zeta \sim \zeta'$.*
2. *If $\eta \sim \eta'$, then $A_n(\eta) = A_n(\eta'), S_n(\eta) = S_n(\eta')$ for all $n \in \mathbb{Z}^S$.*
3. *In the finite volume case one can prove that every equivalence class in $\mathbb{Z}^V/\Delta^V\mathbb{Z}^V$ contains exactly one recurrent configuration, that is, $\eta, \zeta \in \mathcal{R}_V$ and*

$$\eta = \zeta + \Delta^V m$$

imply $\eta = \zeta$. This is no longer true in infinite volume. As an example we take $S = \mathbb{Z} \times \{1, 2\}, \gamma = 4$. Then the recurrent configurations $\eta(x) = 3$ for all x and $\zeta(x) = 4$ for all x (denoted by $\overline{3}$ and $\overline{4}$) are equivalent:

$$\zeta = \eta + \Delta m$$

where $m(x) = 1$ for all x.

We can now introduce the addition operator on classes: take the class $[\eta]$ containing the recurrent configuration η, let $\xi \in A_n(\eta)$ and define

$$\prod_{x \in S} a_x^{n_x}[\eta] = [\xi]$$

Notice that if $\eta \in \overline{\Omega}$ (that is, $\eta \in \mathcal{R}$ is such that a_x is the limit of $a_{x,V}$ on η), then

$$a_x[\eta] = [a_x \eta] \tag{3.19}$$

Proposition 3.20 *Assume* (2.12). *$\mathcal{R}/\sim$ is a compact metric space.*

Proof. It suffices to show that equivalence classes are closed. Suppose we have sequences $(\eta_k), (\xi_k)$ of recurrent configurations with $\eta_k \sim \xi_k$, $\eta_k \to \eta$, $\xi_k \to \xi$. Then, there exist $m_k \in [-M, M]^S$ with $M = 2\gamma \sup_{x \in S} \sum_{y \in S} G(x, y)$ such that

$$\eta_k = \xi_k + \Delta m_k \tag{3.21}$$

We can choose a subsequence $k_i \to +\infty$ such that $m_{k_i} \to m$. Taking limits along this subsequence in (3.21) yields

$$\eta = \xi + \Delta m,$$

giving $\eta \sim \xi$. ∎

By point 2 of Remark 3.18 the addition of equivalence classes of configurations in $\mathcal{R}$ is well-defined.

Definition 3.22 *Assume* (2.12). *For* $[\eta], [\xi]$ *in* $\mathcal{R}/\sim$ *we define*

$$[\eta] \oplus [\xi]$$

to be the class which contains $A_\xi(\eta)$.

Theorem 3.23 $(\mathcal{R}/\sim, \oplus)$ *is a compact abelian group, hence it admits a unique Haar measure.*

Proof. The group property is immediate; the compactness follows from Proposition 3.20. For the consequence see e.g. [7] p. 31. ∎

The next result shows that from a measure theoretic perspective, there is no difference between classes of the relation $\sim$ and recurrent configurations. As a corollary, we obtain that the set $\mathcal{I}$ of possible weak limit points of the finite volume stationary measures is a singleton.

Proposition 3.24 *For every* $\mu \in \mathcal{I}$ *there exists a set* $\widehat{\Omega} \subset \mathcal{R}$ *of* μ*-measure one such that for all* $\eta \in \widehat{\Omega}$, $[\eta] = \{\eta\}$.

Before proving the proposition, we state and prove

Theorem 3.25 *The set* $\mathcal{I}$ *is a singleton.*

Proof. Suppose that $\mathcal{I}$ contains two different measures μ, ν. Then there exists a measurable subset A such that

$$\mu(A) \neq \nu(A).$$

μ and ν are lifted to $\mathcal{R}/\sim$ via

$$\bar{\mu}([A]) = \mu(\cup_{\eta \in A}[\eta])$$

Using Proposition 3.24

$$\begin{aligned}
\bar{\mu}([A]) &= \mu\left(\cup_{\eta \in A}[\eta]\right) \\
&= \mu\left((\cup_{\eta \in A}[\eta]) \cap \widehat{\Omega}\right) \\
&= \mu\left(\cup_{\eta \in A}\{\eta\}\right) \\
&= \mu(A).
\end{aligned} \tag{3.26}$$

Analogously $\bar{\nu}([A]) = \nu(A)$. Hence $\bar{\mu}$ and $\bar{\nu}$ are different. Because μ and ν are invariant under the action of the addition operators a_x, it follows that $\bar{\mu}$ and $\bar{\nu}$ are different and invariant under the group action. This contradicts the uniqueness of the Haar measure. ∎

Proof. [Proposition 3.24]. Let the set $\widehat{\Omega}$ consist of recurrent configurations η that satisfy

1. For all $x \in S$, a_x and a_x^{-1} are well defined as limits of the corresponding finite volume operators, and $a_x a_x^{-1}\eta = a_x^{-1} a_x \eta = \eta$ (that is $\eta \in \overline{\Omega}$).

2. For all finite volumes V_0, there is a volume $\Lambda, V_0 \subset \Lambda$ so that, whenever W is a finite set outside $\Lambda, W \cap \Lambda = \emptyset$ and for all $n \in [-B, B]^S$

$$\prod_{x \in W} a_x^{n_x} \eta(y) = \eta(y), \qquad \text{for all } y \in V_0 \tag{3.27}$$

That $\mu(\widehat{\Omega}) = 1$ follows from the same kind of arguments as for $\mu(\overline{\Omega}) = 1$ in lemma 3.4. Moreover, $\mu(a_x \widehat{\Omega}) = \mu(a_x^{-1}\widehat{\Omega}) = 1$ by invariance. Consider an arbitrary finite volume V and abbreviate $V_1 = V \cup \partial V$, $V_2 = V_1 \cup \partial V_1$. By the closure relation for the infinite volume addition operators, see (2.7), we have the identity

$$\prod_{x \in V_1} \prod_{y \in V_2} a_y^{\Delta_{xy} n_x} = \text{ id}$$

This gives

$$\begin{aligned}
\prod_{y \in V} a_y^{\Delta n(y)} &= \prod_{y \in V} \prod_{x \in V_1} a_y^{\Delta_{xy} n_x} \\
&= \prod_{x \in V_1} \prod_{y \in V} a_y^{\Delta_{xy} n_x} \\
&= \prod_{x \in V_1} \prod_{y \in V_2} a_y^{\Delta_{xy} n_x} \prod_{x \in V_1} \prod_{y \in V_2 \setminus V} a_y^{-\Delta_{xy} n_x} \\
&= \prod_{x \in V_1} \prod_{y \in V_2 \setminus V} a_y^{-\Delta_{xy} n_x}
\end{aligned}$$

Therefore, from (3.27) it follows that for every $n \in [-B, B]^S$,

$$\lim_{p \uparrow \infty} \left(\prod_{x \in V_p} a_x^{\Delta n(x)} \right) (\eta) = \eta \tag{3.28}$$

along some sequence of increasing volumes. Therefore, if $\eta, \xi \in \widehat{\Omega}$ satisfy

$$\eta + \Delta n = \xi \tag{3.29}$$

then, using (3.28) and (3.29):

$$\eta = \lim_{p \to \infty} \prod_{x \in V_p} a_x^{\Delta n(x)}(\eta) = \xi$$

which shows the desired property of the set $\widehat{\Omega}$. ∎

From now on we denote by μ the unique element of $\mathcal{I}$ as well as the Haar measure.

4 Infinite volume dynamics

From the previous sections we know that $\mathcal{I}$ contains a unique element μ and that addition operators as well as their inverses are well-defined on μ-typical configurations. This measure μ is the natural candidate for a stationary measure of a Markov process on infinite volume recurrent configurations. The construction of this Markov process is completely identical to what was done in [10]. We therefore state the results on existence and Poisson representation of this process without proofs, in the following section, and proceed in section 4.2 to the proof of its ergodic properties, which was open in [10].

4.1 Infinite volume Markov process

For the unique $\mu \in \mathcal{I}$ we can construct a stationary Markov process on μ-typical infinite volume configurations, as in [10].

We assume that the addition rate function φ introduced in (2.4) satisfies

$$\sup_{y\in S} \sum_{x\in S} \varphi(x) G(y,x) < \infty \tag{4.1}$$

This condition ensures that the number of topplings at any site $x \in S$ remains finite almost surely in any finite interval of time when grains are added at intensity φ. Notice that for dissipative systems, by (2.12), we can take the addition rate function constant.

To each site $x \in S$ we associate a Poisson process $N_\varphi^{t,x}$ (for different sites these Poisson processes are mutually independent) with rate $\varphi(x)$. At the event times of $N_\varphi^{t,x}$ we "add a grain" at x, that is, we apply the addition operator a_x to the configuration. For every finite volume $V \in \mathcal{S}$, the natural extension of (2.4)

$$L_V^\varphi = \sum_{x\in V} \varphi(x)(a_x - I) \tag{4.2}$$

is the $L^p(\mu)$ generator of the stationary pure jump process on Ω with semigroup

$$S_V^\varphi(t) = \exp(tL_V^\varphi) f = \int \left(\prod_{x\in V} a_x^{N_\varphi^{t,x}} f \right) d\mathbb{P}, \tag{4.3}$$

where $\mathbb{P}$ denotes the joint distribution of the independent Poisson processes $\{N_\varphi^{t,x}\}$, and $f \in L^p(\mu)$. The following theorems can be derived directly from the techniques developed in [10].

Theorem 4.4 *If φ satisfies condition (4.1), then*

1. *The semigroups $S_V^\varphi(t)$ converge strongly in $L^1(\mu)$ to a semigroup $S_\varphi(t)$.*

2. *$S_\varphi(t)$ is the $L^1(\mu)$ semigroup of a stationary Markov process $\{\eta_t : t \geq 0\}$ on Ω.*

3. For any $f \in \mathcal{L}$,

$$\lim_{t \downarrow 0} \frac{S_\varphi(t)f - f}{t} = L^\varphi f = \sum_{x \in S} \varphi(x)[a_x f - f],$$

where the limit is taken in $L^1(\mu)$.

4. The process $\{\eta_t : t \geq 0\}$ admits a càdlàg version (right-continuous with left limits).

The intuitive description of the process $\{\eta_t : t \geq 0\}$ is correct under condition (4.1), that is, the process has a representation in terms of Poisson processes:

Theorem 4.5 *Assume (4.1). For $\mu \times \mathbb{P}$ almost every (η, ω) the limit*

$$\lim_{V \uparrow S} \prod_{x \in V} a_x^{N_\varphi^{t,x}(\omega)} \eta = \eta_t$$

exists. The process $\{\eta_t : t \geq 0\}$ is a version of the process of Theorem 4.4, that is, its $L^1(\mu)$ semigroup coincides with $S_\varphi(t)$.

To formulate next theorem we need a partial order on configurations, functions, and probability measures on Ω. For $\eta, \xi \in \Omega$, $\eta \leq \xi$ if $\eta(x) \leq \xi(x)$ for all $x \in S$. A function $f : \Omega \to \mathbb{R}$ is *monotone* if $\eta \leq \xi$ implies $f(\eta) \leq f(\xi)$, for all $\eta, \xi \in \Omega$. For two probability measures ν, ν' on Ω, $\nu \leq \nu'$ if $\nu(f) \leq \nu'(f)$ for all monotone bounded Borel measurable function f.

Theorem 4.6 *Let $\nu \leq \mu$. For $\nu \times \mathbb{P}$ almost every (η, ω) the limit*

$$\lim_{V \uparrow S} \prod_{x \in V} a_x^{N_\varphi^{t,x}(\omega)} \eta = \eta_t$$

exists. The process $\{\eta_t : t \geq 0\}$ is Markovian with η_0 distributed according to ν.

Remark 4.7 *Theorem 4.6 implies that $\eta \equiv 1$ can be taken as initial configuration.*

4.2 An ergodic theorem

In the rest of this section, we assume for simplicity that the rate function $\varphi \equiv 1$ is constant, and we write $S(t)$ (see Theorem 4.4), $N^{t,x}$, L and L_V^0 (see (2.4)) without subscript φ.
We investigate the convergence of $\nu S(t)$ to μ for a probability measure $\nu \leq \mu$.
Before we give the statement and its proof, observe that the role of the dissipativity parameter γ here is double. First, the approximation (and even the existence) of the infinite volume process by finite volume ones gets nicer and easier to prove when γ increases. It is essentially based on the dissipativity condition (2.12). On the other hand, in finite volume, the exponential relaxation to the stationary measure μ_V also

depends on γ and in fact, becomes slower for larger γ. This can be seen from ignoring (as would be reasonable for very large γ and dimension d) the interaction with other sites: we then have essentially a one site dynamics by which at exponential times one grain is added to the site until the latter reaches a height $\gamma+1$, after which it topples to height 1 and so on. The relaxation time of this dynamics being clearly proportional to γ, the convergence is slower for larger γ.

Theorem 4.8 *Suppose ν is a probability measure on Ω such that $\nu \leq \mu$. There is a constant $C_2 > 0$ so that for all $f \in \mathcal{L}$, there exists $C_f < +\infty$ such that*

$$\left| \int S(t) f d\nu - \int f d\mu \right| \leq C_f \exp(-C_2 t) \tag{4.9}$$

In particular, $\nu S(t)$ converges weakly to μ, uniformly in $\nu \leq \mu$ and exponentially fast.

Proof. The idea is to approximate $S(t)$ by finite volume semigroups, and to estimate the speed of convergence as a function of the volume. More precisely, we split

$$\left| \int S(t) f d\nu - \int f d\mu \right| \leq A_t^V(f) + B_t^V(f) + C_V(f) \tag{4.10}$$

with

$$\begin{aligned} A_t^V(f) &= \left| \int S(t) f d\nu - \int S_V(t) f d\nu_V \right| \\ B_t^V(f) &= \left| \int S_V(t) f d\nu_V - \int f d\mu_V \right| \\ C_V(f) &= \left| \int f d\mu_V - \int f d\mu \right|. \end{aligned}$$

where ν_V is the restriction of ν to V and

$$S_V(t) f(\eta) = \int f \left(\prod_{x \in V} a_{x,V}^{N^{t,x}} \eta \right) d\mathbb{P}$$

By Theorem 3.25,

$$\lim_{V \uparrow S} C_V(f) = 0. \tag{4.11}$$

For the first term in the right-hand side of (4.10) we write

$$A_t^V(f) = \left| \int \int \left(f \left(\prod_{x \in S} a_x^{N^{t,x}} \eta \right) - f \left(\prod_{x \in V} a_{x,V}^{N^{t,x}} \eta \right) \right) d\mathbb{P} d\nu \right| \tag{4.12}$$

The integrand of the right hand side is zero if no avalanche from V^c has influenced sites of D_f during the interval $[0,t]$, otherwise it is bounded by $2\|f\|_\infty$. Therefore, since $N^{t,x}$ are rate one Poisson processes:

$$A_t^V(f) \leq 2\|f\|_\infty t \sum_{y \in D_f} \sum_{x \in V^c} G(x,y) \tag{4.13}$$

The second term in the right hand side of (4.10) is estimated by the relaxation to equilibrium of the finite volume dynamics. The generator L_V^0 has the eigenvalues

$$\sigma(L_V^0) = \left\{ \sum_{x\in V} \left(\exp\left(2\pi i \sum_{y\in V} G_V(x,y) n_y \right) - 1 \right) : n \in \mathbb{Z}^V/\Delta^V\mathbb{Z}^V \right\} \tag{4.14}$$

The eigenvalue 0 corresponding to the stationary state arises from the choice $n = \overline{0}$. For the speed of relaxation to equilibrium we are interested in the minimum absolute value of the real part of the non-zero eigenvalues. More precisely:

$$B_t^V(f) \leq C_f \exp(-\lambda_V t)$$

where

$$\begin{aligned} \lambda_V &= \inf\left\{ |Re(\lambda)| : \lambda \in \sigma(L_V^0) \setminus \{0\} \right\} \\ &= 2\inf\left\{ \sum_{x\in V} \sin^2\left(\pi \sum_{y\in V} G_V(x,y) n_y \right) : n \in \mathbb{Z}^V/\Delta^V\mathbb{Z}^V, n \neq \overline{0} \right\} \end{aligned}$$

by (4.14). Since there is a constant c so that for all real numbers r

$$\sin^2(\pi r) \geq c(\min\{|r-k| : k\in\mathbb{Z}\})^2$$

we get

$$\sum_{x\in V} \sin^2\left(\pi((\Delta^V)^{-1}n)_x\right) \geq c \inf\left\{ \|(\Delta^V)^{-1}n - k\|^2 : n \in \mathbb{Z}^V/\Delta^V\mathbb{Z}^V, n\neq \overline{0}, k\in\mathbb{Z}^V \right\} \tag{4.15}$$

where $\|\cdot\|$ represents the euclidian norm in $\mathbb{Z}^V$ that we estimate by

$$\|(\Delta^V)^{-1}n - k\|^2 = \|(\Delta^V)^{-1}(n - \Delta^V k)\|^2 \geq \|\Delta^V\|^{-2}$$

For any regular volume we have

$$\|\Delta^V\| \leq \sqrt{2\gamma^2 + 16d^2}$$

This gives

$$B_t^V(f) \leq C_f \exp(-Ct) \tag{4.16}$$

where $C > 0$ is independent of V.
The statement of the theorem now follows by combining (4.11), (4.13), (4.16). ∎

Remark 4.17 *When we restrict ourselves to the case where*

$$\sum_{x\in S} \varphi(x) = M < \infty, \tag{4.18}$$

L^φ becomes a bounded operator, hence it generates a pure jump process which is a continuous time random walk on the group $(\mathcal{R}/\sim, \oplus)$. By the ergodic properties of random walks on compact groups we then obtain that

$$\lim_{t\to\infty} \nu S_\varphi(t) = \mu.$$

for every measure ν on $\mathcal{R}/\sim$ (see Theorems 2.5.14, 2.6.2 and Corollary 2.6.4 in [7] for details).

4.3 Mixing property

To the stationary process defined in Theorem 4.4, we associate the process on $\mathcal{R}/\sim$ by putting

$$[\eta]_t = [\eta_t]. \tag{4.19}$$

For that, it is important to notice that the equivalence of recurrent configurations is preserved in time (by Theorem 4.5, and points 1,2 of Remark 3.18): when $\eta \sim \xi$, then $\eta_t \sim \xi_t$ with $\mathbb{P}$-probability one. For the following theorem, we abbreviate without consequences $[\eta]_t = \eta_t$.

Theorem 4.20 *The process $\{\eta_t : t \geq 0\}$ is mixing, that is, for all $f, g \in \mathcal{L}$:*

$$\lim_{t\to\infty} \int (S(t)f)\, g d\mu = \int f d\mu \int g d\mu \tag{4.21}$$

Proof. Since the semigroup is a normal operator on $L^2(\mu)$, ergodicity of the process implies the mixing property by [13]. It thus suffices to prove that for a bounded non-negative function f such that $\int f d\mu > 0$ and $Lf = 0$, then μ-a.s. $f = \int f d\mu$. By the invariance of μ under a_x,

$$0 = -2 \int (fLf)\; d\mu = \sum_{x\in S} \int (a_x f - f)^2 d\mu$$

which implies $a_x f = f$ for all a_x, μ-a.s. Hence, the measure

$$d\nu_f = \frac{f d\mu}{\int f d\mu}$$

is invariant under the action of a_x, thus under the group action. By uniqueness of the Haar measure, we conclude $\nu_f = \mu$. ∎

5 Decay of correlations

In this section we prove that the infinite volume measure μ has exponential decay of correlations under a condition of "strong dissipativity". That means for the model (2.1) with $S = \mathbb{Z}^d$ that γ must be sufficiently large, e.g. $\gamma > 13$ for $d = 2$; for the strips $S = \mathbb{Z} \times \{1, \ldots, \ell\}$ with finite ℓ it always suffices that $\gamma > 3$.
In [11] the exponential decay between very special local observables (indicators of so-called weakly allowed clusters) is also obtained in the case where the Green function decays exponentially. However, the technique developed in that paper does not apply to all local functions.

5.1 Decoupling argument

We start with the heuristics of the main ingredient in the proof of exponential decay of correlations. The rest is based on quite general stochastic-geometric methods that are reviewed in [6].

To be specific, suppose that $S = \mathbb{Z}^2$ and $\gamma > 4$ (in (2.1)). Then, for every volume $V \in \mathcal{S}$,

$$\mu_V(\eta(x) = a|\eta(z) = c) = \mu_{V\setminus z}(\eta(x) = a) \tag{5.1}$$

for all $a, c \in \{1, \ldots, \gamma\}, c > 4, x \neq z$ in V, because, by the burning algorithm, we can burn away the sites on which we know that the configuration is sufficiently large. Instead of fixing in (5.1) the height value at one site z, we could do the same thing on some region $C \subset V$ that does not contain x, see Lemma 5.4. On the other hand, if sites x and y are not very close to each other, we can find volumes $\Lambda_x, \Lambda_y \subset V$ that contain x, respectively y, that also do not touch (more precisely, that satisfy $(\Lambda_x \cup \partial\Lambda_x) \cap \Lambda_y = \emptyset$). Then, see Lemma 5.7,

$$\mu_{\Lambda_x \cup \Lambda_y}(\eta(x) = a, \eta(y) = b) = \mu_{\Lambda_x}(\eta(x) = a)\mu_{\Lambda_y}(\eta(y) = b) \tag{5.2}$$

The combination of (5.1) with (5.2) yields conditional independence of two events that are separated by a region C where the configuration is sufficiently high, see Lemma 5.9.

Definition 5.3 *Let $V \in \mathcal{S}$, $C \subset V$, $\sigma \in \Omega_V$. The subconfiguration σ_C is V-burnable if there exists a bijection $f : \{1, \ldots, n\} \to C$ such that*

$$\mathcal{N}_V(f(1)) < \sigma(f(1)),$$

and for every $j = 1, \ldots, n-1$,

$$\mathcal{N}_{V\setminus\{f(1),\ldots,f(j)\}}(f(j+1)) < \sigma(f(j+1)).$$

As an example, on $\mathbb{Z}^2$ with maximal height $\gamma = \Delta_{xx} = 5$, every closed curve along which the heights are at least 4 and containing at least one point with height 5 is burnable.

Lemma 5.4 *Let $V = \Lambda \cup C \in \mathcal{S}, \Lambda \cap C = \emptyset$ and fix an arbitrary configuration $\sigma \in \Omega_V$ so that σ_C is V-burnable. Put*

$$E_C = \{\eta \in \Omega_V : \eta_C = \sigma_C\}$$

Then, for all events A that depend only on the configuration in Λ (that is, $A \in \mathcal{F}_\Lambda$),

$$\mu_V(A|E_C) = \mu_\Lambda(A) \tag{5.5}$$

Proof. By the burning algorithm, $\eta \in \mathcal{R}_V \cap E_C$ if and only if $\eta_\Lambda \in \mathcal{R}_\Lambda$ and $\eta_C = \sigma_C$. Therefore,

$$\begin{aligned} \mu_V(A|E_C) &= \frac{\sum_{\eta \in \mathcal{R}_V} I(\eta \in A) I(\eta \in E_C)}{\sum_{\eta \in \mathcal{R}_V} I(\eta \in E_C)} \\ &= \frac{\sum_{\eta_\Lambda \in \mathcal{R}_\Lambda} \sum_{\eta_C \in \mathcal{R}_C} I(\eta \in A) I(\eta \in E_C)}{\sum_{\eta_C \in \mathcal{R}_C} I(\eta \in E_C) |\mathcal{R}_\Lambda|} \\ &= \mu_\Lambda(A). \end{aligned}$$

∎

Remark 5.6 *We do not need to condition on one fixed burnable configuration. The Lemma and its proof above remain unchanged when taking*

$$E_C = \{\eta \in \Omega_V : \eta_C \text{ is } V\text{-burnable}\}$$

the event that we can burn away the sites of C first.

Lemma 5.7 *Let $\Lambda_1, \Lambda_2 \in \mathcal{S}$ with*

$$(\Lambda_1 \cup \partial \Lambda_1) \cap \Lambda_2 = \emptyset. \tag{5.8}$$

For $A \in \mathcal{F}_{\Lambda_1}, B \in \mathcal{F}_{\Lambda_2}$,

$$\mu_{\Lambda_1 \cup \Lambda_2}(A \cap B) = \mu_{\Lambda_1 \cup \Lambda_2}(A) \mu_{\Lambda_1 \cup \Lambda_2}(B)$$

Proof. We have $\eta \in \mathcal{R}_{\Lambda_1 \cup \Lambda_2}$ if and only if $\eta_{\Lambda_1} \in \mathcal{R}_{\Lambda_1}$ and $\eta_{\Lambda_2} \in \mathcal{R}_{\Lambda_2}$. The rest is writing out expectations as in the proof of Lemma 5.4. ∎

We now state the conditional independence.

Lemma 5.9 *For $V \in \mathcal{S}$ and $C \subset V$, suppose that $V \setminus C = \Lambda_1 \cup \Lambda_2$ with Λ_1, Λ_2 satisfying* (5.8). *Then, for all $A \in \mathcal{F}_{\Lambda_1}, B \in \mathcal{F}_{\Lambda_2}$*

$$\mu_V(A \cap B|E_C) = \mu_V(A|E_C)\mu_V(B|E_C)$$

Proof. By Lemma 5.4,

$$\mu_V(A \cap B|E_C) = \mu_{V \setminus C}(A \cap B)$$

and continuing via Lemma 5.7

$$\mu_V(A \cap B|E_C) = \mu_{\Lambda_1 \cup \Lambda_2}(A) \mu_{\Lambda_1 \cup \Lambda_2}(B). \tag{5.10}$$

The proof is finished by applying again Lemma 5.4 to the two factors in the right-hand side of (5.10). ∎

The conditional independence (5.10) is reminiscent of the situation for Markov random fields. Here μ_V is not Markovian but nevertheless for all $A \in \mathcal{F}_\Lambda, \Lambda \subset V$ the conditional probability of A given the configuration in $V \setminus \Lambda$ is

$$\mu_V(A|\eta_{V \setminus \Lambda}) = \mu_\Lambda(A) \tag{5.11}$$

whenever $\eta_{\partial \Lambda \cap V}$ is V-burnable. In particular, this conditional probability (5.11) is then independent of the particular $\eta_{V \setminus \Lambda}$.

5.2 Geometric argument

From the previous decoupling argument, it is clear how to proceed for the proof of decay of correlations. What needs to be established is that there will be typically some "circuit" C, separating two far away dependence sets, where the configuration is burnable. We thus basically end up with a stochastic-geometric or percolation-like argument as also reviewed in [6]. The first thing to see is that burnability is sufficiently probable. We do that first for the strip in Lemma 5.12 and then for the full lattice in Lemma 5.16.

Lemma 5.12 *Let* $V = \{(x, y) \in \mathbb{Z}^2 : |x| \leq k, y = 1, \dots \ell\}$ *and* $\gamma \geq 4$ *in* (2.1)*. Fix some* $x_1, |x_1| \leq k$ *and let* $C = \{(x_1, y) \in V : y = 1, \dots \ell\}$*. There is* $p = p(\ell, \gamma) > 0$ *(uniformly in* k*) such that for all events* $E(x_1)$ *that only depend on the heights* $\eta(x, y)$ *with* $(x, y) \notin C$*,*

$$\mu_V(\eta(x, y) \geq 4 \text{ for all } (x, y) \in C | E(x_1)) \geq p > 0 \tag{5.13}$$

Proof. Via Bayes'rule,

$$\begin{aligned} \mu_V(\eta(x, y) \geq 4 \text{ for all } (x, y) \in C | E(x_1)) &= \\ \mu_V(E(x_1) | \eta(x, y) \geq 4 \text{ for all } (x, y) \in C) & \frac{\mu_V(\eta(x, y) \geq 4 \text{ for all } (x, y) \in C)}{\mu_V(E(x_1))} \end{aligned} \tag{5.14}$$

If $\eta(x, y) \geq 4$ for the points $(x, y) \in C$, then η_C is V-burnable, and by Lemma 5.4

$$\mu_V(E(x_1) | \eta(x_1, y_1) \geq 4 \text{ whenever } |y_1| \leq \ell) = \mu_{V \setminus C}(E(x_1))$$

On the other hand, by counting,

$$\frac{\mu_{V \setminus C}(E(x_1))}{\mu_V(E(x_1))} \geq \frac{|\mathcal{R}_V|}{|\mathcal{R}_{V \setminus C}||\mathcal{R}_C|}$$

and

$$\mu_V(\eta(x, y) \geq 4 \text{ for all } (x, y) \in C) = (\gamma - 4 + 1)^\ell \frac{|\mathcal{R}_{V \setminus C}|}{|\mathcal{R}_V|}$$

As a consequence we can take

$$0 < p \leq \frac{(\gamma - 4 + 1)^\ell}{|\mathcal{R}_C|}$$

∎

Remark 5.15 *Obviously,* $p \downarrow 0$ *as* $\ell \uparrow \infty$*.*

For the regular lattice $S = \mathbb{Z}^d$ we have:

Lemma 5.16 *Consider the model (2.1) with $S = \mathbb{Z}^d$. For all $\varepsilon > 0$, there is a $\gamma_0 < +\infty$ so that for all $V \in \mathcal{S}$, all $x \in V$ and all events $E \in \mathcal{F}_{V\setminus x}$,*

$$\mu_V(\eta(x) > 2d|E) > 1 - \varepsilon$$

whenever $\gamma \geq \gamma_0$.

Proof. We can repeat the steps of the proof in Lemma 5.12. At the end we must estimate the number of burnable heights at x divided by the number of configurations at x. That is,

$$\mu_V(\eta(x) > 2d|E) > \frac{\gamma - 2d}{\gamma}$$

It thus suffices that $2d < \gamma\varepsilon$. ∎

We need one more lemma before giving the geometric argument, because the latter requires stochastic domination by Bernoulli measure.

Lemma 5.17 *The invariant probability measure μ_V for the sandpile dynamics in V is irreducible,that is, for two given recurrent configurations η, η', there is a sequence $\eta_0 = \eta, \dots, \eta_m = \eta'$ of recurrent configurations such that η_i and η_{i+1} differ only in one site.*

Proof. Since η' can be reached from η by a finite number of additions, it is enough to show that for any $x \in V$, there is such a sequence from η to $a_x\eta$. Let

$$\Gamma_x^+(\eta) = \{y \in V; (a_x\eta)(y) > \eta(y)\},$$

$$\Gamma_x^-(\eta) = \{y \in V; (a_x\eta)(y) < \eta(y)\}.$$

We first add sand grains one by one on the sites $y \in \Gamma_x^+(\eta)$, to reach the value $(a_x\eta)(y)$. Each step leads to a configuration larger than η, thus recurrent. We denote by $\eta^+ = \eta_0^+$ the recurrent configuration $(a_x\eta)_{\Gamma_x^+(\eta)}\eta_{[\Gamma_x^+(\eta)]^c}$. If $\Gamma_x^-(\eta) = \emptyset$ we are finished. If not, we write $\Gamma_x^-(\eta) = \{z_1, \dots, z_n\}$ and we pass from $\eta_k^+ = (a_x\eta)_{\{z_1,\dots,z_k\}}\eta^+_{\{z_1,\dots,z_k\}^c}$ to η_{k+1}^+ for every $1 \leq k \leq n$ ($\eta_k^+ \geq a_x\eta$ is recurrent and differs from η_{k-1}^+ in one site) to reach $\eta_n^+ = a_x\eta$. ∎

We are now in a position to give the main stochastic-geometric argument leading to exponential decay of correlations. It copies the proof that complete analyticity for Markovian random fields follows from absence of disagreement percolation, as done in [2], see Theorem 7.1 in [6], except that we can replace the Markov property by the decoupling property (5.11).

Let $p_c(d)$ denote the percolation threshold for Bernoulli site percolation on $\mathbb{Z}^d$. Let $V \in \mathcal{S}$ and let $\Lambda \subset V$ on which we fix two arbitrary height configurations η_Λ and η'_Λ to consider two conditional probabilities $\mu_1 = \mu_V(\cdot|\eta_\Lambda)$ and $\mu_2 = \mu_V(\cdot|\eta'_\Lambda)$.

Theorem 5.18 *Suppose that $\gamma \geq 4$ for $S = \mathbb{Z} \times \{1, \ldots, \ell\}$ or $4d < \gamma p_c(d)$ for $S = \mathbb{Z}^d$ in* (2.1)*. There exist constants $\alpha > 0, C < +\infty$ so that for all $V \in \mathcal{S}, \Lambda \subset V, W \subset V \setminus \Lambda, \eta \in \mathcal{R}$ and for every event $A \in \mathcal{F}_W$,*

$$|\mu_1(A) - \mu_2(A)| \leq Ce^{-\alpha dist\ (W,\Lambda)} \tag{5.19}$$

where $dist(\cdot, \cdot)$ is the nearest neighbor distance between the two subsets.

Proof. We give the proof for the lattice $S = \mathbb{Z}^d$. The case of the strip is analogous but a little simpler (using Lemma 5.12).
We use a coupling argument. First, we introduce some linear ordering on $V \setminus \Lambda$. We construct via iteration a coupling between μ_1 and μ_2 which is a random field (X, X'). We start by setting $X(x) = \eta(x), X'(x) = \eta'(x)$ on Λ. Suppose that we have already realized the coupling as $(X, X') = (\eta, \eta')$ on all sites outside some non-empty set $T \subset V \setminus \Lambda$. Consider then the conditional distributions $\mu_V(\cdot|\eta_{V\setminus T})$ and $\mu_V(\cdot|\eta'_{V\setminus T})$. One possibility is that on the external boundary both $\eta_{\partial T}$ and $\eta'_{\partial T}$ are V-burnable. But then, via (5.11), these two conditional probabilities are equal on T and we can take the optimal coupling for which $X = X'$ on T. Alternatively, we choose the smallest site $x \in T$ having a nearest neighbor $y \in V \setminus T$, for which $X(y) \leq 2d$ or $X'(y) \leq 2d$ and we find the value $(\eta(x), \eta'(x))$ for the coupling at x from sampling the optimal coupling between the single site distributions $\mu_V(X(x) = \cdot|\eta_{V\setminus T})$ and $\mu_V(X'(x) = \cdot|\eta'_{V\setminus T})$. At this step the coupling is defined outside $T \setminus x$ and we can repeat the iteration giving us a coupling between μ_1 and μ_2.
From the above construction, it is possible that in the coupling $X(x) \neq X'(x)$ at some $x \in W$, only if there is a nearest neighbor path from x to Λ on which $X(y) \leq 2d$ or $X'(y) \leq 2d$. On the other hand, no matter what we fix off y,

$$P(X(y) \leq 2d \text{ or } X'(y) \leq 2d|\eta(z), \eta'(z), z \in V \setminus y) \leq 2(1 - \mu_V(\eta(y) > 2d|\eta(z), z \neq y)) \tag{5.20}$$

For γ large enough (from Lemma 5.16), this is bounded by $p_c(d)$. The proof is then concluded via an application of stochastic domination with the Bernoulli product measure (thanks to Lemma 5.17, see Theorem 4.8 in [6]) and using that the cluster-diameter in sub-critical Bernoulli site percolation has an exponential tail. ∎

Examples.

1. The dissipative system in dimension 2: we have $p_c(2) = 0.5927$ (as numerical result). Thus we need to take $\gamma > 13$ so that $8 < \gamma p_c(2)$.

2. The dissipative system in high dimension. Since $p_c(d) \simeq 1/(2d)$ for large d, we conclude exponential decay of correlations as soon as $\gamma > 8d^2$.

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