UNIFORM POINCARÉ INEQUALITIES FOR UNBOUNDED CONSERVATIVE SPIN SYSTEMS: THE NON–INTERACTING CASE

PIETRO CAPUTO

Abstract. We prove a uniform Poincaré inequality for non–interacting unbounded spin systems with a conservation law, when the single–site potential is a bounded perturbation of a convex function. The result is then applied to Ginzburg-Landau processes to show diffusive scaling of the associated spectral gap.

2000 MSC: 60K35
Key words: Conservative spin systems, Poincaré inequality, Ginzburg-Landau process, spectral gap.

1. Introduction and main result

Consider a probability measure μ on $\mathbb{R}$ of the form

$$\mu(\mathrm{d}\eta) = \frac{e^{-V(\eta)}}{Z}\,\mathrm{d}\eta\,, \tag{1.1}$$

with $Z = \int e^{-V(\eta)}\mathrm{d}\eta$. Denote by μ_N the N-fold product measure obtained by tensorization of μ on $\mathbb{R}^N$, $N \in \mathbb{N}$. The canonical Gibbs measure with density $\rho \in \mathbb{R}$ is defined by conditioning μ_N on the $N-1$ dimensional hyperplane $\sum_{i=1}^N \eta_i = \rho N$, i.e.

$$\nu_{N,\rho} = \mu_N\Big(\;\cdot\;\Big|\;\sum_{i=1}^N \eta_i = \rho N\Big)\,. \tag{1.2}$$

We are going to give sufficient conditions on the potential V in order that the canonical measures $\nu_{N,\rho}$ satisfy a Poincaré inequality, uniformly in ρ and N.

For any probability measure ν we write $\nu(F) = \int F\mathrm{d}\nu$ for the mean of a function F and $\mathrm{Var}_\nu(F)$ for the variance $\nu(F^2) - \nu(F)^2$. For any smooth function F on $\mathbb{R}^N$ we write $\partial_i F$ for the partial gradient along the i-th coordinate. We say that a measure ν on $\mathbb{R}^N$ satisfies a Poincaré inequality if there exists a finite constant γ such that

$$\mathrm{Var}_\nu(F) \leqslant \gamma \sum_{i=1}^N \nu\big[(\partial_i F)^2\big]$$

holds for every smooth, real function F. Specializing to the canonical Gibbs measures (1.2) we define the quadratic form

$$\mathcal{E}_{N,\rho}(F) = \sum_{i=1}^N \nu_{N,\rho}\big[(\partial_i F)^2\big]\,.$$

Date: February 4, 2002.

For every $N \in \mathbb{N}$ and $\rho \in \mathbb{R}$ the Poincaré constant is given by

$$\gamma(N,\rho) = \sup_F \frac{\mathrm{Var}_{\nu_{N,\rho}}(F)}{\mathcal{E}_{N,\rho}(F)} \tag{1.3}$$

where the supremum is carried over all smooth, real functions F on $\mathbb{R}^N$. We say that a uniform Poincaré inequality holds whenever

$$\sup_{N\in\mathbb{N}} \sup_{\rho\in\mathbb{R}} \gamma(N,\rho) \, < \, \infty \, . \tag{1.4}$$

The main result of this paper states that such an estimate holds when V is a bounded perturbation of a strictly convex function.

Theorem 1.1. *Let V be a measurable function of the form $V = \varphi + \psi$ with $\varphi \in \mathcal{C}^2(\mathbb{R})$ and*

$$\inf_{x\in\mathbb{R}} \varphi''(x) \, > \, 0 \, , \qquad \sup_{x\in\mathbb{R}} |\psi(x)| \, < \, \infty \, .$$

Then the measures $\nu_{N,\rho}$ satisfy a uniform Poincaré inequality.

The proof of Theorem 1.1 will be given in the next two sections. It relies on a simple but powerful idea recently introduced by Carlen, Carvalho and Loss in [9]. A similar technique was then used also in [8] to study the relaxation to equilibrium for a conservative lattice gas dynamics. The argument of [9] essentially shows that in view of the symmetry of the measures (1.2) one can reduce the problem to the analysis of a one–dimensional process. The latter will be studied by means of a local limit theorem expansion.

Poincaré inequalities for conservative systems are usually studied on the level of the corresponding Ginzburg-Landau or Kawasaki dynamics, [16, 1, 2, 6]. This is an ergodic diffusion process on the hyperplane $\sum_{i=1}^N \eta_i = \rho N$, with $\nu_{N,\rho}$ as reversible invariant measure, and Dirichlet form of the type

$$\mathcal{D}_{N,\rho}(F) = \sum_{i=1}^{N-1} \nu_{N,\rho}\big[(\partial_{i+1}F - \partial_i F)^2\big] \, .$$

In this context the Poincaré inequality becomes a statement about the gap in the spectrum of the associated self adjoint Markov generator, or equivalently about the rate of convergence to equilibrium in the $L^2(\nu_{N,\rho})$–norm. In section 4 we shall see that an immediate corollary of Theorem 1.1 is an estimate of the form

$$\mathrm{Var}_{\nu_{N,\rho}}(F) \leqslant k \, N^2 \, \mathcal{D}_{N,\rho}(F) \tag{1.5}$$

for all smooth functions F with a constant k independent of the density ρ. This says that the spectral gap scales diffusively with the size of the system, uniformly in the density. Such estimates are usually a key step in establishing hydrodynamical limits, see [13]. The question of the generality under which estimate (1.5) holds was already raised in [17]. It was pointed out that when V is a strictly convex function then (1.5) holds. Indeed, in this case a general argument based on the Bakry–Emery criterium applies, see [7, 10]. More directly, when there is no perturbation ($\psi = 0$), Theorem 1.1 becomes an immediate consequence of the Brascamp–Lieb inequality [5]. On the other hand the extension to bounded perturbations of a strictly convex function proved to be rather challenging. We refer the reader to [4, 12, 15] and references therein to get an idea of the difficulties one has to face when leaving the purely convex setting. Recently it was shown in [14] that (1.5) holds when V is of the form $V(x) = a\,x^2 + \psi(x)$, $a > 0$ and ψ a bounded function. The authors prove the statement (1.5) by adapting the martingale method originally introduced in [16]. They also prove that the stronger logarithmic Sobolev inequality holds. The recent paper [10] gives further development along the same lines by slightly improving the hypothesis on the potential V. These results seem to rely strongly on the fact that V is essentially quadratic. Our approach is substantially simpler and covers a wider class of potentials. On the other hand it is based on the permutation symmetry (exchangeability) of the canonical measure and it might be difficult to adapt to truly interacting non-product cases.

The paper is organized as follows. In the next section we use a uniform local central limit theorem expansion to study a one–dimensional process which plays a key role in the iterative proof of Theorem 1.1. The latter is given in section 3. In section 4 we discuss the application to spectral gap estimates for Ginzburg-Landau processes.

Acknowledgments. I am greatly indebt to Fabio Martinelli. Many ideas and techniques involved in this note emerged from discussions with him during our joint work [8]. I also would like to thank Filippo Cesi, Cyril Roberto and Sebastiano Carpi for several interesting conversations.

2. Uniform local CLT and an application

We assume throughout that V is a potential satisfying the hypothesis of Theorem 1.1. Namely, $V(x) = \varphi(x) + \psi(x)$ and $\varphi'' \geqslant \delta > 0$, $|\psi|_\infty < \infty$. Given $\rho \in \mathbb{R}$ define the probability density

$$h^\rho(x) = \frac{e^{-V(x+\rho)-\lambda(\rho)x}}{Z_\rho}\,, \tag{2.1}$$

with $Z_\rho = \int e^{-V(x+\rho)-\lambda(\rho)x}\mathrm{d}x$. The parameter $\lambda = \lambda(\rho) \in \mathbb{R}$, the so-called chemical potential, is uniquely determined by ρ through the condition

$$\int x\, h^\rho(x)\,\mathrm{d}x = 0\,. \tag{2.2}$$

We write $\sigma^2 = \sigma_\rho^2$ for the variance

$$\sigma^2 = \int x^2\, h^\rho(x)\,\mathrm{d}x\,.$$

Unless otherwise specified all integrals here and below are understood to range over the real line. We call μ_ρ the probability measure with density $h^\rho(\cdot - \rho)$. If $\mu_{N,\rho}$ denotes the product $\mu_\rho \otimes \cdots \otimes \mu_\rho$ (N times), $N \in \mathbb{N}$, then the canonical measure $\nu_{N,\rho}$ can be equivalently obtained as in (1.2) with μ_N replaced by $\mu_{N,\rho}$.

2.1. Local central limit theorem.

Let π_i be the canonical projection of $\mathbb{R}^N$ onto $\mathbb{R}$ given by $\pi_i\eta = \eta_i$, $\eta = (\eta_1,\dots,\eta_N) \in \mathbb{R}^N$. We call $\nu^1_{N,\rho}$ the one-site marginal of $\nu_{N,\rho}$, i.e. $\nu^1_{N,\rho} = \nu_{N,\rho}\circ\pi_1^{-1}$ is the distribution of η_1 under $\nu_{N,\rho}$. By permutation symmetry all one-site marginals coincide. The density $g_{N,\rho}$ of $\nu^1_{N,\rho}$ can be written in the form

$$g_{N,\rho}(x) = \frac{h^\rho(x-\rho)G^\rho_{N-1}(x-\rho)}{G^\rho_N(0)}\,, \tag{2.3}$$

with

$$G_N^\rho(x) = \int \mathrm{d}\eta_1 \cdots \int \mathrm{d}\eta_N \Big(\prod_{i=1}^N h^\rho(\eta_i)\Big)\delta\Big(\sum_{i=1}^N \eta_i + x\Big)\,. \tag{2.4}$$

Here we are using Dirac's notation

$$f(0) = \int \mathrm{d}x\, f(x)\,\delta(x)\,.$$

Note that if we consider independent random variables η_i with common distribution defined by the density h^ρ, then the normalized sum

$$\frac{1}{\sigma\sqrt{N}}\sum_{i=1}^N \eta_i$$

has a density given by

$$F_N^\rho(z) = \sigma\sqrt{N}G_N^\rho(-z\sigma\sqrt{N})\,.$$

We shall use the classical local central limit theorem expansion for the density F_N^ρ. Introduce the centered moments $m_k = m_k(\rho)$:

$$m_k = \int x^k\, h^\rho(x)\,\mathrm{d}x\,, \qquad k = 1, 2, \dots$$

so that in particular $m_1 = 0$, $m_2 = \sigma^2$. A well known estimate (see [11], chap. XVI, Theorem 2) gives

$$F_N^\rho(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\Big(1 + \frac{P_3(z)}{\sqrt{N}} + \frac{P_4(z)}{N}\Big) + O\big(N^{-\frac{3}{2}}\big) \tag{2.5}$$

where P_3, P_4 are the polynomials

$$P_3(z) = \frac{m_3}{6\sigma^3}(z^3 - 3z)\,, \qquad P_4(z) = \frac{m_3^2}{72\sigma^6}(z^3 - 3z) + \frac{m_4 - 3\sigma^4}{24\sigma^4}(z^4 - 6z^2 + 3)\,.$$

The above expansion holds as soon as the fifth moment m_5 exists. What is important for us is that (2.5) holds uniformly in $z \in \mathbb{R}$ and $\rho \in \mathbb{R}$. This in turn follows directly from a uniform bound on the normalized moments m_k/σ^k, $k \leqslant 5$, which we prove below as a consequence of a more general estimate. Note that since $|m_{k+1}| \leqslant \sigma\sqrt{m_{2k}}$ we can restrict to even powers.

Lemma 2.1. *Assume $V = \varphi + \psi$ with $\varphi'' \geqslant 0$, $|\psi|_\infty < \infty$. Then there exists $k = k(|\psi|_\infty) < \infty$ such that for every $n \geqslant 2$*

$$\sup_{\rho\in\mathbb{R}} \Big|\frac{m_{2n}}{\sigma^{2n}}\Big| \leqslant \prod_{\ell=2}^n (1 + k\ell^2)\,. \tag{2.6}$$

Proof. We first establish a bound for the Poincaré constant in terms of the variance σ^2. We denote by μ_ρ the probability measure with density $(\mathcal{Z}_\rho)^{-1}e^{-V(x)-\lambda(\rho)x}$ and by $\tilde\mu_\rho$ the probability measure with density $(\tilde{\mathcal{Z}}_\rho)^{-1}e^{-\varphi(x)-\lambda(\rho)x}$. Clearly $\mathcal{Z}_\rho = e^{-\rho\lambda(\rho)}Z_\rho$ (cf. (2.1)) and

$$\tilde{\mathcal{Z}}_\rho = \int e^{-\varphi(x)-\lambda(\rho)x}\mathrm{d}x \in \big[e^{-|\psi|_\infty}\mathcal{Z}_\rho, e^{|\psi|_\infty}\mathcal{Z}_\rho\big]\,.$$

Let γ_ρ and $\tilde\gamma_\rho$ be the Poincaré constants associated to μ_ρ and $\tilde\mu_\rho$ respectively. Let also $\tilde\sigma^2$ denote the variance of $\tilde\mu_\rho$, $\tilde\sigma^2 = \tilde\mu_\rho(x^2) - \tilde\mu_\rho(x)^2$. We shall use here a result derived by Bobkov in [3], which says that since the density of $\tilde\mu_\rho$ is log-concave one has the bound

$$\tilde\gamma_\rho \leqslant 12\,\tilde\sigma^2 \tag{2.7}$$

It is not difficult to establish a similar bound for μ_ρ. Namely, for any smooth function f such that $\mu_\rho(f) = 0$ we write

$$\begin{aligned}\mu_\rho(f^2) &\leqslant \mu_\rho\big[(f-\tilde\mu_\rho(f))^2\big] \leqslant e^{2|\psi|_\infty}\tilde\mu_\rho\big[(f-\tilde\mu_\rho(f))^2\big]\\ &\leqslant e^{2|\psi|_\infty}\,\tilde\gamma_\rho\,\tilde\mu_\rho\big[(f')^2\big] \leqslant e^{4|\psi|_\infty}\,\tilde\gamma_\rho\,\mu_\rho\big[(f')^2\big]\,,\end{aligned}$$

so that

$$\gamma_\rho \leqslant e^{4|\psi|_\infty}\tilde\gamma_\rho\,.$$

On the other hand

$$\tilde\sigma^2 \leqslant \tilde\mu_\rho\big[(x-\rho)^2\big] \leqslant e^{2|\psi|_\infty}\mu_\rho\big[(x-\rho)^2\big] = e^{2|\psi|_\infty}\sigma^2\,.$$

From (2.7) we obtain

$$\gamma_\rho \leqslant 12\,e^{6|\psi|_\infty}\,\sigma^2\,. \tag{2.8}$$

Once we have such an estimate the proof of (2.6) is immediate. For every $n\in\mathbb{N}$ we have

$$\begin{aligned}m_{2n} &= \mathrm{Var}_{\mu_\rho}\big[(x-\rho)^n\big] + m_n^2\\ &\leqslant n^2\,\gamma_\rho\,\mu_\rho\big[(x-\rho)^{2(n-1)}\big] + m_n^2 = n^2\,\gamma_\rho\,m_{2(n-1)} + m_n^2\,.\end{aligned}$$

Setting $k = 12\,e^{6|\psi|_\infty}$, from (2.8) we have

$$m_{2n} \leqslant k\,n^2\sigma^2\,m_{2(n-1)} + m_n^2\,.$$

This implies the claim for $n=2$ and using $m_{n+1}^2 \leqslant \sigma^2 m_{2n}$ one easily obtains the rest through induction. □

2.2. **The operator $\mathcal{K}$.** Here we introduce the relevant one–dimensional process and prove a key spectral estimate, see Theorem 2.2. Let $\mathcal{H}$ denote the Hilbert space $L^2(\mathbb{R},\nu^1_{N,\rho})$ and use the symbol $\langle\cdot,\cdot\rangle$ for the corresponding scalar product $\langle f,g\rangle = \nu_{N,\rho}[(\bar f\circ\pi_1)(g\circ\pi_1)]$, with $\bar f$ denoting the complex conjugate function. Write also $\langle f\rangle$ for the mean of a function $f\in\mathcal{H}$ w.r.t. $\nu^1_{N,\rho}$. We write $\mathcal{H}_0$ for the subspace of $f\in\mathcal{H}$ such that $\langle f\rangle = 0$. We define the stochastic self–adjoint operator $\mathcal{K}:\mathcal{H}\to\mathcal{H}$ by the sesquilinear form:

$$\langle f,\mathcal{K}g\rangle = \nu_{N,\rho}\big[(\bar f\circ\pi_1)(g\circ\pi_2)\big]\,,\qquad f,g\in\mathcal{H}\,. \tag{2.9}$$

Let ξ_ρ be the linear function $\xi_\rho(x) = x-\rho$. A simple computation shows that

$$\mathcal{K}\xi_\rho = -\frac{1}{N-1}\,\xi_\rho \tag{2.10}$$

for every $\rho\in\mathbb{R}$. Thus the spectrum of $\mathcal{K}$ always contains the eigenvalues $-(N-1)^{-1}$ and 1. We prove below that the rest of the spectrum is confined around zero within a neighbourhood of radius $O(N^{-\frac{3}{2}})$.

Theorem 2.2. *There exists $C<\infty$ independent of ρ and N such that for every $f\in\mathcal{H}_0$ satisfying $\langle f,\xi_\rho\rangle = 0$ one has*

$$\big|\langle f,\mathcal{K}f\rangle\big| \leqslant C\,N^{-\frac{3}{2}}\,\langle f,f\rangle\,. \tag{2.11}$$

The rest of this section deals with the proof of Theorem 2.2. The idea is essentially the same as in [8] where this type of result has been established for a discrete lattice gas model.

Denote by $\tilde g_{N,\rho}(x,y)$ the density of the joint distribution of (η_1,η_2) under $\nu_{N,\rho}$. This is given by

$$\tilde g_{N,\rho}(x,y) = \frac{h^\rho(x-\rho)h^\rho(y-\rho)G^\rho_{N-2}((x-\rho)+(y-\rho))}{G^\rho_N(0)}\,. \tag{2.12}$$

When f is real and $\langle f\rangle = 0$ we write

$$\langle f, \mathcal{K} f\rangle = \int\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)Q_{N,\rho}(x,y)f(x)f(y) \tag{2.13}$$

where we introduced the kernel

$$Q_{N,\rho}(x,y) = \frac{\tilde g_{N,\rho}(x,y) - g_{N,\rho}(x)g_{N,\rho}(y)}{g_{N,\rho}(x)g_{N,\rho}(y)}\,. \tag{2.14}$$

Define the set

$$\mathcal{B}_\rho = \{(x,y)\in\mathbb{R}^2 :\ \ |x-\rho|+|y-\rho| \leqslant\ B\,\sigma\log N\,\} \tag{2.15}$$

where B is a constant to be fixed later on. The following expansion is the key step in the proof of Theorem 2.2.

Lemma 2.3. *For every $B<\infty$, there exists $C<\infty$ such that*

$$\sup_{\rho\in\mathbb{R}}\ \sup_{(x,y)\in\mathcal{B}_\rho}\left| Q_{N,\rho}(x,y) + \frac{\xi_\rho(x)\,\xi_\rho(y)}{\sigma^2 N}\right| \leqslant C\,N^{-\frac{3}{2}} \tag{2.16}$$

Proof. In order to simplify notations we shall simply write

$$\bar x = \xi_\rho(x) = x-\rho\,,\qquad \bar y = \xi_\rho(y) = y-\rho\,.$$

Using (2.3) and (2.12) we rewrite

$$Q_{N,\rho}(x,y) = \frac{G^\rho_{N-2}(\bar x+\bar y)G^\rho_N(0) - G^\rho_{N-1}(\bar x)G^\rho_{N-1}(\bar y)}{G^\rho_{N-1}(\bar x)G^\rho_{N-1}(\bar y)}\,. \tag{2.17}$$

We now use the expansion (2.5). With the change of variable

$$\sigma\sqrt{N}G^\rho_N(\bar x) = F^\rho_N(-\bar x/\sigma\sqrt{N})$$

we see that the only terms which are not negligeable w.r.t. $O(N^{-\frac{3}{2}})$ in the range $|\bar x|\leqslant B\,\sigma\log N$ are given by the constant term in P_4 and the linear terms in P_3 and P_4. This implies

$$\sup_{\rho\in\mathbb{R}}\ \sup_{|\bar x|\,\leqslant\,B\,\sigma\log N}\left|\sigma\sqrt{N}G^\rho_N(\bar x) - \frac{e^{-\frac{\bar x^2}{2\sigma^2 N}}}{\sqrt{2\pi}}\Big(1+\frac{\alpha+\beta\bar x}{N}\Big)\right| \leqslant C\,N^{-\frac{3}{2}} \tag{2.18}$$

$$\alpha := \frac{m_4-3\sigma^4}{8\sigma^4}\,,\qquad \beta := \frac{m_3}{2\sigma^4}+\frac{m_3^2}{24\sigma^7\sqrt{N}}\,.$$

Note that by Lemma 2.1 α is uniformly bounded and $\beta\bar x$ is bounded by $C\,\log N$ in the range $|\bar x|\leqslant B\,\sigma\log N$ for some uniform $C<\infty$.

We introduce the following convention. We call $\varepsilon(N)$ anything which vanishes at least as $O(N^{-\frac{3}{2}})$ uniformly in $(x,y) \in \mathcal{B}_\rho$. Thus the result (2.18) will be used in the form

$$\sigma\sqrt{N}G^\rho_N(\bar{x}) = \frac{e^{-\frac{\bar{x}^2}{2\sigma^2 N}}}{\sqrt{2\pi}}\Big(1+\frac{\alpha+\beta\bar{x}}{N}\Big) + \varepsilon(N)\,. \tag{2.19}$$

Use now (2.19) to write

$$\begin{aligned}
&2\pi\sigma^2(N-1)G^\rho_{N-1}(\bar{x})G^\rho_{N-1}(\bar{y}) \\
&\qquad = e^{-\frac{\bar{x}^2+\bar{y}^2}{2\sigma^2(N-1)}}\Big(1+\frac{\alpha+\beta\bar{x}}{N-1}\Big)\Big(1+\frac{\alpha+\beta\bar{y}}{N-1}\Big) + \varepsilon(N) \\
&\qquad = e^{-\frac{\bar{x}^2+\bar{y}^2}{2\sigma^2 N}}\Big(1+\frac{2\alpha+\beta(\bar{x}+\bar{y})}{N}\Big) + \varepsilon(N)\,.
\end{aligned}$$

Furthermore, writing $q(N) = (N-1)/\sqrt{N(N-2)}$ and observing that $q(N) = 1+O(N^{-2})$, one has

$$\begin{aligned}
&2\pi\sigma^2(N-1)G^\rho_{N-2}(\bar{x}+\bar{y})G^\rho_N(0) \\
&\qquad = q(N)\, e^{-\frac{(\bar{x}+\bar{y})^2}{2\sigma^2(N-2)}}\Big(1+\frac{\alpha+\beta(\bar{x}+\bar{y})}{N-2}\Big)\Big(1+\frac{\alpha}{N}\Big) + \varepsilon(N) \\
&\qquad = e^{-\frac{(\bar{x}+\bar{y})^2}{2\sigma^2 N}}\Big(1+\frac{2\alpha+\beta(\bar{x}+\bar{y})}{N}\Big) + \varepsilon(N) \\
&\qquad = e^{-\frac{\bar{x}^2+\bar{y}^2}{2\sigma^2 N}}\Big(1-\frac{\bar{x}\bar{y}}{\sigma^2 N}\Big)\Big(1+\frac{2\alpha+\beta(\bar{x}+\bar{y})}{N}\Big) + \varepsilon(N)\,.
\end{aligned}$$

Inserting in (2.17) we have obtained

$$Q_{N,\rho}(x,y) = -\frac{\bar{x}\bar{y}}{\sigma^2 N} + \varepsilon(N)\,.$$

□

Next we need to control the atypical region $\mathcal{B}^c_\rho$. We shall use $\mathbf{1}_{\mathcal{B}_\rho}$ and $\mathbf{1}_{\mathcal{B}^c_\rho}$ to denote the indicator function of $\mathcal{B}_\rho$ and its complement respectively.

Lemma 2.4. *There exist constants $C, B < \infty$ such that uniformly in ρ*

$$\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)\,\big|Q_{N,\rho}(x,y)\big|^2\,\mathbf{1}_{\mathcal{B}^c_\rho}(x,y) \leqslant C\,N^{-3}\,. \tag{2.20}$$

Proof. A first estimate gives

$$g_{N,\rho}(x)g_{N,\rho}(y)\big|Q(x,y)\big|^2 \leqslant 2g_{N,\rho}(x)g_{N,\rho}(y) + 2\frac{\tilde{g}_{N,\rho}(x,y)^2}{g_{N,\rho}(x)g_{N,\rho}(y)}\,. \tag{2.21}$$

Observe that by (2.5) – see also (2.18) – we have

$$\frac{G^\rho_{N-1}(x-\rho)}{G^\rho_N(0)} = e^{-\frac{(x-\rho)^2}{2\sigma^2(N-1)}} + O\big(N^{-\frac{1}{2}}\big) \tag{2.22}$$

uniformly in $x \in \mathbb{R}$ and $\rho \in \mathbb{R}$. In particular by (2.3) we have $g_{N,\rho}(x) \leqslant C\,h^\rho(x-\rho)$ uniformly. Then

$$\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)\,\mathbf{1}_{\mathcal{B}^c_\rho}(x,y) = \big[\nu^1_{N,\rho}\otimes\nu^1_{N,\rho}\big]\big(\mathbf{1}_{\mathcal{B}^c_\rho}\big) \leqslant C^2\,\big[\mu_\rho\otimes\mu_\rho\big]\big(\mathbf{1}_{\mathcal{B}^c_\rho}\big)$$

Now the measure μ_ρ with density $h^\rho(\cdot-\rho)$ has exponential tails and satisfies

$$\big[\mu_\rho\otimes\mu_\rho\big]\big(\mathbf{1}_{\mathcal{B}^c_\rho}\big) \leqslant C\,N^{-3}\,, \tag{2.23}$$

if B is large enough, uniformly in ρ. To prove (2.23) observe that if γ_ρ denotes the Poincaré constant of μ_ρ (see the proof of Lemma 2.1) then

$$\mu_\rho\Big[\exp\frac{|x-\rho|}{\sqrt{\gamma_\rho}}\Big] < \infty$$

uniformly in $\rho \in \mathbb{R}$. The latter is obtained by iteration, applying Poincaré inequality to the functions $e^{\pm(x-\rho)/2}$. On the other hand by (2.8) one has $\gamma_\rho \leqslant k\sigma^2$ for some uniform constant $k < \infty$. This implies (2.23) and gives the correct estimate for the first term in (2.21).

To handle the second term in (2.21) we observe that

$$\frac{\tilde g_{N,\rho}(x,y)}{g_{N,\rho}(y)} = g_{N-1,\rho_y}(x)\,, \qquad \rho_y := \frac{\rho N - y}{N-1}\,.$$

This can be checked using (2.3), (2.12) and the fact that for any ρ'

$$g_{N,\rho}(x) = \frac{h^{\rho'}(x-\rho')G^{\rho'}_{N-1}((x-\rho')+(\rho'-\rho)N)}{G^{\rho'}_N((\rho'-\rho)N)}$$

and similarly

$$\tilde g_{N,\rho}(x,y) = \frac{h^{\rho'}(x-\rho')h^{\rho'}(y-\rho')G^{\rho'}_{N-2}((x-\rho')+(y-\rho')+(\rho'-\rho)N)}{G^{\rho'}_N((\rho'-\rho)N)}\,.$$

The integral of the second term in (2.21) is then estimated as before observing that $|x-\rho_y|+|y-\rho_x| \geqslant (|x-\rho|+|y-\rho|)/2$ as soon as $N \geqslant 3$. Thus we have again the right tail estimate as above. $\square$

We are now able to finish the proof of Theorem 2.2. Let us go back to (2.13) and split the integral there as

$$\begin{aligned}\langle f, \mathcal{K} f\rangle &= \int\!\!\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)Q_{N,\rho}(x,y)f(x)f(y)\mathbf{1}_{\mathcal{B}_\rho}(x,y)\\ &\quad+\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)Q_{N,\rho}(x,y)f(x)f(y)\mathbf{1}_{\mathcal{B}_\rho^c}(x,y)\,.\end{aligned}$$

The second term here is estimated by Schwarz inequality and Lemma 2.4 so that

$$\begin{aligned}&\Big|\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)Q_{N,\rho}(x,y)f(x)f(y)\mathbf{1}_{\mathcal{B}_\rho^c}(x,y)\Big|\\ &\qquad\leqslant \langle f,f\rangle\Big(\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)\,\big|Q_{N,\rho}(x,y)\big|^2\,\mathbf{1}_{\mathcal{B}_\rho^c}(x,y)\Big)^{\frac12}\\ &\qquad\leqslant C\,N^{-\frac32}\,\langle f,f\rangle\,.\end{aligned}\tag{2.24}$$

The other term can be written as

$$\begin{aligned}&-\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)\,\frac{\xi_\rho(x)\xi_\rho(y)}{\sigma^2 N}\,f(x)f(y)\mathbf{1}_{\mathcal{B}_\rho}(x,y)\\ &+\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)\Big[Q_{N,\rho}(x,y)+\frac{\xi_\rho(x)\xi_\rho(y)}{\sigma^2 N}\Big]\,f(x)f(y)\mathbf{1}_{\mathcal{B}_\rho}(x,y)\end{aligned}$$

A Scwharz inequality and Lemma 2.3 imply that the second term above is bounded by $C\,N^{-\frac{3}{2}}\,\langle f,f\rangle$. Since by assumption $\langle f,\xi_\rho\rangle = 0$ we rewrite the first term above as

$$\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)\,\frac{\xi_\rho(x)\xi_\rho(y)}{\sigma^2 N}\,f(x)f(y)\mathbf{1}_{\mathcal{B}_\rho^c}(x,y)\,.$$

We estimate the absolute value of this expression by

$$\langle f,f\rangle\,\frac{1}{\sigma^2 N}\int_{|x-\rho|\,\geqslant\,\frac{B}{2}\,\sigma\log N}\mathrm{d}x\, g_{N,\rho}(x)\xi_\rho(x)^2\,.$$

Finally the last integral can be estimated using $g_{N,\rho}(x)\leqslant C\,h^\rho(x-\rho)$ as in Lemma 2.4 and the bound of Lemma 2.1:

$$\int_{|x-\rho|\,\geqslant\,\frac{B}{2}\,\sigma\log N}\mathrm{d}x\, g_{N,\rho}(x)\xi_\rho(x)^2\leqslant C\,\sqrt{m_4}\,\Big(\int_{|x-\rho|\,\geqslant\,\frac{B}{2}\,\sigma\log N}\mathrm{d}x\, h^\rho(x-\rho)\Big)^{\frac{1}{2}}\leqslant C\,\sigma^2 N^{-\frac{1}{2}}$$

Collecting these estimates we have obtained

$$\Big|\int\!\!\int \mathrm{d}x\,\mathrm{d}y\, g_{N,\rho}(x)g_{N,\rho}(y)Q_{N,\rho}(x,y)f(x)f(y)\mathbf{1}_{\mathcal{B}_\rho}(x,y)\Big|\leqslant C\,N^{-\frac{3}{2}}\,\langle f,f\rangle\,. \tag{2.25}$$

Estimates (2.24) and (2.25) imply the Theorem.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the recursive inequality presented in the theorem below. Recall the definition (1.3) of the Poincaré constant $\gamma(N,\rho)$, and set

$$\gamma(N)=\sup_{\rho\in\mathbb{R}}\gamma(N,\rho)\,. \tag{3.1}$$

Theorem 3.1. *There exist constants $C<\infty$ and $N_0\in\mathbb{N}$ such that for every $N>N_0$*

$$\gamma(N)\leqslant\big[1+CN^{-\frac{3}{2}}\big]\,\gamma(N-1)\,. \tag{3.2}$$

Proof. Take an arbitrary real smooth function F on $\mathbb{R}^N$. For simplicity we drop all subscripts and simply write $\mathrm{Var}(F)$ for $\mathrm{Var}_{\nu_{N,\rho}}(F)$ and $\mathcal{E}(F)$ for $\mathcal{E}_{N,\rho}(F)$. Let $\mathcal{F}_k$ denote the $\sigma-$algebra generated by the one–site variables η_k, $k=1,\dots,,N$. $\mathrm{Var}(F\,|\,\mathcal{F}_k)$ denotes the $\mathcal{F}_k$-measurable random variable $\nu_{N,\rho}(F^2\,|\,\mathcal{F}_k)-\nu_{N,\rho}(F\,|\,\mathcal{F}_k)^2$. We use the notation

$$\mathcal{E}^{(k)}(F)=\sum_{i:\,i\neq k}\nu_{N,\rho}\big((\partial_i F)^2\big)\,.$$

Note that

$$\sum_{k=1}^N\mathcal{E}^{(k)}(F)=(N-1)\mathcal{E}(F)\,.$$

For any F one has the decomposition

$$\mathrm{Var}(F)=\frac{1}{N}\sum_{k=1}^N\nu\big(\mathrm{Var}(F\,|\,\mathcal{F}_k)\big)+\frac{1}{N}\sum_{k=1}^N\mathrm{Var}\big(\nu(F\,|\,\mathcal{F}_k)\big)\,. \tag{3.3}$$

By definition (3.1), for each k we have

$$\nu\big(\mathrm{Var}(F\,|\,\mathcal{F}_k)\big)\leqslant\gamma(N-1)\mathcal{E}^{(k)}(F)\,.$$

Summing over k gives

$$\frac{1}{N}\sum_{k=1}^{N}\nu\big(\mathrm{Var}(F\,|\,\mathcal{F}_k)\big)\leqslant\frac{N-1}{N}\,\gamma(N-1)\,\mathcal{E}(F)\,. \tag{3.4}$$

We turn to estimate the second term in (3.3). Here comes the idea of [9]. Namely assume without loss that $\nu(F)=0$ and write the quadratic form

$$\frac{1}{N}\sum_{k=1}^{N}\mathrm{Var}\big(\nu(F\,|\,\mathcal{F}_k)\big)=\nu\big(F\mathcal{P}F\big)$$

where the stochastic operator $\mathcal{P}:L^2(\nu)\to L^2(\nu)$ is defined by

$$\mathcal{P}F=\frac{1}{N}\sum_{k=1}^{N}\nu(F\,|\,\mathcal{F}_k)\,.$$

In this way (3.3) and (3.4) give

$$\nu\big(F(\mathbf{1}-\mathcal{P})F\big)\leqslant\frac{N-1}{N}\,\gamma(N-1)\,\mathcal{E}(F)\,. \tag{3.5}$$

We need a spectral gap estimate for the generator $\mathbf{1}-\mathcal{P}$. We are going to prove

$$\nu\big(F(\mathbf{1}-\mathcal{P})F\big)\geqslant\frac{N-1}{N}\big[1-CN^{-\frac{3}{2}}\big]\,\nu\big(F^2\big)\,, \tag{3.6}$$

for all real $F\in L^2(\nu)$ such that $\nu(F)=0$ with a uniform constant $C<\infty$ independent of the density ρ. Together with (3.5) this will complete the proof of the theorem.

Recalling the notation introduced in the previous section we define the closed subspace Γ of $L^2(\nu)$ consisting of sums of functions of a single variable:

$$\Gamma=\Big\{F\in L^2(\nu):\ F=\sum_{k=1}^{N}f_k\circ\pi_k\,,\ \ f_1,\dots,f_N\in\mathcal{H}_0\,,\ \Big\} \tag{3.7}$$

Since $\mathcal{P}F\in\Gamma$ for every $F\in L^2(\nu)$ we may restrict to $F\in\Gamma$ to prove (3.6). For $F\in\Gamma$, $F=\sum_k f_k\circ\pi_k$, we define $\Phi_F=\sum_k f_k$, a function in $\mathcal{H}_0$. Taking any real $F\in\Gamma$, a simple computation shows that

$$\nu\big(F^2\big)=\langle\Phi_F,\mathcal{K}\Phi_F\rangle+\sum_k\langle f_k,(\mathbf{1}-\mathcal{K})f_k\rangle\,, \tag{3.8}$$

where $\mathcal{K}$ is the operator defined in (2.9). Similarly one computes

$$\begin{aligned}\nu\big(F(\mathbf{1}-\mathcal{P})F\big)=&\ \frac{N-2}{N}\,\langle\Phi_F,\mathcal{K}(\mathbf{1}-\mathcal{K})\Phi_F\rangle\\&+\frac{1}{N}\sum_k\langle f_k,(\mathbf{1}-\mathcal{K})[(N-1)\mathbf{1}+\mathcal{K}]f_k\rangle\,.\end{aligned} \tag{3.9}$$

Consider now the subspace $\mathcal{S}\subset\Gamma$ of symmetric functions:

$$\mathcal{S}=\Big\{F\in L^2(\nu):\ F=\sum_{k=1}^{N}f\circ\pi_k\,,\ \ f\in\mathcal{H}_0\Big\}\,. \tag{3.10}$$

Since $\mathcal{S}$ is invariant for $\mathcal{P}$, i.e. $\mathcal{P}\mathcal{S} \subset \mathcal{S}$ we may consider separately the cases $F \in \mathcal{S}$ and $F \in \mathcal{S}^\perp$, with $\mathcal{S}^\perp$ denoting the orthogonal complement in Γ. When $F \in \mathcal{S}$ we have $\Phi_F = Nf$ and rearranging terms in (3.8) and (3.9) we obtain

$$\nu\big(F^2\big) = N(N-1)\,\langle f, [\mathcal{K} + \frac{\mathbf{1}}{N-1}]f\rangle \tag{3.11}$$

$$\nu\big(F(\mathbf{1}-\mathcal{P})F\big) = (N-1)^2\langle f, [\mathbf{1}-\mathcal{K}][\mathcal{K} + \frac{\mathbf{1}}{N-1}]f\rangle \tag{3.12}$$

By Theorem 2.2 we see that $\mathcal{K}+\frac{\mathbf{1}}{N-1}$ is non-negative on the whole subspace $\mathcal{H}_0$. Moreover by (2.10) and (3.11) we see that $\nu(F^2) = 0$ when f is a multiple of ξ_ρ. We may then restrict to the case $\langle f, \xi_\rho\rangle = 0$. Writing $\tilde{f} = [\mathcal{K} + \frac{\mathbf{1}}{N-1}]^{\frac{1}{2}} f$ and observing that $\langle \tilde{f}\rangle = 0$ and $\langle \tilde{f}, \xi_\rho\rangle = 0$, Theorem 2.2 yields the estimate

$$\begin{aligned}\nu\big(F(\mathbf{1}-\mathcal{P})F\big) &\geqslant (N-1)^2\big[1 - CN^{-\frac{3}{2}}\big]\,\langle \tilde{f}, \tilde{f}\rangle \\ &= \frac{N-1}{N}\big[1 - CN^{-\frac{3}{2}}\big]\,\nu\big(F^2\big)\,, \qquad F \in \mathcal{S}\,.\end{aligned} \tag{3.13}$$

We turn to study the case $F \in \mathcal{S}^\perp$. Let us first observe that in the definition (3.7) of Γ one can assume without loss that $\sum_k \langle f_k, \xi_\rho\rangle = 0$. Indeed if $c = (N\langle \xi_\rho, \xi_\rho\rangle)^{-1}\sum_k\langle f_k, \xi_\rho\rangle$ and $g_k = f_k - c\xi_\rho$, we have $\sum_k g_k \circ \pi_k = \sum_k f_k \circ \pi_k$ in $L^2(\nu)$ since by the conservation law $\sum_k \xi_\rho \circ \pi_k = 0$. Therefore $\langle \Phi_F, \xi_\rho\rangle = 0$ may be assumed from the start. Now, for every $G \in \mathcal{S}$, $G = \sum_k g \circ \pi_k$, with $g \in \mathcal{H}_0$ one has

$$\nu(FG) = (N-1)\,\langle \Phi_F, [\mathcal{K} + \frac{\mathbf{1}}{N-1}]g\rangle\,.$$

Thus $F \in \mathcal{S}^\perp$ implies that $[\mathcal{K}+\frac{\mathbf{1}}{N-1}]\Phi_F$ is a constant in $\mathcal{H}$. Since $\langle\Phi_F\rangle = 0$ and $\langle \Phi_F, \xi_\rho\rangle = 0$, Theorem 2.2 implies $\Phi_F = 0$. Writing $\hat{f}_k = (\mathbf{1}-\mathcal{K})^{\frac{1}{2}} f_k$, then (3.8) and (3.9) imply

$$\nu\big(F^2\big) = \sum_k \langle \hat{f}_k, \hat{f}_k\rangle \tag{3.14}$$

$$\nu\big(F(\mathbf{1}-\mathcal{P})F\big) = \frac{1}{N}\sum_k\langle \hat{f}_k, [(N-1)\mathbf{1} + \mathcal{K}]\hat{f}_k\rangle \tag{3.15}$$

Since $\langle \hat{f}_k\rangle = 0$ for all k we may use Theorem 2.2 to estimate

$$\langle \hat{f}_k, \mathcal{K}\hat{f}_k\rangle \geqslant -\frac{1}{N-1}\langle \hat{f}_k, \hat{f}_k\rangle\,.$$

From (3.14) and (3.15) we obtain

$$\nu\big(F(\mathbf{1}-\mathcal{P})F\big) \geqslant \frac{N-2}{N-1}\nu\big(F^2\big) = \frac{N-1}{N}\big[1 - \frac{1}{(N-1)^2}\big]\nu\big(F^2\big)\,.$$

This ends the proof of the claim (3.6). □

Once we have Theorem 3.1 the conclusion of Theorem 1.1 is straightforward. Indeed,

$$\prod_{N=N_0+1}^{\infty}\ \big[1 + CN^{-\frac{3}{2}})\big] \leqslant C' \tag{3.16}$$

for some uniform constant C' and Theorem 3.1 yields

$$\gamma(N) \leqslant C'\,\gamma(N_0)\,, \qquad N \geqslant N_0 + 1\,.$$

The uniform Poincaré inequality of Theorem 1.1 then follows from the fact that $\gamma(N_0)$ is indeed finite.

Lemma 3.2. *For every $N_0 \geqslant 2$*

$$\sup_{\rho\in\mathbb{R}} \gamma(N_0,\rho) < \infty\,. \tag{3.17}$$

Proof. Let $\tilde\nu_{N,\rho}$ denote the canonical measure obtained in (1.2) where the potential V is replaced by its convex component φ. Let also $\tilde\gamma(N,\rho)$ denote the corresponding Poincaré constant. Since $\varphi'' \geqslant \delta > 0$ one can use the Brascamp–Lieb inequality [5] to prove $\tilde\gamma(N,\rho) \leqslant \delta^{-1}$, uniformly in N and ρ, see also [7]. A standard argument (as in the proof of Lemma 2.1) on the other hand gives $\gamma(N,\rho) \leqslant e^{4N|\psi|_\infty}\tilde\gamma(N,\rho)$, for every $N\in\mathbb{N}$ and $\rho\in\mathbb{R}$. This gives, uniformly in ρ

$$\gamma(N,\rho) \leqslant \delta^{-1} e^{4N|\psi|_\infty}\,.$$

□

4. Ginzburg-Landau processes

We consider the discrete lattice $\mathbb{Z}^d$, with $d \geqslant 1$ an integer. Given a finite subset $\Lambda\subset\mathbb{Z}^d$, we denote by Λ^* the set of oriented bonds b contained in Λ, i.e. the couples $b=(x,y)$, $x,y\in\Lambda$ with $x=y+e$, e a unit vector in $\mathbb{Z}^d$. Denoting $\Lambda_L=\{1,2,\dots,L\}^d$, the $L-$hypercube in $\mathbb{Z}^d$, we define the product measure $\mu_{\Lambda_L,\rho}$ as the usual grand canonical measure $\mu_{N,\rho}$ with $N=L^d$. Then $\nu_{\Lambda_L,\rho}$ stands for the probability measure obtained from $\mu_{\Lambda_L,\rho}$ by conditioning on $L^{-d}\sum_{x\in\Lambda_L}\eta_x=\rho$. The Ginzburg-Landau dynamics is defined by the Dirichlet form

$$\mathcal{D}_{L,\rho}(F) = \frac{1}{2}\sum_{b\in\Lambda_L^*}\nu_{\Lambda_L,\rho}\big[(\nabla_b F)^2\big] \tag{4.1}$$

where we used the notation

$$\nabla_b F = \partial_y F - \partial_x F\,,\quad b=(x,y)\,.$$

The inverse of the spectral gap associated to $\mathcal{D}_{L,\rho}$ is given by

$$\chi(L,\rho) = \sup_F \frac{\mathrm{Var}_{\nu_{\Lambda_L,\rho}}(F)}{\mathcal{D}_{L,\rho}(F)} \tag{4.2}$$

with the supremum ranging over all real smooth functions on $\mathbb{R}^{\Lambda_L}$. As already observed in [7] we have a simple upper bound on $\chi(L,\rho)$ in terms of $\gamma(N,\rho)$ with $N=L^d$.

Lemma 4.1. *There exists a constant C only depending on d such that*

$$\chi(L,\rho) \leqslant C\,L^2\,\gamma(L^d,\rho) \tag{4.3}$$

Proof. We first make some observations about paths in Λ_L. We denote $\mathcal{C}_{xy}(L)$ the set of all paths γ_{xy} connecting sites $x,y\in\Lambda_L$, which use only bonds in Λ_L^*. The length of a path, denoted $|\gamma_{xy}|$ is the number of bonds composing it. Given $x,y\in\Lambda_L$ we need a rule to select a single path γ_{xy} from $\mathcal{C}_{xy}(L)$. We may choose γ_{xy} as follows. Fix $x,y\in\Lambda_L$ and define points $x^{(i)}$, $i=0,\dots,d$, such that $x^{(0)}=x$, $x^{(d)}=y$, and when $i=1,\dots,d-1$

$$x_j^{(i)} = \begin{cases} y_j & j=1,\dots,i \\ x_j & j=i+1,\dots,d \end{cases}$$

Call $\gamma^{(i)}$, $i = 1, \ldots, d$, the straight line parallel to the i-th axis joining sites $x^{(i-1)}$ and $x^{(i)}$. The path γ_{xy} is given by $\gamma^{(1)} \cup \cdots \cup \gamma^{(d)}$. It is not difficult to prove the following properties: there exists a finite constant k only depending on d such that

- for every $x, y \in \Lambda_L$, $|\gamma_{xy}| \leqslant kL$, and
- for every $b \in \Lambda_L^*$, $\sum_{x,y\in\Lambda_L} 1_{\{b\in\gamma_{xy}\}} \leqslant kL^{d+1}$

When we write γ_{xy} below we always assume that this path has been chosen according to the above rule.

Given $\eta \in \mathbb{R}^{\Lambda_L}$, $y \in \Lambda_L$ we write $\eta^{(y)}$ for the configuration

$$\eta_x^{(y)} = \begin{cases} \eta_x & x \neq y \\ \rho L^d - \sum_{z\neq y} \eta_z & x = y \end{cases} .$$

For $F : \mathbb{R}^{\Lambda_L} \to \mathbb{R}$, we denote $F_y(\eta)$ the function $\eta \to F(\eta^{(y)})$. Clearly, for any $y \in \Lambda_L$ we have

$$\mathrm{Var}_{\nu_{\Lambda_L,\rho}}(F) = \mathrm{Var}_{\nu_{\Lambda_L,\rho}}(F_y) . \tag{4.4}$$

It is then sufficient to show

$$L^{-d} \sum_{y\in\Lambda_L} \sum_{x\in\Lambda_L} \nu_{\Lambda_L,\rho}\big[(\partial_x F_y)^2\big] \leqslant C\, L^2\, \mathcal{D}_{L,\rho}(F) . \tag{4.5}$$

For any $x, y \in \Lambda_L$ we have

$$\partial_x F_y = \big(\partial_x F\big)_y - \big(\partial_y F\big)_y$$

and therefore

$$\nu_{\Lambda_L,\rho}\big[(\partial_x F_y)^2\big] = \nu_{\Lambda_L,\rho}\big[(\partial_x F - \partial_y F)^2\big] .$$

We write

$$\partial_x F - \partial_y F = \sum_{b\in\gamma_{yx}} \nabla_b F .$$

Since $|\gamma_{yx}| \leqslant kL$, Schwarz' inequality gives

$$\big(\partial_x F - \partial_y F\big)^2 \leqslant kL \sum_{b\in\Lambda_L^*} \big(\nabla_b F\big)^2 1_{\{b\in\gamma_{yx}\}} . \tag{4.6}$$

From the second property of our paths we see that (4.5) with $C = k^2$ follows from (4.6) when summing over x, y and dividing by L^d. $\square$

From Theorem 1.1 we obtain

Corollary 4.2. *Assume $V = \varphi + \psi$ with $\varphi \in \mathcal{C}^2$, $\varphi'' \geqslant \delta > 0$ and $|\psi|_\infty < \infty$. Then there exists $C = C(\delta, |\psi|_\infty) < \infty$ such that for every $\rho \in \mathbb{R}$ and $L \in \mathbb{N}$*

$$\mathrm{Var}_{\nu_{\Lambda_L,\rho}}(F) \leqslant C\, L^2\, \mathcal{D}_{L,\rho}(F) \tag{4.7}$$

holds for every smooth function F on $\mathbb{R}^{\Lambda_L}$.

References

[1] L. Bertini, B. Zegarlinski, *Coercive inequalities for Kawasaki dynamics: The product case.* Markov Proc. Rel. Fields **5**, 125-162, 1999

[2] L. Bertini, B. Zegarlinski, *Coercive inequalities for Gibbs measures* J. Funct. Anal. **162**, 257-289, 1999

[3] S. G. Bobkov, *Isoperimetric and analytic inequalities for log-concave probability measures.* Ann. Probab. **27**, 1903–1921, 1999

[4] T. Bodineau, B. Helffer, *On Log-Sobolev inequalities for unbounded spin systems.* J. Funct. Anal. **166**, 168-178, 1999

[5] H. J. Brascamp, E. H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation.* J. Funct. Anal. **22**, 366–389, 1976

[6] N. Cancrini, F. Martinelli, *On the spectral gap of Kawasaki dynamics under a mixing condition revisited.* J. Math. Phys. **41**, no. 3, 1391-1423, 2000

[7] P. Caputo, *A remark on spectral gap and logarithmic Sobolev inequalities for conservative spin systems.* Unpublished note, 2001, archived as `mp_arc 01-71`

[8] P. Caputo, F. Martinelli, *Relaxation time of anisotropic simple exclusion processes and quantum Heisenberg models*, `preprint` 2002

[9] E. Carlen, M.C. Carvalho, M. Loss, *Many–Body Aspects of Approach to Equilibrium.* Séminaire: Equations aux Dérivées Partielles, Exp. No. XIX, Ecole Polytech., Palaiseau, 2001. Also available at `http://www.math.gatech.edu/ carlen/preprints/index.html`

[10] D. Chafai, *Glauber versus Kawasaki for spectral gap and logarithmic Sobolev inequalities of some unbounded conservative spin systems*, `preprint` 2002, archived as `mp_arc 02-30`

[11] W. Feller, *An introduction to probability theory and its applications.* vol. 2, Wiley, 1971

[12] I. Gentil, C. Roberto, *Spectral gap for spin systems: some non convex phase examples.* J. Funct. Anal. **180**, 66–84, 2001

[13] C. Kipnis, C. Landim, *Scaling limits of interacting particle systems.* Grundlehren der Mathematischen Wissenschaften, 320. Springer-Verlag, Berlin, 1999

[14] C. Landim, G. Panizo, H. T. Yau, Spectral gap and logarithmic Sobolev inequality for unbounded conservative spin systems, to appear in Ann. Inst. H. Poincaré

[15] M. Ledoux, *Logarithmic Sobolev inequalities for unbounded spin systems revisited.* Séminaire de Probabilités, XXXV, 167–194, Lecture Notes in Math., 1755, Springer, Berlin, 2001

[16] S. T. Lu, H. T. Yau, *Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics.* Comm. Math. Phys. **156**, 399-433, 1993

[17] S. R. S. Varadhan, *Nonlinear diffusion limit for a system with nearest neighbor interactions. II* Asymptotic problems in probability theory: stochastic models and diffusions on fractals (Sanda/Kyoto, 1990), 75–128, Pitman Res. Notes Math. Ser., 283, Longman Sci. Tech. , 1993.

Dip. Matematica, Universita' di Roma Tre, L.go S. Murialdo 1, 00146 Roma, Italy.

E-mail address: `caputo@mat.uniroma3.it`