Scattering for the Forced Non-Linear Schrödinger Equation with a Potential on the Half-Line *

Ricardo Weder
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas,
Universidad Nacional Autónoma de México,
Apartado Postal 20-726, México D.F. 01000.
E-Mail: weder@servidor.unam.mx †

Abstract

In this paper we construct the scattering operator for the forced non-linear Schrödinger equation with a potential on the half-line. Moreover, in the case where the force is zero, and the solutions satisfy the homogeneous Dirichlet boundary condition at zero, we prove that the scattering operator determines uniquely the potential and the non-linearity and we give a method for the reconstruction of both.

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†Fellow, Sistema Nacional de Investigadores.

1 Introduction

In this paper we construct the scattering operator for the forced non-linear Schrödinger equation with a potential on the half-line (FNLSP),

$$i\frac{\partial}{\partial t}u(x,t) = -\frac{d^2}{dx^2}u(x,t)+V_0(x)u(x,t)+F(x,t,u),\ u(0,t) = f(t),\ u(x,0) = \phi(x), \text{ with } \phi(0) = f(0), \tag{1.1}$$

where $F(x,t,u)$ is a complex-valued function of $x \in \mathbf{R}^+ := (0,\infty)$, $t \in \mathbf{R}$, $u \in \mathbf{C}$. To solve this problem we consider the linear Schrödinger equation on the half-line with homogeneous Dirichlet boundary condition at zero that is obtained linearizing (1.1) with force identically zero. Namely,

$$i\frac{\partial}{\partial t}u(x,t) = H\,u(x,t),\ u(0,t) = 0,\ u(x,0) = \phi(x), \tag{1.2}$$

where $x \in \mathbf{R}^+$, $t \in \mathbf{R}$. The Hamiltonian, H, is the following operator,

$$H := -\frac{d^2}{dx^2} + V_0(x), \tag{1.3}$$

with domain,

$$D(H) := \left\{\phi \in L^2 : \phi, \frac{d}{dx}\phi \text{ absolutely continuous in } \mathbf{R}^+, \left(-\frac{d^2}{dx^2} + V_0(x)\right)\phi \in L^2, \phi(0) = 0\right\}, \tag{1.4}$$

where as usual, L^2 denotes the Hilbert space of square-integrable functions on $\mathbf{R}^+$. We assume that the potential, V_0, is real valued and that it satisfies the following condition,

$$\int_0^\infty x\,|V_0(x)|\,dx < \infty. \tag{1.5}$$

The Hamiltonian, H, is a self-adjoint operator in L^2 (see Section 2). It is the self-adjoint realization of the differential expression $-\frac{d^2}{dx^2} + V_0(x)$ with homogeneous Dirichlet boundary condition, $\phi(0) = 0$, at zero. By $W_{l,p}, l = 0,1,2,\cdots, 1 \leq p \leq \infty$, we designate the standard Sobolev spaces [1] in $\mathbf{R}^+$ and by $W^{(0)}_{l,p}, 1 \leq p < \infty$, the completion of $C_0^\infty(\mathbf{R}^+)$ in the norm of $W_{l,p}$. The functions in $W^{(0)}_{l,p}, l \geq 1$, satisfy the homogeneous Dirichlet boundary condition at zero, $\frac{d^j}{dx^j}u(0) = 0, j = 0,1,\cdots,l-1$. In the case $l = 0$ we use the standard notation, $W_{0,p} = W^{(0)}_{0,p} = L^p, 1 \leq p < \infty$. For $l \geq 1$ we use the notation, $W^{(0)}_{l,\infty} := \{\phi \in W_{l,\infty} : \frac{d^j}{dx^j}\phi(0) = 0, j = 0,1,2,\cdots,l-1\}$. By H_0 we designate the self-adjoint realization of $-\frac{d^2}{dx^2}$ with domain $W_{2,2} \cap W_{1,2}^{(0)}$, i.e., the self-adjoint realization with homogeneous Dirichlet boundary condition at zero. H_0 is, clearly, the particular case of H when $V_0 \equiv 0$. The wave operators corresponding to the pair (H_0, H) are defined as follows,

$$W_{\pm} := \mathrm{s} - \lim_{t \to \pm\infty} e^{itH} \, e^{-itH_0}, \tag{1.6}$$

where the limit is taken in the strong topology in L^2. It is proven in [7] that the wave operators exist and are asymptotically complete, i.e., their range is equal to the subspace of continuity of H, $\mathcal{H}_c$ (there is no singular-continuous spectrum). $\mathcal{H}_c$ is the orthogonal complement in L^2 to the subspace generated by all eigenvectors of H. The corresponding linear scattering operator is defined as,

$$S_L := W_+^* \, W_-. \tag{1.7}$$

To state our results on scattering for (1.1) we first introduce some notations and definitions. As is costumary, we say that $F(x,t,u)$ is a C^k function of u in the real sense if for each $x \in \mathbf{R}^+, t \in \mathbf{R}$, $\mathrm{Re}F$ and $\mathrm{Im}F$ are C^k functions with respect to the real and imaginary parts of u. Below we assume that F is C^2 in the real sense and that $\left(\frac{\partial}{\partial x} F\right)(x,t,u)$ is C^1 in the real sense. If $F = F_1 + iF_2$ with F_1, F_2 real-valued, and $u = r + is, r, s \in \mathbf{R}$ we denote,

$$F^{(2)}(x,t,u) := \sum_{j=1}^{2} \left[\left| \frac{\partial^2}{\partial r^2} F_j(x,t,u) \right| + \left| \frac{\partial^2}{\partial r \partial s} F_j(x,t,u) \right| + \left| \frac{\partial^2}{\partial s^2} F_j(x,t,u) \right| \right], \tag{1.8}$$

$$\left(\frac{\partial}{\partial x} F \right)^{(1)} (x,t,u) := \sum_{j=1}^{2} \left[\left| \frac{\partial}{\partial r} \left(\frac{\partial}{\partial x} F_j \right) (x,t,u) \right| + \left| \frac{\partial}{\partial s} \left(\frac{\partial}{\partial x} F_j \right) (x,t,u) \right| \right]. \tag{1.9}$$

Assumption A

We assume that F is a C^2 function of u in the real sense, that $F(x,t,0) = 0$, and that for each fixed $x \in \mathbf{R}^+, t \in \mathbf{R}$ all the first order derivatives, in the real sense, of F vanish at $u = 0$. Moreover, suppose that $\frac{\partial}{\partial x} F$ is C^1 in the real sense. We assume that the following estimates hold,

$$F^{(2)}(x,t,u) = O\left(|u|^{p-2}\right), \; \left(\frac{\partial}{\partial x} F \right)^{(1)} (x,t,u) = O\left(|u|^{p-1}\right), \; u \to 0, \text{ uniformly for } x \in \mathbf{R}^+, t \in \mathbf{R}, \tag{1.10}$$

for some $\rho < p < \infty$, and where ρ is the positive root of $\frac{1}{2} \frac{\rho - 1}{\rho + 1} = \frac{1}{\rho}$. Note that $\rho \approx 3.56$. Furthermore, if the force f in (1.1) is not identically zero we suposse that for each $R > 0$ there is a constant C_R such that,

$$\left| \frac{\partial}{\partial t} F(0,t,u) \right| \leq C_R \, |u|, t \in \mathbf{R}, \; |u| \leq R. \tag{1.11}$$

∎

Let us designate,

$$N := \left\{ f \in C^2(\mathbf{R}, \mathbf{C}) : \sup_{t\in\mathbf{R}} (1+|t|)^{dp} \left(|f(t)| + \left| \frac{d}{dt} f(t) \right| + \left| \frac{d^2}{dt^2} f(t) \right| \right) < \infty \right\},$$

$$\text{with norm : } \|f\|_N := \sup_{t\in\mathbf{R}} (1+|t|)^{dp} \left(|f(t)| + \left| \frac{d}{dt} f(t) \right| + \left| \frac{d^2}{dt^2} f(t) \right| \right), \qquad (1.12)$$

where $d := \frac{1}{2}(p-1)/(p+1)$ and p is as in Assumption A.

Assumption B

We assume that V_0 satisfies,

$$\int_0^\infty (1+x) |V_0(x)| \, dx < \infty, \qquad (1.13)$$

$$\sup_{x\in\mathbf{R}} \int_x^{x+1} |V_0(y)|^2 \, dy < \infty, \qquad (1.14)$$

and that $f \in N$. If f is not identically zero suppose that $V_0 \in W_{1,2}((0,\gamma))$ for some $\gamma > 0$.

∎

We denote,

$$M := \left\{ u \in C(\mathbf{R}, W_{1,p+1}) : \sup_{t\in\mathbf{R}} (1+|t|)^d \|u\|_{W_{1,p+1}} < \infty \right\},$$

$$\text{with norm } \|u\|_M := \sup_{t\in\mathbf{R}} (1+|t|)^d \|u\|_{W_{1,p+1}}. \qquad (1.15)$$

Under condition (1.5) H has no positive or zero eigenvalues, it has a finite number of negative eigenvalues, it has no singular-continuos spectrum, and the absolutely continuous spectrum consists of $[0,\infty)$. For these results see [5] and [7]. In what follows we designate by $u(t)$ the function $u(\cdot,t)$, where $u(x,t)$ is defined in $\mathbf{R}^+ \times \mathbf{R}$.

THEOREM 1.1. *Suppose that Assumptions A and B are satisfied and that H has no negative eigenvalues. Then, there is a $\delta > 0$ such that for all $f \in N$ and all $\phi_- \in W_{2,2} \cap W^{(0)}_{1,1+\frac{1}{p}}$ with $\|f\|_N + \|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W^{(0)}_{1,1+\frac{1}{p}}} \leq \delta$ there is a unique solution, u, to (1.1) such that $u \in C(\mathbf{R}, W_{1,2}) \cap M$, $u(0,t) = f(t)$ and,*

$$\lim_{t\to-\infty} \|u(t) - e^{-itH}\phi_-\|_{W_{1,2}} = 0. \qquad (1.16)$$

Moreover, there is a unique $\phi_+ \in W_{1,2}^{(0)}$ *such that,*

$$\lim_{t\to\infty} \|u(t) - e^{-itH}\phi_+\|_{W_{1,2}} = 0. \tag{1.17}$$

Furthermore, $e^{-itH}\phi_\pm \in M$ *and*

$$\left\|u - e^{-itH}\phi_\pm\right\|_M \leq C\left\{\left\|e^{-itH}\phi_\pm\right\|_M^p + \|f\|_N + \|f\|_N^{2p}\right\}, \tag{1.18}$$

$$\|\phi_+ - \phi_-\|_{W_{1,2}^{(0)}} \leq C\left\{\left[\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W^{(0)}_{1,1+\frac{1}{p}}}\right]^p + \|f\|_N + \|f\|_N^{2p}\right\}. \tag{1.19}$$

The scattering operator, $S_{V_0,f} : \phi_- \hookrightarrow \phi_+$ *is one to one.*

■

The scattering operator that relates incoming and outgoing asymptotic states that are solutions to the free linear Schrödinger equation (1.2) with potential $V_0 \equiv 0$ is defined as follows,

$$S := W_+^* \, S_{V_0,f} \, W_-. \tag{1.20}$$

In the case where $f \equiv 0$ we use the notation $S_{V_0} := S_{V_0,0}$. The following theorem allows us to uniquely reconstruct the scattering operator for the linearized problem, S_L (1.7), from the non-linear scattering operator, S.

THEOREM 1.2. *Suppose that the conditions of Theorem 1.1 are satisfied with force* $f \equiv 0$*. Then, for every* $\phi_- \in W_{2,2} \cap W^{(0)}_{1,1+\frac{1}{p}}$,

$$\frac{d}{d\epsilon} S(\epsilon\phi)\bigg|_{\epsilon=0} = S_L\phi, \tag{1.21}$$

where the derivative in the left-hand side of (1.21) exists in the strong convergence in $W_{1,2}^{(0)} \cap W_{1,p+1}^{(0)}$.

COROLLARY 1.3. *Under the conditions of Theorem 1.2 the scattering operator,* S*, determines uniquely the potential* V_0.

Proof: It follows from Theorem 1.2 that S determines uniquely S_L. From S_L we uniquely reconstruct the scattering phase shift, see (2.10) in Section 2. Finally, we uniquely reconstruct V_0 from the scattering phase shift [5].

$\blacksquare$

As we show below, in the case where $F(x,t,u) = \sum_{j=1}^{\infty} V_j(x,t)\,|u|^{2(j_0+j)}u$ we also uniquely reconstruct the $V_j, j = 1, 2, \cdots$.

THEOREM 1.4. *Suppose that the conditions of Theorem 1.2 are satisfied, and furthermore, that* $F(x,t,u) = \sum_{j=1}^{\infty} V_j(x,t)|u|^{2(j_0+j)}u,$ *for* $|u| \leq \eta$, *for some* $\eta > 0$, *with* j_0 *an integer such that,* $j_0 \geq (p-3)/2$, *and where* V_j *is a jointly continuous function of* x, t, *such that for each fixed* $t \in \mathbf{R}$, $V_j(\cdot,t) \in W_{1,\infty}$ *and* $\sup_{t\in\mathbf{R}} \|V_j(\cdot,t)\|_{W_{1,\infty}} \leq C^j, j = 1, 2, \cdots,$ *for some constant* C. *Then, for any* $\phi \in W_{2,2} \cap W^{(0)}_{1,1+\frac{1}{p}}$ *there is an* $\epsilon_0 > 0$ *such that for all* $0 < \epsilon < \epsilon_0$:

$$i\left((S_{V_0} - I)(\epsilon\phi),\, \phi\right)_{L^2} = \sum_{j=1}^{\infty} \epsilon^{2(j_0+j)+1}\left[\int\int_{\mathbf{R}\times\mathbf{R}^+} dt\, dx\, V_j(x,t)\left|e^{-itH}\phi\right|^{2(j_0+j+1)} + Q_j\right], \tag{1.22}$$

where $Q_1 = 0$ *and* $Q_j, j > 1$, *depends only on* ϕ *and on* V_k *with* $k < j$. *Furthermore , for any* $\acute{x} \in \mathbf{R}^+, \acute{t} \in \mathbf{R}$, *and any* $\lambda > 0$, *we denote,* $\phi_\lambda(x) := e^{i\acute{t}H}\,\hat{\phi}$, *with* $\hat{\phi}(x) := \tilde{\phi}(\lambda(x - \acute{x}))$, *where we define,* $\tilde{\phi}(x) := \phi(x), x \geq 0$, *and* $\tilde{\phi}(x) := -\phi(-x), x \leq 0$. *If moreover,* ϕ *has compact support in* $[0,\infty)$, *and it is not identically zero,*

$$V_j(\acute{x},\acute{t}) = \frac{\lim_{\lambda\to\infty} \lambda^3 \int\int_{\mathbf{R}\times\mathbf{R}^+} dt\, dx\, V_j(x,t)\left|e^{-itH}\phi_\lambda\right|^{2(j_0+j+1)}}{2\int\int_{\mathbf{R}\times\mathbf{R}^+} dt\, dx\, \left|e^{-itH_0}\phi\right|^{2(j_0+j+1)}}. \tag{1.23}$$

$\blacksquare$

In the case where the potentials $V_j, j = 1, 2, \cdots,$ are time independent we just take $\acute{t} = 0$ in the definition of ϕ_λ.

COROLLARY 1.5. *If the conditions of Theorem 1.4 are satisfied the scattering operator,* S, *determines uniquely the potentials* $V_j, j = 0, 1, \cdots$.

Proof: We know from Corollary 1.3 that S determines uniquely V_0. Then the wave operators, $W_\pm$, are uniquely determined (see the stationary formulae (2.9) for the wave operators in Section 2). It follows from the definition (1.20), that from S we uniquely reconstruct S_{V_0}. Finally, we uniquely reconstruct $V_j, j = 1, 2, \cdots$ in a recursive way using (1.22) and (1.23).

$\blacksquare$

For related results for the non-linear Schrödinger equation with a potential in $\mathbf{R}^n, n = 1, 2, \cdots$, see [13], [14], and for the case of the non-linear Klein-Gordon equation with a potential in $\mathbf{R}^n, n = 1, 2, \cdots$, see [12], [15], as well as the references quoted in these papers. For the integrable case where (1.1) can be studied with inverse scattering transform methods see [3] and the references quoted there.

Our proofs give a method for the unique reconstruction of all $V_j, j = 0, 1, 2, \cdots$, from the scattering operator S. First, we uniquely reconstruct S_L from S using (1.21). From S_L we compute the phase shift by means of (2.10). Then, from the phase shift we uniquely reconstruct V_0 (see [5]). Once V_0 is known we compute the wave operatos $W_\pm$, for example using (2.9). Then, we compute S_{V_0} by means of (1.20). Finally we uniquely reconstruct $V_j, j = 1, 2, \cdots$, using equations (1.22) and (1.23).

The technical tools that made possible to prove the results above are the $L^p - L^{\acute{p}}$ estimate for the linear Schrödinger equation (1.2) that we proved in [16], and the boundedness of the wave operators on the spaces $W_{1,p}^{(0)}, 1 < p < \infty$, that we prove in Section 2 below. We denote by P_c the orthogonal projector onto $\mathcal{H}_c$. For any pair of Banach spaces, X, Y, we designate by $\mathcal{B}(X, Y)$ the Banach space of bounded operators from X into Y. when $X = Y$ we use the notation $\mathcal{B}(X)$.

THEOREM 1.6. *(The $L^p - L^{\acute{p}}$ estimate [16]). Suppose that V_0 satisfies (1.5) then,*

$$\left\| e^{-itH} \, P_c \right\|_{\mathcal{B}\left(L^p, L^{\acute{p}}\right)} \leq C \frac{1}{|t|^{(1/p-1/2)}}, \; 1/p + 1/\acute{p} = 1, \; 1 \leq p \leq 2. \tag{1.24}$$

If furthermore, $V_0 \in L^1$,

$$\left\| e^{-itH} \, P_c \right\|_{\mathcal{B}\left(W_{1,p}^{(0)}, W_{1,\acute{p}}^{(0)}\right)} \leq C \frac{1}{|t|^{(1/p-1/2)}}, \; 1/p + 1/\acute{p} = 1, \; 1 \leq p \leq 2. \tag{1.25}$$

The $L^p - L^{\acute{p}}$ estimate expresses the smoothing properties of the linear Schrödinger equation (1.2) in a quantitative way. It also exibits the dispersive nature of this equation. It is the main reason why there is scattering for the FNLSP (1.1). Our result in the boundedness of the wave operators is the following theorem.

THEOREM 1.7. *Suppose that (1.5) holds. Then, the wave operators $W_\pm$ and their adjoints $W_\pm^*$, originally defined in $L^2 \cap L^p$, extend to bounded operators on $L^p, 1 < p < \infty$. If, furthermore, $V_0 \in L^1$, the $W_\pm$ and the $W_\pm^*$, originally defined on $L^2 \cap W_{1,p}^{(0)}$, extend to bounded operators on $W_{1,p}^{(0)}, 1 < p < \infty$.*

The importance of Theorem 1.7 lies on the fact that it allows us to prove that if H has no bound states the norm $\|(I+H)^{1/2}\phi\|_{L^p}$ is equivalent to the norm of $W^{(0)}_{1,p}, 1 < p < \infty$. For this purpose we use the intertwining relations,

$$f(H)P_c = W_{\pm} f(H_0) W^*_{\pm}, \text{ and } f(H_0) = W^*_{\pm} f(H) W_{\pm}, \tag{1.26}$$

that hold for any Borel function f. The norm $\|(I+H)^{1/2}\phi\|_{L^p}$ is very convenient because it is defined by means of the weight $(I+H)^{1/2}$ that commutes with the operator e^{-itH} that gives the solution to the initial-boundary value problem (1.2) that is obtained by linearizing (1.1) with force identically zero.

2 The Linear Schrödinger Equation on The Half-Line

We first state a number of results on the linear Schrödinger equation on the half-line,

$$\left(-\frac{d^2}{dx^2} + V_0(x)\right)\phi(x) = k^2\phi(x), x > 0, k \in \mathbf{C}, \tag{2.1}$$

that we use below. Let us denote by $f(k,x), k \in \mathbf{C}, \mathrm{Im\ k} \geq 0$, the Jost solution to (2.1) (see [2], [5] and [6]). It is the solution to (2.1) that satisfies $f(k,x) \sim e^{ikx}$, $x \to \infty$. The Jost solution is not required to satisfy the homogeneous Dirichlet boundary condition at zero. It only satisfies it if k^2 is an eigenvalue of H. Let us designate, $\sigma(x) := \int_x^\infty |V_0(y)|\, dy$ and $\sigma_1(x) := \int_x^\infty \sigma(y)\, dy$. Condition (1.5) is equivalent to $\sigma_1(0) < \infty$. The Jost solution can be represented as follows [5],

$$f(k,x) = e^{ikx} + \int_x^\infty K(x,y)\, e^{iky}\, dy, \tag{2.2}$$

where the kernel $K(x,y), x, y \in \mathbf{R}^+$ is real valued and it satisfies the inequality,

$$|K(x,y)| \leq \frac{1}{2}\sigma\left(\frac{x+y}{2}\right) 0 \exp\left(\sigma_1(x) - \sigma_1\left(\frac{x+y}{2}\right)\right). \tag{2.3}$$

Moreover, $K(x,y) = 0, y < x$. We extend K to a function defined on $\mathbf{R}^+ \times \mathbf{R}$ by seting $K(x,y) = 0$ for $y < 0$. Note that $\overline{f(k,x)} = f(-k,x)$ for $k \in \mathbf{R}$. For any $k \in \mathbf{C}$ equation (2.1) has also the regular solution, $\phi(k,x)$, that satisfies $\phi(k,0) = 0$, $\acute{\phi}(k,0) = 1$. Then, it follows from Theorem 5.8 of [17] that H is self-adjoint in the domain (1.4).

Let us designate,

$$\psi(k,x) := \sqrt{\frac{2}{\pi}}\frac{k}{|F(k)|}\phi(k,x), k \in \mathbf{R}\setminus 0, \tag{2.4}$$

where $F(k)$ is the Jost function, $F(k) := f(k,0)$. We have that, $F(k) \neq 0$ for $k \neq 0$. Then, [2], [5], [6], [17], [7], the operator,

$$(U\phi)(k) := \mathrm{s} - \lim_{b\to\infty}\int_0^b \psi(k,x)\,\phi(x)\,dx, \tag{2.5}$$

where the limit is in the strong topology in L^2, is unitary from $\mathcal{H}_c$ onto L^2 and its adjoint is given by,

$$(U^*\phi)(x) := \mathrm{s} - \lim_{b\to\infty}\int_0^b \psi(k,x)\,\phi(k)\,dk. \tag{2.6}$$

Moreover, for any Borel function f we have that, $f(H)P_c = Uf(H_0)U^*$. Let us denote by U_0 the operator (2.5) corresponding to the free case $V_0 \equiv 0$. It is just the sine transform,

$$(U_0\phi)(k) := \mathrm{s} - \lim_{b\to\infty}\sqrt{\frac{2}{\pi}}\int_0^b \sin(kx)\phi(x)\,dx, \tag{2.7}$$

and

$$(U_0^*\phi)(k) := \mathrm{s} - \lim_{b\to\infty}\sqrt{\frac{2}{\pi}}\int_0^b \sin(kx)\phi(k)\,dk. \tag{2.8}$$

The scattering phase shift, $\delta(k)$, is defined in terms of the phase of the Jost function as $F(k) := e^{-i\delta(k)}$, where $\delta(k)$ is the continuous branch of the phase function such that $\delta(\infty) = 0$. The following stationary formulae for the wave operators hold [7],

$$W_\pm = U^*\, e^{\mp i\delta(k)}U_0. \tag{2.9}$$

It follows from (2.9) that the linear scatttering operator, S_L, is given by,

$$S_L := U_0^*\, e^{2i\delta(k)}\, U_0. \tag{2.10}$$

Hence, in momentum space the linear scattering operator corresponding to the pair (H_0, H) is just multiplication by $S(k) := e^{2i\delta(k)}$. Note that, $S(-k) = \overline{S(k)}$.

If $V \in L^1$ - and also under weaker conditions- for any $\epsilon > 0$ there is a constant K_ϵ such that,

$$\int_0^\infty |V(x)|\,|\phi|^2\,dx \leq \epsilon \left\|\frac{\partial}{\partial x}\phi\right\|_{L^2}^2 + K_\epsilon\,\|\phi\|_{L^2}^2\,,\ \phi \in W_{1,2}. \tag{2.11}$$

For this result see Proposition 2.2 of [16]. Equation (2.11) implies that the quadratic form, $\mathcal{H}(\phi,\psi) := (\acute{\phi},\acute{\psi})_{L^2} + (V_0\phi,\psi)_{L^2}$ with domain $W_{1,2}^{(0)}$ is closed and bounded below. Furthermore, H is the associated self-adjoint operator (cf Theorem X.17 of [10]). Hence, the form domain of H is $W_{1,2}^{(0)}$, and, $D\left(\sqrt{H+N}\right) = W_{1,2}^{(0)}$, where N is so large that, $H+N>0$. It follows from (2.11) and as $(H+N)^{-1/2}$ is bounded from L^2 into $W_{1,2}^{(0)}$ that the norm $\left\|\sqrt{H+N}\,\phi\right\|_{L^2}$ is equivalent to the norm of $W_{1,2}^{(0)}$. We use this equivalence below without further comment.

Proof of Theorem 1.7: The generalized eigenfunction $\psi(k,x)$ (2.4) is represented in terms of the Jost solution as follows [2], [6]

$$\psi(k,x) = \frac{i}{\sqrt{2\pi}}\, e^{-i\delta(k)}\, \left[f(-k,x) - S(k) f(k,x)\right]. \tag{2.12}$$

By (2.9) and (2.12) the $W_\pm$ are integral operators with kernel, $w_\pm(x,y)$, given by,

$$w_\pm(x,y) := \frac{i}{\pi}\int_0^\infty \left[f(-k,x) - S(k) f(k,x)\right]\, S_\pm(k)\, \sin(ky)\, dk, \tag{2.13}$$

where $S_+(k) := \overline{S(k)}$ and $S_-(k) := 1$. We give the proof in the case of W_+. The case of W_- follows in the same way, and actually it is a bit simpler.

Let us denote,

$$d(k,x) := f(k,x) - e^{ikx}. \tag{2.14}$$

It follows from (2.2) that,

$$K(x,y) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-iky}\, d(k,x)\, dk. \tag{2.15}$$

We decompose W_+ as follows,

$$W_+ = A\,\frac{1+i\mathcal{H}}{2}\, E_0. \tag{2.16}$$

By E_0 we denote the operator of odd-extension of functions defined on $(0,\infty)$ into functions defined on $\mathbf{R}$, i.e., $(E_0\phi)(x) := \phi(x), x \geq 0$, and $(E_0\phi)(x) := -\phi(-x), x < 0$. The operator E_0 is bounded from L^p into $L^p(\mathbf{R})$, and from $W_{1,p}^{(0)}$ into $W_{1,p}^{(0)}(\mathbf{R}), 1 \leq p \leq \infty$. $\mathcal{H}$ is the Hilbert transform,

$$(\mathcal{H}\phi)(x) := \frac{1}{\pi}\text{P.V.}\int_{-\infty}^\infty \frac{1}{y}\phi(x-y)\, dy = F^{-1}(-i\,\text{sign}k)F\phi, \tag{2.17}$$

where $P.V.$ means that the integral is taken in principal value sense and with F the Fourier transform,

$$(F\phi)(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \phi(x)\, dx. \tag{2.18}$$

The Hilbert transform is a bounded operator on $W_{l,p}(\mathbf{R}), l = 0, 1, \cdots, 1 < p < \infty$ [11]. The spaces $W_{l,p}(\mathbf{R})$ and $W_{l,p}^{(0)}(\mathbf{R})$ are defined as the spaces $W_{l,p}, W_{l,p}^{(0)}$, but with $\mathbf{R}$ instead of $\mathbf{R}^+$ [1]. Furthermore, A is the integral operator from $L^2(\mathbf{R})$ into L^2 with kernel, $a(x, y), x \in \mathbf{R}^+, y \in \mathbf{R}$, defined as,

$$a(x, y) := -\frac{1}{2\pi} \int_{-\infty}^{\infty} [f(-k, x) - S(k) f(k, x)]\, S_+(k)\, e^{-iky}\, dk. \tag{2.19}$$

Using (2.15) we further decompose A as follows,

$$A = -(\chi_{\mathbf{R}^+} + \tilde{K})(I + B)P + \chi_{\mathbf{R}^+} + \tilde{K}, \tag{2.20}$$

with $\chi_{\mathbf{R}^+}$ the characteristic function of $\mathbf{R}^+$, $P\phi(x) := \phi(-x)$, and where,

$$B := F^{-1}\, T(\cdot)\, F, \text{ with } T(k) := S(k) - 1. \tag{2.21}$$

Moreover, $\tilde{K}$ is the integral operator from $L^2(\mathbf{R})$ into L^2 with kernel $K(x, y), x \in \mathbf{R}^+, y \in \mathbf{R}$. It follows from the extension of Wiener's theorem to the Fourier transform (see the Corollary to Theorem 4.204 in page 154 of [4]) and the argument given in pages 212 and 213 of [5] that if (1.5) holds the Fourier transform of $T(k)$ is integrable on $\mathbf{R}$. Then, $B \in \mathcal{B}\left(W_{l,p}(\mathbf{R})\right), l = 0, 1, \cdots, 1 \leq p \leq \infty$. By (2.3),

$$\sup_{x \in \mathbf{R}^+} \int_{-\infty}^{\infty} |K(x, y)|\, dy < \infty, \ \sup_{y \in \mathbf{R}} \int_0^{\infty} |K(x, y)|\, dx < \infty, \tag{2.22}$$

and then $\tilde{K}$ is a bounded operator from $L^p(\mathbf{R})$ into $L^p, 1 \leq p \leq \infty$. It follows that $W_+ \in \mathcal{B}(L^p), 1 < p < \infty$. By duality, W_+^* is also bounded on $L^p, 1 < p < \infty$.

We designate,

$$h(u, v) := K(u - v, u + v), u, v \geq 0, u \geq v. \tag{2.23}$$

It is proven in [5], page 176, that $h(u, v)$ is the unique solution to the following equation,

$$h(u, v) = \frac{1}{2} \int_u^{\infty} V_0(y)\, dy + \int_u^{\infty} dx \int_0^v V_0(x - y)\, h(x, y)\, dy. \tag{2.24}$$

Let us designate,

$$q(u, v) := \frac{1}{2} \sigma(u) \exp\left(\sigma_1(u - v) - \sigma_1(u)\right), u \geq v. \tag{2.25}$$

We have that [5],

$$|h(u,v)| \leq q(u,v). \tag{2.26}$$

As $q(u,v)$ is a non-increasing function of u, and $q(u,v)$ is non-decreasing on v, we have that,

$$\left|\frac{\partial}{\partial u} h(u,v)\right| \leq \frac{1}{2} |V_0(u)| + \sigma(u-v)\, q(u,v), \tag{2.27}$$

and,

$$\left|\frac{\partial}{\partial v}\, h(u,v)\right| \leq \sigma(u-v)\, q(u,v). \tag{2.28}$$

As $K(x,y) = h(\frac{x+y}{2}, \frac{y-x}{2})$ it follows from (2.27) and (2.28) that if (1.5) is satisfied and if moreover, $V_0 \in L^1$,

$$\sup_{x \in \mathbf{R}^+} \int_{-\infty}^{\infty} \left|\frac{\partial}{\partial x} K(x,y)\right| dy < \infty, \ \sup_{y \in \mathbf{R}} \int_0^{\infty} \left|\frac{\partial}{\partial x} K(x,y)\right| dx < \infty, \tag{2.29}$$

and then $\tilde{K}$ is a bounded operator from $L^p(\mathbf{R})$ into $W_{1,p}, 1 \leq p \leq \infty$. In consequence $W_+ \in \mathcal{B}(W_{1,p}^{(0)}, W_{1,p}), 1 < p < \infty$. By the intertwining relations (1.26), $\sqrt{H+M}\, W_\pm = W_\pm \sqrt{H_0 + M}$, and it follows that $W_\pm$ are bounded on $W_{1,2}^{(0)}$. Given $\phi \in W_{1,p}^{(0)}, 1 < p < \infty$, take $\phi_n \in C_0^\infty(\mathbf{R}^+)$ that converges to ϕ in $W_{1,p}^{(0)}$. Then, as $W_+ \in \mathcal{B}(W_{1,p}^{(0)}, W_{1,p})$ we have that, $W_+\phi_n \to W_+\phi$ in $W_{1,p}$. Furthermore, as $W_+\phi_n \in W_{1,2}^{(0)}$, $(W_+\phi_n)(0) = 0$. Moreover, by Sobolev's imbedding theorem, $W_+\phi_n$ converges uniformly to $W_+\phi$, and then, $(W_+\phi)(0) = 0$, and it follows that $W_+\phi \in W_{1,p}^{(0)}$. This completes the proof that $W_+ \in \mathcal{B}\left(W_{1,p}^{(0)}\right), 1 < p < \infty$. Taking the adjoint of (2.16) and of (2.20) we prove in the same way that $W_+^* \in \mathcal{B}\left(W_{1,p}^{(0)}\right), 1 < p < \infty$.

∎

3 Scattering Theory

In [11] it is proven that the following norm is equivalent to the norm of $W_{1,p}(\mathbf{R}), 1 < p < \infty$,

$$\left\| F^{-1}(1+k^2)^{1/2} F\phi \right\|_{L^p(\mathbf{R})}. \tag{3.1}$$

By extending $\phi \in W_{1,p}^{(0)}$ to and odd function in $W_{1,p}^{(0)}(\mathbf{R})$, we prove that the norm,

$$\left\| U_0^{-1}(1+k^2)^{1/2} U_0\phi \right\|_{L^p}, \tag{3.2}$$

is equivalent to the norm of $W^{(0)}_{1,p}, 1 < p < \infty$. It follows from Theorem 1.7 and equations (1.26), that if (1.5) holds, $V_0 \in L^1$ and H has no negative eigenvalues, then the following norm is equivalent to the norm of $W^{(0)}_{1,p}, 1 < p < \infty$,

$$\left\| (I+H)^{1/2} \phi \right\|_{L^p}. \tag{3.3}$$

Furthermore, by Proposition 2.2 of [16] if (1.14) is satisfied, for any $\epsilon > 0$ there is a constant K_ϵ such that,

$$\| V_0 \phi \|^2_{L^2} \leq \epsilon \, \| H_0 \phi \|^2_{L^2} + K_\epsilon \, \| \phi \|^2_{L^2}, \phi \in D(H_0). \tag{3.4}$$

Equation (3.4) implies that V_0 is relatively bounded with respect to H_0 with relative bound zero. Then, $D(H) = W_{2,2} \cap W^{(0)}_{1,2}$ and $\|(I+H)\phi\|_{L^2}$ defines a norm that is equivalent to the norm of $W_{2,2} \cap W^{(0)}_{1,2}$. Below we use these equivalences without further comments, and we will always assume that Assumptions A and B are satisfied.

Let us denote by $W_{-1,2}$ the dual of $W^{(0)}_{1,2}$ with the duality pairing given by the scalar product of L^2. Recall (see Section 2) that the quadratic form domain of H is $W^{(0)}_{1,2}$. Then, H extends to a bounded operator from $W^{(0)}_{1,2}$ into $W_{-1,2}$. Furthermore , e^{-itH} is a bounded operator from $W^{(0)}_{1,2}$ into $C\left(\mathbf{R}, W^{(0)}_{1,2}\right) \cap C^1\left(\mathbf{R}, W_{-1,2}\right)$ and, $i\frac{\partial}{\partial t} e^{-itH}\,\phi = He^{-itH}\phi = e^{-itH}H\phi$. Let $u(x,t) \in C\left(\mathbf{R}, W_{1,2}\right)$ be a solution to (1.1). We designate, $v(x,t) := u(x,t) - r(x,t)$, where $r(x,t) := [f(t) + \frac{1}{2}x^2(V(0)f(t) + F(0,t,f(t)) - i\acute{f}(t))]g(x)$, with $g \in C^2_0([0,\infty))$, $g(x) = 1, 0 \leq x \leq \delta/2$, with support contained in $[0,\delta)$, and with δ as in Assumption B. Then, $v(x,t) \in C\left(\mathbf{R}, W^{(0)}_{1,2}\right)$ is a solution to

$$i\frac{\partial}{\partial t} v(x,t) = Hv(x,t) + F_1(x,t,v), \; v(0,t) = 0, \tag{3.5}$$

where,

$$F_1(x,t,v) := F(x,t,v+r) - i\frac{\partial}{\partial t} r + V(x)\, r - \frac{\partial^2}{\partial x^2} r, \tag{3.6}$$

and with, $v(x,0) = v_0(x) := \phi(x) - r(x,0)$. It follows from the compatibility condition, $\phi(0) = f(0)$, that $v_0 \in W^{(0)}_{1,2}$. Equations (1.1) and (3.5 are, of course, equivalent. Note that $u \in C(\mathbf{R}, W_{1,2}) \cap M$ is a solution to (1.1), were instead of fixing the initial value at $t = 0$ we give it at $t = -\infty$ by requiring that condition (1.16) be satisfied (Cauchy problem at $-\infty$), if and only if $v \in C(\mathbf{R}, W^{(0)}_{1,2}) \cap M_0$ is a solution to (3.5) such that,

$$\lim_{t\to-\infty} \|v(t) - e^{-itH}\phi_-\|_{W^{(0)}_{1,2}} = 0, \tag{3.7}$$

where,

$$M_0 := \left\{ v \in C(\mathbf{R}, W^{(0)}_{1,p+1}) : \sup_{t\in\mathbf{R}} (1+|t|)^d \, \|v\|_{W^{(0)}_{1,p+1}} < \infty \right\},$$
$$\text{with norm, } \|v\|_{M_0} := \sup_{t\in\mathbf{R}} (1+|t|)^d \, \|v\|_{W^{(0)}_{1,p+1}}. \tag{3.8}$$

We prepare some results that we need in order to write down an integral equation that is equivalent to (3.5)-(3.7). For $v \in M_0$ let us designate,

$$\mathcal{Q}v(t) := \frac{1}{i}\int_{-\infty}^{t} e^{-i(t-\tau)H} F_1(x,t,v(\tau))\, d\tau. \tag{3.9}$$

By the $L^p - L^{\acute{p}}$ estimate (1.25), and as $W^{(0)}_{1,p+1}$ is continuously imbedded in L^∞ it follows from a simple calculation that,

$$\|\mathcal{Q}v(t)\|_{W^{(0)}_{1,p+1}} \leq C\,(1+|t|)^{-d}[(\|v\|_{M_0} + \|f\|_N)^p + \|f\|_N], \tag{3.10}$$

and that

$$\|\mathcal{Q}v_1(t) - \mathcal{Q}v_2(t)\|_{W^{(0)}_{1,p+1}} \leq C\,(1+|t|)^{-d}\left[(\|v_1\|_{M_0} + \|v_2\|_{M_0} + \|f\|_N)^{p-1}\,\|v_1 - v_2\|_{M_0}\right]. \tag{3.11}$$

Here we used that $p\,d > 1$. We can take the constant C in (3.10), (3.11) uniform in closed balls in M_0. In the following calculation we use (3.10).

$$\|\mathcal{Q}v(t)\|^2_{W^{(0)}_{1,2}} \leq C\mathrm{Re}\int_{-\infty}^{t} d\tau\, \left(\sqrt{I+H}F_1(x,\tau,v(\tau)), \sqrt{I+H}\,\mathcal{Q}v(\tau)\right)_{L^2} \leq C\int_{-\infty}^{t} d\tau\, \|F_1(x,\tau,v(\tau))\|_{W^{(0)}_{1,1+1/p}} \times$$
$$(1+|\tau|)^{-d}[(\|v\|_{M_0} + \|f\|_N)^p + \|f\|_N] \leq C\int_{-\infty}^{t} d\tau\, [(\|v\|_{M_0} + \|f\|_N)^p + \|f\|_N]^2\,(1+|\tau|)^{-d(p+1)}$$
$$\leq C(1+\max[0,-t])^{-(d+dp-1)}\,[(\|v\|_{M_0} + \|f\|_N)^p + \|f\|_N]^2. \tag{3.12}$$

Equation (3.10) implies that the integral in the right-hand side of (3.9) converges in $W^{(0)}_{1,p+1}$ and that it defines a function in M_0. Also, (3.12) implies that this integral converges in $W^{(0)}_{1,2}$. It follows that $v \in C(\mathbf{R}, W^{(0)}_{1,2}) \cap M_0$ is a solution to (3.5) satisfying (3.7) if and only if it is a solution to the following integral equation (for a similar argument in the case of the Cauchy problem at finite time see [16]),

$$v = e^{-itH}\phi_- + \mathcal{Q}\, v. \tag{3.13}$$

We prove below Theorem 1.1 solving equation (3.13). Note that, $W_{2,2} \cap W^{(0)}_{1,1+\frac{1}{p}} \subset W^{(0)}_{1,2}$.

Proof of Theorem 1.1: We first prove the uniqueness. It is enough to prove that (3.13) has a unique solution in $C\left(\mathbf{R}, W^{(0)}_{1,2}\right) \cap M_0$. Suppose that there are two, v_1 and v_2. Let us denote $v_{j,T} := \chi_{(-\infty,-T)}(t) v_j(t), j = 1, 2, T > 0$, where by $\chi_{(-\infty,-T)}(t)$ we designate the characteristic function of $(-\infty, -T)$. Then,

$$v_{1,T}(t) - v_{2,T}(t) = \mathcal{Q} v_{1,T}(t) - \mathcal{Q} v_{2,T}(t), \text{ for } t \leq -T. \tag{3.14}$$

Let us now define $\tilde{M}_0$ as M_0 but replacing in (3.8) d by $\tilde{d}$ where $\tilde{d} < d$ and $\tilde{d}p > 1$. Note that $\tilde{M}_0 \subset M_0$. Since $v_1, v_2 \in M_0$ and arguing as in the proof of (3.11) we prove that

$$\|v_{1,T} - v_{2,T}\|_{\tilde{M}_0} \leq C \frac{1}{(1+T)^{\epsilon}} \|v_{1,T} - v_{2,T}\|_{\tilde{M}_0}, \tag{3.15}$$

with $\epsilon > 0$. Then, taking T large enough, we have that $v_1(t) = v_2(t)$ for $t \leq -T$. Hence, by the uniqueness of the Cauchy problem with data at finite time (see Theorem 3.1 and Remark 3.3 of [16]), $v_1(t) = v_2(t), t \in \mathbf{R}$.

It follows from Sobolev's imbedding theorem that,

$$\|e^{-itH}\phi_-\|_{W^{(0)}_{1,p+1}} \leq C \|e^{-itH}\phi_-\|_{W_{2,2}} \leq C \left\|(I+H) e^{-itH}\phi_-\right\|_{L^2} = C \left\|(I+H)\phi_-\right\|_{L^2} \leq C \left\|\phi_-\right\|_{W_{2,2}}. \tag{3.16}$$

Hence, it follows from (1.25), and (3.16) that,

$$\left\|e^{-itH}\phi_-\right\|_{M_0} \leq C \left\{ \|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W^{(0)}_{1,1+\frac{1}{p}}} \right\}. \tag{3.17}$$

We designate, $M_{0,R} := \{v \in M_0 : \|v\|_{M_0} \leq R\}, R > 0$. By (3.10), (3.11) and (3.17) we take can R and δ so small that the map $v \hookrightarrow P(v) := e^{-itH}\phi_- + \mathcal{Q}\, v$ is a contraction from $M_{0,R}$ into $M_{0,R}$ -with contraction ratio smaller than 1/2- for all $\phi_- \in W_{2,2} \cap W^{(0)}_{1,1+\frac{1}{p}}$ and for all $f \in N$ with $\|\phi_-\|_{W_{2,2}} + \|\phi_-\|_{W^{(0)}_{1,1+\frac{1}{p}}} + \|f\|_N \leq \delta$. By the contraction mapping theorem P has a unique fixed point that is the unique solution to (3.13) in $M_{0,R}$. We also have that,

$$\|v\|_{M_0} \leq \left\| e^{-itH}\phi_- \right\|_{M_0} + C[\|f\|_N + \|f\|_N^p] + \frac{1}{2}\|v\|_{M_0}, \tag{3.18}$$

and we obtain that,

$$\|v\|_{M_0} \leq C\left[\left\| e^{-itH}\phi_- \right\|_{M_0} + \|f\|_N + \|f\|_N^p \right]. \tag{3.19}$$

Equation(1.18) for ϕ_- follows from (3.10), (3.13) and (3.19). Equation (3.12) implies that $v \in C(\mathbf{R}, W_{1,2}^{(0)})$ and using also (3.13) we prove that (1.16) holds. We now define,

$$\phi_+ := \phi_- + \frac{1}{i}\int_{-\infty}^{\infty} e^{i\tau H} F_1(x,t,v(\tau))\, d\tau. \tag{3.20}$$

The argument given in (3.12) implies that $\phi_+ \in W_{1,2}^{(0)}$ and furthermore, that

$$\|\phi_+ - \phi_-\|_{W_{1,2}^{(0)}} \leq C(\|v\|_{M_0} + \|f\|_N)^p. \tag{3.21}$$

We now prove (1.19) using (3.17), (3.19) and (3.21). By the definition of ϕ_+ and (3.13),

$$v(t) = e^{-itH}\phi_+ - \frac{1}{i}\int_t^{\infty} e^{-i(t-\tau)H} F_1(x,t,v(\tau))\, d\tau, \tag{3.22}$$

and equation (1.17) follows as above. The uniqueness of ϕ_+ is immediate since e^{-itH} is unitary.

Furthermore, arguing as above and using (3.22) we prove that $e^{-itH}\phi_+ \in M_0$ and that (1.18) holds for ϕ_+. Suppose that $S_{V_0,f}\,\phi_1 = S_{V_0,f}\,\phi_2$, and let $v_j, j = 1,2$ be the solutions to (3.13) with, respectively, $\phi_- = \phi_1$ and $\phi_- = \phi_2$. Then, denoting $v_3(t) := v_1(t) - v_2(t)$, by (3.22),

$$v_3(t) = -\frac{1}{i}\int_t^{\infty} e^{-i(t-\tau)H} F_1(x,t,v_1(\tau))\, d\tau + \frac{1}{i}\int_t^{\infty} e^{-i(t-\tau)H} F_1(x,t,v_2(\tau))\, d\tau. \tag{3.23}$$

Estimating as in the proof of uniqueness given above we prove that $v_1(t) = v_2(t), t \geq T$, for T positive enough. But then by the uniqueness result for the Cauchy problem with initial data at finite time given in Theorem 3.1 and Remark 3.3 of [16], $v_1(t) = v_2(t), t \in \mathbf{R}$ and by (1.16), $\phi_1 = \phi_2$. This proves that $S_{V_0,f}$ is one to one.

Proof of Theorem 1.2: Since the force f is identically zero, we have that $S(0) = 0$. By the intertwining relations (1.26) $W_\pm$ and $W_\pm^*$ are bounded operators on $W_{2,2} \cap W_{1,2}^{(0)} = D(H)$. Then, it is enough to prove that

$$s - \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(S_{V_0}(\epsilon\phi) - \epsilon\phi \right) = 0. \tag{3.24}$$

By (3.17) and (3.19) with ϕ_- replaced by $\epsilon\phi$:

$$\|v\|_{M_0} \leq C|\epsilon| \left[\|\phi_-\|_{W_{2,2}} + \|\phi_-\|^{(0)}_{W_{1,1+\frac{1}{p}}}\right]. \tag{3.25}$$

Equation (3.24) follows estimating the integral in the right-hand side of (3.20) as in the proof of (3.10), and (3.12), and using (3.25) .

■

Proof of Theorem 1.4: Equation (1.22) follows from (3.20) and since the contraction mapping theorem implies that,

$$v = e^{-itH}\epsilon\phi + \sum_{n=1}^{\infty} \mathcal{Q}^n e^{-itH}\epsilon\phi. \tag{3.26}$$

To prove (1.23) we consider the integral,

$$I_j := \lambda^3 \int\int_{\mathbf{R}\times\mathbf{R}^+} dtdx V_j(x,t) \left|e^{-itH}\phi_\lambda\right|^{2(j_0+j+1)}. \tag{3.27}$$

In order to use a theorem on strong resolvent convergence of operators defined in a unique Hilbert space [9], we find it convenient to integrate x over $\mathbf{R}$ in the right-hand side of (3.27). For this purpose, we define $V_j, j = 0, 1, \cdots$, to be zero for $x < 0$, and we denote, $H_D := -\Delta_D + V_0$, where Δ_D is the self-adjoint realization of the Laplacian in $L^2(\mathbf{R})$ with homogeneous Dirichlet boundary condition at zero. Note that, $H_D = -\Delta_- \oplus H$, where Δ_- is the self-adjoint realization of the Laplacian in $L^2((-\infty, 0))$ with homogeneous Dirichlet boundary condition at zero. Below we always assume that λ is so large that, $\hat{\phi}(x) := \tilde{\phi}(\lambda(x - \acute{x})) = 0$ for $x \leq 0$ (recall that ϕ has compact support). As e^{-itH} commutes with H, we have that, $e^{-itH} \in \mathcal{B}\left(W^{(0)}_{1,1+\frac{1}{p}}\right)$. Then, $\phi_\lambda := e^{i\acute{t}H}\hat{\phi} \in W_{2,2} \cap W^{(0)}_{1,1+\frac{1}{p}}$. Let us define, $\psi_\lambda(x) := \phi_\lambda(x), x \geq 0$, and $\psi_\lambda(x) := 0, x \leq 0$. Hence,

$$e^{-itH_D}\,\psi_\lambda = 0 \oplus e^{-itH}\phi_\lambda, \tag{3.28}$$

and we have that,

$$I_j = \lambda^3 \int\int_{\mathbf{R}\times\mathbf{R}} dt\, dx\, V_j(x,t) \left|e^{-itH_D}\psi_\lambda\right|^{2(j_0+j+1)}. \tag{3.29}$$

We designate by $\Delta^{(\lambda)}$ the self-adjoint realization of the Laplacian in $L^2(\mathbf{R})$ with homogeneous Dirichlet boundary condition at $x = -\lambda\acute{x}$. Denote, $H^{(\lambda)} := -\Delta^{(\lambda)} + V_\lambda$ where, $V_\lambda := \frac{1}{\lambda^2}V_0(\frac{x}{\lambda} + \acute{x})$. As $H > 0$, we have that, $H^{(\lambda)} > 0$. Let us designate, $\tilde{t} := \lambda^2(t-\acute{t})$ and $\tilde{x} := \lambda(x-\acute{x})$. Then,

$$\left(e^{-i\tilde{t}H^{(\lambda)}}\tilde{\phi}\right)(\tilde{x}) = \left(e^{-itH_D}\psi_\lambda\right)(x). \tag{3.30}$$

Equation (3.30) is proven as follows. Denote $f(\tilde{x},\tilde{t}) := \left(e^{-itH_D}\psi_\lambda\right)(x)$. Then, we have that $f(\tilde{x},\tilde{t}) \in D(H_\lambda)$ and furthermore,

$$i\frac{\partial}{\partial\tilde{t}}f(\tilde{x},\tilde{t}) = H^{(\lambda)}\, f(\tilde{x},\tilde{t}),\ f(\tilde{x},0) = \tilde{\phi}. \tag{3.31}$$

As the solution to (3.31) is unique, equation (3.30) holds. Then,

$$I_j = \int\int_{\mathbf{R}\times\mathbf{R}} d\tilde{t}\, d\tilde{x} V_j\left(\frac{\tilde{x}}{\lambda}+\acute{x}, \frac{\tilde{t}}{\lambda^2}+\acute{t}\right)\left|e^{-i\tilde{t}H^{(\lambda)}}\tilde{\phi}\right|^{2(j_0+j+1)}. \tag{3.32}$$

Let Δ be the self-adjoint realization of the Laplacian in $L^2(\mathbf{R})$ with domain $W_{2,2}(\mathbf{R})$. We denote $D := (-\Delta - i)C_0^\infty(\mathbf{R})$. As Δ is essentially self-adjoint in $C_0^\infty(\mathbf{R})$, D is dense in $L^2(\mathbf{R})$. Hence, for $\chi = (-\Delta-i)\psi, \psi \in C_0^\infty(\mathbf{R})$,

$$\lim_{\lambda\to\infty}\left[(H^{(\lambda)}-i)^{-1}-(-\Delta-i)^{-1}\right]\chi = \lim_{\lambda\to\infty}(H^{(\lambda)}-i)^{-1}(\Delta^{(\lambda)}-V_\lambda-\Delta)\psi = 0, \tag{3.33}$$

where we used that $\psi(-\lambda\acute{x}) = 0$ if λ is large enough. Then, by Theorem VIII.21 in page 286 of [9],

$$\mathrm{s}-\lim_{\lambda\to\infty} e^{-i\tilde{t}H^{(\lambda)}} = e^{i\tilde{t}\Delta}, \tag{3.34}$$

where the limit is taken in the strong topology in $L^2(\mathbf{R})$. Using (3.30), and changing variables from $\tilde{x}$ to x , we obtain that,

$$\left(V_\lambda e^{-i\tilde{t}H^{(\lambda)}}\tilde{\phi}, e^{-i\tilde{t}H^{(\lambda)}}\tilde{\phi}\right) \leq \frac{1}{\lambda}\|V_0\|_{L^1}\|e^{-itH}\phi_\lambda\|^2_{L^\infty}. \tag{3.35}$$

Let us denote by $W^{(0)}_{s,2}, 0 < s < 1$, the completion of C_0^∞ in the norm

$$\|\phi\|_{W^{(0)}_{s,2}} := \left\|U_0^{-1}(1+k^2)^{s/2}U_0\phi\right\|_{L^2}, 0 < s < 1. \tag{3.36}$$

Note that $W^{(0)}_{1,2} \subset W^{(0)}_{s,2}, 0 < s < 1$. By (1.26) the following norm is equivalent ot the norm of $W^{(0)}_{s,2}, 0 < s < 1$,

$$\left\|(I+H)^{s/2}\phi\right\|_{L^2}. \tag{3.37}$$

Then, by (3.35), and as $W^{(0)}_{s,2}, 1/2 < s < 1$, is continuously imbedded in L^∞,

$$\left(V_\lambda e^{-i\tilde{t}H^{(\lambda)}}\tilde{\phi}, e^{-i\tilde{t}H^{(\lambda)}}\tilde{\phi}\right) \leq \frac{C}{\lambda}\,\|\hat{\phi}\|^2_{W^{(0)}_{s,2}} \leq \frac{C}{\lambda^{2-2s}}, \tag{3.38}$$

where we used that, $\|\hat{\phi}\|_{W^{(0)}_{s,2}} \leq C/\lambda^{1/2-s}$. This estimate follows as $\|\hat{\phi}\|_{L^2} \leq C/\lambda^{1/2}$, $\|\hat{\phi}\|_{W^{(0)}_{1,2}} \leq C\lambda^{1/2}$, and by Hölder's inequality. Hence, by (3.38),

$$\left\| \frac{\partial}{\partial \tilde{x}} e^{-i\tilde{t}H^{(\lambda)}} \tilde{\phi} \right\|^2_{L^2(\mathbf{R})} = \left\| \sqrt{H^{(\lambda)}} e^{-i\tilde{t}H^{(\lambda)}} \tilde{\phi} \right\|^2_{L^2(\mathbf{R})} - \left(V_\lambda e^{-i\tilde{t}H^{(\lambda)}} \tilde{\phi}, e^{-i\tilde{t}H^{(\lambda)}} \tilde{\phi} \right)$$
$$\leq \left\| \sqrt{H^{(\lambda)}} \tilde{\phi} \right\|^2_{L^2(\mathbf{R})} + \frac{C}{\lambda^{2-2s}} \leq \left\| \frac{\partial}{\partial \tilde{x}} \tilde{\phi} \right\|^2_{L^2(\mathbf{R})} + \frac{C}{\lambda^{2-2s}}, \tilde{t} \in \mathbf{R}, \tag{3.39}$$

and moreover,

$$\lim_{\lambda\to\infty} \left\| \frac{\partial}{\partial \tilde{x}} e^{-i\tilde{t}H^{(\lambda)}} \tilde{\phi} \right\|^2_{L^2(\mathbf{R})} = \lim_{\lambda\to\infty} \left[\left(\frac{\partial}{\partial \tilde{x}} e^{-i\tilde{t}H^{(\lambda)}} \tilde{\phi}, \frac{\partial}{\partial \tilde{x}} e^{-i\tilde{t}H^{(\lambda)}} \tilde{\phi} \right) + \left(V_\lambda e^{-i\tilde{t}H^{(\lambda)}} \tilde{\phi}, e^{-i\tilde{t}H^{(\lambda)}} \tilde{\phi} \right) \right] =$$
$$\lim_{\lambda\to\infty} \left\| \sqrt{H^{(\lambda)}} e^{-i\tilde{t}H^{(\lambda)}} \tilde{\phi} \right\|^2_{L^2(\mathbf{R})} = \lim_{\lambda\to\infty} \left\| \sqrt{H^{(\lambda)}} \tilde{\phi} \right\|^2_{L^2(\mathbf{R})} = \lim_{\lambda\to\infty} \left[\left(\frac{\partial}{\partial \tilde{x}} \tilde{\phi}, \frac{\partial}{\partial \tilde{x}} \tilde{\phi} \right) + \left(V_\lambda \tilde{\phi}, \tilde{\phi} \right) \right] =$$
$$\left\| \frac{\partial}{\partial \tilde{x}} \tilde{\phi} \right\|^2_{L^2(\mathbf{R})} = \left\| \frac{\partial}{\partial \tilde{x}} e^{i\tilde{t}\Delta} \tilde{\phi} \right\|^2_{L^2(\mathbf{R})}. \tag{3.40}$$

Furthermore, by (3.34), and (3.39) we have that $\frac{\partial}{\partial \tilde{x}} e^{-i\tilde{t}H^{(\lambda)}} \tilde{\phi}$ converges weakly in $L^2(\mathbf{R})$ to $\frac{\partial}{\partial \tilde{x}} e^{i\tilde{t}\Delta} \tilde{\phi}$, and since by (3.40) the norms converge, it follows that we have strong convergence in $L^2(\mathbf{R})$. Then, (3.34) holds in the strong convergence in $W_{1,2}(\mathbf{R})$ and by Sobolev's imbedding theorem, (3.34) also holds in the strong convergence in $L^q(\mathbf{R}), 1 \leq q \leq \infty$, and furthermore by (3.39),

$$\left\| e^{-i\tilde{t}H^{(\lambda)}} \tilde{\phi} \right\|_{L^q(\mathbf{R})} \leq C \left\| e^{-i\tilde{t}H^{(\lambda)}} \tilde{\phi} \right\|_{W_{1,2}(\mathbf{R})} \leq C, \lambda \geq 1, \tilde{t} \in \mathbf{R}, 1 \leq q \leq \infty. \tag{3.41}$$

It follows from (1.24) and (3.30) that,

$$\left\| e^{-i\tilde{t}H_\lambda} \tilde{\phi} \right\|^{p+1}_{L^{p+1}(\mathbf{R})} = \lambda \left\| e^{-itH_D} \psi_\lambda \right\|^{p+1}_{L^{p+1}(\mathbf{R})} \leq C \frac{1}{|t-\acute{t}|^{d(p+1)}} \lambda \left\| \hat{\phi} \right\|^{p+1}_{L^{1+1/p}} = C \frac{1}{\tilde{t}^{d(p+1)}} \left\| \tilde{\phi} \right\|^{p+1}_{L^{1+1/p}}, \tag{3.42}$$

where $d := \frac{1}{2}\frac{p-1}{p+1}$. The equation (1.23) follows from (3.32), (3.34) with the convergence in the $L^q, 1 \leq q \leq \infty$, norm , (3.41), (3.42) and the dominated convergence theorem. Note that $2(j_0+j+1) \geq p+1$, that $d(p+1) > 1$, that V_j is continuous, and that

$$\int \int_{\mathbf{R}\times\mathbf{R}} dt\, dx \left| e^{it\Delta} \tilde{\phi} \right|^{2(j_0+j+1)} = 2 \int \int_{\mathbf{R}\times\mathbf{R}^+} dt\, dx \left| e^{-itH_0} \phi \right|^{2(j_0+j+1)}. \tag{3.43}$$

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