MATRIX-VALUED GENERALIZATIONS OF THE THEOREMS OF BORG AND HOCHSTADT

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Dedicated with great pleasure to Jerry Goldstein and Rainer Nagel on the occasion of their 60th birthdays

Abstract. We prove a generalization of the well-known theorems by Borg and Hochstadt for periodic self-adjoint Schrödinger operators without a spectral gap, respectively, one gap in their spectrum, in the matrix-valued context. Our extension of the theorems of Borg and Hochstadt replaces the periodicity condition of the potential by the more general property of being reflectionless (the resulting potentials then automatically turn out to be periodic and we recover Després' matrix version of Borg's result). In addition, we assume the spectra to have uniform maximum multiplicity (a condition automatically fulfilled in the scalar context considered by Borg and Hochstadt). Moreover, the connection with the stationary matrix KdV hierarchy is established.

The methods employed in this paper rely on matrix-valued Herglotz functions, Weyl–Titchmarsh theory, pencils of matrices, and basic inverse spectral theory for matrix-valued Schrödinger operators.

1. Introduction

In a previous paper, [27], two of us constructed a class of self-adjoint $m \times m$ matrix-valued Schrödinger operators $H(\Sigma_n) = -d^2/dx^2 \mathcal{I}_m + \mathcal{Q}(\Sigma_n, \cdot)$ in $L^2(\mathbb{R})^{m\times m}$, $m \in \mathbb{N}$, with prescribed absolutely continuous finite-band spectrum Σ_n of the type

$$\Sigma_n = \left\{ \bigcup_{j=0}^{n-1} [E_{2j}, E_{2j+1}] \right\} \cup [E_{2n}, \infty), \quad n \in \mathbb{N}_0 \tag{1.1}$$

of uniform spectral multiplicity $2m$. Here

$$\{E_\ell\}_{0\le \ell\le 2n} \subseteq \mathbb{R},\ n \in \mathbb{N},\ \text{with } E_\ell < E_{\ell+1},\ 0 \le \ell \le 2n-1, \tag{1.2}$$

and hence $H(\Sigma_n)$ satisfies

$$\operatorname{spec}(H(\Sigma_n)) = \Sigma_n. \tag{1.3}$$

Throughout this paper all matrices will be considered over the field of complex numbers $\mathbb{C}$, and the corresponding linear space of $k \times \ell$ matrices will be denoted by $\mathbb{C}^{k\times\ell}$, $k, \ell \in \mathbb{N}$. Moreover, $\mathcal{I}_k$ denotes the identity matrix in $\mathbb{C}^{k\times k}$ for $k \in \mathbb{N}$, $\mathcal{M}^*$ the adjoint (i.e., complex conjugate transpose), $\mathcal{M}^t$ the transpose of a matrix $\mathcal{M}$, $\operatorname{diag}(m_1, \dots, m_k) \in \mathbb{C}^{k\times k}$ a diagonal $k \times k$ matrix, and $\mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$ denotes the set of locally absolutely continuous functions on $\mathbb{R}$. The spectrum, point spectrum

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(the set of eigenvalues), essential spectrum, absolutely continuous spectrum, and singularly continuous spectrum of a self-adjoint linear operator T in a separable complex Hilbert space are denoted by $\operatorname{spec}(T)$, $\operatorname{spec}_{\mathrm{p}}(T)$, $\operatorname{spec}_{\mathrm{ess}}(T)$, $\operatorname{spec}_{\mathrm{ac}}(T)$, and $\operatorname{spec}_{\mathrm{sc}}(T)$, respectively.

The constructed matrix potentials $\mathcal{Q}_{\Sigma_n}$ (resp., $H(\Sigma_n)$) turns out to be *reflectionless* in the sense discussed in [16], [29], and [47] (cf. [17], [18], [46], [76] in the scalar context $m = 1$), that is, the half-line Weyl–Titchmarsh matrices $\mathcal{M}_\pm(\Sigma_n, z, x)$ associated with $H(\Sigma_n)$, the half-lines $[x, \pm\infty)$, and a Dirichlet boundary condition at $x \in \mathbb{R}$, satisfy

$$\lim_{\varepsilon\downarrow 0} \mathcal{M}_+(\Sigma_n, \lambda + i\varepsilon, x) = \lim_{\varepsilon\downarrow 0} \mathcal{M}_-(\Sigma_n, \lambda - i\varepsilon, x), \tag{1.4}$$
$$\lambda \in \bigcup_{j=0}^{n-1} (E_{2j}, E_{2j+1}) \cup (E_{2n}, \infty), \ x \in \mathbb{R}.$$

Especially, $\mathcal{M}_+(\Sigma_n, \cdot, x)$ is the analytic continuation of $\mathcal{M}_-(\Sigma_n, \cdot, x)$ through the set Σ_n, and vice versa. In other words, $\mathcal{M}_+(\Sigma_n, \cdot, x)$ and $\mathcal{M}_-(\Sigma_n, \cdot, x)$ are the two branches of an analytic matrix-valued function $\mathcal{M}(\Sigma_n, \cdot, x)$ on the two-sheeted Riemann surface of $\big(\prod_{\ell=0}^{2n}(z - E_\ell)\big)^{1/2}$. The reflectionless property (1.4) then implies the absolute continuity of the spectrum of H_{Σ_n} and its uniform (maximal) multiplicity $2m$.

In this sequel of paper [27], we focus on the two special cases $n = 0$ and $n = 1$ and derive matrix-valued extensions of the well-known theorems by Borg [3] and Hochstadt [35], respectively. Before describing our principal new results and the contents of each section, we briefly recall the classical results by Borg and Hochstadt in the scalar case $m = 1$.

In 1946 Borg [3] proved, among a variety of other inverse spectral theorems, the following result.

Theorem 1.1 ([3]). *Let $\Sigma_0 = [E_0, \infty)$ for some $E_0 \in \mathbb{R}$ and $q_0 \in L^1_{\mathrm{loc}}(\mathbb{R})$ be real-valued and periodic. Suppose that $h_0 = -\frac{d^2}{dx^2} + q_0$ is the associated self-adjoint Schrödinger operator in $L^2(\mathbb{R})$ (cf. (1.9) for $m = 1$) and assume that*

$$\operatorname{spec}(h_0) = \Sigma_0. \tag{1.5}$$

Then

$$q_0(x) = E_0 \text{ for a.e. } x \in \mathbb{R}. \tag{1.6}$$

Traditionally, uniqueness results such as Theorem 1.1 are called Borg-type theorems. (However, this terminology is not uniquely adopted and hence a bit unfortunate. Indeed, inverse spectral results on finite intervals recovering the potential coefficient(s) from several spectra, were also pioneered by Borg in his celebrated paper [3], and hence are also coined Borg-type theorems in the literature, see, e.g., [55, Sect. 6].) Actually, Borg assumed $q_0 \in L^2_{\mathrm{loc}}(\mathbb{R})$, but that seems a minor detail.

Remark 1.2.
(i) A closer examination of the short proof of (an extension of) Theorem 1.1 provided in [16] shows that periodicity of q_0 is not the point for the uniqueness result (1.6). The key ingredient (besides $\operatorname{spec}(h_0) = [E_0, \infty)$ and q_0 real-valued) is clearly the fact that q_0 is reflectionless in the sense of (1.4).
(ii) Real-valued periodic potentials are known to satisfy (1.4), but so are certain classes of real-valued quasi-periodic and almost-periodic potentials q_0 (see, e.g., [17],

[18], [43], [44], [45], [46], [47], [76]). *In particular, the class of real-valued algebro-geometric finite-gap potentials* q_0 (*a subclass of the set of real-valued quasi-periodic potentials*) *is a prime example satisfying* (1.4) *without necessarily being periodic.*
(*iii*) *We note that real-valuedness of* q_0 *is an essential assumption in Theorem* 1.1. *Indeed, it is well-known that* $q(x) = \exp(ix)$, $x \in \mathbb{R}$, *leads to the half-line spectrum* $[0, \infty)$. *A detailed treatment of a class of examples of this type can be found in* [21], [22], [31], [63], [64]. *Moreover, the example of complete exponential localization of the spectrum of a discrete Schrödinger operator with a quasi-periodic real-valued potential having two basic frequencies and no gaps in its spectrum* [12] *illustrates the importance of the reflectionless property of* q_0 *in Theorem* 1.1.

Next we recall Hochstadt's theorem [35] from 1965.

Theorem 1.3 ([35]). *Let* $\Sigma_1 = [E_0, E_1] \cup [E_2, \infty)$ *for some* $E_0 < E_1 < E_2$ *and* $q_1 \in L^1_{\text{loc}}(\mathbb{R})$ *be real-valued and periodic. Suppose that* $h_1 = -\frac{d^2}{dx^2} + q_1$ *is the associated self-adjoint Schrödinger operator in* $L^2(\mathbb{R})$ (*cf.* (1.9) *for* $m = 1$) *and assume that*

$$\operatorname{spec}(h_1) = \Sigma_1. \tag{1.7}$$

Then

$$q_1(x) = C_0 + 2\wp(x + \omega_3 + \alpha) \text{ for some } \alpha \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}. \tag{1.8}$$

Here $\wp(\cdot) = \wp(\cdot; \omega_1; \omega_3)$ denotes the elliptic Weierstrass function with half-periods $\omega_1 > 0$ and $-i\omega_3 > 0$ (cf. [1, Ch. 18]).

Remark 1.4. *Again it will turn out that periodicity of* q_1 *is not the point for the uniqueness result* (1.8). *The key ingredient* (*besides* $\operatorname{spec}(h_1) = \Sigma_1$ *and* q_1 *real-valued*) *is again the fact that* q_1 *is reflectionless in the sense of* (1.4). *Similarly, Remarks* 1.2 (*ii*), (*iii*) *apply of course in the present context.*

The principal results of this paper then read as follows.

Theorem 1.5. *Let* $m \in \mathbb{N}$, *suppose* $\mathcal{Q}_\ell = \mathcal{Q}_\ell^* \in L^1_{\text{loc}}(\mathbb{R})^{m\times m}$ *and assume that the differential expressions* $-\mathcal{I}_m \frac{d^2}{dx^2} + \mathcal{Q}_\ell$, $\ell = 0, 1$, *are in the limit point case at* $\pm\infty$. *Define the self-adjoint Schrödinger operators* H_ℓ *in* $L^2(\mathbb{R})^{m\times m}$

$$\begin{aligned} &H_\ell = -\mathcal{I}_m \frac{d^2}{dx^2} + \mathcal{Q}_\ell, \quad \ell = 0, 1, \\ &\operatorname{dom}(H_\ell) = \{g \in L^2(\mathbb{R})^m \mid g, g' \in \text{AC}_{\text{loc}}(\mathbb{R})^m;\, (-g'' + \mathcal{Q}_\ell g) \in L^2(\mathbb{R})^m\} \end{aligned} \tag{1.9}$$

and assume that $\mathcal{Q}_\ell$ *is reflectionless* (*cf.* (1.4)).
(*i*) *Let* $\Sigma_0 = [E_0, \infty)$ *for some* $E_0 \in \mathbb{R}$ *and suppose that* H_0 *has spectrum*

$$\operatorname{spec}(H_0) = \Sigma_0. \tag{1.10}$$

Then

$$\mathcal{Q}_0(x) = E_0\, \mathcal{I}_m \text{ for a.e. } x \in \mathbb{R}. \tag{1.11}$$

(*ii*) *Let* $\Sigma_1 = [E_0, E_1] \cup [E_2 \infty)$ *for some* $E_0 < E_1 < E_2$ *and suppose that* H_1 *has spectrum*

$$\operatorname{spec}(H_1) = \Sigma_1. \tag{1.12}$$

Then

$$\begin{aligned}\mathcal{Q}_1(x) &= (1/3)(E_0+E_1+E_2)\mathcal{I}_m \\ &\quad + 2\mathcal{U}\,\mathrm{diag}(\wp(x+\omega_3+\alpha_1),\dots,\wp(x+\omega_3+\alpha_m))\mathcal{U}^{-1} \\ &\quad \text{for some } \alpha_j\in\mathbb{R},\ 1\le j\le m \text{ and a.e. } x\in\mathbb{R},\end{aligned} \tag{1.13}$$

where $\mathcal{U}$ is an $m\times m$ unitary matrix independent of $x\in\mathbb{R}$. In particular, $\mathcal{Q}_1$ satisfies the stationary KdV equation

$$\mathcal{Q}_1''' - 3(\mathcal{Q}_1^2)' + 2(E_0+E_1+E_2)\mathcal{Q}_1' = 0. \tag{1.14}$$

As shown in [16], periodic Schrödinger operators in $L^2(\mathbb{R})^{m\times m}$ with spectra of uniform (maximal) multiplicity $2m$ are reflectionless in the sense that (1.4) holds for all λ in the open interior of the spectrum. Hence one obtains the following result.

Theorem 1.6. *Let $m\in\mathbb{N}$, suppose $\mathcal{Q}_\ell=\mathcal{Q}_\ell^*\in L^1_{\mathrm{loc}}(\mathbb{R})^{m\times m}$ is periodic and define the self-adjoint Schrödinger operators H_ℓ, $\ell=0,1$ in $L^2(\mathbb{R})^{m\times m}$ as in* (1.9)*. Assume*

$$\mathrm{spec}(H_\ell)=\Sigma_\ell,\quad \ell=0,1, \tag{1.15}$$

and suppose that H_ℓ, $\ell=0,1$ has uniform (maximal) spectral multiplicity $2m$. Then $\mathcal{Q}_\ell$, $\ell=0,1$, are reflectionless and hence the assertions (1.11), (1.13), *and* (1.14) *of Theorem* 1.5 *hold.*

Remark 1.7.
(*i*) *The assumption of uniform (maximal) spectral multiplicity* $2m$ *in Theorem* 1.6 (*i*) *is an essential one. Otherwise, one can easily construct nonconstant potentials $\mathcal{Q}$ such that the associated Schrödinger operator $H_{\mathcal{Q}}$ has overlapping band spectra and hence spectrum equal to a half-line. For such a construction it suffices to consider the case in which $\mathcal{Q}$ is a diagonal matrix. In the special scalar case $m=1$, reflectionless potentials automatically give rise to maximum uniform spectral multiplicity* 2 *for the corresponding scalar Schrödinger operator in $L^2(\mathbb{R})$.*
(*ii*) *Theorem* 1.6 (*i*) *assuming $\mathcal{Q}_0\in L^\infty(\mathbb{R})^{m\times m}$ to be periodic has been proved by Deprés* [19] *using an entirely different approach based on a detailed Floquet analysis. Deprés' result was reproved in* [16] *under the current general assumptions on $\mathcal{Q}_0$ using methods based on matrix-valued Herglotz functions and trace formulas.*
(*iii*) *For different proofs of Borg's Theorem* 1.1 *in the scalar case $m=1$ we refer to* [35], [39], [40], [43].
(*iv*) *Without loss of generality we focus on the limit point case of the differential expression $-\mathcal{I}_m\frac{d^2}{dx^2}+\mathcal{Q}$ at $\pm\infty$ in this paper. In fact, by a result originally due to Povzner* [65]*, scalar Schrödinger differential expressions leading to minimal operators bounded from below are in the limit point case at $\pm\infty$. Povzner's result was later also proved by Wienholtz* [78] *and is reproduced as Theorem 35 in* [30, p. 58]. *As shown in* [15]*, Wienholtz's proof extends to the matrix case at hand. Since the spectra Σ_ℓ, $\ell=0,1$ are bounded from below, the limit point assumption is justified (and natural).*

This paper is another modest contribution to the inverse spectral theory of matrix-valued Schrödinger (and Dirac-type) operators and part of a recent program in this area (cf. [13], [14], [15], [16], [25], [26], [27], [28], and [29]). For other relevent recent literature in this context we refer, for instance, to [5], [6], [7], [8],

[9], [10], [11], [37], [38], [41], [56], [57], [69], [73], [74], [75]. For the applicability of this circle of ideas to the nonabelian Korteweg–de Vries hierarchy we refer to [27] and the references cited therein.

In Section 2 we recall basic facts on Weyl–Titchmarsh theory and pencils of matrices as needed in the remainder of this paper. Section 3 summarizes the principal results of paper [27]. Finally, in Section 4 we present the matrix extensions of Borg's and Hochstadt's theorem and the corresponding connection with the stationary KdV hierarchy.

2. Preliminaries on Weyl–Titchmarsh Theory and Pencils of Matrices

The basic assumption for this section will be the following.

Hypothesis 2.1. *Fix $m \in \mathbb{N}$, suppose $\mathcal{Q} = \mathcal{Q}^* \in L^1_{\text{loc}}(\mathbb{R})^{m\times m}$, introduce the differential expression*

$$\mathcal{L} = -\mathcal{I}_m \frac{d^2}{dx^2} + \mathcal{Q}, \quad x \in \mathbb{R}. \tag{2.1}$$

and suppose $\mathcal{L}$ is in the limit point case at $\pm\infty$.

Given Hypothesis 2.1 we consider the matrix-valued Schrödinger equation

$$-\psi''(z,x) + \mathcal{Q}(x)\psi(z,x) = z\psi(z,x) \text{ for a.e. } x \in \mathbb{R}, \tag{2.2}$$

where $z \in \mathbb{C}$ plays the role of a spectral parameter and ψ is assumed to satisfy

$$\psi(z,\cdot), \psi'(z,\cdot) \in \text{AC}_{\text{loc}}(\mathbb{R})^{m\times m}. \tag{2.3}$$

Throughout this paper, x-derivatives are abbreviated by a prime $'$.

Let $\Psi(z,x,x_0)$ be a $2m \times 2m$ normalized fundamental system of solutions of (2.2) at some $x_0 \in \mathbb{R}$ which we partition as

$$\Psi(z,x,x_0) = \begin{pmatrix} \theta(z,x,x_0) & \phi(z,x,x_0) \\ \theta'(z,x,x_0) & \phi'(z,x,x_0) \end{pmatrix}. \tag{2.4}$$

Here $'$ denotes d/dx, $\theta(z,x,x_0)$ and $\phi(z,x,x_0)$ are $m \times m$ matrices, entire with respect to $z \in \mathbb{C}$, and normalized according to $\Psi(z,x_0,x_0) = \mathcal{I}_{2m}$.

By Hypothesis 2.1, the $m\times m$ Weyl–Titchmarsh matrices associated with $\mathcal{L}$, the half-lines $[x,\pm\infty)$, and a Dirichlet boundary condition at x, are given by (c.f. [32], [33], [34], [42], [62], [70], [71], [72])

$$\mathcal{M}_\pm(z,x) = \Psi'_\pm(z,x,x_0)\Psi_\pm(z,x,x_0)^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{2.5}$$

where $\Psi_\pm(z,\cdot,x_0)$ satisfy $(\mathcal{L} - z\mathcal{I}_m)\Psi_\pm(z,\cdot,x_0) = 0$ and

$$\Psi_\pm(z,\cdot,x_0) \in L^2([x_0,\pm\infty))^{m\times m}. \tag{2.6}$$

Next, we recall the definition of matrix-valued Herglotz function.

Definition 2.2. *A map $\mathcal{M}\colon \mathbb{C}_+ \to \mathbb{C}^{n\times n}$, $n \in \mathbb{N}$, extended to $\mathbb{C}_-$ by $\mathcal{M}(\bar z) = \mathcal{M}(z)^*$ for all $z \in \mathbb{C}_+$, is called an $n \times n$ Herglotz matrix if it is analytic on $\mathbb{C}_+$ and $\operatorname{Im}(\mathcal{M}(z)) \geq 0$ for all $z \in \mathbb{C}_+$.*

Here we denote $\operatorname{Im}(\mathcal{M}) = (\mathcal{M} - \mathcal{M}^*)/2i$ and $\operatorname{Re}(\mathcal{M}) = (\mathcal{M} + \mathcal{M}^*)/2$.

$\pm\mathcal{M}_\pm(\cdot,x)$ are $m \times m$ Herglotz matrices of rank m and hence admit the representations

$$\pm\mathcal{M}_\pm(z,x) = \operatorname{Re}(\pm\mathcal{M}_\pm(\pm i,x)) + \int_{\mathbb{R}} d\Omega_\pm(\lambda,x)\left((\lambda - z)^{-1} - \lambda(1+\lambda^2)^{-1}\right), \tag{2.7}$$

where

$$\int_{\mathbb{R}} \|d\Omega_\pm(\lambda,x)\|_{\mathbb{C}^{m\times m}} \, (1+\lambda^2)^{-1} < \infty \tag{2.8}$$

and

$$\Omega_\pm((\lambda,\mu],x) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\int_{\lambda+\delta}^{\mu+\delta} d\nu \, \mathrm{Im}(\pm\mathcal{M}_\pm(\nu+i\varepsilon,x)). \tag{2.9}$$

Necessary and sufficient conditions for $\mathcal{M}_\pm(\cdot,x_0)$ to be the half-line $m\times m$ Weyl–Titchmarsh matrix associated with a Schrödinger operator on $[x_0,\pm\infty)$ in terms of the corresponding measures $\Omega_\pm(\cdot,x_0)$ in the Herglotz representation (2.7) of $\mathcal{M}_\pm(\cdot,x_0)$ can be derived using the matrix-valued extension of the classical inverse spectral theory approach due to Gelfand and Levitan [24], as worked out by Rofe-Beketov [66]. The following result describes sufficient conditions for a monotonically nondecreasing matrix function to be the matrix spectral function of a half-line Schrödinger operator. It extends well-known results in the scalar case $m=1$ (cf. [52, Sects. 2.5, 2.9], [53], [61, Sect. 26.5], [77]).

Theorem 2.3 ([66]). *Suppose $\Omega_+(\cdot,x_0)$ is a monotonically nondecreasing $m\times m$ matrix-valued function on $\mathbb{R}$. Then $\Omega_+(\cdot,x_0)$ is the matrix spectral function of a self-adjoint Schrödinger operator H_+ in $L^2([x_0,\infty))^m$ associated with the $m\times m$ matrix-valued differential expression $\mathcal{L}_+ = -d^2/dx^2\mathcal{I}_m+\mathcal{Q}$, $x>x_0$, with a Dirichlet boundary condition at x_0, a self-adjoint boundary condition at ∞ (if necessary), and a self-adjoint potential matrix $\mathcal{Q}$ with $\mathcal{Q}^{(r)}\in L^1([x_0,R])^{m\times m}$ for all $R>x_0$ if and only if the following two conditions hold.*
(i) Whenever $f\in C([x_0,\infty))^{m\times 1}$ with compact support contained in $[x_0,\infty)$ and

$$\int_{\mathbb{R}} F(\lambda)^* d\Omega_+(\lambda,x_0)\, F(\lambda) = 0, \ \text{ then } f=0 \ \text{ a.e.}, \tag{2.10}$$

where

$$F(\lambda) = \lim_{R\uparrow\infty}\int_{x_0}^{R} dx\, \frac{\sin(\lambda^{1/2}(x-x_0))}{\lambda^{1/2}} f(x), \quad \lambda\in\mathbb{R}. \tag{2.11}$$

(ii) Define

$$\widetilde\Omega_+(\lambda,x_0) = \begin{cases} \Omega_+(\lambda,x_0) - \frac{2}{3\pi}\lambda^{3/2}, & \lambda\geq 0 \\ \Omega_+(\lambda,x_0), & \lambda<0 \end{cases} \tag{2.12}$$

and assume the limit

$$\lim_{R\uparrow\infty}\int_{-\infty}^{R} d\widetilde\Omega_+(\lambda,x_0)\, \frac{\sin(\lambda^{1/2}(x-x_0))}{\lambda^{1/2}} = \Phi(x) \tag{2.13}$$

exists and $\Phi\in L^\infty([x_0,R])^{m\times m}$ for all $R>x_0$. Moreover, suppose that for some $r\in\mathbb{N}_0$, $\Phi^{(r+1)}\in L^1([x_0,R])^{m\times m}$ for all $R>x_0$, and $\Phi(x_0)=0$.

Assuming Hypothesis 2.1, we next introduce the self-adjoint Schrödinger operator H in $L^2(\mathbb{R})^m$ by

$$\begin{aligned} &H = -\mathcal{I}_m\frac{d^2}{dx^2}+\mathcal{Q}, \\ &\mathrm{dom}(H) = \{g\in L^2(\mathbb{R})^m \mid g,g'\in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^m;\, (-g''+\mathcal{Q}g)\in L^2(\mathbb{R})^m\}. \end{aligned} \tag{2.14}$$

The resolvent of H then reads

$$((H-z)^{-1}f)(x) = \int_{\mathbb{R}} dx'\, \mathcal{G}(z,x,x')f(x'), \quad z\in\mathbb{C}\backslash\mathbb{R},\ f\in L^2(\mathbb{R})^m, \tag{2.15}$$

with the Green's matrix $\mathcal{G}(z,x,x')$ of H given by

$$\mathcal{G}(z,x,x') = \Psi_{\mp}(z,x,x_0)[\mathcal{M}_-(z,x_0)-\mathcal{M}_+(z,x_0)]^{-1}\Psi_{\pm}(\overline{z},x',x_0)^*, \\ x \lesseqgtr x',\ z\in\mathbb{C}\backslash\mathbb{R}. \tag{2.16}$$

Introducing

$$\mathcal{N}_{\pm}(z,x) = \mathcal{M}_-(z,x)\pm\mathcal{M}_+(z,x), \quad z\in\mathbb{C}\backslash\mathbb{R},\ x\in\mathbb{R}, \tag{2.17}$$

the $2m\times 2m$ Weyl–Titchmarsh function $\mathcal{M}(z,x)$ associated with H on $\mathbb{R}$ is then given by

$$\begin{aligned}\mathcal{M}(z,x) &= \big(\mathcal{M}_{p,q}(z,x)\big)_{p,q=1,2} \\ &= \begin{pmatrix} \mathcal{M}_{\pm}(z,x)\mathcal{N}_-(z,x)^{-1}\mathcal{M}_{\mp}(z,x) & \mathcal{N}_-(z,x)^{-1}\mathcal{N}_+(z,x)/2 \\ \mathcal{N}_+(z,x)\mathcal{N}_-(z,x)^{-1}/2 & \mathcal{N}_-(z,x)^{-1}\end{pmatrix}, \\ &\qquad z\in\mathbb{C}\backslash\mathbb{R},\ x\in\mathbb{R}.\end{aligned} \tag{2.18}$$

Then $\mathcal{M}(z,x)$ is a $2m\times 2m$ matrix-valued Herglotz function of rank $2m$ with representations

$$\mathcal{M}(z,x) = \mathrm{Re}(\mathcal{M}(i,x)) + \int_{\mathbb{R}} d\Omega(\lambda,x)\,\big((\lambda-z)^{-1}-\lambda(1+\lambda^2)^{-1}\big) \tag{2.19}$$

$$= \exp\left(\mathcal{C}(x) + \int_{\mathbb{R}} d\lambda\,\Upsilon(\lambda,x)\big((\lambda-z)^{-1}-\lambda(1+\lambda^2)^{-1}\big)\right), \tag{2.20}$$

where

$$\int_{\mathbb{R}} \|d\Omega(\lambda,x)\|_{\mathbb{C}^{2m\times 2m}}\,(1+\lambda^2)^{-1}<\infty, \tag{2.21}$$

$$\mathcal{C}(x)=\mathcal{C}(x)^*, \quad 0\le\Upsilon(\cdot,x)\le\mathcal{I}_{2m} \text{ a.e.} \tag{2.22}$$

and

$$\Omega((\lambda,\mu],x) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda+\delta}^{\mu+\delta} d\nu\,\mathrm{Im}(\mathcal{M}(\nu+i\varepsilon,x)), \tag{2.23}$$

$$\Upsilon(\lambda,x) = \lim_{\varepsilon\downarrow 0}\pi^{-1}\mathrm{Im}(\ln(\mathcal{M}(\lambda+i\varepsilon,x))) \text{ for a.e. } \lambda\in\mathbb{R}. \tag{2.24}$$

The Herglotz, and particularly exponential Herglotz property (cf. [2], [4], [29]) of the diagonal Green's function of H,

$$\mathfrak{g}(z,x) = \mathcal{G}(z,x,x), \quad z\in\mathbb{C}\backslash\mathrm{spec}(H),\ x\in\mathbb{R}, \tag{2.25}$$

will be of particular importance in Section 4 and hence we note for subsequent purpose,

$$\mathfrak{g}(z,x) = \exp\left(\mathfrak{C}(x) + \int_{\mathbb{R}} d\lambda\,\Xi(\lambda,x)\big((\lambda-z)^{-1}-\lambda(1+\lambda^2)^{-1}\big)\right), \tag{2.26}$$

where

$$\mathfrak{C}(x)=\mathfrak{C}(x)^*, \quad 0\le\Xi(\cdot,x)\le\mathcal{I}_m \text{ a.e.}, \tag{2.27}$$

$$\Xi(\lambda,x) = \lim_{\varepsilon\downarrow 0}\pi^{-1}\mathrm{Im}(\ln(\mathfrak{g}(\lambda+i\varepsilon,x))) \text{ for a.e. } \lambda\in\mathbb{R}. \tag{2.28}$$

We also recall the following characterization of $\mathcal{M}(z,x_0)$ to be used later. In the scalar context $m=1$, this characterization has been used by Rofe-Beketov [67], [68] (see also [52, Sect. 7.3]).

Theorem 2.4 ([67], [68]). *Assume Hypothesis* 2.1, *suppose that* $z\in\mathbb{C}\backslash\mathbb{R}$, $x_0\in\mathbb{R}$, *and let* $\ell,r\in\mathbb{N}_0$. *Then the following assertions are equivalent.*
(i) $\mathcal{M}(z,x_0)$ *is the* $2m\times 2m$ *Weyl–Titchmarsh matrix associated with a Schrödinger operator* H *in* $L^2(\mathbb{R})^m$ *of the type* (2.14) *with an* $m\times m$ *matrix-valued potential* $\mathcal{Q}\in L^1_{\text{loc}}(\mathbb{R})$ *and* $\mathcal{Q}\in C^\ell((-\infty,x_0))$ *and* $\mathcal{Q}\in C^r((x_0,\infty))$.
(ii) $\mathcal{M}(z,x_0)$ *is of the type* (2.18) *with* $\mathcal{M}_\pm(z,x_0)$ *being half-line* $m\times m$ *Weyl–Titchmarsh matrices on* $[x_0,\pm\infty)$ *corresponding to a Dirichlet boundary condition at* x_0 *and a self-adjoint boundary condition at* $-\infty$ *and/or* ∞ *(if any) which are associated with an* $m\times m$ *matrix-valued potential* $\mathcal{Q}$ *satisfying* $\mathcal{Q}\in C^\ell((-\infty,x_0))$ *and* $\mathcal{Q}\in C^r((x_0,\infty))$, *respectively.*
If (i) or (ii) holds, then the $2m\times 2m$ *matrix-valued spectral measure* $\Omega(\cdot,x_0)$ *associated with* $\mathcal{M}(z,x_0)$ *is determined by* (2.18) *and* (2.23).

Next, we consider variations of the reference point $x\in\mathbb{R}$. Since $\Psi_\pm$ satisfies the second-order linear $m\times m$ matrix-valued differential equation (2.2), $\mathcal{M}_\pm$ in (2.5) satisfies the matrix-valued Riccati-type equation (independently of any limit point assumptions at $\pm\infty$)

$$\mathcal{M}_\pm'(z,x)+\mathcal{M}_\pm(z,x)^2=\mathcal{Q}(x)-z\mathcal{I}_m,\quad x\in\mathbb{R},\ z\in\mathbb{C}\backslash\mathbb{R}. \tag{2.29}$$

The asymptotic high-energy behavior of $\mathcal{M}_\pm(z,x)$ as $|z|\to\infty$ has recently been determined in [13] under minimal smoothness conditions on $\mathcal{Q}$ and without assuming that $\mathcal{L}$ is in the limit point case at $\pm\infty$. Here we recall just a special case of the asymptotic expansion proved in [13] which is most suited for our discussion at hand. We denote by $C_\varepsilon\subset\mathbb{C}_+$ the open sector with vertex at zero, symmetry axis along the positive imaginary axis, and opening angle ε, with $0<\varepsilon<\pi/2$.

Theorem 2.5 ([13]). *Fix* $x_0\in\mathbb{R}$ *and let* $x\geq x_0$. *In addition to Hypothesis* 2.1 *suppose that* $\mathcal{Q}\in C^\infty([x_0,\pm\infty))^{m\times m}$ *and that* $\mathcal{L}$ *is in the limit point case at* $\pm\infty$. *Let* $\mathcal{M}_\pm(z,x)$, $x\geq x_0$, *be defined as in* (2.5). *Then, as* $|z|\to\infty$ *in* C_ε, $\mathcal{M}_\pm(z,x)$ *has an asymptotic expansion of the form* ($\operatorname{Im}(z^{1/2})>0$, $z\in\mathbb{C}_+$)

$$\mathcal{M}_\pm(z,x)\underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=}\pm i\mathcal{I}_m z^{1/2}+\sum_{k=1}^N\mathcal{M}_{\pm,k}(x)z^{-k/2}+o(|z|^{-N/2}),\quad N\in\mathbb{N}. \tag{2.30}$$

The expansion (2.30) *is uniform with respect to* $\arg(z)$ *for* $|z|\to\infty$ *in* C_ε *and uniform in* x *as long as* x *varies in compact subsets of* $[x_0,\infty)$. *The expansion coefficients* $\mathcal{M}_{\pm,k}(x)$ *can be recursively computed from*

$$\begin{aligned}
&\mathcal{M}_{\pm,1}(x)=\mp\frac{i}{2}\mathcal{Q}(x),\quad \mathcal{M}_{\pm,2}(x)=\frac{1}{4}\mathcal{Q}'(x),\\
&\mathcal{M}_{\pm,k+1}(x)=\pm\frac{i}{2}\bigg(\mathcal{M}_{\pm,k}'(x)+\sum_{\ell=1}^{k-1}\mathcal{M}_{\pm,\ell}(x)\mathcal{M}_{\pm,k-\ell}(x)\bigg),\quad k\geq 2.
\end{aligned} \tag{2.31}$$

The asymptotic expansion (2.30) *can be differentiated to any order with respect to* x.

If one only assumes Hypothesis 2.1 (*i.e.,* $\mathcal{Q} \in L^1([x_0,R])^{m\times m}$ *for all* $R > x_0$), *then*

$$\mathcal{M}_\pm(z,x) \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} \pm i\mathcal{I}_m z^{1/2} + o(1). \tag{2.32}$$

Remark 2.6. *Due to the recursion relation* (2.31), *the coefficients* $\mathcal{M}_{\pm,k}$ *are universal polynomials in* $\mathcal{Q}$ *and its* x*-derivatives* (*i.e., differential polynomials in* $\mathcal{Q}$).

Finally, in addition to (2.16) (still assuming Hypothesis 2.1), one infers for the $2m\times 2m$ Weyl–Titchmarsh function $\mathcal{M}(z,x)$ associated with H on $\mathbb{R}$ in connection with arbitrary half-lines $[x,\pm\infty)$, $x\in\mathbb{R}$,

$$\mathcal{M}(z,x) = \big(\mathcal{M}_{j,j'}(z,x)\big)_{j,j'=1,2}, \quad z\in\mathbb{C}\backslash\mathbb{R}, \tag{2.33}$$

$$\begin{aligned}\mathcal{M}_{1,1}(z,x) &= \mathcal{M}_\pm(z,x)[\mathcal{M}_-(z,x)-\mathcal{M}_+(z,x)]^{-1}\mathcal{M}_\mp(z,x)\\ &= \psi_+'(z,x,x_0)[\mathcal{M}_-(z,x_0)-\mathcal{M}_+(z,x_0)]^{-1}\psi_-'(\overline{z},x,x_0)^*,\end{aligned} \tag{2.34}$$

$$\begin{aligned}\mathcal{M}_{1,2}(z,x) &= 2^{-1}[\mathcal{M}_-(z,x)-\mathcal{M}_+(z,x)]^{-1}[\mathcal{M}_-(z,x)+\mathcal{M}_+(z,x)]\\ &= \psi_+(z,x,x_0)[\mathcal{M}_-(z,x_0)-\mathcal{M}_+(z,x_0)]^{-1}\psi_-'(\overline{z},x,x_0)^*,\end{aligned} \tag{2.35}$$

$$\begin{aligned}\mathcal{M}_{2,1}(z,x) &= 2^{-1}[\mathcal{M}_-(z,x)+\mathcal{M}_+(z,x)][\mathcal{M}_-(z,x)-\mathcal{M}_+(z,x)]^{-1}\\ &= \psi_+'(z,x,x_0)[\mathcal{M}_-(z,x_0)-\mathcal{M}_+(z,x_0)]^{-1}\psi_-(\overline{z},x,x_0)^*,\end{aligned} \tag{2.36}$$

$$\begin{aligned}\mathcal{M}_{2,2}(z,x) &= [\mathcal{M}_-(z,x)-\mathcal{M}_+(z,x)]^{-1}\\ &= \psi_+(z,x,x_0)[\mathcal{M}_-(z,x_0)-\mathcal{M}_+(z,x_0)]^{-1}\psi_-(\overline{z},x,x_0)^*.\end{aligned} \tag{2.37}$$

Introducing the convenient abbreviation,

$$\mathcal{M}(z,x) = \begin{pmatrix} \mathfrak{h}(z,x) & -\mathfrak{g}_2(z,x) \\ -\mathfrak{g}_1(z,x) & \mathfrak{g}(z,x)\end{pmatrix}, \quad z\in\mathbb{C}\backslash\mathbb{R},\ x\in\mathbb{R}, \tag{2.38}$$

one then verifies from (2.33)–(2.38) and from $\mathcal{M}(\overline{z},x)^* = \mathcal{M}(z,x)$, $\mathcal{M}_\pm(\overline{z},x)^* = \mathcal{M}_\pm(z,x)$ that

$$\mathfrak{g}(\overline{z},x)^* = \mathfrak{g}(z,x), \quad \mathfrak{g}_2(\overline{z},x)^* = \mathfrak{g}_1(z,x), \quad \mathfrak{h}(\overline{z},x)^* = \mathfrak{h}(z,x), \tag{2.39}$$

$$\mathfrak{g}(z,x)\mathfrak{g}_1(z,x) = \mathfrak{g}_2(z,x)\mathfrak{g}(z,x), \tag{2.40}$$

$$\mathfrak{h}(z,x)\mathfrak{g}_2(z,x) = \mathfrak{g}_1(z,x)\mathfrak{h}(z,x), \tag{2.41}$$

$$\mathfrak{g}(z,x) = [\mathcal{M}_-(z,x)-\mathcal{M}_+(z,x)]^{-1}, \tag{2.42}$$

$$\mathfrak{g}(z,x)\mathfrak{h}(z,x) - \mathfrak{g}_2(z,x)^2 = -(1/4)\mathcal{I}_m, \tag{2.43}$$

$$\mathfrak{h}(z,x)\mathfrak{g}(z,x) - \mathfrak{g}_1(z,x)^2 = -(1/4)\mathcal{I}_m, \tag{2.44}$$

$$\mathcal{M}_\pm(z,x) = \mp(1/2)\mathfrak{g}(z,x)^{-1} - \mathfrak{g}(z,x)^{-1}\mathfrak{g}_2(z,x) \tag{2.45}$$

$$\mp(1/2)\mathfrak{g}(z,x)^{-1} - \mathfrak{g}_1(z,x)\mathfrak{g}(z,x)^{-1}, \tag{2.46}$$

assuming Hypothesis 2.1. Moreover, (2.39)–(2.46) and the Riccati-type equations (2.29) imply the following results for $z \in \mathbb{C}\backslash\mathbb{R}$ and a.e. $x \in \mathbb{R}$,

$$\mathfrak{g}' = -(\mathfrak{g}_1 + \mathfrak{g}_2), \tag{2.47}$$

$$\mathfrak{g}_1' = -(\mathcal{Q} - z\mathcal{I}_m)\mathfrak{g} - \mathfrak{h} \tag{2.48}$$

$$= (-\mathfrak{g}'' + \mathfrak{g}\mathcal{Q} - \mathcal{Q}\mathfrak{g})/2, \tag{2.49}$$

$$\mathfrak{g}_2' = -\mathfrak{g}(\mathcal{Q} - z\mathcal{I}_m) - \mathfrak{h} \tag{2.50}$$

$$= (-\mathfrak{g}'' + \mathcal{Q}\mathfrak{g} - \mathfrak{g}\mathcal{Q})/2, \tag{2.51}$$

$$\mathfrak{h}' = -\mathfrak{g}_1(\mathcal{Q} - z\mathcal{I}_m) - (\mathcal{Q} - z\mathcal{I}_m)\mathfrak{g}_2, \tag{2.52}$$

$$\mathfrak{h} = [\mathfrak{g}'' - \mathfrak{g}(\mathcal{Q} - z\mathcal{I}_m) - (\mathcal{Q} - z\mathcal{I}_m)\mathfrak{g}]/2 \tag{2.53}$$

if $\mathcal{Q} \in L^1_{\text{loc}}(\mathbb{R})^{m\times m}$, and

$$\mathfrak{g}_1'' = -2(\mathcal{Q} - z\mathcal{I}_m)\mathfrak{g}' - \mathcal{Q}'\mathfrak{g} + \mathfrak{g}_1\mathcal{Q} - \mathcal{Q}\mathfrak{g}_1, \tag{2.54}$$

$$\mathfrak{g}_2'' = -2\mathfrak{g}'(\mathcal{Q} - z\mathcal{I}_m) - \mathfrak{g}\mathcal{Q}' + \mathcal{Q}\mathfrak{g}_2 - \mathfrak{g}_2\mathcal{Q} \tag{2.55}$$

if in addition $\mathcal{Q}' \in L^1_{\text{loc}}(\mathbb{R})^{m\times m}$.

We conclude this section by recalling the definition of reflectionless matrix-valued potentials as discussed in [16], [26], [29], and [47]. We follow the corresponding notion introduced in connection with scalar Schrödinger operators and refer to [17], [18], [46], [76] for further details in this context.

Definition 2.7. *Assume Hypothesis* 2.1 *and define the self-adjoint Schrödinger operator H in $L^2(\mathbb{R})^{m\times m}$ as in* (2.14). *Suppose that* $\text{spec}_{\text{ess}}(H) \neq \emptyset$ *and let* Ξ *be defined by* (2.28). *Then* $\mathcal{Q}$ *is called reflectionless if for all* $x \in \mathbb{R}$,

$$\Xi(\lambda, x) = (1/2)\mathcal{I}_m \text{ for a.e. } \lambda \in \text{spec}_{\text{ess}}(H). \tag{2.56}$$

Explicit examples of reflectionless potentials will be discussed in Section 3. If $\mathcal{Q}$ is reflectionless we will sometimes slightly abuse notation and also call the corresponding Schrödinger operator H in $L^2(\mathbb{R})^{m\times m}$ reflectionless.

3. A Class of Matrix-Valued Schrödinger Operators with Prescribed Finite-Band Spectra

Given the preliminaries of Section 2, we now recall the construction of a class of matrix-valued Schrödinger operators with a prescribed finite-band spectrum of uniform maximum multiplicity, the principal result of [27]. Let

$$\{E_\ell\}_{0\leq\ell\leq 2n} \subseteq \mathbb{R},\ n \in \mathbb{N},\ \text{ with } E_\ell < E_{\ell+1},\ 0 \leq \ell \leq 2n-1, \tag{3.1}$$

and introduce the polynomial

$$R_{2n+1}(z) = \prod_{\ell=0}^{2n}(z - E_\ell), \quad z \in \mathbb{C}. \tag{3.2}$$

Moreover, we define the square root of R_{2n+1} by

$$R_{2n+1}(\lambda)^{1/2} = \lim_{\varepsilon\downarrow 0} R_{2n+1}(\lambda + i\varepsilon)^{1/2}, \quad \lambda \in \mathbb{R}, \tag{3.3}$$

and

$$R_{2n+1}(\lambda)^{1/2} = |R_{2n+1}(\lambda)^{1/2}| \begin{cases} (-1)^n i & \text{for } \lambda \in (-\infty, E_0), \\ (-1)^{n+j} i & \text{for } \lambda \in (E_{2j-1}, E_{2j}),\ j = 1, \dots, n, \\ (-1)^{n+j} & \text{for } \lambda \in (E_{2j}, E_{2j+1}),\ j = 0, \dots, n-1, \\ 1 & \text{for } \lambda \in (E_{2n}, \infty), \end{cases} \quad \lambda \in \mathbb{R} \tag{3.4}$$

and analytically continue $R_{2n+1}^{1/2}$ from $\mathbb{R}$ to all of $\mathbb{C}\backslash\Sigma_n$, where Σ_n is defined by

$$\Sigma_n = \left\{ \bigcup_{j=0}^{n-1} [E_{2j}, E_{2j+1}] \right\} \cup [E_{2n}, \infty). \tag{3.5}$$

In this context we also mention the useful formula

$$\overline{R_{2n+1}(\overline{z})^{1/2}} = -R_{2n+1}(z)^{1/2}, \quad z \in \mathbb{C}_+. \tag{3.6}$$

Theorem 3.1 ([27]). *Let $z \in \mathbb{C}\backslash\Sigma_n$ and $n \in \mathbb{N}$. Define $R_{2n+1}^{1/2}$ as in (3.1)–(3.4) followed by an analytic continuation to $\mathbb{C}\backslash\Sigma_n$. Moreover, let F_n and H_{n+1} be two monic polynomials of degree n and $n+1$, respectively. Then $iR_{2n+1}(z)^{-1/2}F_n(z)$ is a Herglotz function if and only if all zeros of F_n are real and there is precisely one zero in each of the intervals $[E_{2j-1}, E_{2j}]$, $1 \leq j \leq n$. Moreover, if $iR_{2n+1}^{-1/2}F_n$ is a Herglotz function, then it can be represented in the form*

$$\frac{iF_n(z)}{R_{2n+1}(z)^{1/2}} = \frac{1}{\pi} \int_{\Sigma_n} \frac{F_n(\lambda)d\lambda}{R_{2n+1}(\lambda)^{1/2}} \frac{1}{\lambda - z}, \quad z \in \mathbb{C}\backslash\Sigma_n. \tag{3.7}$$

Similarly, $iR_{2n+1}(z)^{-1/2}H_{n+1}(z)$ is a Herglotz function if and only if all zeros of H_{n+1} are real and there is precisely one zero in each of the intervals $(-\infty, E_0]$ and $[E_{2j-1}, E_{2j}]$, $1 \leq j \leq n$. Moreover, if $iR_{2n+1}^{-1/2}H_{n+1}$ is a Herglotz function, then it can be represented in the form

$$\begin{aligned} \frac{iH_{n+1}(z)}{R_{2n+1}(z)^{1/2}} &= \operatorname{Re}\left(\frac{iH_{n+1}(i)}{R_{2n+1}(i)^{1/2}} \right) \\ &\quad + \frac{1}{\pi} \int_{\Sigma_n} \frac{H_{n+1}(\lambda)d\lambda}{R_{2n+1}(\lambda)^{1/2}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right), \quad z \in \mathbb{C}\backslash\Sigma_n. \end{aligned} \tag{3.8}$$

Actually, Theorem 4.1 can be improved by invoking ideas developed in the Appendix of [48] (cf. also [76]). Since this appears to be of independent interest we provide a brief discussion.

We start with the elementary observation that the Herglotz function

$$m(z) = \begin{cases} \frac{z-\beta}{z-\alpha}, & -\infty < \alpha < \beta < \infty, \\ z - \beta, & \alpha = -\infty,\ \beta \in \mathbb{R}, \\ \frac{-1}{z-\alpha}, & \alpha \in \mathbb{R},\ \beta = +\infty, \end{cases} \quad z \in \mathbb{C}_+, \tag{3.9}$$

admits the (exponential) representation

$$m(z) = C(\alpha, \beta) \exp \int_\alpha^\beta d\lambda \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right), \quad z \in \mathbb{C}_+, \tag{3.10}$$

where

$$C(\alpha,\beta)=\begin{cases}\left(\frac{1+\beta^2}{1+\alpha^2}\right)^{1/2}, & -\infty<\alpha<\beta<\infty,\\ 1, & \alpha=-\infty,\ \beta\in\mathbb{R}\text{ or }\alpha\in\mathbb{R},\ \beta=+\infty.\end{cases} \tag{3.11}$$

Theorem 3.2. *Let $z\in\mathbb{C}\backslash\Sigma_n$, $n\in\mathbb{N}$, and define $R_{2n+1}^{1/2}$ as in* (3.1)–(3.4) *followed by an analytic continuation to $\mathbb{C}\backslash\Sigma_n$. Suppose M is a Herglotz function such that*

$$\lim_{\varepsilon\downarrow 0} M(\lambda+i\varepsilon)\in i\mathbb{R} \quad \textit{for a.e. } \lambda\in\Sigma_n \tag{3.12}$$

and assume in addition that M is real-valued on $\mathbb{C}\backslash\Sigma$. Then M is either of the form

$$M(z)=\frac{i\widehat{F}_n(z)}{R_{2n+1}(z)^{1/2}}, \tag{3.13}$$

where $\widehat{F}_n$ is a polynomial of degree n (not necessarily monic), positive on the semi-axis (E_{2n},∞), with precisely one zero in each of the intervals $[E_{2j-1},E_{2j}]$, $1\leq j\leq n$, or else, M is of the form

$$M(z)=\frac{i\widehat{H}_{n+1}(z)}{R_{2n+1}(z)^{1/2}}, \tag{3.14}$$

where $\widehat{H}_{n+1}$ is a polynomial of degree $n+1$ (not necessarily monic), positive on the semi-axis (E_{2n},∞), with precisely one zero in each of the intervals $(-\infty,E_0]$ and $[E_{2j-1},E_{2j}]$, $1\leq j\leq n$.

Moreover, if $i\widehat{F}_n/R_{2n+1}^{1/2}$ is a Herglotz function, it can be represented in the form (3.7). *Similarly, if $i\widehat{H}_{n+1}/R_{2n+1}^{1/2}$ is a Herglotz function, it can be represented in the form* (3.8).

Proof. The Herglotz function M admits the exponential representation (cf. [2])

$$M(z)=K\exp\left(\frac{1}{2}\int_{\Sigma_n}d\lambda\left(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\right)+\int_{\Sigma_{n,-}}d\lambda\left(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\right)\right), \tag{3.15}$$

where $K>0$ and

$$\Sigma_{n,-}=\{\lambda\in\mathbb{R}\backslash\Sigma_n\,|\,M(\lambda)<0\}. \tag{3.16}$$

Since the Herglotz function M is strictly monotonically increasing on $\mathbb{R}\backslash\Sigma_n$, M can have at most one zero in each interval $(-\infty,E_0)$, (E_{2j-1},E_{2j}), $j=1,\dots,n$. Moreover, $M(E_0-0)$, $M(E_{2j}-0)\in\{+\infty,0\}$, $M(E_{2j-1}+0)\in\{-\infty,0\}$, $j=1,\dots,n$. Thus, depending on whether or not $M\big|_{(-\infty,E_0)}\geq 0$, the set $\Sigma_{n,-}$ admits one of the following two representations

$$\Sigma_{n,-}=\bigcup_{j=1}^{n}(E_{2j-1},\mu_j) \tag{3.17}$$

or

$$\Sigma_{n,-}=(-\infty,\nu_0)\cup\bigcup_{j=1}^{n}(E_{2j-1},\nu_j) \tag{3.18}$$

for some $\nu_0 \in (-\infty, E_0]$ and some $\mu_j, \nu_j \in [E_{2j-1}, E_{2j}]$, $1 \leq j \leq n$. Repeated use of (3.10) then proves the representation

$$\begin{aligned} M(z) &= C_1 \bigg(\prod_{j=0}^{n-1} \frac{(E_{2j+1} - z)}{(E_{2j} - z)} \frac{1}{(E_{2n} - z)} \bigg)^{1/2} \prod_{j=1}^{n} \frac{(z - \mu_j)}{(z - E_{2j-1})} \\ &= C_1 \frac{\prod_{j=1}^{n} (z - \mu_j)}{\Big(\prod_{\ell=0}^{2n} (z - E_\ell)\Big)^{1/2}} = \frac{i\widehat{F}_n(z)}{R_{2n+1}(z)^{1/2}}, \end{aligned} \tag{3.19}$$

where

$$\widehat{F}_n(z) = C_1 \prod_{j=1}^{n} (z - \mu_j) \tag{3.20}$$

for some $C_1 > 0$, whenever (3.17) holds, and

$$M(z) = C_2 (z - \nu_0) \bigg(\prod_{j=0}^{n-1} \frac{E_{2j+1} - z}{E_{2j} - z} \frac{1}{E_{2n} - z} \bigg)^{1/2} \prod_{j=1}^{n} \frac{z - \nu_j}{z - E_{2j-1}} = \frac{i\widehat{H}_{n+1}(z)}{R_{2n+1}(z)^{1/2}}, \tag{3.21}$$

where

$$\widehat{H}_{n+1}(z) = C_2 \prod_{k=0}^{n} (z - \nu_k) \tag{3.22}$$

for some $C_2 > 0$, whenever (3.18) holds. $\square$

Given $m \in \mathbb{N}$, we denote by

$$\mathcal{A}(z) = \sum_{k=0}^{n} \mathcal{A}_k z^k, \quad \mathcal{A}_k \in \mathbb{C}^{m \times m}, \, 1 \leq k \leq n, \, z \in \mathbb{C}, \tag{3.23}$$

a polynomial pencil of $m \times m$ matrices (in short, a pencil) in the following. $\mathcal{A}$ is called of degree $n \in \mathbb{N}_0$ if $\mathcal{A}_n \neq 0$ and monic if $\mathcal{A}_n = \mathcal{I}_m$.

Definition 3.3. *Let A be a pencil of the type* (3.23).
(i) The pencil $\mathcal{A}$ is called self-adjoint if $\mathcal{A}_k = \mathcal{A}_k^*$ for all $1 \leq k \leq n$ (i.e., $\mathcal{A}(\overline{z})^* = \mathcal{A}(z)$ for all $z \in \mathbb{C}$).
(ii) A self-adjoint pencil $\mathcal{A}$ is called weakly hyperbolic if $\mathcal{A}_n > 0$ and for all $f \in \mathbb{C}^m \backslash \{0\}$, the roots of the polynomial $(f, \mathcal{A}(\cdot) f)_{\mathbb{C}^m}$ are real. If in addition all these zeros are distinct, the pencil $\mathcal{A}$ is called hyperbolic.
(iii) Let $\mathcal{A}$ be a weakly hyperbolic pencil and denote by $\{p_j(\mathcal{A}, f)\}_{1 \leq j \leq n}$,

$$p_j(\mathcal{A}, f) \leq p_{j+1}(\mathcal{A}, f), \; 1 \leq j \leq n-1, \; f \in \mathbb{C}^m \backslash \{0\}, \tag{3.24}$$

the roots of the polynomial $(f, \mathcal{A}(\cdot) f)_{\mathbb{C}^m}$ ordered in magnitude. The range of the roots $p_j(\mathcal{A}, f)$, $f \in \mathbb{C}^m \backslash \{0\}$ is denoted by $\Delta_j(\mathcal{A})$ and called the jth root zone of $\mathcal{A}$.
(iv) A hyperbolic pencil $\mathcal{A}$ is called strongly hyperbolic if $\overline{\Delta_j(\mathcal{A})}$ and $\overline{\Delta_k(\mathcal{A})}$ are mutually disjoint for $j \neq k$, $1 \leq j, k \leq n$.

For details on spectral theory of polynomial matrix (in fact, operator) pencils we refer, for instance, to [58], [59], [60].

Corollary 3.4 ([27]). *Let $z \in \mathbb{C}\backslash\Sigma_n$ and $m,n \in \mathbb{N}$. Define $R_{2n+1}^{1/2}$ as in (3.1)–(3.4) followed by an analytic continuation to $\mathbb{C}\backslash\Sigma_n$. Moreover let $\mathcal{F}_n$ and $\mathcal{H}_{n+1}$ be two monic $m\times m$ matrix pencils of degree n and $n+1$, respectively. Then $(i/2)R_{2n+1}^{-1/2}\mathcal{F}_n$ is a Herglotz matrix if and only if the root zones $\Delta_j(\mathcal{F}_n)$ of $\mathcal{F}_n$ satisfy*

$$\Delta_j(\mathcal{F}_n) \subseteq [E_{2j-1}, E_{2j}], \quad 1 \leq j \leq n. \tag{3.25}$$

Analogously, $(i/2)R_{2n+1}^{-1/2}\mathcal{H}_{n+1}$ is a Herglotz matrix if and only if the root zones $\Delta_j(\mathcal{H}_{n+1})$ of $\mathcal{H}_{n+1}$ satisfy

$$\Delta_0(\mathcal{H}_{n+1}) \subset (-\infty, E_0], \quad \Delta_j(\mathcal{H}_{n+1}) \subseteq [E_{2j-1}, E_{2j}], \quad 1 \leq j \leq n. \tag{3.26}$$

If (3.25) (resp., (3.26)) holds, then $\mathcal{F}_n$ (resp., $\mathcal{H}_{n+1}$) is a strongly hyperbolic pencil.

Next, we define the following $2m \times 2m$ matrix $\mathcal{M}_{\Sigma_n}(z,x_0)$ which will turn out to be the underlying Weyl–Titchmarsh matrix associated with the class of $m \times m$ matrix-valued Schrödinger operators with prescribed finite-band spectrum Σ_n of maximal multiplicity. We introduce, for fixed $x_0 \in \mathbb{R}$,

$$\begin{aligned}\mathcal{M}_{\Sigma_n}(z,x_0) &= \big(\mathcal{M}_{\Sigma_n,p,q}(z,x_0)\big)_{1\leq p,q\leq 2} \\ &= \frac{i}{2R_{2n+1}(z)^{1/2}}\begin{pmatrix} \mathcal{H}_{n+1,\Sigma_n}(z,x_0) & -\mathcal{G}_{2,n-1,\Sigma_n}(z,x_0) \\ -\mathcal{G}_{1,n-1,\Sigma_n}(z,x_0) & \mathcal{F}_{n,\Sigma_n}(z,x_0)\end{pmatrix}, \; z \in \mathbb{C}\backslash\Sigma_n.\end{aligned} \tag{3.27}$$

Here $R_{2n+1}(z)^{1/2}$ is defined as in (3.1)–(3.4) followed by analytic continuation into $\mathbb{C}\backslash\Sigma$ and the polynomial matrix pencils $\mathcal{F}_{n,\Sigma_n}$, $\mathcal{G}_{1,n-1,\Sigma_n}$, $\mathcal{G}_{2,n-1,\Sigma_n}$, and $\mathcal{H}_{n+1,\Sigma_n}$ are introduced as follows:

(i) $\mathcal{F}_{n,\Sigma_n}(\cdot,x_0)$ is an $m \times m$ monic matrix pencil of degree n, that is, $\mathcal{F}_{n,\Sigma}(\cdot,x_0)$ is of the type

$$\mathcal{F}_{n,\Sigma_n}(z,x_0) = \sum_{\ell=0}^{n} \mathcal{F}_{n-\ell,\Sigma_n}(x_0)z^\ell, \quad \mathcal{F}_{0,\Sigma_n}(x_0) = \mathcal{I}_m, \; z \in \mathbb{C} \tag{3.28}$$

and

$$\frac{i}{2R_{2n+1}^{1/2}}\mathcal{F}_{n,\Sigma_n}(\cdot,x_0) \text{ is assumed to be an } m \times m \text{ Herglotz matrix.} \tag{3.29}$$

Hence $\mathcal{F}_{n,\Sigma_n}(\cdot,x_0)$ is a self-adjoint (in fact, strongly hyperbolic) pencil,

$$\mathcal{F}_{n,\Sigma_n}(\overline{z},x_0)^* = \mathcal{F}_{n,\Sigma_n}(z,x_0), \quad z \in \mathbb{C} \tag{3.30}$$

and $(i/2)R_{2n+1}^{-1/2}\mathcal{F}_{n,\Sigma_n}$ and $2iR_{2n+1}^{1/2}\mathcal{F}_{n,\Sigma_n}^{-1}$ admit the Herglotz representations

$$\frac{i}{2R_{2n+1}(z)^{1/2}}\mathcal{F}_{n,\Sigma_n}(z,x_0) = \frac{1}{2\pi}\int_{\Sigma_n}\frac{d\lambda}{R_{2n+1}(\lambda)^{1/2}}\mathcal{F}_{n,\Sigma_n}(\lambda,x_0)\frac{1}{\lambda-z}, \quad z \in \mathbb{C}\backslash\Sigma_n, \tag{3.31}$$

$$\begin{aligned}&iR_{2n+1}(z)^{1/2}\mathcal{F}_{n,\Sigma_n}(z,x_0)^{-1} \\ &= \frac{1}{\pi}\int_{\Sigma_n} d\lambda\, R_{2n+1}(\lambda)^{1/2}\mathcal{F}_{n,\Sigma_n}(\lambda,x_0)^{-1}\left(\frac{1}{\lambda-z} - \frac{\lambda}{1+\lambda^2}\right) \\ &\quad + \Gamma_{\Sigma_n,0}(x_0) - \sum_{k=1}^{N}(z-\mu_k(x_0))^{-1}\Gamma_{\Sigma_n,k}(x_0), \\ &\qquad z \in \mathbb{C}\backslash\{\Sigma_n \cup \{\mu_k(x_0)\}_{1\leq k\leq N}\},\end{aligned} \tag{3.32}$$

where

$$\Gamma_{\Sigma_n,0}(x_0)=\Gamma_{\Sigma_n,0}(x_0)^*\in\mathbb{C}^{m\times m},\quad 0\le\Gamma_{\Sigma_n,k}(x_0)\in\mathbb{C}^{m\times m},\ 1\le k\le N,$$
$$\sum_{k=1}^N \operatorname{rank}(\Gamma_{\Sigma_n,k}(x_0))\le mn,\quad \mu_k(x_0)\in\bigcup_{j=1}^n[E_{2j-1},E_{2j}],\quad 1\le k\le N. \tag{3.33}$$

In fact, there are precisely m numbers $\mu_k(x_0)$ in $[E_{2j-1},E_{2j}]$ for each $1\le j\le n$, counting multiplicity (they are the points z where $\mathcal{F}_{n,\Sigma_n}(z,x_0)$ is not invertible).

(ii) Given these facts we now define

$$\mathcal{G}_{1,n-1,\Sigma_n}(z,x_0)=\bigg(\sum_{k=1}^N\frac{\varepsilon_k(x_0)}{z-\mu_k(x_0)}\Gamma_{\Sigma_n,k}(x_0)\bigg)\mathcal{F}_{n,\Sigma_n}(z,x_0), \tag{3.34}$$
$$\mathcal{G}_{2,n-1,\Sigma_n}(z,x_0)=\mathcal{F}_{n,\Sigma_n}(z,x_0)\bigg(\sum_{k=1}^N\frac{\varepsilon_k(x_0)}{z-\mu_k(x_0)}\Gamma_{\Sigma_n,k}(x_0)\bigg), \tag{3.35}$$
$$\varepsilon_k(x_0)\in\{1,-1\},\ 1\le k\le N,\quad z\in\mathbb{C}\backslash\{\mu_k(x_0)\}_{1\le k\le N}, \tag{3.36}$$

and

$$\begin{aligned}\mathcal{H}_{n+1,\Sigma_n}(z,x_0)&=R_{2n+1}(z)\mathcal{F}_{n,\Sigma_n}(z,x_0)^{-1} \qquad (3.37)\\ &+\bigg(\sum_{k=1}^N\frac{\varepsilon_k(x_0)}{z-\mu_k(x_0)}\Gamma_{\Sigma_n,k}(x_0)\bigg)\mathcal{F}_{n,\Sigma_n}(z,x_0)\bigg(\sum_{\ell=1}^N\frac{\varepsilon_\ell(x_0)}{z-\mu_\ell(x_0)}\Gamma_{\Sigma_n,\ell}(x_0)\bigg),\\ &\qquad z\in\mathbb{C}\backslash\{\mu_k(x_0)\}_{1\le k\le N}.\end{aligned}$$

Lemma 3.5 ([27])**.** *Let $z\in\mathbb{C}\backslash\{\mu_k(x_0)\}_{1\le k\le N}$. $\mathcal{G}_{p,n-1,\Sigma_n}(\cdot,x_0)$, $p=1,2$, are $m\times m$ polynomial matrix pencils of equal degree at most $n-1$ and $\mathcal{H}_{n+1,\Sigma_n}(\cdot,x_0)$ is a strongly hyperbolic* (*and hence self-adjoint*) *$m\times m$ monic matrix pencil of degree $n+1$. Moreover, the following identities hold.*

$$\mathcal{G}_{2,n-1,\Sigma_n}(\overline{z},x_0)^*=\mathcal{G}_{1,n-1,\Sigma_n}(z,x_0), \tag{3.38}$$
$$\mathcal{F}_{n,\Sigma_n}(z,x_0)\mathcal{G}_{1,n-1,\Sigma_n}(z,x_0)=\mathcal{G}_{2,n-1,\Sigma_n}(z,x_0)\mathcal{F}_{n,\Sigma_n}(z,x_0), \tag{3.39}$$
$$\mathcal{H}_{n+1,\Sigma_n}(z,x_0)\mathcal{G}_{2,n-1,\Sigma_n}(z,x_0)=\mathcal{G}_{1,n-1,\Sigma_n}(z,x_0)\mathcal{H}_{n+1,\Sigma_n}(z,x_0), \tag{3.40}$$
$$\mathcal{F}_{n,\Sigma_n}(z,x_0)\mathcal{H}_{n+1,\Sigma_n}(z,x_0)-\mathcal{G}_{2,n-1,\Sigma_n}(z,x_0)^2=R_{2n+1}(z)\mathcal{I}_m, \tag{3.41}$$
$$\mathcal{H}_{n+1,\Sigma_n}(z,x_0)\mathcal{F}_{n,\Sigma_n}(z,x_0)-\mathcal{G}_{1,n-1,\Sigma_n}(z,x_0)^2=R_{2n+1}(z)\mathcal{I}_m. \tag{3.42}$$

Next, introducing

$$\begin{aligned}&\mathcal{M}_{\pm,\Sigma_n}(z,x_0)\\ &=\pm iR_{2n+1}(z)^{1/2}\mathcal{F}_{n,\Sigma_n}(z,x_0)^{-1}-\mathcal{G}_{1,n-1,\Sigma_n}(z,x_0)\mathcal{F}_{n,\Sigma_n}(z,x_0)^{-1} \qquad (3.43\text{a})\\ &=\pm iR_{2n+1}(z)^{1/2}\mathcal{F}_{n,\Sigma_n}(z,x_0)^{-1}-\mathcal{F}_{n,\Sigma_n}(z,x_0)^{-1}\mathcal{G}_{2,n-1,\Sigma_n}(z,x_0), \qquad (3.43\text{b})\\ &\qquad z\in\mathbb{C}\backslash\{\Sigma_n\cup\{\mu_k(x_0)\}_{1\le k\le N}\},\end{aligned}$$

$\pm\mathcal{M}_{\pm,\Sigma_n}(\cdot,x_0)$ are $m\times m$ Herglotz matrices with representations

$$\begin{aligned}\pm\mathcal{M}_{\pm,\Sigma_n}(z,x_0)&=\frac{1}{\pi}\int_{\Sigma_n} d\lambda\, R_{2n+1}(\lambda)^{1/2}\mathcal{F}_{n,\Sigma_n}(\lambda,x_0)^{-1}\bigg(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\bigg)\\&\quad+\Gamma_{\Sigma_n,0}(x_0)-\sum_{k=1}^{N}\frac{1\pm\varepsilon_k(x_0)}{z-\mu_k(x_0)}\Gamma_{\Sigma_n,k}(x_0), \qquad (3.44)\\&\qquad z\in\mathbb{C}\backslash\{\Sigma_n\cup\{\mu_k(x_0)\}_{1\leq k\leq N}\}.\end{aligned}$$

Applying Theorem 2.3, $\mathcal{M}_{\pm,\Sigma_n}(z,x_0)$ are seen to be the half-line Weyl–Titchmarsh matrices uniquely associated with a potential $\mathcal{Q}_{\Sigma_n}\in L^1_{\text{loc}}(\mathbb{R})^{m\times m}$. In addition, we denote by $\psi_{\pm,\Sigma_n}(z,x,x_0)$ the Weyl solutions (2.5), (2.6) associated with $\mathcal{Q}_{\Sigma_n}$,

$$\psi_{\pm,\Sigma_n}(z,x,x_0)=\theta_{\Sigma_n}(z,x,x_0)+\phi_{\Sigma_n}(z,x,x_0)\mathcal{M}_{\pm,\Sigma_n}(z,x_0),\quad z\in\mathbb{C}\backslash\Sigma_n, \qquad (3.45)$$

where, in obvious notation, $\theta_{\Sigma_n}(z,x,x_0)$, $\phi_{\Sigma_n}(z,x,x_0)$ denote the fundamental system (2.4) corresponding to $\mathcal{Q}_{\Sigma_n}$. The $2m\times 2m$ Weyl–Titchmarsh matrix associated with $\mathcal{Q}_{\Sigma_n}$ on $\mathbb{R}$ is then given by

$$\begin{aligned}\mathcal{M}_{\Sigma_n}(z,x)&=\big(\mathcal{M}_{\Sigma,p,q}(z,x)\big)_{1\leq p,q\leq 2} \qquad (3.46)\\&=\frac{i}{2R_{2n+1}(z)^{1/2}}\begin{pmatrix}\mathcal{H}_{n+1,\Sigma_n}(z,x) & -\mathcal{G}_{2,n-1,\Sigma_n}(z,x)\\ -\mathcal{G}_{1,n-1,\Sigma_n}(z,x) & \mathcal{F}_{n,\Sigma_n}(z,x)\end{pmatrix},\quad z\in\mathbb{C}\backslash\Sigma,\end{aligned}$$

where we abbreviated

$$\begin{aligned}\mathcal{F}_{n,\Sigma_n}(z,x)&=\theta_{\Sigma_n}(z,x,x_0)\mathcal{F}_{n,\Sigma_n}(z,x_0)\theta_{\Sigma_n}(\overline{z},x,x_0)^*\\&\quad+\phi_{\Sigma_n}(z,x,x_0)\mathcal{H}_{n+1,\Sigma_n}(z,x_0)\phi_{\Sigma_n}(\overline{z},x,x_0)^*\\&\quad-\phi_{\Sigma_n}(z,x,x_0)\mathcal{G}_{1,n-1,\Sigma_n}(z,x_0)\theta_{\Sigma_n}(\overline{z},x,x_0)^*\\&\quad-\theta_{\Sigma_n}(z,x,x_0)\mathcal{G}_{2,n-1,\Sigma_n}(z,x_0)\phi_{\Sigma_n}(\overline{z},x,x_0)^*, \qquad (3.47)\\ \mathcal{G}_{1,n-1,\Sigma_n}(z,x)&=-\theta'_{\Sigma_n}(z,x,x_0)\mathcal{F}_{n,\Sigma_n}(z,x_0)\theta_{\Sigma_n}(\overline{z},x,x_0)^*\\&\quad-\phi'_{\Sigma_n}(z,x,x_0)\mathcal{H}_{n+1,\Sigma_n}(z,x_0)\phi_{\Sigma_n}(\overline{z},x,x_0)^*\\&\quad+\phi'_{\Sigma_n}(z,x,x_0)\mathcal{G}_{1,n-1,\Sigma_n}(z,x_0)\theta_{\Sigma_n}(\overline{z},x,x_0)^*\\&\quad+\theta'_{\Sigma_n}(z,x,x_0)\mathcal{G}_{2,n-1,\Sigma_n}(z,x_0)\phi_{\Sigma_n}(\overline{z},x,x_0)^*, \qquad (3.48)\\ \mathcal{G}_{2,n-1,\Sigma_n}(z,x)&=-\theta_{\Sigma_n}(z,x,x_0)\mathcal{F}_{n,\Sigma_n}(z,x_0)\theta'_{\Sigma_n}(\overline{z},x,x_0)^*\\&\quad-\phi_{\Sigma_n}(z,x,x_0)\mathcal{H}_{n+1,\Sigma_n}(z,x_0)\phi'_{\Sigma_n}(\overline{z},x,x_0)^*\\&\quad+\phi_{\Sigma_n}(z,x,x_0)\mathcal{G}_{1,n-1,\Sigma_n}(z,x_0)\theta'_{\Sigma_n}(\overline{z},x,x_0)^*\\&\quad+\theta_{\Sigma_n}(z,x,x_0)\mathcal{G}_{2,n-1,\Sigma_n}(z,x_0)\phi'_{\Sigma_n}(\overline{z},x,x_0)^*, \qquad (3.49)\\ \mathcal{H}_{n+1,\Sigma_n}(z,x)&=\theta'_{\Sigma_n}(z,x,x_0)\mathcal{F}_{n,\Sigma_n}(z,x_0)\theta'_{\Sigma_n}(\overline{z},x,x_0)^*\\&\quad+\phi'_{\Sigma_n}(z,x,x_0)\mathcal{H}_{n+1,\Sigma_n}(z,x_0)\phi'_{\Sigma_n}(\overline{z},x,x_0)^*\\&\quad-\phi'_{\Sigma_n}(z,x,x_0)\mathcal{G}_{1,n-1,\Sigma_n}(z,x_0)\theta'_{\Sigma_n}(\overline{z},x,x_0)^*\\&\quad-\theta'_{\Sigma_n}(z,x,x_0)\mathcal{G}_{2,n-1,\Sigma_n}(z,x_0)\phi'_{\Sigma_n}(\overline{z},x,x_0)^*. \qquad (3.50)\end{aligned}$$

Considerations of this type can be found in [52, Sect. 8.2] in the special scalar case $m=1$ and in the matrix context $m\in\mathbb{N}$ in [72, Sect. 9.4].

One then infers

$$\mathcal{F}'_{n,\Sigma_n} = -(\mathcal{G}_{1,n-1,\Sigma_n} + \mathcal{G}_{2,n-1,\Sigma_n}), \tag{3.51}$$

$$\mathcal{G}'_{1,n-1,\Sigma_n} = -(Q_{\Sigma_n} - z\mathcal{I}_m)\mathcal{F}_{n,\Sigma_n} - \mathcal{H}_{n+1,\Sigma_n} \tag{3.52}$$

$$= (-\mathcal{F}''_n + \mathcal{F}_n\mathcal{Q}_{\Sigma_n} - \mathcal{Q}_{\Sigma_n}\mathcal{F}_{n,\Sigma_n})/2, \tag{3.53}$$

$$\mathcal{G}''_{1,n-1,\Sigma_n} = -2(\mathcal{Q}_{\Sigma_n} - z\mathcal{I}_m)\mathcal{F}'_{n,\Sigma_n} - \mathcal{Q}'_{\Sigma_n}\mathcal{F}_{n,\Sigma_n} + \mathcal{G}_{1,n-1,\Sigma_n}\mathcal{Q}_{\Sigma_n} - \mathcal{Q}_{\Sigma_n}\mathcal{G}_{1,n-1,\Sigma_n}, \tag{3.54}$$

$$\mathcal{G}'_{2,n-1,\Sigma_n} = -\mathcal{F}_{n,\Sigma_n}(Q_{\Sigma_n} - z\mathcal{I}_m) - \mathcal{H}_{n+1,\Sigma_n} \tag{3.55}$$

$$= (-\mathcal{F}''_{n,\Sigma_n} + \mathcal{Q}_{\Sigma_n}\mathcal{F}_{n,\Sigma_n} - \mathcal{F}_{n,\Sigma_n}\mathcal{Q}_{\Sigma_n})/2, \tag{3.56}$$

$$\mathcal{G}''_{2,n-1,\Sigma_n} = -2\mathcal{F}'_{n,\Sigma_n}(\mathcal{Q}_{\Sigma_n} - z\mathcal{I}_m) - \mathcal{F}_{n,\Sigma_n}\mathcal{Q}'_{\Sigma_n} + \mathcal{Q}_{\Sigma_n}\mathcal{G}_{2,n-1,\Sigma_n} - \mathcal{G}_{2,n-1,\Sigma_n}\mathcal{Q}_{\Sigma_n}, \tag{3.57}$$

$$\mathcal{H}'_{n+1,\Sigma_n} = -\mathcal{G}_{1,n-1,\Sigma_n}(Q_{\Sigma_n} - z\mathcal{I}_m) - (Q_{\Sigma_n} - z\mathcal{I}_m)\mathcal{G}_{2,n-1,\Sigma_n}, \tag{3.58}$$

$$\mathcal{H}_{n+1,\Sigma_n} = [\mathcal{F}''_{n,\Sigma_n} - \mathcal{F}_{n,\Sigma_n}(\mathcal{Q}_{\Sigma_n} - z\mathcal{I}_m) - (\mathcal{Q}_{\Sigma_n} - z\mathcal{I}_m)\mathcal{F}_{n\Sigma_n}]/2 \tag{3.59}$$

and

$$\mathcal{F}_{n,\Sigma_n}(\overline{z},x)^* = \mathcal{F}_{n,\Sigma_n}(z,x), \quad \mathcal{H}_{n+1,\Sigma_n}(\overline{z},x)^* = \mathcal{H}_{n+1,\Sigma_n}(z,x),$$

$$\mathcal{G}_{2,n-1,\Sigma}(\overline{z},x)^* = \mathcal{G}_{1,n-1,\Sigma_n}(z,x), \tag{3.60}$$

$$\mathcal{F}_{n,\Sigma_n}(z,x)\mathcal{G}_{1,n-1,\Sigma_n}(z,x) = \mathcal{G}_{2,n-1,\Sigma_n}(z,x)\mathcal{F}_{n,\Sigma_n}(z,x), \tag{3.61}$$

$$\mathcal{H}_{n+1,\Sigma_n}(z,x)\mathcal{G}_{2,n-1,\Sigma_n}(z,x) = \mathcal{G}_{1,n-1,\Sigma_n}(z,x)\mathcal{H}_{n+1,\Sigma_n}(z,x), \tag{3.62}$$

$$\mathcal{H}_{n+1,\Sigma_n}(z,x)\mathcal{F}_{n,\Sigma_n}(z,x) - \mathcal{G}_{1,n-1,\Sigma_n}(z,x)^2 = R_{2n+1}(z)\mathcal{I}_m, \tag{3.63}$$

$$\mathcal{F}_{n,\Sigma_n}(z,x)\mathcal{H}_{n+1,\Sigma_n}(z,x) - \mathcal{G}_{2,n-1,\Sigma_n}(z,x)^2 = R_{2n+1}(z)\mathcal{I}_m. \tag{3.64}$$

Combining (2.33)–(2.37) and (3.46) then yields

$$\mathcal{M}_{\pm,\Sigma_n}(z,x)$$

$$= \pm iR_{2n+1}(z)^{1/2}\mathcal{F}_{n,\Sigma_n}(z,x)^{-1} - \mathcal{G}_{1,n-1,\Sigma_n}(z,x)\mathcal{F}_{n,\Sigma_n}(z,x)^{-1} \tag{3.65a}$$

$$= \pm iR_{2n+1}(z)^{1/2}\mathcal{F}_{n,\Sigma_n}(z,x)^{-1} - \mathcal{F}_{n,\Sigma_n}(z,x)^{-1}\mathcal{G}_{2,n-1,\Sigma_n}(z,x), \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.65b}$$

One observes that for each $x \in \mathbb{R}$, $\mathcal{M}_{+,\Sigma_n}(\cdot,x)$ is the analytic continuation of $\mathcal{M}_{-,\Sigma_n}(\cdot,x)$ through the set Σ_n, and vice versa,

$$\lim_{\varepsilon\downarrow 0}\mathcal{M}_{+,\Sigma_n}(\lambda+i\varepsilon,x) = \lim_{\varepsilon\downarrow 0}\mathcal{M}_{-,\Sigma_n}(\lambda-i\varepsilon,x) = \lim_{\varepsilon\downarrow 0}\mathcal{M}_{-,\Sigma_n}(\lambda+i\varepsilon,x)^*, \tag{3.66}$$

$$\lambda \in \bigcup_{j=0}^{n-1}(E_{2j},E_{2j+1}) \cup (E_{2n},\infty), \ x \in \mathbb{R}.$$

In other words, for each $x \in \mathbb{R}$, $\mathcal{M}_{+,\Sigma_n}(\cdot,x)$ and $\mathcal{M}_{-,\Sigma_n}(\cdot,x)$ are the two branches of an analytic matrix-valued function $\mathcal{M}_{\Sigma_n}(\cdot,x)$ on the two-sheeted Riemann surface of $R_{2n+1}^{1/2}$. This implies that $\mathcal{Q}_{\Sigma_n}$ is reflectionless as will be discussed in Lemma 3.7. In addition, it is worthwhile to emphasize that in the present case of reflectionless potentials, $\mathcal{F}_{n,\Sigma_n}(z,x_0)$ and $\mathcal{G}_{1,n-1,\Sigma_n}(z,x_0)$ for some fixed $x_0 \in \mathbb{R}$, uniquely determine $\mathcal{M}_{\pm,\Sigma_n}(z,x_0)$ and hence $\mathcal{Q}_{\Sigma_n}(x)$ for all $x \in \mathbb{R}$.

Introducing the open interior Σ_n^o of Σ_n defined by $\Sigma_n^o = \bigcup_{j=0}^{n-1}(E_{2j},E_{2j+1}) \cup (E_{2n},\infty)$, one obtains the following results.

Theorem 3.6 ([27]). *Let $z \in \mathbb{C}\backslash\mathbb{R}$ and $x \in \mathbb{R}$. Then*
(i) $\mathcal{F}_{n,\Sigma_n}(\cdot,x)$ and $\mathcal{H}_{n+1,\Sigma_n}(\cdot,x)$ are strongly hyperbolic (and hence self-adjoint) $m\times m$ monic matrix pencils of degree n and $n+1$, respectively, and $\mathcal{G}_{p,n-1,\Sigma_n}(\cdot,x)$, $p=1,2$, are $m\times m$ matrix pencils of degree $n-1$.
(ii) The differential expression $\mathcal{L}_{\Sigma_n} = -\mathcal{I}_m\frac{d^2}{dx^2} + \mathcal{Q}_{\Sigma_n}$ is in the limit point case at $\pm\infty$.
(iii) $\mathcal{M}_{\pm,\Sigma_n}(z,\cdot)$ in (3.65) satisfy the matrix-valued Riccati-type equation

$$\mathcal{M}'_{\pm,\Sigma_n}(z,x) + \mathcal{M}_{\pm,\Sigma_n}(z,x)^2 = \mathcal{Q}_{\Sigma_n}(x) - z\mathcal{I}_m, \quad x \in \mathbb{R},\ z \in \mathbb{C}\backslash\mathbb{R}. \tag{3.67}$$

Moreover, $\mathcal{M}_{\pm,\Sigma_n}(z,x)$ in (3.65) are the $m\times m$ Weyl–Titchmarsh matrices associated with self-adjoint operators $H^D_{\pm,x,\Sigma_n}$ in $L^2([x,\pm\infty))^m$, with a Dirichlet boundary condition at the point x and an $m\times m$ matrix-valued potential $\mathcal{Q}_{\Sigma_n}$ satisfying

$$\mathcal{Q}_{\Sigma_n} = \mathcal{Q}^*_{\Sigma_n} \in C^\infty(\mathbb{R})^{m\times m}, \quad \mathcal{Q}^{(r)}_{\Sigma_n} \in L^\infty(\mathbb{R}) \text{ for all } r \in \mathbb{N}_0. \tag{3.68}$$

In addition, $\mathcal{Q}_{\Sigma_n}$ is analytic in a neighborhood of the real axis. $H^D_{\pm,x,\Sigma_n}$ is given by

$$\begin{aligned} &H^D_{\pm,x,\Sigma_n} = -\mathcal{I}_m\frac{d^2}{dx^2} + \mathcal{Q}_{\Sigma_n}, \\ &\mathrm{dom}(H^D_{\pm,x,\Sigma_n}) = \{g \in L^2((x,\pm\infty))^m \mid g,g' \in AC([x,c])^m \text{ for all } c \gtrless x; \\ &\qquad \lim_{\varepsilon\downarrow 0} g(x\pm\varepsilon) = 0;\ (-g'' + \mathcal{Q}_{\Sigma_n}g) \in L^2((x,\pm\infty))^m\}. \end{aligned} \tag{3.69}$$

(iv) For each $x\in\mathbb{R}$, $\mathcal{M}_{\Sigma_n}(z,x)$ in (3.46) is a $2m\times 2m$ Weyl–Titchmarsh matrix associated with the self-adjoint operator H_{Σ_n} in $L^2(\mathbb{R})^m$ defined by

$$\begin{aligned} &H_{\Sigma_n} = -\mathcal{I}_m\frac{d^2}{dx^2} + \mathcal{Q}_{\Sigma_n}, \\ &\mathrm{dom}(H_{\Sigma_n}) = \{g \in L^2(\mathbb{R})^m \mid g,g' \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^m;\ (-g'' + \mathcal{Q}_{\Sigma_n}g) \in L^2(\mathbb{R})^m\}. \end{aligned} \tag{3.70}$$

In particular, $\mathcal{M}_{\Sigma_n}(\cdot,x)$ is a $2m\times 2m$ Herglotz matrix of H_{Σ_n} admitting a representation of the type (2.19), with measure $\Omega_{\Sigma_n}(\cdot,x)$ given by

$$d\Omega_{\Sigma_n}(\lambda,x) = \begin{cases} \frac{1}{2\pi R_{2n+1}(\lambda)^{1/2}}\begin{pmatrix} \mathcal{H}_{n+1,\Sigma_n}(\lambda,x) & -\mathcal{G}_{2,n-1,\Sigma_n}(\lambda,x) \\ -\mathcal{G}_{1,n-1,\Sigma_n}(\lambda,x) & \mathcal{F}_{n,\Sigma_n}(\lambda,x)\end{pmatrix} d\lambda, & \lambda \in \Sigma_n^o, \\ 0, & \lambda \in \mathbb{R}\backslash\Sigma_n. \end{cases} \tag{3.71}$$

(v) H_{Σ_n} has purely absolutely continuous spectrum Σ_n,

$$\mathrm{spec}(H_{\Sigma_n}) = \mathrm{spec}_{ac}(H_{\Sigma_n}) = \Sigma_n, \quad \mathrm{spec}_{\mathrm{p}}(H_{\Sigma_n}) = \mathrm{spec}_{\mathrm{sc}}(H_{\Sigma_n}) = \emptyset, \tag{3.72}$$

with $\mathrm{spec}(H_{\Sigma_n})$ of uniform spectral multiplicity $2m$.

It should be emphasized that the construction of $\mathcal{Q}_{\Sigma_n}$ in the scalar case $m=1$ is due to Levitan [49] (see also [50], [51], [52, Ch. 8], [54]).

That $\mathcal{Q}_{\Sigma_n}$ is reflectionless is an elementary consequence of (3.66) as discussed next.

Lemma 3.7 ([27]). *Denote by $\mathfrak{g}_{\Sigma_n}(z,x) = \mathcal{G}_{\Sigma_n}(z,x,x)$, $z\in\mathbb{C}_+$, $x \in\mathbb{R}$, the diagonal Green's function of H_{Σ_n}. Then*

$$\lim_{\varepsilon\downarrow 0}\mathfrak{g}_{\Sigma_n}(\lambda+i\varepsilon,x) = -\lim_{\varepsilon\downarrow 0}\mathfrak{g}_{\Sigma_n}(\lambda+i\varepsilon,x)^* \text{ for all } \lambda\in\Sigma_n^o \tag{3.73}$$

and hence $\mathcal{Q}_{\Sigma_n}$ is reflectionless.

Proof. Since $\mathfrak{g}(z,x) = (\mathcal{M}_-(z,x) - \mathcal{M}_+(z,x))^{-1}$, (3.66) implies (3.73). The latter implies that $\lim_{\varepsilon\downarrow 0} \mathfrak{g}(\lambda + i\varepsilon, x) = iG_{\Sigma_n}(\lambda, x)$ for all $\lambda \in \Sigma_n^o$ for some $m \times m$ matrix $G_{\Sigma_n}(\lambda, x) > 0$. This in turn implies

$$\Xi_{\Sigma_n}(\lambda, x) = \lim_{\varepsilon\downarrow 0} \pi^{-1}\mathrm{Im}(\ln(iG_{\Sigma_n}(\lambda + i\varepsilon, x))) = (1/2)\mathcal{I}_m \text{ for all } \lambda \in \Sigma^o, \tag{3.74}$$

and hence $\mathcal{Q}_{\Sigma_n}$ is reflectionless by Definition 2.7. □

Next, we briefly turn to the stationary matrix Korteweg–de Vries (KdV) hierarchy (cf. [20, Ch. 15], [23]) and show that the finite-band potential $\mathcal{Q}_\Sigma$ satisfies some (and hence infinitely many) equations of the stationary KdV equations.

Assuming $\mathcal{Q} = \mathcal{Q}^* \in C^\infty(\mathbb{R})^{m\times m}$, we recall the expansions (cf. Theorem 2.5)

$$\mathfrak{g}(z,x) = [\mathcal{M}_-(z,x) - \mathcal{M}_+(z,x)]^{-1} \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} \frac{i}{2z^{1/2}} \sum_{k=0}^{\infty} \widehat{\mathfrak{R}}_k(x) z^{-k} \tag{3.75}$$

for some coeficients $\widehat{\mathfrak{R}}_k$. Explicitly, one obtains

$$\widehat{\mathfrak{R}}_0 = \mathcal{I}_m, \quad \widehat{\mathfrak{R}}_1 = \tfrac{1}{2}\mathcal{Q}, \quad \widehat{\mathfrak{R}}_2 = -\tfrac{1}{8}\mathcal{Q}'' + \tfrac{3}{8}\mathcal{Q}^2, \text{ etc.} \tag{3.76}$$

The stationary KdV hierarchy is then given by

$$\text{s-KdV}_k(\mathcal{Q}) = -2\sum_{\ell=0}^{k} c_{k-\ell}\widehat{\mathfrak{R}}'_{\ell+1}(\mathcal{Q}, \dots) = 0, \quad k \in \mathbb{N}_0, \tag{3.77}$$

where $\{c_\ell\}_{\ell=1,\dots,k} \subset \mathbb{C}$, $c_0 = 1$ denotes a set of constants.
By Remark 2.6, each $\widehat{\mathfrak{R}}_\ell$ is a differential polynomial in $\mathcal{Q}$ and next we slightly abuse notation and indicate this by writing $\widehat{\mathfrak{R}}_\ell(\mathcal{Q}, \dots)$ for $\widehat{\mathfrak{R}}_\ell(x)$, $\widehat{\mathfrak{R}}'_{\ell+1}(\mathcal{Q}, \dots)$ for $\widehat{\mathfrak{R}}'_{\ell+1}(x)$, etc.

Theorem 3.8 ([27])**.** *The self-adjoint finite-band potential $\mathcal{Q}_{\Sigma_n} \in C^\infty(\mathbb{R})^{m\times m}$, discussed in Theorem* 3.6*, is a stationary KdV solution satisfying*

$$\text{s-KdV}_n(\mathcal{Q}_{\Sigma_n}) = -2\sum_{\ell=0}^{n} c_{n-\ell}(\underline{E})\widehat{\mathfrak{R}}'_{\ell+1}(\mathcal{Q}_{\Sigma_n}, \dots) = 0. \tag{3.78}$$

Here $c_\ell(\underline{E})$ are given by

$$\begin{aligned} &c_0(\underline{E}) = 1, \\ &c_k(\underline{E}) = -\sum_{\substack{j_0,\dots,j_{2n}=0 \\ j_0+\cdots+j_{2n}=k}}^{k} \frac{(2j_0)!\cdots(2j_{2n})!}{2^{2k}(j_0!)^2\cdots(j_{2n}!)^2(2j_0-1)\cdots(2j_{2n}-1)} E_0^{j_0}\cdots E_{2n}^{j_{2n}}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad k = 1,\dots,n. \end{aligned} \tag{3.79}$$

4. Matrix Extensions of Borg's and Hochstadt's Theorems

In this our principal section, we now prove Theorem 1.5, the matrix extension of Borg's and Hochstadt's theorem, Theorems 1.1 and 1.3. Our strategy of proof will be the following: First we show that the (reflectionless) Schrödinger operators H_{Σ_ℓ} constructed in our previous Section 3 with spectrum Σ_ℓ, satisfy the conclusions (1.11) and (1.13) for $\ell = 0, 1$, respectively. Then, in a second step, we will prove that any reflectionless Schrödinger operator with spectrum given by Σ_ℓ, $\ell = 0, 1$, is precisely of the form H_{Σ_ℓ} as constructed in Section 3.

Theorem 4.1. *Let $\ell = 0, 1$ and $\mathcal{Q}_{\Sigma_\ell}$ be the finite-band potentials constructed in Section 3, with $\operatorname{spec}(H_{\Sigma_\ell}) = \Sigma_\ell$ (cf. Theorem 3.6). Then*

$$\mathcal{Q}_{\Sigma_0}(x) = E_0 \mathcal{I}_m \text{ for a.e. } x \in \mathbb{R} \tag{4.1}$$

and

$$\begin{aligned}\mathcal{Q}_{\Sigma_1}(x) &= (1/3)(E_0 + E_1 + E_2)\mathcal{I}_m \\ &\quad + 2\mathcal{U} \operatorname{diag}(\wp(x + \omega_3 + \alpha_1), \dots, \wp(x + \omega_3 + \alpha_m))\mathcal{U}^{-1} \\ &\quad \text{for some } \alpha_j \in \mathbb{R},\ 1 \le j \le m \text{ and a.e. } x \in \mathbb{R},\end{aligned} \tag{4.2}$$

where $\mathcal{U}$ is an $m \times m$ unitary matrix independent of $x \in \mathbb{R}$. Moreover, $\mathcal{Q}_{\Sigma_0}$ satisfies the first element of the KdV hierarchy,

$$\mathcal{Q}'_{\Sigma_0} = 0, \tag{4.3}$$

and $\mathcal{Q}_{\Sigma_1}$ satisfies the stationary KdV equation

$$\mathcal{Q}'''_{\Sigma_1} - 3(\mathcal{Q}^2_{\Sigma_1})' + 2(E_0 + E_1 + E_2)\mathcal{Q}'_{\Sigma_1} = 0. \tag{4.4}$$

Proof. We consider the elementary case $\ell = 0$ first. Then the explicit expressions,

$$\mathcal{F}_{0,\Sigma_0}(z,x) = \mathcal{I}_m, \quad \mathcal{G}_{p,-1,\Sigma_0}(z,x) = 0,\ p = 1,2, \quad \mathcal{H}_{1,\Sigma_0}(z,x) = (z - E_0)\mathcal{I}_m, \tag{4.5}$$

$$\mathcal{M}_{\pm,\Sigma_0}(z,x) = \pm i(z - E_0)^{1/2}\mathcal{I}_m, \quad \mathfrak{g}_{\Sigma_0}(z,x) = (i/2)(z - E_0)^{-1/2}\mathcal{I}_m, \text{ etc.}, \tag{4.6}$$

immediately imply (4.1) and (4.3). Hence we turn to the case $\ell = 1$. In this case one obtains,

$$\mathcal{F}_{1,\Sigma_1}(z,x) = z\mathcal{I}_m + (1/2)\mathcal{Q}_{\Sigma_1}(x) + c_1\mathcal{I}_m, \tag{4.7}$$

$$\mathcal{G}_{p,0,\Sigma_1}(z,x) = -(1/4)\mathcal{Q}'_{\Sigma_1}(x),\ p = 1,2, \tag{4.8}$$

$$\mathcal{H}_{2,\Sigma_1}(z,x) = z^2\mathcal{I}_m + z(-(1/2)\mathcal{Q}_{\Sigma_1} + c_1\mathcal{I}_m) + (1/4)\mathcal{Q}''_{\Sigma_1} - (1/2)\mathcal{Q}^2_{\Sigma_1} - c_1\mathcal{Q}_{\Sigma_1}, \tag{4.9}$$

$$\mathcal{M}_{\pm,\Sigma_1}(z,x) = \pm iR_3(z)^{1/2}\mathcal{F}_{1,\Sigma_1}(z,x)^{-1} - \mathcal{G}_{1,0,\Sigma_1}(z,x)\mathcal{F}_{1,\Sigma_1}(z,x)^{-1}, \tag{4.10}$$

$$\mathfrak{g}_{\Sigma_1}(z,x) = (i/2)R_3(z)^{-1/2}(z\mathcal{I}_m + (1/2)\mathcal{Q}_{\Sigma_1}(x) + c_1\mathcal{I}_m)^{-1}, \text{ etc.}, \tag{4.11}$$

abbreviating

$$c_1 = -(1/2)(E_0 + E_1 + E_2). \tag{4.12}$$

Combining (3.61), (4.7), and (4.8), $\mathcal{Q}_{\Sigma_1}(x)$ and $\mathcal{Q}'_{\Sigma_1}(x)$ commute and hence one obtains for each $x \in \mathbb{R}$,

$$[\mathcal{Q}^{(r)}_{\Sigma_1}(x), \mathcal{Q}^{(s)}_{\Sigma_1}(x)] = 0 \text{ for all } r, s \in \mathbb{N}_0. \tag{4.13}$$

Since $\mathcal{Q}_{\Sigma_1}$ and all its derivatives are self-adjoint, one can simultaneously diagonalize the family of matrices $\{\mathcal{Q}^{(r)}_{\Sigma_1}(x_0)\}_{r\in\mathbb{N}_0}$ by a fixed unitary $m \times m$ matrix $\mathcal{U}$. By (4.7)–(4.10), this also shows that $\mathcal{F}_{1,\Sigma_1}(z,x_0)$, $\mathcal{G}_{p,0,\Sigma_1}(z,x_0)$, $p = 1,2$, $\mathcal{H}_{2,\Sigma_1}(z,x_0)$, and $\mathcal{M}_{\pm,\Sigma_1}(z,x_0)$ can all be simultaneously diagonalized by $\mathcal{U}$. In particular, the spectral measure in the Herglotz representation (3.44) of $\mathcal{M}_{\pm,\Sigma_1}(z,x_0)$ can be diagonalized by $\mathcal{U}$. After diagonalization with $\mathcal{U}$, the inverse spectral approach in [66] (i.e., the matrix-valued extension of the scalar Gelfand–Levitan method [24], [52,], [77]) then yields a diagonal matrix potential of the type

$$\operatorname{diag}(q_{1,\Sigma_1}(x), \dots, q_{m,\Sigma_1}(x)), \tag{4.14}$$

and hence $\mathcal{Q}_{\Sigma_1}$ itself is of the form

$$\mathcal{Q}_{\Sigma_1}(x) = \mathcal{U}\,\mathrm{diag}(q_{1,\Sigma_1}(x), \dots, q_{m,\Sigma_1}(x))\mathcal{U}^{-1}. \tag{4.15}$$

In order to determine the scalar potentials q_{k,Σ_1}, $1 \le k \le m$, it now suffices to solve the corresponding scalar problem ($m = 1$). But then Hochstadt's result [35] immediately yields

$$q_{k,\Sigma_1}(x) = (1/3)(E_0 + E_1 + E_2) + 2\wp(x + \omega_3 + \alpha_k), \quad 1 \le k \le m, \tag{4.16}$$

for some $\{\alpha_k\}_{1\le k\le m} \subset \mathbb{R}$, and hence (4.2). Finally, (4.4) is a consequence of (3.78), (3.79), taking $n = 1$. $\square$

To complete the proof of Theorem 1.5, we next will prove that reflectionless Schrödinger operators with spectrum equal to Σ_ℓ, $\ell = 0, 1$, in fact, coincide with some element of the family H_{Σ_ℓ}, described in Section 3.

Theorem 4.2. *Suppose $\mathcal{Q}_\ell$, $\ell = 0, 1$ satisfies Hypothesis* 2.1*, define H_ℓ as in* (2.14)*, and suppose* $\mathrm{spec}(H_\ell) = \Sigma_\ell$, $\ell = 0, 1$*. In addition, assume that Q_ℓ, $\ell = 0, 1$, is reflectionless. Then H_ℓ coincides with an element of the family of operators H_{Σ_ℓ} parametrized by a choice of $\mathcal{F}_{\ell,\Sigma_\ell}(z, x_0)$, $\mathcal{G}_{1,\ell-1,\Sigma_\ell}(z, x_0)$, $\ell = 0, 1$*[1].

Proof. We start with the case $\ell = 0$. Recalling $\Sigma_0 = [E_0, \infty)$, and $R_1(z)^{1/2} = (z - E_0)^{1/2}$ defined as in (3.3),(3.4), followed by an analytic continuation from $\mathbb{R}$ to $\mathbb{C}\backslash\Sigma_0$, we denote by $\mathfrak{g}(\Sigma_0, z, x)$ the diagonal Green's function of H_0 (cf. (2.25)),

$$\mathfrak{g}(\Sigma_0, z, x) = (\mathcal{M}_-(\Sigma_0, z, x) - \mathcal{M}_+(\Sigma_0, z, x))^{-1}, \tag{4.17}$$

where in obvious notation $\mathcal{M}_\pm(\Sigma_0, z, x)$ denote the Weyl-Titchmarsh matrices associated with $\mathcal{Q}_0$. Since $\mathcal{Q}_0$ is reflectionless, the matrix $\Xi(\Sigma_0, \cdot, x)$ in its associated exponential Herglotz representation (2.26) satisfies,

$$\Xi(\Sigma_0, \lambda, x) = \begin{cases} (1/2)\mathcal{I}_m & \text{for a.e. } \lambda \in (E_0, \infty), \\ 0 & \text{for a.e. } \lambda \in (-\infty, E_0). \end{cases} \tag{4.18}$$

Insertion of (4.18) into (2.26) then yields

$$\mathfrak{g}(\Sigma_0, z, x) = i(z - E_0)^{-1/2} \exp(\mathfrak{C}_0(x)), \quad z \in \mathbb{C}\backslash\Sigma_0. \tag{4.19}$$

A comparison with the high-energy asymptotics of $\mathfrak{g}$ implied by (2.32) and (4.17) yields

$$\mathfrak{g}(\Sigma_0, z, x) \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} (i/2)\mathcal{I}_m z^{-1/2} + o(1) \tag{4.20}$$

and hence $\mathfrak{C}_0(x) = -\ln(2)\mathcal{I}_m$ implying

$$\mathfrak{g}(\Sigma_0, z, x) = (i/2)(z - E_0)^{-1/2}\mathcal{I}_m, \quad z \in \mathbb{C}\backslash\Sigma_0. \tag{4.21}$$

Thus,

$$\mathcal{Q}_0(x) = \mathcal{Q}_{\Sigma_0}(x) = E_0\,\mathcal{I}_m, \quad x \in \mathbb{R}, \tag{4.22}$$

with $\mathcal{Q}_{\Sigma_0}$ constructed in Section 3 (cf. also (4.5), (4.6)).
Next we turn to the case $\ell = 1$. Recalling $\Sigma_1 = [E_0, E_1] \cup [E_2, \infty)$, and $R_3(z)^{1/2} =$

[1] More precisely, a choice of $\mathcal{F}_{0,\Sigma_\ell}(z, x_0) = \mathcal{I}_m$ for $\ell = 0$ and a choice of $\mathcal{F}_{1,\Sigma_\ell}(z, x_0)$ and a set of signs $\varepsilon_k(x_0) \in \{1, -1\}$, $k = 1, \dots, N$ (cf. (4.55)) for $\ell = 1$.

$[(z-E_0)(z-E_1)(z-E_2)]^{1/2}$ defined as in (3.3), (3.4), followed by an analytic continuation from $\mathbb{R}$ to $\mathbb{C}\backslash\Sigma_1$, we introduce

$$\mathfrak{g}(\Sigma_1,z,x)=(\mathcal{M}_-(\Sigma_1,z,x)-\mathcal{M}_+(\Sigma_1,z,x))^{-1}, \tag{4.23}$$

where in obvious notation $\mathcal{M}_\pm(\Sigma_1,z,x)$ denote the Weyl-Titchmarsh matrices associated with $\mathcal{Q}_1$, and note that $\mathfrak{g}(\Sigma_1,\lambda,x)$ is a self-adjoint matrix for $\lambda\in(-\infty,E_0)\cup(E_1,E_2)$, that is, for all $x\in\mathbb{R}$,

$$\mathfrak{g}(\Sigma_1,\lambda,x)=\mathfrak{g}(\Sigma_1,\lambda,x)^*,\ \lambda\in(-\infty,E_0)\cup(E_1,E_2). \tag{4.24}$$

Since by hypothesis $\mathcal{Q}_1$ is reflectionless, one also has for all $x\in\mathbb{R}$,

$$\lim_{\varepsilon\downarrow 0}\mathfrak{g}(\Sigma_1,\lambda+i\varepsilon,x)^*=-\lim_{\varepsilon\downarrow 0}\mathfrak{g}(\Sigma_1,\lambda+i\varepsilon,x) \text{ for a.e. } \lambda\in(E_0,E_1)\cup(E_2,\infty). \tag{4.25}$$

Combining (4.24) and (4.25) with the properties of $R_3(z)^{1/2}$ as discussed in (3.4), one infers that for all $x\in\mathbb{R}$,

$$-i\lim_{\varepsilon\downarrow 0}R_3(\lambda+i\varepsilon)^{1/2}\mathfrak{g}(\Sigma_1,\lambda+i\varepsilon,x) \text{ is self-adjoint for a.e. } \lambda\in\mathbb{R}. \tag{4.26}$$

Next, we will take a closer look at $\mathfrak{g}(\Sigma_1,\cdot)$ and show that (4.26) in fact holds for all $\lambda\in\mathbb{R}$. Since $\mathcal{Q}_1$ is reflectionless, the matrix $\Xi(\Sigma_1,\cdot,x)$ in its associated exponential Herglotz representation (2.26) satisfies,

$$\Xi(\Sigma_1,\lambda,x)=\begin{cases}(1/2)\mathcal{I}_m & \text{for a.e. } \lambda\in(E_0,E_1)\cup(E_2,\infty),\\ 0 & \text{for a.e. } \lambda\in(-\infty,E_0).\end{cases} \tag{4.27}$$

Insertion of (4.27) into (2.26) then yields

$$\begin{aligned}\mathfrak{g}(\Sigma_1,z,x)&=i\bigg(\frac{1+E_0^2}{1+E_1^2}\bigg)^{1/4}\bigg(\frac{(z-E_1)}{(z-E_0)(z-E_2)}\bigg)^{1/2} \\ &\times\exp\bigg(\mathfrak{C}_1(x)+\int_{E_1}^{E_2}d\lambda\,\Xi(\Sigma_1,\lambda,x)\big((\lambda-z)^{-1}-\lambda(1+\lambda^2)^{-1}\big)\bigg), \quad z\in\mathbb{C}\backslash\Sigma_1.\end{aligned} \tag{4.28}$$

As a consequence, one obtains

$$\begin{aligned}-iR_3(z)^{1/2}\mathfrak{g}(\Sigma_1,z,x)&=\bigg(\frac{1+E_0^2}{1+E_1^2}\bigg)^{1/4}(z-E_1) \\ &\times\exp\bigg(\mathfrak{C}_1(x)+\int_{E_1}^{E_2}d\lambda\,\Xi(\Sigma_1,\lambda,x)\big((\lambda-z)^{-1}-\lambda(1+\lambda^2)^{-1}\big)\bigg), \quad z\in\mathbb{C}\backslash\Sigma_1.\end{aligned} \tag{4.29}$$

By (4.29), $-i\lim_{\varepsilon\downarrow 0}R_3(\lambda+i\varepsilon)^{1/2}\mathfrak{g}(\Sigma_1,\lambda+i\varepsilon,x)$ is self-adjoint for $\lambda\in\mathbb{R}\backslash(E_1,E_2)$. However, since $\mathfrak{g}(\Sigma_1,z,x)$ is analytic in $z\in\mathbb{C}\backslash\Sigma_1$, one infers that $-i\lim_{\varepsilon\downarrow 0}R_3(\lambda+i\varepsilon)^{1/2}\mathfrak{g}(\Sigma_1,\lambda+i\varepsilon,x)$ is self-adjoint for $\lambda\in\mathbb{R}\backslash\{E_1,E_2\}$. Next, a comparison with the high-energy asymptotics of $\mathfrak{g}$ implied by (2.32) and (4.23) yields

$$\mathfrak{g}(\Sigma_1,z,x)\underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=}(i/2)\mathcal{I}_m z^{-1/2}+o(1) \tag{4.30}$$

and hence

$$\mathfrak{C}_1(x)=\bigg(\frac{1}{4}\ln\bigg(\frac{1+E_1^2}{1+E_0^2}\bigg)-\ln(2)\bigg)\mathcal{I}_m+\int_{E_1}^{E_2}d\lambda\,\Xi(\Sigma_1,\lambda,x)\frac{\lambda}{1+\lambda^2}. \tag{4.31}$$

Moreover, since $0 \leq \Xi(\Sigma_1, z, x) \leq \mathcal{I}_m$, we obtain for $z = E_1 - \varepsilon$,

$$\begin{aligned}
&(E_1 - z)\exp\bigg(\mathfrak{C}_1(x) + \int_{E_1}^{E_2} d\lambda\, \Xi(\Sigma_1, \lambda, x)\big((\lambda - z)^{-1} - \lambda(1+\lambda^2)^{-1}\big)\bigg) \\
&= \frac{E_2 - E_1 + \varepsilon}{2}\bigg(\frac{(1+E_2^2)^2}{(1+E_0^2)(1+E_1^2)}\bigg)^{1/4} \\
&\quad \times \exp\bigg(-\int_{E_1}^{E_2} d\lambda\, (\mathcal{I}_m - \Xi(\Sigma_1, \lambda, x))(\lambda - E_1 + \varepsilon)^{-1}\bigg)
\end{aligned} \tag{4.32}$$

and hence (4.29) remains bounded at $z = E_1$. Thus, $-i\lim_{\varepsilon\downarrow 0} R_3(\lambda+i\varepsilon)^{1/2}\mathfrak{g}(\Sigma_1, \lambda + i\varepsilon, x)$ is self-adjoint for $\lambda \in \mathbb{R}\backslash\{E_2\}$ and by the Schwartz reflection principle, $-iR_3(z)^{1/2}\mathfrak{g}(\Sigma_1, z, x)$ is analytic for $z \in \mathbb{C}\backslash\{E_2\}$. Finally, if E_2 would be a pole of $-iR_3(z)^{1/2}\mathfrak{g}(\Sigma_1, z, x)$, then $\mathfrak{g}(\Sigma_1, z, x)$ would have a $(z - E_2)^{-(3/2)}$ singularity at E_2, contradicting the Herglotz property of $\mathfrak{g}(\Sigma_1, \cdot, x)$. (Of course, the same argument applies to $z = E_1$.) Thus,

$$-iR_3(z)^{1/2}\mathfrak{g}(\Sigma_1, z, x) \text{ is entire with respect to } z. \tag{4.33}$$

By (4.29), one infers the bound

$$\| - iR_3(z)^{1/2}\mathfrak{g}(\Sigma_1, z, x)\| \leq C(x)|z| \text{ for } |z| > \max(|E_1|, |E_2|) \tag{4.34}$$

for some constant $C(x) > 0$. Thus, $\mathfrak{g}(\Sigma_1, z, x)$ is of the form

$$\mathfrak{g}(\Sigma_1, \lambda, x) = (i/2)R_3(z)^{-1/2}(\mathcal{A}(x)z + \mathcal{B}(x)), \tag{4.35}$$

for some $m \times m$ matrices $\mathcal{A}(x), \mathcal{B}(x) \in \mathbb{C}^{m\times m}$. Hence $\mathfrak{g}(\Sigma_1, z, x)$ has an asymptotic expansion to all orders as $|z| \to \infty$ and an insertion of the asymptotic expansion into the Riccati-type equation (2.29) yields (cf. also (2.31))

$$\mathfrak{g}(\Sigma_1, z, x) = (i/2)R_3(z)^{-1/2}\big(\mathcal{I}_m z + (1/2)\mathcal{Q}_1(x) + c_1\mathcal{I}_m\big), \quad z \in \mathbb{C}\backslash\Sigma_1,\ x \in \mathbb{R}. \tag{4.36}$$

with

$$c_1 = -(E_0 + E_1 + E_2)/2. \tag{4.37}$$

By (4.36), (2.35), (2.36), and (2.47), $\mathcal{Q}_1$ is locally absolutely continuous on $\mathbb{R}$. By (2.48) and (2.50), $\mathcal{Q}_1'$ is locally absolutely continuous on $\mathbb{R}$. Iterating this procedure, using (2.47)–(2.55), one infers inductively that

$$\mathcal{Q}_1 \in C^\infty(\mathbb{R}). \tag{4.38}$$

(2.53) and (4.36) then yield

$$\begin{aligned}
\mathfrak{h}(\Sigma_1, z, x) = (i/2)R_3(z)^{-1/2}\big(&\mathcal{I}_m z^2 + (-(1/2)\mathcal{Q}_1(x) + c_1\mathcal{I}_m)z \\
&+ (1/4)\mathcal{Q}_1''(x) - (1/2)\mathcal{Q}_1(x)^2 - c_1\mathcal{Q}_1(x)\big), \\
&\qquad\qquad z \in \mathbb{C}\backslash\Sigma_1,\ x \in \mathbb{R},
\end{aligned} \tag{4.39}$$

and (4.36), (2.49), and (2.51) prove

$$\mathfrak{g}_{p,0}(\Sigma_1, z, x) = -(i/8)R_3(z)^{-1/2}\big(\mathcal{Q}_1'(x) + \mathcal{C}_p\big), \quad p = 1, 2, \tag{4.40}$$

for some constant matrices $\mathcal{C}_p \in \mathbb{C}^{m\times m}$, $p = 1, 2$. Insertion of (4.36) and (4.40) into (2.45), (2.46), taking into account the asymptotics (2.30), (2.31) of the Weyl–Titchmarsh matrices $\mathcal{M}_\pm(\Sigma_1, z, x)$ associated with $\mathcal{Q}_1$ then shows $\mathcal{C}_p = 0$, $p = 1, 2$, and hence

$$\mathfrak{g}_{p,0}(\Sigma_1, z, x) = -(i/8)R_3(z)^{-1/2}\mathcal{Q}_1'(x), \quad p = 1, 2, \ z \in \mathbb{C}\backslash\Sigma_1, \ x \in \mathbb{R}. \tag{4.41}$$

By (2.40), (4.36), and (4.41), $\mathcal{Q}_1(x)$ and $\mathcal{Q}_1'(x)$ commute and hence one obtains for each $x \in \mathbb{R}$,

$$[\mathcal{Q}_1^{(r)}(x), \mathcal{Q}_1^{(s)}(x)] = 0 \text{ for all } r, s \in \mathbb{N}_0. \tag{4.42}$$

Next, multiplying (2.44) by $R_3(z)$ and collecting the coefficients of z^k, $0 \leq k \leq 2$, yields

$$(1/4)\mathcal{Q}_1''(x) - (3/4)\mathcal{Q}_1(x)^2 - c_1\mathcal{Q}_1(x) + d_1\mathcal{I}_m = 0, \tag{4.43}$$

$$\begin{aligned}&((1/4)\mathcal{Q}_1''(x) - (1/2)\mathcal{Q}_1(x)^2 - c_1\mathcal{Q}_1(x))((1/2)\mathcal{Q}_1(x) + c_1\mathcal{I}_m)\\ &- (1/16)\mathcal{Q}_1'(x)^2 + E_0E_1E_2\mathcal{I}_m = 0,\end{aligned} \tag{4.44}$$

with

$$d_1 = c_1^2 - \sum_{\substack{k_1,k_2=0\\ k_1<k_2}}^{2} E_{k_1}E_{k_2}. \tag{4.45}$$

Eliminating $\mathcal{Q}_1''(x)$ in (4.43), (4.44) finally yields

$$\mathcal{Q}_1'(x)^2 = -16R_3(-(1/2)\mathcal{Q}_1(x) - c_1\mathcal{I}_m). \tag{4.46}$$

Since $\mathcal{Q}_1(x)$ is a self-adjoint $m \times m$ matrix, we may write

$$\mathcal{Q}_1(x) = \sum_{k=1}^{N} q_k(x)\mathcal{P}_k(x), \tag{4.47}$$

where $q_k(x)$ and $\mathcal{P}_k(x)$ denote the eigenvalues and corresponding self-adjoint spectral projections of $\mathcal{Q}_1(x)$, that is,

$$\mathcal{P}_k(x)\mathcal{P}_\ell(x) = \delta_{k,\ell}\mathcal{P}_\ell(x), \quad \sum_{k=1}^{N}\mathcal{P}_k(x) = \mathcal{I}_m. \tag{4.48}$$

Introducing $\mathcal{F}_1(\Sigma_1, z, x)$ by

$$\mathcal{F}_1(\Sigma_1, z, x) = z\mathcal{I}_m + (1/2)\mathcal{Q}_1(x) + c_1\mathcal{I}_m, \tag{4.49}$$

this implies

$$\mathcal{F}_1(\Sigma_1, z, x) = \sum_{k=1}^{N}(z - \mu_k(x))\mathcal{P}_k(x), \quad \mu_k(x) = -(1/2)q_k(x) - c_1. \tag{4.50}$$

Since by (3.4),

$$R_3(\lambda)^{1/2} = |R_3(\lambda)|^{1/2}\begin{cases}-1, & \lambda \in (E_0, E_1),\\ 1, & \lambda \in (E_2, \infty),\end{cases} \tag{4.51}$$

and

$$R_3(\lambda)^{-1/2}\mathcal{F}_1(\Sigma_1, \lambda, x) > 0, \quad \lambda \in \Sigma_1^o, \tag{4.52}$$

one concludes that for fixed $x \in \mathbb{R}$ and all $g \in \mathbb{C}^m$, $(g, \mathcal{F}_1(\Sigma_1, \lambda, x)g)_{\mathbb{C}^m}$ changes sign for $\lambda \in [E_1, E_2]$. Thus,

$$\mu_k(x) \in [E_1, E_2], \quad 1 \leq k \leq N, \tag{4.53}$$

in accordance with (3.33). A comparison with (3.32) then yields

$$\begin{aligned}\Gamma_k(\Sigma_1,x) &= -i \lim_{z\to\mu_k(x)} (z-\mu_k(x))R_3(z)^{1/2}\mathcal{F}_1(\Sigma_1,z,x)^{-1} \\ &= -iR_3(\mu_k(x))^{1/2}\mathcal{P}_k(x).\end{aligned} \tag{4.54}$$

Hence, given a sequence of signs,

$$\varepsilon_k(x) \in \{1,-1\}, \quad 1 \le k \le N, \tag{4.55}$$

and temporarily assuming

$$\mu_k(x) \in (E_1,E_2), \quad 1 \le k \le N, \tag{4.56}$$

one computes

$$\begin{aligned}\mathcal{G}_{1,0}(\Sigma_1,z,x) &= \bigg(\sum_{k=1}^N \frac{\varepsilon_k(x)}{z-\mu_k(x)}\Gamma_k(\Sigma_1,x)\bigg)\mathcal{F}_1(\Sigma_1,z,x) \\ &= -i\sum_{k=1}^n \varepsilon_k(x)R_3(\mu_k(x))^{1/2}\Gamma_k(\Sigma_1,x) \\ &= -iR_3(-(1/2)\mathcal{Q}_1(x)-c_1\mathcal{I}_m)^{1/2}.\end{aligned} \tag{4.57}$$

Here the choice of the matrix square root in (4.57) is a direct consequence of the choice of signs $\varepsilon_k(x)$, $1 \le k \le N$. By equation (4.46), one obtains

$$\mathcal{G}_{1,0}(\Sigma_1,z,x) = -(1/4)\mathcal{Q}_1'(x) \tag{4.58}$$

in accordance with (4.40). In particular, $\mathcal{F}_1(\Sigma_1,z,x)$ in (4.49) and $\mathcal{G}_{1,0}(\Sigma_1,\lambda,x)$ in (4.58) are of the form (4.7) and (4.8). Finally, the temporary restriction (4.56) can be removed by continuity. Summing up,

$$\mathcal{Q}_1(x) = \mathcal{Q}_{\Sigma_1}(x), \quad x \in \mathbb{R}, \tag{4.59}$$

with $\mathcal{Q}_{\Sigma_1}$ constructed as in Section 3 given some $\mathcal{F}_{1,\sigma_1}(z,x_0)$ and some choice of signs $\varepsilon_k(x_0) \in \{1,-1\}$, $1 \le k \le N$ (cf. also (4.7)–(4.12)). $\square$

We note that the case $\ell = 0$ in Theorem 4.2 was originally treated in [16] using a (matrix-valued) trace formula approach.

Combining Theorems 4.1 and 4.2 proves Theorem 1.5.

Since (self-adjoint) periodic potentials $\mathcal{Q}$ which lead to Schrödinger operators with uniform maximum spectral multiplicity are reflectionless in the sense of Definition 2.7 as shown in [16], Theorem 1.6 yields the proper matrix generalizations of Borg's and Hochstadt's results, Theorem 1.1 and 1.3.

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