CRITICAL REGION FOR DROPLET FORMATION IN THE TWO-DIMENSIONAL ISING MODEL

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Abstract: We study the formation/dissolution of equilibrium droplets in finite systems at parameters corresponding to phase coexistence. Specifically, we consider the 2D Ising model in volumes of size L^2, inverse temperature $\beta > \beta_c$ and overall magnetization conditioned to take the value $m^\star L^2 - 2m^\star v_L$, where β_c^{-1} is the critical temperature, $m^\star = m^\star(\beta)$ is the spontaneous magnetization and v_L is a sequence of positive numbers. We find that the critical scaling for droplet formation/dissolution is when $v_L^{3/2} L^{-2}$ tends to a definite limit. Specifically, we identify a dimensionless parameter Δ, proportional to this limit, a non-trivial critical value Δ_c and a function λ_Δ such that the following holds: For $\Delta < \Delta_c$, there are no droplets beyond $\log L$ scale, while for $\Delta > \Delta_c$, there is a single, Wulff-shaped droplet containing a fraction $\lambda_\Delta \geq \lambda_c = 2/3$ of the magnetization deficit and there are no other droplets beyond the scale of $\log L$. Moreover, λ_Δ and Δ are related via a universal equation that apparently is independent of the details of the system.

CONTENTS

1. INTRODUCTION

1.1 Motivation.

The connection between microscopic interactions and pure-phase (bulk) thermodynamics has been understood at a mathematically sophisticated level for many years. However, an analysis of systems at phase coexistence which contain droplets has begun only recently. Over a century ago, Curie [22], Gibbs [30] and Wulff [52] derived from surface-thermodynamical considerations that a single droplet of a particular shape—the *Wulff shape*—will appear in systems that are forced to exhibit a fixed excess of a minority phase. A mathematical proof of this fact starting from a system defined on the microscopic scale has been given in the context of percolation and Ising systems, first in dimension $d = 2$ [4, 24] and, more recently, in all dimensions $d \geq 3$ [11, 18, 19]. Other topics related to the droplet shape have intensively been studied: Fluctuations of a contour line [3, 15–17, 23, 34], wetting phenomena [47] and Gaussian fields near a "wall" [5, 13, 26]. See [12] for a summary of these results and comments on the (recent) history of these developments.

The initial stages of the rigorous "Wulff construction" program have focused on systems in which the droplet subsumes a finite fraction of the available volume. Of no less interest is the situation when the excess represents only a vanishing fraction of the total volume. In [25], substantial progress has been made on these questions in the context of the Ising model at low temperatures. Subsequent developments [35, 36, 45, 46] have allowed the extension, in $d = 2$, of the aforementioned results up to the critical point [37]. Specifically, what has so far been shown is as follows: For two-dimensional volumes Λ_L of side L and $\delta > 0$ arbitrarily small, if the magnetization deficit exceeds $L^{4/3+\delta}$, then a Wulff droplet accounts, pretty much, for all the deficit, while if the magnetization deficit is bounded by $L^{4/3-\delta}$, there are no droplets beyond the scale of $\log L$. The preceding are of course asymptotic statements that hold with probability tending to one as $L \to \infty$.

The focus of this paper is the intermediate regime, which has not yet received appropriate attention. Assuming the magnetization deficit divided by $L^{4/3}$ tends to a definite limit, we define a dimensionless parameter, denoted by Δ, which is proportional to this limit. (A precise definition of Δ is provided in (1.10).) Our principal result is as follows: There is a critical value Δ_c such that for $\Delta < \Delta_c$, there are no large droplets (again, nothing beyond $\log L$ scale), while for $\Delta > \Delta_c$, there is a single, large droplet of a diameter of the order $L^{2/3}$. However, in contrast to all situations that have previously been analyzed, this large droplet only accounts for a finite fraction, $\lambda_\Delta < 1$, of the magnetization deficit, which, in addition, does *not* tend to zero as $\Delta \downarrow \Delta_c$! (Indeed, $\lambda_\Delta \downarrow \lambda_c$, with $\lambda_c = 2/3$.) Whenever the droplet appears, its interior is representative of the minus phase, its shape is close to the optimal (Wulff) shape and its volume is tuned to contain the λ_Δ-fraction of the deficit magnetization. Furthermore, for all values of Δ, there is at most one droplet of size $L^{2/3}$ and nothing else beyond the scale $\log L$. At $\Delta = \Delta_c$ the situation is not completely resolved. However, there are only two possibilities: Either there is one droplet of linear size $L^{2/3}$ or no droplet at all.

The above transition is the result of a competition between two mechanisms for coping with a magnetization deficit in the system: Absorption of the deficit by the ambient *fluctuations* or the formation of a *droplet*. The results obtained in [24,25] and [37] deal with the situations when one of the two mechanisms completely dominates the other. As is seen by a simple-minded comparison of the exponential costs of the two mechanisms, $L^{4/3}$ is the only conceivable scaling of the magnetization deficit where these are able to coexist. (This is the core of the heuristic approach outlined in [7,8,43].) However, at the point where the droplets first appear, one can envision alternate scenarios involving complicated fluctuations and/or a multitude of droplets with effective interactions ranging across many scales. To rule out such possibilities it is necessary to demonstrate the absence of these "intermediate-sized" droplets and the insignificance—or absence—of large fluctuations. This was argued on a heuristic level in [9] and will be proven rigorously here.

Thus, instead of blending into each other through a series of intermediate scales, the droplet-dominated and the fluctuation-dominated regimes meet—literally—at a single point. Furthermore, all essential system dependence is encoded into one dimensionless parameter Δ and the transition between the Gaussian-dominated and the droplet-dominated regimes is thus characterized by a *universal* constant Δ_c. In addition, the relative fraction λ_Δ of the deficit "stored" in the droplet depends on Δ via a *universal* equation which is apparently independent of the details of the system [9]. At this point we would like to stress that, even though the rigorous results presented here are restricted to the case of the two-dimensional Ising model, we expect that their validity can be extended to a much larger class of models and the universality of the dependence on Δ will become the subject of a *mathematical* statement. Notwithstanding the rigorous analysis, this universal setting offers the possibility of fitting experimental/numerical data from a variety of systems onto a single curve.

A practical understanding of how droplets disappear is by no means an esoteric issue. Aside from the traditional, i.e., three-dimensional, setting, there are experimental realizations which are effectively two-dimensional (see [39] and references therein). Moreover, there are purported applications of Ising systems undergoing "fragmentation" in such diverse areas as nuclear physics and adatom formation [33]. From the perspective of statistical physics, perhaps more important are the investigations of small systems at parameter values corresponding to a first order transition in the bulk. In these situations, non-convexities appear in finite-volume thermodynamic functions [33, 40, 41, 48], which naturally suggest the appearance of a droplet. Several papers have studied the disappearance of droplets and reported intriguing finite-size characteristics [7, 8, 39, 42, 43, 48, 49]. It is hoped that the results established here will shed some light in these situations.

1.2 The model.

The primary goal of this paper is a detailed description of the above droplet-formation phenomenon in the Ising model. In general dimension, this system is defined by the formal Hamiltonian

$$\mathcal{H} = -\sum_{\langle x,y\rangle} \sigma_x\sigma_y, \tag{1.1}$$

where $\langle x,y\rangle$ denotes a nearest-neighbor pair on $\mathbb{Z}^d$ and where $\sigma_x \in \{-1,+1\}$ denotes an Ising spin. To define the Hamitonian in a finite volume $\Lambda \subset \mathbb{Z}^d$, we use $\partial\Lambda$ to denote the external boundary of Λ, $\partial\Lambda = \{x \notin \Lambda : \text{ there exists a bond } \langle x,y\rangle \text{ with } y \in \Lambda\}$, fix a collection of boundary spins $\sigma_{\partial\Lambda} = (\sigma_x)_{x\in\partial\Lambda}$ and restrict the sum in (1.1) to bonds $\langle x,y\rangle$ such that $\{x,y\}\cap\Lambda \neq \emptyset$. We denote this finite-volume Hamiltonian by $\mathcal{H}_\Lambda(\sigma_\Lambda,\sigma_{\partial\Lambda})$. The special choices of the boundary configurations such that $\sigma_x = +1$, resp., $\sigma_x = -1$ for all $x \in \partial\Lambda$ will be referred to as plus, resp., minus boundary conditions.

The Hamitonian gives rise to the concept of a finite-volume *Gibbs measure* (also known as *Gibbs state*) which is a measure assigning each configuration $\sigma_\Lambda = (\sigma_x)_{x\in\Lambda} \in \{-1,+1\}^\Lambda$ the probability

$$P_\Lambda^{\sigma_{\partial\Lambda},\beta}(\sigma_\Lambda) = \frac{e^{-\beta\mathcal{H}_\Lambda(\sigma_\Lambda,\sigma_{\partial\Lambda})}}{Z_\Lambda^{\sigma_{\partial\Lambda}}(\beta)}. \tag{1.2}$$

Here $\beta \geq 0$ denotes the inverse temperature, $\sigma_{\partial\Lambda}$ is an arbitrary boundary configuration and $Z_\Lambda^{\sigma_{\partial\Lambda}}(\beta)$ is the partition function. Most of this work will concentrate on squares of $L\times L$ sites, which we will denote by Λ_L, and the plus boundary conditions. In this case we denote the above probability by $P_L^{+,\beta}(-)$ and the associated expectation by $\langle - \rangle_L^{+,\beta}$. As the choice of the signs in (1.1–1.2) indicates, the measure $P_L^{+,\beta}$ with $\beta > 0$ tends to favor alignment of neighboring spins with an excess of plus spins over minus spins.

Remark 1. As is well known, the Ising model is equivalent to a model of a lattice gas where at most one particle is allowed to occupy each site. In our case, the sites occupied by a particle are represented by minus spins, while the plus spins correspond to the sites with no particles. In the particle distribution induced by $P_L^{+,\beta}$, the total number of particles is not fixed; hence, we will occasionally refer to this measure as the "grandcanonical" ensemble. On the other hand, if the number of minus spins is fixed (by conditioning on the total magnetization, see Section 1.3), the resulting measure will sometimes be referred to as the "canonical" ensemble.

The Ising model has been studied very extensively by mathematical physicists in the last 20-30 years and a lot of interesting facts have been rigorously established. We proceed by listing the properties of the *two-dimensional* model which will ultimately be needed in this paper. For general overviews of various aspects mentioned below we refer to, e.g., [12, 28, 29, 51]. The readers familiar with the background (and the standard notation) should feel free to skip the remainder of this section and go directly to Section 1.3 where we discuss the main results of the present paper.

• *Bulk properties.* For all $\beta \geq 0$, the measure $P_L^{+,\beta}$ has a unique infinite volume (weak) limit $P^{+,\beta}$ which is a translation-invariant, ergodic, extremal Gibbs state for the interaction (1.1). Let $\langle - \rangle^{+,\beta}$ denote the expectation with respect to $P^{+,\beta}$. The persistence of the plus-bias in the thermodynamic limit, characterized by the *magnetization*

$$m^\star(\beta) = \langle \sigma_0 \rangle^{+,\beta}, \tag{1.3}$$

marks the region of phase coexistence in this model. Indeed, there is a non-trivial critical value $\beta_c \in (0, \infty)$—known [1, 6, 38, 44] to satisfy $e^{2\beta_c} = 1 + \sqrt{2}$—such that for $\beta > \beta_c$, we have $m^\star(\beta) > 0$ and there are multiple infinite-volume Gibbs states, while for $\beta \leq \beta_c$, the magnetization vanishes and there is a unique infinite-volume Gibbs state for the interaction (1.1). Further, using $\langle A; B \rangle^{+,\beta}$ to denote the truncated correlation function $\langle AB \rangle^{+,\beta} - \langle A \rangle^{+,\beta} \langle B \rangle^{+,\beta}$, the magnetic *susceptibility*, defined by

$$\chi(\beta) = \sum_{x \in \mathbb{Z}^2} \langle \sigma_0; \sigma_x \rangle^{+,\beta}, \tag{1.4}$$

is finite for all $\beta > \beta_c$, see [21, 50]. By the GHS or FKG inequalities, we have $\chi(\beta) \geq 1 - m^\star(\beta)^2 > 0$ for all $\beta \in [0, \infty)$.

• *Peierls' contours.* Our next requisite item is a description of the Ising configurations in terms of Peierls' contours. Given an Ising configuration in Λ with plus boundary conditions, we consider the set of dual bonds intersecting direct bonds that connect a plus spin with a minus spin. These dual bonds will be assembled into contours as follows: First we note that only an even number of dual bonds meet at each site of the dual lattice. When two bonds meet at a single dual site, we simply connect them. When four bonds are incident with one dual lattice site, we apply the rounding rule "south-east/north-west" to resolve the "cross" into two curves "bouncing" off each other (see, e.g., [24, 46] or Figure 1). Using these rules consistently, the aforementioned set of dual bonds decomposes into a set of non self-intersecting polygons with rounded corners. These are our *contours*.

Each contour γ is a boundary of a bounded subset of $\mathbb{R}^2$, which we denote by $V(\gamma)$. We will also need a symbol for the set of sites in the interior of γ; we let $\mathbb{V}(\gamma) = V(\gamma) \cap \mathbb{Z}^2$. The *diameter* of a contour γ is defined as the diameter of the set $V(\gamma)$ in the ℓ_2-metric on $\mathbb{R}^2$. In the thermodynamic interpretation used in Section 1.1, contours represent microscopic boundaries of droplets. The advantage of the contour language is that it permits the identification of a sharp boundary between two phases; the disadvantage is that, in order to study the typical shape (and other properties) of large droplets, one has to first resum over small fluctuations of this boundary.

• *Surface tension.* In order to study droplet equilibrium, we need to introduce the concept of microscopic surface tension. Following [4, 45], on $\mathbb{Z}^2$ we can conveniently use *duality*. Given a $\beta > \beta_c$, let $\beta^* = \frac{1}{2} \log \coth \beta$ denote the *dual temperature*. For any $(k_1, k_2) \in \mathbb{Z}^2$ and $k =$

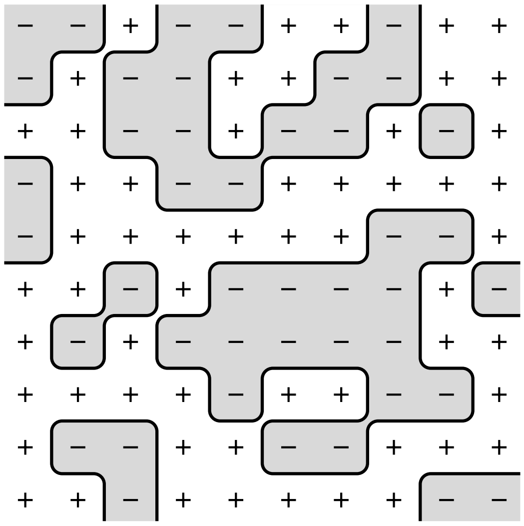

FIGURE 1. An example of an Ising spin configuration and its associated Peierls' contours. In general, a contour consists of a string of dual lattice bonds that bisect a direct bond between a plus spin and a minus spin. When four such dual bonds meet at a single (dual) lattice site, an ambiguity is resolved by applying the south-east/north-west rounding rule. (The remaining corners are rounded just for æsthetic reasons.) The shaded areas correspond to the part of $V(\gamma)$ occupied by the minus spins.

$(k_1^2+k_2^2)^{1/2}$, let $\boldsymbol{n} = (k_1/k, k_2/k) \in \mathcal{S}_1 = \{x \in \mathbb{R}^2\colon \|x\| = 1\}$. (Here $\|x\|$ is the Euclidean norm of x.) Then the limit

$$\tau_\beta(\boldsymbol{n}) = \lim_{N\to\infty} \frac{1}{Nk} \log\langle \sigma_0 \sigma_{Nk\boldsymbol{n}}\rangle^{+,\beta^*}, \tag{1.5}$$

where $Nk\boldsymbol{n} = (k_1N, k_2N) \in \mathbb{Z}^2$, exists independently of what integers k_1 and k_2 we chose to represent $\boldsymbol{n}$ and defines a function on a dense subset of $\mathcal{S}_1$. It turns out that this function can be continuously extended to all $\boldsymbol{n} \in \mathcal{S}_1$. We call the resulting quantity $\tau_\beta(\boldsymbol{n})$ the *surface tension* in direction $\boldsymbol{n}$ at inverse temperature β. As is well known, $\boldsymbol{n} \mapsto \tau_\beta(\boldsymbol{n})$ is invariant under rotations of $\boldsymbol{n}$ by integer multiples of $\frac{\pi}{2}$ and $\tau_{\text{min}} = \inf_{\boldsymbol{n}\in\mathcal{S}_1} \tau_\beta(\boldsymbol{n}) > 0$ for all $\beta > \beta_{\text{c}}$ [45]. Informally, the quantity $\tau_\beta(\boldsymbol{n})N$ represents the statistical-mechanical cost of a (fluctuating) contour line connecting two sites at distance N on a straight line with direction (or normal vector) $\boldsymbol{n}$.

Remark 2. Our definition of the surface tension differs from the standard definition by a factor of β^{-1}. In particular, the physical units of τ_β are length^{-1} rather than energy$\times$length^{-1}. The present definition eliminates the need for an explicit occurrence of β in many expressions throughout this paper and, as such, is notationally more convenient.

• *Surface properties.* On the level of macroscopic thermodynamics, it is obvious that when a droplet of the minority phase is present in the system, it is pertinent to minimize the total surface cost. By our previous discussion, the cost per unit length is given by the surface tension $\tau_\beta(\boldsymbol{n})$. Thus, one is naturally led to the functional $\mathcal{W}_\beta(\gamma)$ that assigns the number

$$\mathcal{W}_\beta(\gamma) = \int_\gamma \tau_\beta(\boldsymbol{n}_t)\mathrm{d}t \tag{1.6}$$

to each rectifiable, closed curve $\gamma = (\gamma_t)$ in $\mathbb{R}^2$. Here $\boldsymbol{n}_t$ denotes the normal vector at γ_t. The goal of the resulting variational problem is to minimize $\mathcal{W}_\beta(\partial D)$ over all $D \subset \mathbb{R}^2$ with rectifiable boundary subject to the constraint that the volume of D coincides with that of the droplet. The classic solution, due to Wulff [52], is that $\mathcal{W}_\beta(\partial D)$ is minimized by the shape

$$D_W = \bigl\{\boldsymbol{r} \in \mathbb{R}^2 \colon \boldsymbol{r} \cdot \boldsymbol{n} \le \tau_\beta(\boldsymbol{n}),\, \boldsymbol{n} \in \mathcal{S}_1\bigr\} \tag{1.7}$$

rescaled to contain the appropriate volume. (Here $\boldsymbol{r} \cdot \boldsymbol{n}$ denotes the dot product in $\mathbb{R}^2$.) We will use W to denote the shape D_W scaled to have a *unit* (Lebesgue) volume. It follows from (1.7) that W is a convex set in $\mathbb{R}^2$. We define

$$w_1(\beta) = \mathcal{W}_\beta(\partial W) \tag{1.8}$$

and note that $w_1(\beta) > 0$ once $\beta > \beta_\mathrm{c}$.

Our preliminary arsenal is now complete and we are prepared to discuss the main results.

1.3 Main results.

Recall the notation Λ_L for a square of $L \times L$ sites in $\mathbb{Z}^2$. Consider the Ising model in volume Λ_L with plus boundary condition and inverse temperature β. Let us define the total magnetization (of a configuration σ) in Λ_L by the formula

$$M_L = \sum_{x \in \Lambda_L} \sigma_x. \tag{1.9}$$

Let $(v_L)_{L\ge 1}$ be a sequence of positive numbers, with $v_L \to \infty$ as $L \to \infty$, such that $m^\star\,|\Lambda_L| - 2m^\star\,v_L$ is an allowed value of M_L for all $L \ge 1$. Our first result concerns the decay rate of the probability that $M_L = m^\star\,|\Lambda_L| - 2m^\star\,v_L$ in the “grandcanonical” ensemble $P_L^{+,\beta}$:

Theorem 1.1 *Let $\beta > \beta_\mathrm{c}$ and let $m^\star = m^\star(\beta)$, $\chi = \chi(\beta)$, and $w_1 = w_1(\beta)$ be as above. Suppose that the limit*

$$\Delta = 2\frac{(m^\star)^2}{\chi w_1} \lim_{L\to\infty} \frac{v_L^{3/2}}{|\Lambda_L|} \tag{1.10}$$

exists with $\Delta \in (0,\infty)$. Then

$$\lim_{L\to\infty} \frac{1}{\sqrt{v_L}} \log P_L^{+,\beta}\bigl(M_L = m^\star\,|\Lambda_L| - 2m^\star\,v_L\bigr) = -w_1 \inf_{0\le\lambda\le 1} \Phi_\Delta(\lambda), \tag{1.11}$$

where

$$\Phi_\Delta(\lambda) = \sqrt{\lambda} + \Delta(1-\lambda)^2, \qquad 0 \le \lambda \le 1. \tag{1.12}$$

The proof of Theorem 1.1 is a direct consequence of Theorems 3.1 and 4.1; the actual proof comes in Section 5. We proceed with some remarks:

Remark 3. Note that, by our choice of the deviation scale, the term $m^\star(\beta)|\Lambda_L|$ can be replaced by the mean value $\langle M_L\rangle_L^{+,\beta}$ in all formulas; see Lemma 2.9 below. The motivation for introducing the factor "$2m^\star$" on the left-hand-side of (1.11) is that then v_L represents the volume of a droplet that must be created in order to achieve the required value of the overall magnetization (provided the magnetization outside, resp., inside the droplet is $m^\star$, resp., $-m^\star$).

Remark 4. The quantity λ that appears in (1.11–1.12) represents the *trial fraction* of the deficit magnetization which might go into a large-scale droplet. (So, by our convention, the volume of such a droplet is just λv_L.) The core of the proof of Theorem 1.1, roughly speaking, is that the probability of seeing a droplet of this size tends to zero as $\exp\{-w_1\sqrt{v_L}\Phi_\Delta(\lambda)\}$. Evidently, a large deviation principle for the size of such a droplet is satisfied with rate $L^{2/3}$ and a rate function proportional to Φ_Δ. However, we will not attempt to make this statement mathematically rigorous.

Next we shall formulate our main result on the asymptotic form of typical configurations in the "canonical" ensemble described by the conditional measure $P_L^{+,\beta}(\,\cdot\,|M_L = m^\star\,|\Lambda_L| - 2m^\star\,v_L)$. For any two sets $A, B\subset\mathbb{R}^2$, let $d_{\mathrm{H}}(A,B)$ denote the Hausdorff distance between A and B,

$$d_{\mathrm{H}}(A,B) = \max\bigl\{\sup_{x\in A}\operatorname{dist}(x,B),\sup_{y\in B}\operatorname{dist}(y,A)\bigr\}, \tag{1.13}$$

where $\operatorname{dist}(x,A)$ is the Euclidean distance of x and A.

Our second main theorem is then as follows:

Theorem 1.2 *Let $\beta>\beta_{\mathrm{c}}$ and suppose that the limit in (1.10) exists with $\Delta\in(0,\infty)$. Recall that W denotes the Wulff shape of a unit volume. Given $\varkappa, s, L\in(0,\infty)$, let $\mathcal{A}_{\varkappa,s,L}$ be the event that any external contour γ for which $\operatorname{diam}\gamma\ge s$ must also satisfy $\operatorname{diam}\gamma\ge\varkappa\sqrt{v_L}$. Next, for each $\epsilon>0$, let $\mathcal{B}_{\epsilon,s,L}$ be the event that there is at most one external contour γ_0 in Λ_L with $\operatorname{diam}\gamma_0\ge s$ and, whenever such a contour γ_0 exists, it satisfies the conditions*

$$\inf_{z\in\mathbb{R}^2} d_{\mathrm{H}}\bigl(V(\gamma_0), z+\sqrt{|V(\gamma_0)|}\,W\bigr)\le\sqrt{\epsilon v_L} \tag{1.14}$$

and

$$\Phi_\Delta\bigl(v_L^{-1}|V(\gamma_0)|\bigr)\le\inf_{0\le\lambda'\le1}\Phi_\Delta(\lambda')+\epsilon. \tag{1.15}$$

In addition, the event $\mathcal{B}_{\epsilon,s,L}$ also requires that the magnetization inside γ_0 obeys the constraint

$$\Bigl|\sum_{x\in\mathbb{V}(\gamma_0)}(\sigma_x+m^\star)\Bigr|\le\epsilon v_L. \tag{1.16}$$

There exists a constant $\varkappa_0>0$ such that for each $\zeta>0$ and each $\epsilon>0$ there exist numbers $K_0<\infty$ and $L_0<\infty$ such that

$$P_L^{+,\beta}\bigl(\mathcal{A}_{\varkappa,s,L}\cap\mathcal{B}_{\epsilon,s,L}\big|M_L = m^\star\,|\Lambda_L| - 2m^\star\,v_L\bigr)\ge 1-L^{-\zeta} \tag{1.17}$$

holds provided $\varkappa \le \varkappa_0$ *and* $s = K \log L$ *with* $K \ge K_0$ *and* $L \ge L_0$, .

Thus, simply put, whenever there is a large droplet in the system, its shape rarely deviates from that of the Wulff shape and its volume (in units of v_L) is almost always given by a value of λ nearly minimizing Φ_Δ. Moreover, all other droplets in the system are at most of a logarithmic size.

Most of the physically interesting behavior of this system is simply a consequence of where Φ_Δ achieves its minimum and how this minimum depends on Δ. The upshot, which is stated concisely in Proposition 2.1 below, is that there is a critical value of Δ, given by

$$\Delta_{\mathrm{c}} = \frac{1}{2}\Bigl(\frac{3}{2}\Bigr)^{3/2}, \tag{1.18}$$

such that if $\Delta < \Delta_{\mathrm{c}}$, then Φ_Δ has the unique minimizer at $\lambda = 0$, while for $\Delta > \Delta_{\mathrm{c}}$, the unique minimizer of Φ_Δ is nontrivial. More explicitly, for $\Delta \ne \Delta_{\mathrm{c}}$, the function Φ_Δ is minimized by

$$\lambda_\Delta = \begin{cases} 0, & \text{if } \Delta < \Delta_{\mathrm{c}}, \\ \lambda_+(\Delta), & \text{if } \Delta > \Delta_{\mathrm{c}}, \end{cases} \tag{1.19}$$

where $\lambda_+(\Delta)$ is the maximal positive solution to the equation

$$4\Delta\sqrt{\lambda}(1-\lambda) = 1. \tag{1.20}$$

The reason for the changeover is that, as Δ increases through Δ_{c}, a local minimum becomes a global minimum, see the proof of Proposition 2.1. As a consequence, the minimizing fraction λ does *not* tend to zero as $\Delta \downarrow \Delta_{\mathrm{c}}$; in particular, it tends to $\lambda_{\mathrm{c}} = 2/3$.

Using the information about the unique minimizer of Φ_Δ for $\Delta \ne \Delta_{\mathrm{c}}$, it is worthwhile to reformulate Theorem 1.2 as follows:

Corollary 1.3 *Let* $\beta > \beta_{\mathrm{c}}$ *and suppose that the limit in (1.10) exists with* $\Delta \in (0,\infty)$. *Let* Δ_{c} *and* λ_Δ *be as in (1.18) and (1.19), respectively. Let* K *be sufficiently large (i.e.,* $K \ge K_0$, *where* K_0 *is as in Theorem 1.2). Considering the conditional distribution* $P_L^{+,\beta}(\,\cdot\,|M_L = m^\star|\Lambda_L| - 2m^\star v_L)$, *the following holds with probability tending to one as* $L \to \infty$:

1. *If* $\Delta < \Delta_{\mathrm{c}}$, *then all contours* γ *in* Λ_L *satisfy* $\operatorname{diam}\gamma \le K\log L$.
2. *If* $\Delta > \Delta_{\mathrm{c}}$, *then there is exactly one external contour* γ_0 *with* $\operatorname{diam}\gamma_0 > K\log L$ *and all other external contours* γ *satisfy* $\operatorname{diam}\gamma \le K\log L$. *Moreover, the unique "large" external contour* γ_0 *asymptotically satisfies the bounds (1.14–1.16) for all* $\epsilon > 0$. *In particular,* $|V(\gamma_0)| = v_L(\lambda_\Delta + o(1))$ *with probability tending to one as* $L \to \infty$.

We remark that although the situation at $\Delta = \Delta_{\mathrm{c}}$ is not fully resolved, we must have either a single large droplet or no droplet at all; i.e., the outcome must mimic the case $\Delta > \Delta_{\mathrm{c}}$ or $\Delta < \Delta_{\mathrm{c}}$. A better understanding of the case $\Delta = \Delta_{\mathrm{c}}$ will certainly require a more refined analysis; e.g., the second-order large-deviation behavior of the measure $P_L^{+,\beta}(\cdot)$.

Remark 5. We note that in the course of this work, the phrase "$\beta > \beta_{\mathrm{c}}$" appears in three disparate meanings. First, for $\beta > \beta_{\mathrm{c}}$, the magnetization is positive, second, for $\beta > \beta_{\mathrm{c}}$, the surface tension is positive and third, for $\beta > \beta_{\mathrm{c}}$, truncated correlations decay exponentially. The facts that the transition temperatures associated with these properties all coincide *and* that β_c is given by the self-dual condition plays no essential role in are arguments. Nor are any other particulars of the square lattice really used. Thus, we believe that our results could be extended to other planar lattices without much modification. However, in the cases where the coincidence has not yet been (or cannot be) established, we would need to define "β_c" so as to satisfy all three criteria.

1.4 Discussion and outline.

The mechanism which drives the droplet formation/dissolution phenomenon described in the above theorems is not difficult to understand on a heuristic level. This heuristic derivation (which applies to all dimensions $d \geq 2$) has been discussed in detail elsewhere [9], so we will be correspondingly brief. The main ideas are best explained in the context of the large-deviation theory for the "grandcanonical" distribution and, as a matter of fact, the actual proof also follows this path.

Consider the Ising model in the box Λ_L and suppose we wish to observe a magnetization deficiency $\delta M = 2m^\star v_L$ from the nominal value of $m^\star|\Lambda_L|$. Of course, this can be achieved in one shot by the formation of a Wulff droplet at the cost of about $\exp\{-w_1\sqrt{v_L}\}$. Alternatively, if we demand that this deficiency emerges out of the background fluctuations, we might guess on the basis of fluctuation-dissipation arguments that the cost would be of the order

$$\exp\Bigl\{-\frac{(\delta M)^2}{2\text{Var}(M_L)}\Bigr\} \approx \exp\Bigl\{-2\frac{(m^\star\, v_L)^2}{\chi|\Lambda_L|}\Bigr\}, \tag{1.21}$$

where χ is the susceptibility and $\text{Var}(M_L) = (\chi + o(1))|\Lambda_L|$ is the variance of M_L in distribution $P_L^{+,\beta}$. Obviously, the former mechanism dominates when $\sqrt{v_L} \ll v_L^2/|\Lambda_L|$, i.e., when $v_L \gg L^{4/3}$, while the latter dominates under the opposite extreme conditions, i.e., when $v_L \ll L^{4/3}$. (These are exactly the regions previously treated in [25, 37] where the corresponding statements have been established in full rigor.) In the case when $v_L/L^{4/3}$ tends to a finite limit we now find that the two terms are comparable. This is the basis of our parameter Δ defined in (1.10).

Assuming $v_L^{3/2}/|\Lambda_L|$ is essentially at its limit, let us instead try a droplet of volume λv_L, where $0 \leq \lambda \leq 1$. The droplet cost is now reduced to

$$\exp\bigl\{-w_1\sqrt{\lambda}\sqrt{v_L}\bigr\}, \tag{1.22}$$

but we still need to account for the remaining fraction of the deficiency. Assuming the fluctuation-dissipation reasoning can still be applied, this is now

$$\exp\Bigl\{-2\frac{(m^\star\, v_L)^2}{\chi|\Lambda_L|}(1-\lambda)^2\Bigr\} = \exp\bigl\{-w_1\sqrt{v_L}(1-\lambda)^2\Delta\bigr\}. \tag{1.23}$$

Putting these together we find that the total cost of achieving the deficiency $\delta M = 2m^\star v_L$ using a droplet of volume λv_L is given in the leading order by

$$\exp\bigl\{-w_1\Phi_\Delta(\lambda)\sqrt{v_L}\bigr\}. \tag{1.24}$$

An optimal droplet size is then found by minimizing $\Phi_\Delta(\lambda)$ over λ. This is exactly the content of Theorem 1.1. We remark that even on the level of heuristic understanding, some justification is required for the decoupling of these two mechanisms. In [9], we have argued this case on a heuristic level; in the present work, we simply provide a complete proof.

The pathway of the proof is as follows: The approximate equalities (1.22–1.24) must be proved in the form of upper and lower bounds which agree in the $L \to \infty$ limit. (Of course, we never actually have to go through the trouble of establishing the formulas involving $\Phi_\Delta(\lambda)$ for non-optimal values of λ.) For the lower bound (see Theorem 3.1) we simply shoot for the minimum of $\Phi_\Delta(\lambda)$: We produce a near-Wulff droplet of the desired area and, on the complementary region, allow the background fluctuations to account for the rest. Here, as a bound, we are permitted to use a contour ensemble with restriction to contours of *logarithmic* size which ensures the desired Gaussian behavior.

The upper bound requires considerably more effort. The key step is to show that, with probability close to one, there are no droplets at any scale larger than $\log L$ or smaller than $\sqrt{v_L}$. Notwithstanding the technical difficulties, the result (Theorem 4.1) is of independent interest because it applies for all $\Delta \in (0,\infty)$, including the case $\Delta = \Delta_{\text{c}}$. Once the absence of these "intermediate" contour scales has been established, the proof of the main results directly follow.

We finish with a brief outline of the remainder of this paper. In the next section we collect the necessary technical statements needed for the proof of both the upper and lower bound. Specifically, in Section 2.1 we discuss in detail the minimizers of Φ_Δ, in Section 2.2 we introduce the concept of skeletons and in Section 2.3 we list the needed properties of the logarithmic contour ensemble. Section 3 contains the proof of the lower bound, while Section 4 establishes the absence of contour on scales between $\log L$ and the anticipated droplet size. Section 5 assembles these ingredients together into the proofs of the main results.

2. TECHNICAL INGREDIENTS

This section contains three subsections: Section 2.1 presents the solution of the variational problem for function Φ_Δ on the right-hand side of (1.12), while Sections 2.2 and 2.3 collect the necessary technical lemmas concerning the skeleton calculus and the small-contour ensemble. Readers are invited to skip the entire section on a preliminary run-through and return to it only after getting to the proofs in Sections 3–5.

2.1 Variational problem.

Here we investigate the global minima of the function Φ_Δ that was introduced in (1.12). Since the general picture is presumably applicable in higher dimensions as well (certainly at the level of heuristic arguments, see [9]), we might as well carry out the analysis in the case of a general dimension $d \geq 2$. For the purpose of this subsection, let

$$\Phi_\Delta(\lambda) = \lambda^{\frac{d-1}{d}} + \Delta(1-\lambda)^2, \qquad 0 \leq \lambda \leq 1. \tag{2.1}$$

We define

$$\Phi_\Delta^\star = \inf_{0\leq\lambda\leq 1} \Phi_\Delta(\lambda) \tag{2.2}$$

and note that $\Phi_\Delta^\star > 0$ once $\Delta > 0$. Let us introduce the d-dimensional version of (1.18),

$$\Delta_{\rm c} = \frac{1}{d}\Bigl(\frac{d+1}{2}\Bigr)^{\frac{d+1}{d}}. \tag{2.3}$$

The minimizers of Φ_Δ are then characterized as follows:

Proposition 2.1 *Let $d \ge 2$ and, for any $\Delta \ge 0$, let $\mathfrak{M}_\Delta$ denote the set of all global minimizers of Φ_Δ on $[0,1]$. Then we have:*
(1) If $\Delta < \Delta_{\rm c}$, then $\mathfrak{M}_\Delta = \{0\}$.
(2) If $\Delta = \Delta_{\rm c}$, then $\mathfrak{M}_\Delta = \{0, \lambda_{\rm c}\}$, where

$$\lambda_{\rm c} = \frac{2}{d+1}. \tag{2.4}$$

(3) If $\Delta > \Delta_{\rm c}$, then $\mathfrak{M}_\Delta = \{\lambda_0\}$, where λ_0 is the maximal positive solution to the equation

$$\frac{2d}{d-1}\Delta\,\lambda^{\frac{1}{d}}(1-\lambda) = 1. \tag{2.5}$$

In particular, $\Lambda_0 > \lambda_{\rm c}$.

Proof. A simple calculation shows that $\lambda = 0$ is always a (one-sided) local minimum of $\lambda \mapsto \Phi_\Delta(\lambda)$, while $\lambda = 1$ is always a (one-sided) local maximum. Moreover, the stationary points of Φ_Δ in $(0,1)$ have to satisfy (2.5). Consider the quantity

$$q(\lambda) = \frac{1}{\Delta}\bigl(1 - \tfrac{d}{d-1}\lambda^{1/d}\Phi_\Delta'(\lambda)\bigr) = \frac{2d}{d-1}\lambda^{1/d}(1-\lambda), \tag{2.6}$$

i.e., $q(\lambda)$ is essentially the left-hand side of (2.5). A simple calculation shows that $q(\lambda)$ achieves its maximal value on $[0,1]$ at $\lambda = \lambda_d = \frac{1}{d+1}$, where it equals $\Delta_d^{-1} = 2d^2(d^2-1)^{-1}(d+1)^{-1/d}$, and is strictly increasing for $\lambda < \lambda_d$ and strictly decreasing for $\lambda > \lambda_d$. On the basis of these observations, it is easy to verify the following facts:

(1) For $\Delta \le \Delta_d$, we have $\Delta q(\lambda) < 1$ for all $\lambda \in [0,1]$ (except perhaps at $\lambda = \lambda_{\rm c}$ when Δ equals Δ_d). Consequently, $\lambda \mapsto \Phi_\Delta(\lambda)$ is strictly increasing throughout $[0,1]$. In particular, $\lambda = 0$ is the unique global minimum of $\Phi_\Delta(\lambda)$ in $[0,1]$.
(2) For $\Delta > \Delta_d$, (2.5), resp., $\Delta q(\lambda) = 1$ has two distinct solutions in $[0,1]$. Consequently, $\lambda \mapsto \Phi_\Delta(\lambda)$ has two local extrema in $(0,1)$: A local maximum at $\lambda = \lambda_-(\Delta)$ and a local minimum at $\lambda = \lambda_+(\Delta)$, where $\lambda_-(\Delta)$ and $\lambda_+(\Delta)$ are the minimal and maximal positive solutions to (2.5), respectively.

As a simple calculation shows, the function $\Delta \mapsto \lambda_+(\Delta)$ is strictly increasing on its domain with $\lambda_+(\Delta) \sim 1 - \frac{d-1}{2d}\frac{1}{\Delta}$ as $\Delta \to \infty$.

In order to decide which of the two previously described local minima ($\lambda = 0$ or $\lambda = \lambda_+(\Delta)$) gives rise to the global minimum, we first note that, while $\Phi_\Delta(0) = \Delta$ tends to infinity as $\Delta \to \infty$, the above asymptotics of $\lambda_+(\Delta)$ shows that $\Phi_\Delta(\lambda_+(\Delta)) \to 1$ as $\Delta \to \infty$. Hence, $\lambda_+(\Delta)$ is the unique global minimum of Φ_Δ once Δ is sufficiently large. It remains to show that the two local minima interchange their roles at $\Delta = \Delta_c$. To that end we compute

$$\frac{\mathrm{d}}{\mathrm{d}\Delta}\Phi_\Delta\big(\lambda_+(\Delta)\big) = \frac{\partial}{\partial\Delta}\Phi_\Delta\big(\lambda_+(\Delta)\big) = \big(1-\lambda_+(\Delta)\big)^2 > 0, \tag{2.7}$$

where we used that $\lambda_+(\Delta)$ is a stationary point of Φ_Δ to derive the first equality. Comparing this with $\frac{\mathrm{d}}{\mathrm{d}\Delta}\Phi_\Delta(0) = 1$, we see that $\Delta \mapsto \Phi_\Delta(\lambda_+(\Delta))$ increases with Δ strictly slower than $\Delta \mapsto \Phi_\Delta(0)$ on any finite interval of Δ's. Hence, there must be a unique value of Δ for which $\Phi_\Delta(0)$ and $\Phi_\Delta(\lambda_+(\Delta))$ are exactly equal. An elementary computation shows that this happens at $\Delta = \Delta_c$, where Δ_c is given by (2.3). This finishes the proof of (1) and (3); in order to show that also (2) holds, we just need to note that $\lambda_+(\Delta_c)$ is exactly λ_c as given in (2.4). □

Proposition 2.1 allows us to define a quantity λ_Δ by formula (1.19), where now $\lambda_+(\Delta)$ is the maximal positive solution to (2.5). Since $\lim_{\Delta\downarrow\Delta_c}\lambda_\Delta = \lambda_c > 0$, $\Delta \mapsto \lambda_\Delta$ undergoes a jump at Δ_c.

2.2 Skeleton estimates.

In this section we introduce coarse-grained versions of contours called *skeletons*. These objects will be extremely useful whenever an upper bound on the probability of large contours is needed. Indeed, the introduction of skeletons will permit us to effectively integrate out small fluctuations of contour lines and thus express the contour weights directly in terms of the surface tension. Skeletons were first introduced in [4,24]; here we use a modified version of the definition from [37].

2.2.1 Definition and geometric properties. Given a scale $s > 0$, an *s-skeleton* is an n-tuple $(x_1,\dots,x_n)$ of points on the dual lattice, $x_i \in (\mathbb{Z}^2)^*$, such that $n > 1$ and

$$s \le \|x_{i+1}-x_i\| \le 2s, \qquad i = 1,\dots,n. \tag{2.8}$$

Here $\|\cdot\|$ denotes the ℓ^2-distance on $\mathbb{R}^2$ and x_{n+1} is identified with x_1. Given a skeleton S, let $\mathsf{P}(S)$ be the closed *polygonal curve* in $\mathbb{R}^2$ induced by S. We will use $|\mathsf{P}(S)|$ to denote the total length of $\mathsf{P}(S)$, in accord with our general notation for the length of curves.

A contour γ is called *compatible* with an s-skeleton $S = (x_1,\dots,x_n)$, if

(1) Γ, viewed as a simple closed path on $\mathbb{R}^2$, passes through all sites x_i, $i = 1,\dots,n$ in the corresponding order.
(2) $d_{\mathrm{H}}(\gamma,\mathsf{P}(S)) \le s$, where d_{H} is the Hausdorff distance (1.13).

We write $\gamma \sim S$ if γ and S are compatible. For each configuration σ, we let $\Gamma_s(\sigma)$ be the set of all *s-large* contours γ in σ; namely all γ in σ for which there is an s-skeleton S such that $\gamma \sim S$. Given a set of s-skeletons $\mathfrak{S} = (S_1,\dots,S_m)$, we say that a configuration σ is *compatible* with $\mathfrak{S}$, if $\Gamma_s(\sigma) = (\gamma_1,\dots,\gamma_m)$ and $\gamma_k \sim S_k$ for all $k = 1,\dots,m$. We will write $\sigma \sim \mathfrak{S}$ to denote that σ and $\mathfrak{S}$ are compatible.

It is easy to see that $\Gamma_s(\sigma)$ actually consists of all contours γ of the configuration σ such that $\operatorname{diam}\gamma \ge s$. Indeed, $\operatorname{diam}\gamma \ge s$ for every $\gamma \in \Gamma_s(\sigma)$ by the conditions (1) and (2.8) above. On the other hand, for any γ with $\operatorname{diam}\gamma \ge s$, we will construct an s-skeleton by the following procedure: Regard γ as a closed non-self-intersecting curve, $\gamma = (\gamma_t)_{0\le t\le 1}$, where γ_0 is chosen so that $\sup_{x\in\gamma}\|x-\gamma_0\| \ge s$. Then we let $x_1 = \gamma_0$ and $x_2 = \gamma_{t_2}$, where $t_2 =$

$\inf\{t > 0 \colon \|\gamma_t - \gamma_0\| \ge s\}$. Similarly, if t_j has been defined and $x_j = \gamma_{t_j}$, we let $x_{j+1} = \gamma_{t_{j+1}}$, where $t_{j+1} = \inf\{t \in (t_j, 1] \colon \|\gamma_t - \gamma_{t_j}\| \ge s\}$. Note that this definition ensures that (2.8) as well as the conditions (1) and (2) hold. The consequence of this construction is that, via the equivalence relation $\sigma \sim \mathfrak{S}$, the set of all skeletons induces a *covering* of the set of all spin configurations.

Remark 6. The reader familiar with [24, 37] will notice that we explicitly keep the stronger condition (1) from [24]. Without the requirement that contours pass through the skeleton points in the given order, Lemma 2.3 and, more importantly, Lemma 2.4 below would fail to hold.

Next we will discuss some subtleties of the geometry of the skeletons stemming from the fact that the corresponding polygons (unlike contours) may have self-intersections. We will stay rather brief; a detailed account of the topic can be found in [24].

We commence with a few geometric definitions: Let $\mathfrak{P} = \{\mathsf{P}_1, \dots, \mathsf{P}_k\}$ denote a finite collection of polygonal curves. Consider a smooth self-avoiding path $\mathcal{L}$ from a point x to ∞ that is generic with respect to the polygons from $\mathfrak{P}$ (i.e., the path $\mathcal{L}$ has a finite number of intersections with each P_j and this number does not change under small perturbations of $\mathcal{L}$). Let $\#(\mathcal{L} \cap \mathsf{P}_j)$ be the number of intersections of $\mathcal{L}$ with P_j. Then we *define* $V(\mathfrak{P}) \subset \mathbb{R}^2$ to be the set of points $x \in \mathbb{R}^2$ such that the total number of intersections, $\sum_{j=1}^n \#(\mathcal{L} \cap \mathsf{P}_j)$, is odd for any path $\mathcal{L}$ from x to ∞ with the above properties. We will use $|V(\mathfrak{P})|$ to denote the area of $V(\mathfrak{P})$.

If $\mathfrak{P}$ happens to be a collection of skeletons, $\mathfrak{P} = \mathfrak{S}$, the relevant set will be $V(\mathfrak{S})$. If $\mathfrak{P}$ happens to be a collection of Ising contours, $\mathfrak{P} = \Gamma$, the associated $V(\Gamma)$ can be thought of as a union of plaquettes centered at sites of $\mathbb{Z}^2$; we will use $\mathbb{V}(\Gamma) = V(\Gamma) \cap \mathbb{Z}^2$ to denote the relevant set of sites. It is clear that if Γ are the contours associated with a spin configuration σ in Λ and the plus boundary condition on $\partial\Lambda$, then $\mathbb{V}(\Gamma)$ are exactly the sites $x \in \Lambda$ where $\sigma_x = -1$. We proceed by listing a few important estimates concerning compatible collections of contours and their associated skeletons:

Lemma 2.2 *There is a finite geometric constant g_1 such that if Γ is a collection of contours and $\mathfrak{S}$ is a collection of s-skeletons with $\Gamma \sim \mathfrak{S}$, then*

$$\sum_{\gamma \in \Gamma} |\gamma| \le g_1 s \sum_{S \in \mathfrak{S}} \bigl|\mathsf{P}(S)\bigr|. \tag{2.9}$$

In particular, if $\operatorname{diam} \gamma \le \varkappa$ *for all $\gamma \in \Gamma$, then we also have, for some finite constant g_2,*

$$\bigl|V(\Gamma)\bigr| \le g_2 \varkappa \sum_{S \in \mathfrak{S}} \bigl|\mathsf{P}(S)\bigr|. \tag{2.10}$$

Proof. Immediate from the definition of s-skeletons. $\square$

Lemma 2.2 will be useful because of the following observation: Let $\mathfrak{S}$ be a collection of s-skeletons and recall that the minimal value of the surface tension, $\tau_{\min} = \inf_{\boldsymbol{n} \in \mathcal{S}_1} \tau_\beta(\boldsymbol{n})$ is strictly positive, $\tau_{\min} > 0$. Then

$$\sum_{S \in \mathfrak{S}} \mathcal{W}_\beta\bigl(\mathsf{P}(S)\bigr) \ge \tau_{\min} \sum_{S \in \mathfrak{S}} \bigl|\mathsf{P}(S)\bigr|. \tag{2.11}$$

Thus the bounds in (2.9–2.10) will allow us to convert a lower bound on the overall contour surface area/volume into a lower bound on the Wulff functional of the associated skeletons.

A little less trivial is the estimate on the difference between the volumes of $V(\Gamma)$ and $V(\mathfrak{S})$:

Lemma 2.3 *There is a finite geometric constant g_3 such that if Γ is a collection of contours and $\mathfrak{S}$ is a collection of s-skeletons with $\Gamma \sim \mathfrak{S}$, then*

$$\Bigl|\bigl|V(\Gamma)\bigr| - \bigl|V(\mathfrak{S})\bigr|\Bigr| \le \bigl|V(\Gamma)\triangle V(\mathfrak{S})\bigr| \le g_3 s \sum_{S\in\mathfrak{S}} \bigl|\mathsf{P}(S)\bigr|. \tag{2.12}$$

Here $V(\Gamma)\triangle V(\mathfrak{S})$ denotes the symmetric difference of $V(\Gamma)$ and $V(\mathfrak{S})$.

Proof. We will just rephrase the proof of Theorem 5.13 in [24]. Let $\Gamma = (\gamma_1, \dots, \gamma_m)$ and fix $\gamma_k \in \Gamma$. Let $S_k \in \mathfrak{S}$ with $S_k = (x_0, \dots, x_n)$ be the skeleton compatible with γ_k and let $S_{k,j}$ denote the segment of the straight line between x_j and x_{j+1}. Since γ passes through the skeleton points *in the given order*, for each j there is a corresponding piece, $\gamma_{k,j}$, of γ which connects x_j and x_{j+1}.

Let $U_{k,j}$ be the subset of $\mathbb{R}^2$ enclosed "between" $S_{k,j}$ and $\gamma_{k,j}$ (i.e., $U_{k,j}$ is the union of all bounded connected components of $\mathbb{R}^2 \setminus (S_{k,j} \cup \gamma_{k,j})$). We claim that

$$V(\Gamma)\triangle V(\mathfrak{S}) \subseteq \bigcup_{k,j} U_{k,j}. \tag{2.13}$$

Indeed, let $x \in V(\Gamma)\triangle V(\mathfrak{S})$ and let $\mathcal{L}$ be a path connecting x to infinity which is generic with respect to both $\mathfrak{S}$ and Γ. Then $\mathcal{L}$ has the same parity of the number of intersections with γ_k and S_k, *unless* $x \in U_{k,j}$ for some k and j. By inspecting the definitions of $V(\Gamma)$ and $V(\mathfrak{S})$, (2.13) is proved.

Let $U_s(\mathsf{P}(S_k))$ be the s-neighborhood of the polygonal curve $\mathsf{P}(S_k)$. Since $\partial U_{k,j} \subset U_s(\mathsf{P}(S_k))$, by (2.13) we have that $V(\Gamma)\triangle V(\mathfrak{S}) \subseteq \bigcup_k U_s(\mathsf{P}(S_k))$. From here (2.12) directly follows. □

2.2.2 Probabilistic estimates. The main reason why skeletons are useful is the availability of the so called *skeleton upper bound*, originally due to Pfister [45]. Recall that, for each $A \subset \mathbb{Z}^2$, we use $P_A^{+,\beta}$ to denote the probability distribution on spins in A with plus boundary condition on the boundary of A. Given a set of skeletons, we let $P_A^{+,\beta}(\mathfrak{S}) = P_A^{+,\beta}(\{\sigma\colon \sigma \sim \mathfrak{S}\})$ be the probability that $\mathfrak{S}$ is a skeleton of *some* configuration in A. Then we have:

Lemma 2.4 (Skeleton upper bound) *For all $\beta > \beta_{\mathrm{c}}$, all finite $A \subset \mathbb{Z}^2$, all scales s and all collections $\mathfrak{S}$ of s-skeletons in A, we have*

$$P_A^{+,\beta}(\mathfrak{S}) \le \exp\bigl\{-\mathcal{W}_\beta(\mathfrak{S})\bigr\}, \tag{2.14}$$

where

$$\mathcal{W}_\beta(\mathfrak{S}) = \sum_{S\in\mathfrak{S}} \mathcal{W}_\beta\bigl(\mathsf{P}(S)\bigr). \tag{2.15}$$

Proof. This is exactly Eq. (1.3.1) in [37]. The proof goes back to [45], Lemma 6.7. For our purposes, the key "splitting" argument is provided in Lemma 5.4 of [46]. A special case of the key estimate appears in Eq. (5.51) from Lemma 5.5 of [46] with the correct interpretation of the left-hand side. ☐

The bound (2.14) will be used in several ways: First, to show that the $K\log L$-large contours in a box of side-length L are improbable, provided K is large enough; this is a consequence of Lemma 2.5 below. The absence of such contours will be wielded to rule out the likelihood of other improbable scenarios. Finally, after all atypical situations have been dispensed with, the skeleton upper bound will deliver the contribution corresponding to the term $\sqrt{\lambda}$ in (1.11).

An important consequence of the skeleton upper bound is the following generalization of the Peierls estimate, which will be useful at several steps of the proof of our main theorems.

Lemma 2.5 *Let $s = K\log L$ and let $\mathcal{S}_{L,K}$ denote the set of all s-skeletons that arise from contours in Λ_L. For each $\beta > \beta_{\rm c}$ and $\alpha > 0$, there is a $K_0 = K_0(\alpha,\beta) < \infty$, such that*

$$\sum_{\mathfrak{S}\subset\mathcal{S}_{L,K}} \exp\bigl\{-\alpha\mathcal{W}_\beta(\mathfrak{S})\bigr\} \le 1 \tag{2.16}$$

for (all L and) all $K \ge K_0$.

Proof. Let $\mathcal{S}^0_{L,K}$ be the set of all $K\log L$-skeletons S such that $S = (x_1,\dots,x_k)$ with $x_1 = 0$. By translation invariance,

$$\sum_{\mathfrak{S}\subset\mathcal{S}_{L,K}} e^{-\alpha\mathcal{W}_\beta(\mathfrak{S})} \le \sum_{n\ge1}\Bigl(L^2\sum_{S\in\mathcal{S}^0_{L,K}} e^{-\alpha\mathcal{W}_\beta(\mathsf{P}(S))}\Bigr)^n, \tag{2.17}$$

where the prefactor L^2 accounts for the translation entropy of each skeleton within Λ_L. The latter sum can be estimated by mimicking the proof of Peierls' bound, where contour entropy was bounded by that of the simple random walk on $\mathbb{Z}^2$. Indeed, each skeleton can be thought of as a sequence of steps with step-length entropy at most $32s^2$, where $s = K\log L$, and with each step weighted by a factor not exceeding $e^{-\tau_{\min}s}$. This and (2.11) yield

$$\sum_{S\in\mathcal{S}^0_{L,K}} e^{-\alpha\mathcal{W}_\beta(\mathsf{P}(S))} \le \sum_{m\ge1}\bigl(32s^2e^{-\alpha\tau_{\min}s}\bigr)^m. \tag{2.18}$$

By choosing K_0 sufficiently large, the right-hand side is less than $\frac12 L^{-2}$ for all $K \ge K_0$. Using this in (2.17), the claim follows. ☐

Lemmas 2.4 and 2.5 will be used in the form of the following corollary:

Corollary 2.6 *Let $\beta > \beta_{\rm c}$, $L \ge 1$ and $\kappa > 0$ be fixed, and let $\mathcal{A}$ be the set of of configurations σ such that $\mathcal{W}_\beta(\mathfrak{S}) \ge \kappa$ for at least one collection of s-skeletons $\mathfrak{S}$ satisfying $\mathfrak{S} \sim \sigma$. Let $\alpha \in (0,1)$, and let $K_0(\alpha,\beta)$ be as in Lemma 2.5. If $s = K\log L$ with $K \ge K_0(\alpha,\beta)$, then*

$$P_L^{+,\beta}(\mathcal{A}) \le e^{-(1-\alpha)\kappa}. \tag{2.19}$$

Proof. By the assumptions of the Lemma, we have

$$P_L^{+,\beta}(\mathcal{A}) \le \sum_{\substack{\mathfrak{S}\subset\mathcal{S}_{K,L}\\ \mathcal{W}_\beta(\mathfrak{S})\ge\kappa}} P_L^{+,\beta}(\mathfrak{S}), \tag{2.20}$$

where we used the notation $P_L^{+,\beta}(\mathfrak{S}) = P_L^{+,\beta}(\{\sigma\colon \sigma\sim\mathfrak{S}\})$. Lemma 2.4 then implies

$$P_L^{+,\beta}(\mathcal{A}) \le \sum_{\substack{\mathfrak{S}\subset\mathcal{S}_{K,L}\\ \mathcal{W}_\beta(\mathfrak{S})\ge\kappa}} e^{-\mathcal{W}_\beta(\mathfrak{S})} \le e^{-(1-\alpha)\kappa}\sum_{\mathfrak{S}\subset\mathcal{S}_{K,L}} e^{-\alpha\mathcal{W}_\beta(\mathfrak{S})}. \tag{2.21}$$

Here we wrote $e^{-\mathcal{W}_\beta(\mathfrak{S})} = e^{-\alpha\mathcal{W}_\beta(\mathfrak{S})}e^{-(1-\alpha)\mathcal{W}_\beta(\mathfrak{S})}$ and then invoked to bound $\mathcal{W}_\beta(\mathfrak{S})\ge\kappa$ to estimate $e^{-(1-\alpha)\mathcal{W}_\beta(\mathfrak{S})}$ by $e^{-(1-\alpha)\kappa}$. Finally, we dropped the constraint to $\mathcal{W}_\beta(\mathfrak{S})\ge\kappa$ in the last sum. Since $s = K\log L$ with $K\ge K_0(\alpha,\beta)$, the last sum is less than one by Lemma 2.5. □

Ideas similar to those used in the proof of Lemma 2.5 can be used to estimate the probability of the occurrence of an s-large contour:

Lemma 2.7 *For each $\beta>\beta_{\rm c}$, there exist a constant $\alpha(\beta)>0$ such that*

$$P_A^{+,\beta}\bigl(\Gamma_s(\sigma)\ne\emptyset\bigr) \le |A|e^{-\alpha(\beta)s} \tag{2.22}$$

for any finite $A\subset\mathbb{Z}^2$ and any scale $s\ge1$.

Proof. Fix $\alpha>0$ and suppose without loss of generality that $|A|>1$ and $s\ge\alpha^{-1}\log|A|$ for some $\alpha>0$. If $\Gamma_s(\sigma)\ne\emptyset$, the associated s-skeleton must satisfy $\mathcal{W}_\beta(\mathfrak{S})\ge\tau_{\rm min}s$. Invoking (2.14) a variant of the estimate (2.17–2.18) (here is where $s\ge\alpha^{-1}\log|A|$ enters into the play), we show that $P_A^{+,\beta}(\Gamma_s(\sigma)\ne\emptyset)\le C|A|s^2e^{-\frac12\tau_{\rm min}s}$, where $C>0$ is a constant. From here the bound (2.22) follows by absorbing the factor Cs^2 into the exponential. □

2.2.3 Quantitative estimates around Wulff minimum. The existence of a minimum for the functional (1.6) and a coarse-graining scheme supplemented with a bound of the type in (2.14) tell us the following: Consider a collection Γ of contours, all of which are roughly of the same scale and which enclose a fixed total volume, and suppose that the value of the Wulff functional on a $\mathfrak{S}$ with $\mathfrak{S}\sim\Gamma$ is close to the Wulff minimum. Then (1) it must be the case that Γ consists of a single contour and (2) the shape of this contour must be close to the Wulff shape. A quantitative (and mathematically precise) version of this statement is given in the forthcoming lemma:

Lemma 2.8 *Let $\beta>\beta_{\rm c}$. Then there exist constants $\epsilon_0=\epsilon_0(\beta)\in(0,1)$, $c=c(\beta)>0$, and $C=C(\beta)<\infty$ such that the following holds for all $\epsilon\in(0,\epsilon_0)$: Let Γ be a collection of contours such that $\operatorname{diam}\gamma>c\epsilon\sqrt{|V(\Gamma)|}$ for all $\gamma\in\Gamma$ and let s be a scale function satisfying $s\le\epsilon\sqrt{|V(\Gamma)|}$. Let $\mathfrak{S}$ be a collection of s-skeletons compatible with Γ, $\mathfrak{S}\sim\Gamma$, such that*

$$\mathcal{W}_\beta(\mathfrak{S}) \le w_1\sqrt{|V(\Gamma)|}(1+\epsilon). \tag{2.23}$$

Then Γ consists of a single contour, $\Gamma=\{\gamma\}$, and there is an $x\in\mathbb{R}^2$ such that

$$d_{\rm H}\bigl(V(\gamma),\sqrt{|V(\gamma)|}W+x\bigr) \le c\sqrt{\epsilon}\sqrt{|V(\gamma)|}, \tag{2.24}$$

where W is the Wulff shape of unit area centered at the origin. Moreover,

$$\bigl||V(\gamma)|-|V(\mathfrak S)|\bigr|\le C\epsilon|V(\gamma)|. \tag{2.25}$$

Proof. We begin by noting that, by the assumptions of the present Lemma, $|V(\Gamma)|$ and $|V(\mathfrak S)|$ have to be of the same order of magnitude. More precisely, we claim that

$$\bigl||V(\Gamma)|-|V(\mathfrak S)|\bigr|\le C\epsilon\bigl|V(\Gamma)\bigr| \tag{2.26}$$

holds with some $C=C(\beta)<\infty$ independent of Γ, $\mathfrak S$ and ϵ. Indeed, from (2.11) and (2.23) we have

$$\sum_{S\in\mathfrak S}\bigl|\mathsf P(S)\bigr|\le\tau_{\text{min}}^{-1}\mathcal W_\beta(\mathfrak S)\le w_1(1+\epsilon)\tau_{\text{min}}^{-1}\sqrt{|V(\Gamma)|}, \tag{2.27}$$

which, using Lemma 2.3 and the bounds $s\le\epsilon\sqrt{|V(\Gamma)|}$ and $\epsilon\le 1$, gives (2.26) with $C=2g_3w_1\tau_{\text{min}}^{-1}$.

The bound (2.26) essentially allows us to replace $V(\Gamma)$ by $V(\mathfrak S)$ in (2.23). Applying Theorem 2.10 from [24] to the set of skeletons $\mathfrak S$ rescaled by $|V(\mathfrak S)|^{-1/2}$, we can conclude that there is point $x\in\mathbb R^2$ and a skeleton $S_0\in\mathfrak S$ such that

$$d_{\text{H}}\bigl(\mathsf P(S_0),\sqrt{|V(\mathfrak S)|}\partial W+x\bigr)\le\alpha\sqrt{\epsilon}\sqrt{|V(\mathfrak S)|}, \tag{2.28}$$

and

$$\sum_{S\in\mathfrak S\setminus\{S_0\}}\bigl|\mathsf P(S)\bigr|\le\alpha\epsilon\sqrt{|V(\mathfrak S)|}, \tag{2.29}$$

where α is a constant proportional to the ratio of the maximum and the minimum of the surface tension. Using (2.26) once more, we can modify (2.28–2.29) by replacing $V(\mathfrak S)$ on the right-hand sides by $V(\Gamma)$ at the cost of changing α to $\alpha(1+C)$. Moreover, since (2.26) also implies that $|\sqrt{|V(\Gamma)|}-\sqrt{|V(\mathfrak S)|}|\le C\epsilon\sqrt{|V(\Gamma)|}$, we have

$$d_{\text{H}}\bigl(\sqrt{|V(\Gamma)|}\partial W,\sqrt{|V(\mathfrak S)|}\partial W\bigr)\le C\epsilon\operatorname{diam}W\sqrt{|V(\Gamma)|}. \tag{2.30}$$

Let $\gamma\in\Gamma$ be the contour corresponding to S_0. By the definition of skeletons, $d_{\text{H}}(\gamma,\mathsf P(S_0))\le s\le\epsilon\sqrt{|V(\Gamma)|}$. Combining this with (2.30), the modified bound (2.28), and $\epsilon\le 1$, we get

$$d_{\text{H}}\bigl(\gamma,\sqrt{|V(\Gamma)|}\partial W+x\bigr)\le c\sqrt{\epsilon}\sqrt{|V(\Gamma)|} \tag{2.31}$$

for any $c\ge 1+\alpha(1+C)+C\operatorname{diam}W$. (From the properties of W, it is easily shown that $\operatorname{diam}W$ is of the order of unity.)

Let us proceed by proving that $\Gamma=\{\gamma\}$. For any $\gamma'\in\Gamma\setminus\{\gamma\}$, let $S_{\gamma'}$ be the unique skeleton in $\mathfrak S$ such that $\gamma'\sim S_{\gamma'}$. Since $\operatorname{diam}\gamma'\le|\mathsf P(S_{\gamma'})|+s$ and, since also $|\mathsf P(S_{\gamma'})|\ge s$, we have $\operatorname{diam}\gamma'\le 2|\mathsf P(S_{\gamma'})|$. Using the modified bound (2.29), we get

$$\operatorname{diam}\gamma'\le 2\bigl|\mathsf P(S_{\gamma'})\bigr|\le 2\alpha(1+C)\epsilon\sqrt{|V(\Gamma)|}. \tag{2.32}$$

If c also satisfies the inequality $c>2\alpha(1+C)$, then this estimate contradicts the assumption that $\operatorname{diam}\gamma'\ge c\epsilon\sqrt{|V(\Gamma)|}$ for all $\gamma'\in\Gamma$. Hence, $\Gamma=\{\gamma\}$ as claimed.

Thus, $V(\Gamma) = V(\gamma)$ and the bound (2.25) is directly implied by (2.26). Moreover, (2.31) holds with $V(\Gamma)$ replaced by $V(\gamma)$ on both sides. To prove (2.24), it remains to show that the naked γ on the left-hand side of (2.31) can be replaced by $V(\gamma)$. But that is trivial because γ is the boundary of $V(\gamma)$ and the Hausdorff distance of two closed sets in $\mathbb{R}^2$ equals the Hausdorff distance of their boundaries. □

2.3 Small-contour ensemble.

The goal of this section is to collect some estimates for the probability in $P_L^{+,\beta}$ conditioned on the fact that all contours are s-small in the sense that $\Gamma_s(\sigma) = \emptyset$. Most of what is to follow appears, in various guises, in the existing literature (cf Remark 7). For some of the estimates (Lemmas 2.9 and 2.10) we will actually provide a proof, while for others (Lemma 2.11) we can quote directly.

2.3.1 Estimates using the GHS inequality. The principal resource for what follows are two basic properties of the correlation function of Ising spins. Specifically, let $\langle \sigma_x; \sigma_y \rangle_{A,\boldsymbol{h}}^{+,\beta}$ denote the truncated correlation function of the Ising model in a set $A \subset \mathbb{Z}^2$ with plus boundary condition, in non-negative inhomogeneous external fields $\boldsymbol{h} = (h_x)$ and inverse temperature β. Then:

(1) If $\beta > \beta_{\mathrm{c}}$, then the correlations in infinite volume decay exponentially, i.e., we have

$$\langle \sigma_x; \sigma_y \rangle_{\mathbb{Z}^2,\boldsymbol{h}}^{+,\beta} \le e^{-\|x-y\|/\xi} \tag{2.33}$$

for some $\xi = \xi(\beta) < \infty$ and all x and y.

(2) The GHS inequality implies that the finite-volume correlation function, $\langle \sigma_x; \sigma_y \rangle_{A,\boldsymbol{h}}^{+,\beta}$, is dominated by the infinite-volume correlation function at any pointwise-smaller field:

$$0 \le \langle \sigma_x; \sigma_y \rangle_{A,\boldsymbol{h}}^{+,\beta} \le \langle \sigma_x; \sigma_y \rangle_{\mathbb{Z}^2,\boldsymbol{h}'}^{+,\beta} \tag{2.34}$$

for all $A \subset \mathbb{Z}^2$ and all $\boldsymbol{h}' = (h'_x)$ with $h'_x \in [0, h_x]$ for all x.

Note that, via (2.34), the exponential decay (2.33) holds uniformly in $A \subset \mathbb{Z}^2$. Part (1) is a consequence of the main result of [21], see [50]; the GHS inequality from part (2) dates back to [31].

Now we are ready to state the desired estimates. Let $A \subset \mathbb{Z}^2$ be a finite set and let s be a scale function. Let $P_A^{+,\beta,s}$ be the Gibbs measure of the Ising model in $A \subset \mathbb{Z}^2$ conditioned on the event $\{\Gamma_s(\sigma) = \emptyset\}$ and let us use $\langle - \rangle_A^{+,\beta,s}$ to denote the expectation with respect to $P_A^{+,\beta,s}$. Then we have the following bounds:

Lemma 2.9 *For each $\beta > \beta_{\mathrm{c}}$, there exist constants $\alpha_1(\beta)$ and $\alpha_2(\beta)$ such that*

$$\bigl|\langle M_A \rangle_A^{+,\beta,s} - m^\star |A|\bigr| \le \alpha_1(\beta)\bigl(|\partial A| + |A|^2 e^{-\alpha_2(\beta)\, s}\bigr) \tag{2.35}$$

for each finite set $A \subset \mathbb{Z}^2$ and any scaling function s. Moreover, if $A' \subset A$, then

$$\bigl|\langle M_A \rangle_A^{+,\beta,s} - \langle M_{A \smallsetminus A'} \rangle_{A \smallsetminus A'}^{+,\beta,s}\bigr| \le \alpha_1(\beta)\bigl(|A'| + |A|^2 e^{-\alpha_2(\beta)\, s}\bigr). \tag{2.36}$$

Proof. By Lemma 2.7, we have $P_A^{+,\beta}(\Gamma_s(\sigma) \neq \emptyset) \le |A| e^{-\alpha_2 s}$ for some $\alpha_2 > 0$, independent of A. An easy estimate then shows

$$\bigl|\langle M_A\rangle_A^{+,\beta,s} - \langle M_A\rangle_A^{+,\beta}\bigr| \le |A|^2 e^{-\alpha_2 s}. \tag{2.37}$$

Therefore, it suffices to prove the bounds (2.35–2.36) without the restriction to the ensemble of s-small contours. Next we claim that, for any $B \subset \mathbb{Z}^2$ we have

$$0 \le \langle\sigma_x\rangle_B^{+,\beta} - \langle\sigma_x\rangle_{B\cup\{y\}}^{+,\beta} \le e^{-\|x-y\|/\xi}. \tag{2.38}$$

Indeed, the difference of the two expectations can be written as an integral $\int_0^\infty \mathrm{d}h \langle\sigma_x;\sigma_y\rangle_{B\cup\{y\},\boldsymbol{h}}^{+,\beta}$, where $\boldsymbol{h} = (h_z)$ is such that $h_z = h\delta_{y,z}$. By property (2) of the truncated correlation function, we have that $\langle\sigma_x;\sigma_y\rangle_{B\cup\{y\},\boldsymbol{h}}^{+,\beta}$ is non-negative, which proves the left inequality in (2.38), while it is bounded above by the same correlation function with $B = \mathbb{Z}^2$. The integral representation can be used again for the correlation function in the infinite volume with the result

$$\langle\sigma_x\rangle_B^{+,\beta} - \langle\sigma_x\rangle_{B\cup\{y\}}^{+,\beta} \le \frac{\bigl\langle\sigma_x\frac{1+\sigma_y}{2}\bigr\rangle^{+,\beta}}{\bigl\langle\frac{1+\sigma_y}{2}\bigr\rangle^{+,\beta}} - \langle\sigma_x\rangle^{+,\beta} \le \frac{\bigl\langle\sigma_x;\frac{1+\sigma_y}{2}\bigr\rangle^{+,\beta}}{\bigl\langle\frac{1+\sigma_y}{2}\bigr\rangle^{+,\beta}} \le \langle\sigma_x;\sigma_y\rangle^{+,\beta}, \tag{2.39}$$

where we used that $\langle 1+\sigma_y\rangle^{+,\beta} \ge 1$ to derive the last inequality. Using the property (1) above, the right-hand side is bounded by $e^{-\|x-y\|/\xi}$.

The bound (2.38) immediately implies both (2.35) and (2.36). Indeed, using (2.38) for all $x \in A$ and $y \in B \setminus A$, we have for all $A \subseteq B \subseteq \mathbb{Z}^2$ that

$$0 \le \langle M_A\rangle_A^{+,\beta} - \langle M_A\rangle_B^{+,\beta} \le \sum_{x\in A}\sum_{y\in B\setminus A} e^{-\|x-y\|/\xi} \le \alpha_1' |\partial A|, \tag{2.40}$$

where $\alpha_1' = \alpha_1'(\beta) < \infty$. This and (2.37) directly imply (2.35). To get (2.36), we also need to note that $|M_A - M_{A\setminus A'}| \le |A'|$. $\square$

Our next claim concerns an upper bound on the probability that the magnetization in the plus state deviates from its mean by a positive amount:

Lemma 2.10 *Let $\beta > \beta_{\mathrm{c}}$ and let $\chi = \chi(\beta)$ be the susceptibility. Then there exists a constant $K = K(\beta)$ such that*

$$P_A^{+,\beta,s}\bigl(M_A \ge \langle M_A\rangle_A^{+,\beta} + m^\star v\bigr) \le 2e^{-\frac{(vm^\star)^2}{2\chi|A|}} \tag{2.41}$$

for any finite $A \subset \mathbb{Z}^2$, any $v \ge 0$, and any $s \ge K\log|A|$.

Proof. Let $\mathcal{M}$ denote the event $\mathcal{M} = \{\sigma\colon M_A \ge \langle M_A\rangle_A^{+,\beta} + m^\star v\}$. By Lemma 2.7 we have that $P_A^{+,\beta,s}(\mathcal{M}) \le 2P_A^{+,\beta}(\mathcal{M})$, so we just need to estimate $P_A^{+,\beta}(\mathcal{M})$. Consider the cumulant generating function $F_A^{+,\beta}(h) = \log\langle e^{hM_A}\rangle_A^{+,\beta}$. The exponential Chebyshev inequality then gives

$$\log P_A^{+,\beta}(\mathcal{M}) \le F_A^{+,\beta}(h) - h\langle M_A\rangle_A^{+,\beta} - hm^\star v, \qquad h \ge 0. \tag{2.42}$$

By the property (2) of the truncated correlation function, we get

$$\frac{\mathrm{d}^2 F_A^{+,\beta}}{\mathrm{d}h^2}(h) = \langle M_A; M_A\rangle_{A,\boldsymbol{h}}^{+,\beta} \le \langle M_A; M_A\rangle_{A,\mathbf{0}}^{+,\beta}, \tag{2.43}$$

where $\boldsymbol{h} = (h_x)$ with $h_x = h$ for all $x \in \mathbb{Z}^2$ and where $\mathbf{0}$ is the zero field. Since $F_A^{+,\beta}(0) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}h}F_A^{+,\beta}(0) = \langle M_A\rangle_A^{+,\beta}$, we get the bound

$$F_A^{+,\beta}(h) \le h\langle M_A\rangle_A^{+,\beta} + \frac{h^2}{2}\langle M_A; M_A\rangle_{A,\mathbf{0}}^{+,\beta}. \tag{2.44}$$

Now, once more by the property (2) above,

$$|A|^{-1}\langle M_A; M_A\rangle_{A,\mathbf{0}}^{+,\beta} \le |A|^{-1}\langle M_A; M_A\rangle_{\mathbb{Z}^2,\mathbf{0}}^{+,\beta} \le |A|^{-1}\sum_{x\in A}\sum_{y\in\mathbb{Z}^2}\langle\sigma_x;\sigma_y\rangle^{+,\beta} = \chi, \tag{2.45}$$

where the sums converge by the property (1). The claim follows by optimizing over h. □

Remark 7. The bound in Lemma 2.10 corresponds to Eq. (9.33) of Proposition 9.1 in [46] proved with the help of Lemma 5.1 from [45]. Similarly, the estimates in Lemma 2.9 are closely related to the bounds in Lemma 2.2.1 of [37]. We included the proofs of both statements to pinpoint the exact formulation needed for our analysis as well as to reduce the number of extraneous references.

2.3.2 Gaussian control of negative deviations. Our last claim concerns the deviations of the plus magnetization in the *negative* direction. Unlike in the previous Section, here the restriction to the small contour is crucial because, obviously, if the deviation is too large, there is a possibility of forming a droplet which cannot be controlled by bulk estimates.

Let $\beta > \beta_c$ and let v be such that $\langle M_A\rangle_A^{+,\beta,s} - 2m^\star v$ is an allowed value of M_A. Define $\Omega_A^s(v)$ by the expression

$$P_A^{+,\beta,s}\bigl(M_A = \langle M_A\rangle_A^{+,\beta,s} - 2vm^\star\bigr) = \frac{1}{\sqrt{2\pi\chi|A|}}\exp\Bigl\{-2\frac{(m^\star)^2}{\chi|A|}v^2 + \Omega_A^s(v)\Bigr\}. \tag{2.46}$$

Then we have:

Lemma 2.11 (Gaussian estimate) *For each $\beta > \beta_c$ and each set of positive constants a_1, a_2, a_3, there are constants $C < \infty$ and $K < \infty$ such that if $s = K\log L$, then*

$$\bigl|\Omega_A^s(v)\bigr| \le C\Bigl(K\frac{v^2}{L^3}\log L \vee \frac{v^3}{L^4}\Bigr) \tag{2.47}$$

for all allowed values of v such that

$$0 \le v \le a_1\frac{L^2}{\log L} \tag{2.48}$$

and all connected sets $A \subset \mathbb{Z}^2$ such that

$$a_2L^2 \le |A| \le L^2 \quad \text{and} \quad |\partial A| \le a_3 L. \tag{2.49}$$

Proof. This is a reformulation of Lemma 2.3.3 from [37]. □

3. Lower bound

In this Section we establish a lower bound for the asymptotic stated in (1.11). In addition to its contribution to the proof of Theorem 1.1, this lower bound will play an essential role in the proofs of Theorem 1.2 and Corollary 1.3. A considerable part of the proof hinges on the Fortuin-Kasteleyn representation of the Ising (and Potts) models, which makes the technical demands of this section rather different from those of the following sections.

3.1 Large-deviation lower bound.

This section is devoted to the proof of the following theorem:

Theorem 3.1 (Lower bound) *Let $\beta > \beta_{\rm c}$ and let (v_L) be a sequence of positive numbers such that $m^\star\,|\Lambda_L| - 2m^\star\,v_L$ is an allowed value of M_L for all L. Suppose that the limit (1.10) exists with $\Delta \in (0,\infty)$. Then there exists a sequence (ϵ_L) with $\epsilon_L \to 0$ such that*

$$P_L^{+,\beta}\bigl(M_L = m^\star\,|\Lambda_L| - 2m^\star\,v_L\bigr) \ge \exp\bigl\{-w_1\sqrt{v_L}\bigl(\inf_{0\le\lambda\le1}\Phi_\Delta(\lambda)+\epsilon_L\bigr)\bigr\} \tag{3.1}$$

holds for all L.

Remark 8. It is worth noting that, unlike in the corresponding statements of the lower bounds in [24, 37], we do not require any control over how fast the error ϵ_L tends to zero as $L\to\infty$. Indeed, it turns out that in the regime of finite Δ, the simple convergence $\epsilon_L \to 0$ will be enough to prove our main results. However, in the cases when v_L tends to infinity so fast that Δ is infinite, a proof would probably need also *some* information about the rate of the convergence $\epsilon_L \to 0$.

The strategy of the proof will simply be to produce a near-Wulff droplet that comprises a particular fraction of the volume v_L. The droplet will account for its requisite share of the deficit magnetization and we then force the exterior to absorb the rest. The probability of the latter event is estimated by using the truncated contour ensemble.

Let us first attend to the production of the droplet. Consider the Wulff shape W of unit area centered at the origin and a closed, self-avoiding polygonal curve $\mathsf{P} \subset W$. We will assume that the vertices of P have rational coordinates and, if N denotes the number of vertices of P, that each vertex is at most $1/N$ away from the boundary of W. Let $\operatorname{Int}\mathsf{P}$ denote the set of points $x\in\mathbb{R}^2$ surrounded by P. For any $t,r>1$, let $\mathsf{P}_0,\mathsf{P}_1,\mathsf{P}_2,\mathsf{P}_3$ be four magnified copies of P obtained by rescaling P by factors t, $t+r$, $t+2r$, and $t+3r$, respectively. (Thus, for instance, $\mathsf{P}_0 = \{x\in\mathbb{R}^2\colon x/t\in\mathsf{P}\}$.) This yields three "coronas" $K_{t,r}^{\rm I} = \operatorname{Int}\mathsf{P}_1\setminus\operatorname{Int}\mathsf{P}_0$, $K_{t,r}^{\rm II} = \operatorname{Int}\mathsf{P}_2\setminus\operatorname{Int}\mathsf{P}_1$, and $K_{t,r}^{\rm III} = \operatorname{Int}\mathsf{P}_3\setminus\operatorname{Int}\mathsf{P}_2$ surrounding P_0. Let $\mathbb{K}_{t,r}^{\rm I} = K_{t,r}^{\rm I}\cap\mathbb{Z}^2$, and similarly for $\mathbb{K}_{t,r}^{\rm II}$ and $\mathbb{K}_{t,r}^{\rm III}$.

Recall that a $*$-connected circuit in $\mathbb{Z}^2$ is a closed path on vertices of $\mathbb{Z}^2$ whose elementary steps connect either nearest or next-nearest neighbors. Let $\mathcal{E}_{t,r}$ be the set of configurations σ such that $\mathbb{K}_{t,r}^{\rm I}$ contains a $*$-connected circuit of sites $x\in\mathbb{Z}^2$ with $\sigma_x=-1$ and $\mathbb{K}_{t,r}^{\rm III}$ contains a

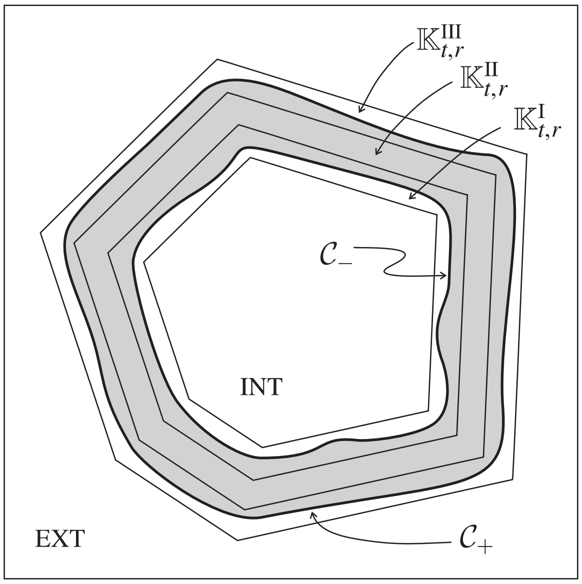

FIGURE 2. An illustration of the "coronas" $\mathbb{K}^{\mathrm{I}}_{t,r}$, $\mathbb{K}^{\mathrm{II}}_{t,r}$, $\mathbb{K}^{\mathrm{III}}_{t,r}$, the sets INT and EXT, and the $*$-connected circuits $\mathcal{C}_+$ and $\mathcal{C}_-$ of plus and minus sites, respectively, which are used in Lemma 3.2 and the proof of Theorem 3.1. Going from inside out, the four polygons correspond to P_0, P_1, P_2 and P_3; the shaded region denotes the set $A_\pm$.

$*$-connected circuit of sites $x \in \mathbb{Z}^2$ with $\sigma_x = +1$. The essential part of our lower bound comes from the following estimate:

Lemma 3.2 *Let $\beta > \beta_{\mathrm{c}}$ and let P be a polygonal curve as specified above. For any pair of sequences (t_L) and (r_L) tending to infinity as $L \to \infty$ in such a way that*

$$t_L L^{-1} \to 0, \quad t_L r_L e^{-r_L \tau_{\min}/3} \to 0 \quad \text{and} \quad r_L t_L^{-1} \to 0, \tag{3.2}$$

there is a sequence (ϵ'_L) with $\epsilon'_L \to 0$ such that

$$P_L^{+,\beta}(\mathcal{E}_{t_L,r_L}) \ge \exp\bigl\{-t_L \mathcal{W}_\beta(\mathsf{P})(1+\epsilon'_L)\bigr\}, \tag{3.3}$$

for all $L \ge 1$.

The proof of this lemma requires some substantial preparations and is therefore deferred to Section 3.2. Using Lemma 3.2, we can prove Theorem 3.1.

Proof of Theorem 3.1. Let us introduce the abbreviation

$$\mathcal{M}_L = \bigl\{\sigma\colon M_L = m^\star\,|\Lambda_L| - 2m^\star\, v_L\bigr\} \tag{3.4}$$

for the central event in question. Suppose first that $\Delta \le \Delta_{\mathrm{c}}$, where Δ_{c} is as in (1.18). Proposition 2.1 then guarantees that $\inf_{0\le\lambda\le1} \Phi_\Delta(\lambda) = \Phi_\Delta(0) = \Delta$. In particular, there is no need to produce a droplet in the system. Let $s = K\log L$. By restricting to the set of configurations $\{\sigma\colon \Gamma_s(\sigma)=\emptyset\}$ we get

$$P_L^{+,\beta}(\mathcal{M}_L) \ge P_L^{+,\beta,s}(\mathcal{M}_L)P_L^{+,\beta}\bigl(\Gamma_s(\sigma)=\emptyset\bigr). \tag{3.5}$$

The resulting lower bound is then a consequence of (2.46), Lemma 2.11 and Lemma 2.7, provided K is sufficiently large.

To handle the remaining cases, $\Delta > \Delta_{\rm c}$, we *will* have to produce a droplet. Fix a polygon P with the above properties, let $\text{Vol}(\mathsf{P})$ denote the two-dimensional Lebesgue volume of its interior, and let $|\mathsf{P}|$ denote the size (i.e., length) of its boundary. Let $\lambda=\lambda_\Delta$, where λ_Δ is as defined in (1.19), and recall that, for this choice of λ, we have $\Phi_\Delta(\lambda)=\inf_{0\le\lambda'\le1}\Phi_\Delta(\lambda')$ and $\lambda\ge\lambda_{\rm c}>0$. Since the goal is to produce a droplet of volume λv_L, we let $t_L=\sqrt{\lambda v_L}$ and pick r_L be such that (3.2) holds as $L\to\infty$. Abbreviating $\mathcal{E}_L=\mathcal{E}_{t_L,r_L}$, we let (ϵ'_L) denote the corresponding sequence from Lemma 3.2. (Note that ϵ'_L may depend on P.)

For configurations in $\mathcal{E}_L$, let $\mathcal{C}_+$ be the innermost $*$-connected circuit of plus spins in $\mathbb{K}_{t,r}^{\rm III}$ and let $\mathcal{C}_-$ denote the outermost $*$-connected circuit of minus spins in $\mathbb{K}_{t,r}^{\rm I}$. Let INT be the set of sites in the interior of $\mathcal{C}_-$ and let EXT be the set of sites in Λ_L that are in the exterior of $\mathcal{C}_+$. (Thus, we have $\text{INT}\cap\mathcal{C}_-=\text{EXT}\cap\mathcal{C}_+=\emptyset$.) Further, let $A_\pm=\Lambda_L\setminus(\text{INT}\cup\text{EXT})$ and use $\sigma_\pm$ to denote the spin configuration on $A_\pm$. Let $M_{\rm INT}$, $M_{\rm EXT}$ and $M_\pm$ denote the overall magnetization in INT, EXT and $A_\pm$, respectively. Finally, let us abbreviate $\mu_{\rm INT}=\lfloor\langle M_{\rm INT}\rangle_{\rm INT}^{+,\beta,s}\rfloor$ and introduce the event $\mathcal{E}'_L=\{\sigma\in\mathcal{E}_L\colon M_{\rm INT}=-\mu_{\rm INT}\}$.

The lower bound on $P_L^{+,\beta}(\mathcal{M}_L)$ will be derived by restricting to the event $\mathcal{E}'_L$, conditioning on $\sigma_\pm$, extracting the probability of having the correct magnetization in $\Lambda_L\setminus A_\pm$, and applying Lemma 2.11 to retrieve the contribution from droplet surface tension. The first two steps of this program give

$$P_L^{+,\beta}(\mathcal{M}_L)\ge P_L^{+,\beta}(\mathcal{M}_L\cap\mathcal{E}'_L)\ge\sum_{\sigma_\pm}P_L^{+,\beta}(\mathcal{M}_L\cap\mathcal{E}'_L|\sigma_\pm)P_L^{+,\beta}(\sigma_\pm). \tag{3.6}$$

Our next goal is to produce a lower bound of the type (3.1) on $P_L^{+,\beta}(\mathcal{M}_L\cap\mathcal{E}'_L|\sigma_\pm)$, uniformly in $\sigma_\pm$. The advantage of conditioning on a fixed configuration is that, if $\mathcal{M}_L\cap\mathcal{E}'_L\cap\{\sigma_\pm\}$ occurs, the overall magnetizations in INT and EXT are fixed. Thus, on $\mathcal{M}_L\cap\mathcal{E}'_L\cap\{\sigma_\pm\}$ we get

$$M_{\rm EXT}=M_L-M_\pm-M_{\rm INT}=\langle M_{\rm EXT}\rangle_{\rm EXT}^{+,\beta,s}-2m^\star v_L\bigl(1-\lambda\text{Vol}(\mathsf{P})-\delta_L\bigr), \tag{3.7}$$

where $\delta_L=\delta_L(\sigma_\pm)$ is given by the equation $2m^\star v_L\delta_L=\text{I}+\text{II}+\text{III}+\text{IV}$ with I–IV defined by

$$\text{I}=\mu_{\rm INT}-m^\star|\text{INT}|,\qquad \text{II}=-\langle M_{\rm EXT}\rangle_{\rm EXT}^{+,\beta,s}+m^\star|\text{EXT}|, \tag{3.8}$$

$$\text{III}=-M_\pm+m^\star|A_\pm|,\qquad \text{IV}=2m^\star\bigl(|\text{INT}|-\lambda\text{Vol}(\mathsf{P})v_L\bigr). \tag{3.9}$$

To estimate I–IV, we first notice the geometric bounds

$$\begin{gathered} t_L^2\text{Vol}(\mathsf{P})-t_L|\mathsf{P}|\le|\text{INT}|\le(t_L+r_L)^2\text{Vol}(\mathsf{P})+(t_L+r_L)|\mathsf{P}|,\\ |A_\pm|\le(t_L+3r_L)^2-t_L^2+(t_L+3r_L)|\mathsf{P}|, \end{gathered} \tag{3.10}$$

and recall that, since both $\mathcal{C}_+$ and $\mathcal{C}_-$ are contained in $A_\pm$, we have $|\mathcal{C}_-|, |\mathcal{C}_+| \le |A_\pm|$. Lemma 2.9 for $s = K \log L$ then allows us to estimate $|\mathrm{I}| \le \alpha_1(\beta)(|A_\pm| + |\mathrm{INT}|^2 L^{-\alpha_2(\beta)K})$ and, similarly, $|\mathrm{II}| \le \alpha_1(\beta)(|A_\pm| + 4L + L^{4-\alpha_2(\beta)K})$, while the remaining two quantities are bounded by invoking $|\mathrm{III}| \le 2|A_\pm|$ and $|\mathrm{IV}| \le 4r_L t_L + 2r_L^2 + 2(t_L + r_L)|\mathsf{P}|$. Using that $r_L = o(\sqrt{v_L})$ and $t_L = O(\sqrt{v_L})$, we have $|A_\pm| = o(v_L)$ as $L \to \infty$. Moreover, if K is so large that $4 - \alpha_2(\beta)K < 4/3$, we also have $|\mathrm{INT}|^2 L^{-\alpha_2(\beta)K} \le L^{4-\alpha_2(\beta)K} = o(v_L)$ as $L \to \infty$. Combining these bounds, it is easy to show that $|\delta_L(\sigma_\pm)| \le \bar\delta_L$ for all $\sigma_\pm$, where $\bar\delta_L$ is a sequence such that $\lim_{L\to\infty} \bar\delta_L = 0$.

Now we are ready to estimate the probability that both INT and EXT produce their share of magnetization deficit. Note first that

$$P_{\mathrm{INT}}^{-,\beta}(M_{\mathrm{INT}} = -\mu_{\mathrm{INT}}) \ge P_{\mathrm{INT}}^{-,\beta,s}(M_{\mathrm{INT}} = -\mu_{\mathrm{INT}}) P_{\mathrm{INT}}^{-,\beta}\bigl(\Gamma_s(\sigma) = \emptyset\bigr). \tag{3.11}$$

Using Lemmas 2.11 and 2.7, we get $P_{\mathrm{INT}}^{-,\beta}(M_{\mathrm{INT}} = -\mu_{\mathrm{INT}}) \ge CL^{-2/3}$ for some $C = C(\beta) > 0$. On the other hand, letting $\mathcal{M}_{\mathrm{EXT}} = \{\sigma\colon M_{\mathrm{EXT}} = \langle M_{\mathrm{EXT}}\rangle_{\mathrm{EXT}}^{+,\beta,s} - 2m^\star v_L(1 - \lambda \mathrm{Vol}(\mathsf{P}) - \delta_L)\}$, a bound similar to (3.11) for $P_{\mathrm{EXT}}^{+,\beta}$ combined with Lemmas 2.11 and 2.7 yields

$$P_{\mathrm{EXT}}^{+,\beta}(\mathcal{M}_{\mathrm{EXT}}) \ge \frac{C'}{\sqrt{|\mathrm{EXT}|}} \exp\Bigl\{-2\frac{(m^\star v_L)^2}{\chi |\mathrm{EXT}|}\bigl(1 - \lambda \mathrm{Vol}(\mathsf{P}) - \delta_L\bigr)^2\Bigr\}, \tag{3.12}$$

where $C' = C'(\beta) > 0$ is independent of $\sigma_\pm$ contributing to (3.6). Combining the previous estimates, we can use Lemma 3.2 to extract the surface energy term. The result is

$$P_L^{+,\beta}(\mathcal{M}_L) \ge C'' L^{-5/3} \exp\bigl\{-w_1 \sqrt{v_L}\, \Phi_L - \epsilon_L' \sqrt{v_L}\bigr\}, \tag{3.13}$$

where $C'' = C''(\beta) > 0$ and where Φ_L stands for the quantity

$$\Phi_L = \frac{\mathcal{W}_\beta(\mathsf{P})}{w_1}\sqrt{\lambda} + \frac{2(m^\star)^2 \chi^{-1} w_1^{-1} v_L^{3/2}}{L^2 - (t_L + r_L)^2}\bigl(1 - \lambda \mathrm{Vol}(\mathsf{P}) + \bar\delta_L\bigr)^2. \tag{3.14}$$

As is clear from our previous reasoning, the quantity Φ_L can be made arbitrary close to $\Phi_\Delta(\lambda)$ by letting $L \to \infty$ and optimizing over P with the above properties. The existence of the desired sequence (ϵ_L) then follows by the definition of the limit. □

3.2 Results using random-cluster representation.

In this section we establish some technical results necessary for the completion of the proof of our lower bound. These results are stated mostly in terms of the random cluster counterpart of the Ising model; the crowning achievement, which is Lemma 3.5, gives immediately in the proof of Lemma 3.2. We remark that the latter is the sum total of what this section contributes to the proof of Theorem 3.1. The uninterested, or well-informed, readers are invited to skip the entire section, provided they are prepared to accept Lemma 3.2 without a proof.

3.2.1 Preliminaries. The *random cluster* representation for the Ising (and Potts) ferromagnets is by now a well established tool. The purpose of the following remarks is to define our notation; for more background and details we refer the reader to, e.g., [10,32] or the excellent review [29].

Let $\mathbb{T} \subset \mathbb{Z}^2$ denote a finite graph. A *bond configuration*, generically denoted by ω, is the assignment of a zero (vacant) or a one (occupied) to each bond in $\mathbb{T}$. The weight of a configuration ω is given, informally, by $R^{|\omega|}q^{C(\omega)}$, where $|\omega|$ denotes the number of occupied bonds and $C(\omega)$ denotes the number of connected components. For the Ising system at hand we have $q = 2$ and $R = e^{2\beta} - 1$. The precise meaning of $C(\omega)$ depends on the boundary conditions; of concern here are the so called *free* and *wired* boundary conditions. In the former, $C(\omega)$ is the usual number of connected components including the isolated sites, while in the latter all clusters touching the bond-complement of $\mathbb{T}$ are identified as a single component.

The free and wired random-cluster measures in Λ_L, denoted by $P_{L,\text{FK}}^{\text{free},\beta}$ and $P_{L,\text{FK}}^{\text{w},\beta}$, respectively, correspond to the free and plus (or minus) boundary conditions in the Ising spin system. Both random-cluster measures enjoy the FKG property and the wired measure stochastically dominates the free measure. The infinite volume limits of these measures also exist; we denote these limiting objects by $P_{\text{FK}}^{\text{free},\beta}$ and $P_{\text{FK}}^{\text{w},\beta}$. The most important type of event we shall consider is the event that sites are connected by paths of occupied bonds. Our notation is as follows: If $x, y \in \mathbb{T}$, we define $\{x \longleftrightarrow y\}$ to be the event that there is such a connection. If we demand the existence of a path using only bonds with both ends in some subgraph $\mathbb{A} \subset \mathbb{T}$, we write $\{x \underset{\mathbb{A}}{\longleftrightarrow} y\}$.

The next concept we need to discuss is *duality*. For any $\mathbb{T} \subset \mathbb{Z}^2$, the *dual* graph $\mathbb{T}^*$ is defined as follows: Each bond of $\mathbb{T}$ is transversal to a bond on $(\mathbb{Z} + \frac{1}{2}) \times (\mathbb{Z} + \frac{1}{2}) = (\mathbb{Z}^2)^*$. These bonds are the bonds of $\mathbb{T}^*$; the sites of $\mathbb{T}^*$ are the endpoints of these bonds. Each configuration ω induces a configuration on the dual graph via the correspondence "direct occupied" with "dual vacant" and *vice versa*. It turns out that, if we start with either free or wired boundary conditions on $\mathbb{T}$, the weights for the dual configurations are also random-cluster weights with parameters $(q^*, R^*) = (q, q/R)$, provided we also interchange the designation of "free" and "wired." Of course, the graph and its dual are not precisely the same. For example, if we examine the relevant graph for the problem dual to the wired system in Λ_L, this consists of an $(L+1) \times (L+1)$ rectangle with the corners missing. Moreover, because the boundary conditions on the dual graph are free, all dual edges touching the boundary sites are occupied independently of the rest of the configuration. Thus, ignoring these decoupled degrees of freedom, the restricted measure is equivalent to a free measure on Λ_{L-1}.

In general, we will use β^* to denote the inverse temperature dual to β, which, for $q = 2$ and the normalization of the Hamiltonian (1.1), is related to β via $\beta^* = \frac{1}{2}\log\coth\beta$. The critical temperature is self dual, i.e., $\beta_\text{c} = \frac{1}{2}\log\coth\beta_\text{c}$. For $\beta > \beta_\text{c}$, the dual model is in the high-temperature phase. Hence, the limiting free and wired measures at β^* coincide and, using the well-known relation between the spin-correlations and the connectivity functions in the FK representation, we have

$$P_{\text{FK}}^{\text{free},\beta^*}(x \longleftrightarrow y) = P_{\text{FK}}^{\text{w},\beta^*}(x \longleftrightarrow y) = \langle\sigma_0\sigma_x\rangle^{+,\beta^*}, \tag{3.15}$$

for all $x, y \in \mathbb{Z}^2$. Thus, the exponential decay of correlations in the spin system at high temperatures, $\langle\sigma_0\sigma_x\rangle^{+,\beta^*} \le e^{-\|x-y\|/\xi}$ where $\xi = \xi(\beta^*)$ is the correlation length, corresponds to an exponential decay of the connectivity probabilities. In particular, the *surface tension* at $\beta > \beta_\text{c}$, as defined in (1.5) for unit vectors $\boldsymbol{n}$ with rationally related components, is the inverse of the correlation length for two point connectivity functions in the direction $\boldsymbol{n}$ at inverse temperature β^*.

3.2.2 Decay estimates. Here we assemble two important ingredients for the proof of Lemma 3.2. We begin by quantifying the decay of the point-to-boundary connectivity function:

Lemma 3.3 *Consider the $q = 2$ random cluster model at $\beta < \beta_{\rm c}$ (which corresponds to the high-temperature phase of the Ising system). Then,*

$$P_{\ell,\text{FK}}^{\text{w},\beta}\bigl(\{0 \longleftrightarrow \partial\Lambda_\ell\}\bigr) \le 4\ell e^{-\ell/\xi} \tag{3.16}$$

for all $\ell \ge 1$.

Proof. This is one portion of the proof of Proposition 4.1 in [20]. □

For the purposes of the next lemma, let $\boldsymbol{n}$ be a unit vector with rationally related components and let $\mathcal{C}(\boldsymbol{n})$ be the set of all pairs (a, b) of positive real numbers such that the $a \times b$ rectangle with side b perpendicular to $\boldsymbol{n}$ can be positioned in $\mathbb{R}^2$ in such a way that all its four corners are in $\mathbb{Z}^2$. We will use $R^{\boldsymbol{n}}_{a,b} \subset \mathbb{Z}^2$ to denote a generic $a \times b$ rectangle with the latter property. If x and y are the two corners along the same b-side of $R^{\boldsymbol{n}}_{a,b}$, we let $\mathcal{B}^{\boldsymbol{n}}_{a,b}$ denote the event $\{x \underset{R^{\boldsymbol{n}}_{a,b}}{\longleftrightarrow} y\}$.

Lemma 3.4 *Let $\beta \in (0, \beta_{\rm c})$ and let $\beta^* = \frac{1}{2}\log\coth\beta$. Let $\boldsymbol{n}$ be a unit vector with rationally related components and suppose that L, a_L and b_L, with $(a_L, b_L) \in \mathcal{C}(\boldsymbol{n})$, tend to infinity in such a way that $a_L/L \to 0$, $b_L/L \to 0$ and* $\operatorname{dist}(R^{\boldsymbol{n}}_{a,b}, \mathbb{Z}^2 \setminus \Lambda_L)/(b_L + \log L) \to \infty$ *as $L \to \infty$. Then*

$$\lim_{L\to\infty} P_{L,\text{FK}}^{\text{free},\beta}\bigl(\mathcal{B}^{\boldsymbol{n}}_{a_L,b_L}\bigr)^{1/b_L} \ge e^{-\tau_{\beta^*}(\boldsymbol{n})}. \tag{3.17}$$

Proof. We will first establish the limit (3.17) for the measure in infinite volume and then show that provided $R^{\boldsymbol{n}}_L$ are well separated from $\mathbb{Z}^2 \setminus \Lambda_L$ as specified, the finite volume effects are not important. Throughout the proof, we will omit the subscript β^* for the surface tension.

Fix $\boldsymbol{n} \in \mathcal{S}_1$ with rationally related components and let $\beta < \beta_{\rm c}$. Let

$$\theta^{\boldsymbol{n}}_{a,b} = P_{\text{FK}}^{\text{w},\beta}\bigl(\mathcal{B}^{\boldsymbol{n}}_{a,b}\bigr), \qquad (a,b) \in \mathcal{C}(\boldsymbol{n}), \tag{3.18}$$

and note that if $(a, b_1) \in \mathcal{C}(\boldsymbol{n})$ and $(a, b_2) \in \mathcal{C}(\boldsymbol{n})$ with $b_2 \ge b_1$, then also $(a, b_1 + b_2) \in \mathcal{C}(\boldsymbol{n})$ and $(a, b_2 - b_1) \in \mathcal{C}(\boldsymbol{n})$. We begin by the claim that the events in question enjoy a subadditive property:

$$\theta^{\boldsymbol{n}}_{a,b_1+b_2} \ge \theta^{\boldsymbol{n}}_{a,b_1}\theta^{\boldsymbol{n}}_{a,b_2}, \qquad (a,b_1),\,(a,b_2) \in \mathcal{C}(\boldsymbol{n}). \tag{3.19}$$

Indeed, we let $R^{\boldsymbol{n}}_{a,b_2}$ be translated relative to $R^{\boldsymbol{n}}_{a,b_1}$ so that the "left" a-side of $R^{\boldsymbol{n}}_{a,b_2}$ coincides with the "right" a-side of $R^{\boldsymbol{n}}_{a,b_1}$. Let x_1 and y_1 be the "left" and "right" bottom corners of $R^{\boldsymbol{n}}_{a,b_1}$ and let x_2 and y_2 be similar corners of $R^{\boldsymbol{n}}_{a,b_2}$. By our construction, y_1 and x_2 coincide. Let $R^{\boldsymbol{n}}_{a,b_1+b_2}$ denote the union $R^{\boldsymbol{n}}_{a,b_1} \cup R^{\boldsymbol{n}}_{a,b_2}$. Then

$$\bigl\{x_1 \underset{R^{\boldsymbol{n}}_{a,b_1+b_2}}{\longleftrightarrow} y_2\bigr\} \supset \bigl\{x_1 \underset{R^{\boldsymbol{n}}_{a,b_1}}{\longleftrightarrow} y_1\bigr\} \cap \bigl\{x_2 \underset{R^{\boldsymbol{n}}_{a,b_2}}{\longleftrightarrow} y_2\bigr\}. \tag{3.20}$$

The inequality (3.19) then follows immediately from the FKG property of the measure $P_{\text{FK}}^{\text{w},\beta}$.

Let $\mathcal{A}(\boldsymbol{n}) = \{a > 0\colon \exists b > 0,\ (a,b) \in \mathcal{C}(\boldsymbol{n})\}$ be the set of allowed values of a. As a consequence of subadditivity, for any $a \in \mathcal{A}(\boldsymbol{n})$ we have the existence of the limit $e^{-\varpi_a(\boldsymbol{n})} = \lim_{b\to\infty}(\theta^{\boldsymbol{n}}_{a,b})^{1/b}$. (Here b only takes values such that $(a,b) \in \mathcal{C}(\boldsymbol{n})$.) Further, if $a_1, a_2 \in \mathcal{A}(\boldsymbol{n})$ with $a_1 \ge a_2$, then there *is* a b such that both $(a_1,b) \in \mathcal{C}(\boldsymbol{n})$ and $(a_2,b) \in \mathcal{C}(\boldsymbol{n})$, and, for any such b, we have $\theta^{\boldsymbol{n}}_{a_1,b} \ge \theta^{\boldsymbol{n}}_{a_2,b}$. Thence $\varpi_{a_1}(\boldsymbol{n}) \le \varpi_{a_2}(\boldsymbol{n})$ whenever $a_1, a_2 \in \mathcal{A}(\boldsymbol{n})$ satisfy $a_1 \ge a_2$. Let $\varpi(\boldsymbol{n}) = \lim_{a\to\infty}\varpi_a(\boldsymbol{n})$, where a's are restricted to $\mathcal{A}(\boldsymbol{n})$. Now the quantity $\theta^{\boldsymbol{n}}_{\infty,b} = \lim_{a\to\infty}\theta^{\boldsymbol{n}}_{a,b}$, where $(a,b) \in \mathcal{C}(\boldsymbol{n})$, still obeys the subadditivity relation (3.19) and, in particular, the *half-space* surface tension $\tau_{\text{h}}(\boldsymbol{n})$ is well defined by the limit

$$e^{-\tau_{\text{h}}(\boldsymbol{n})} = \lim_{b\to\infty}\lim_{\substack{(a,b)\in\mathcal{C}(\boldsymbol{n})\\ a\to\infty}}(\theta^{\boldsymbol{n}}_{a,b})^{1/b}. \tag{3.21}$$

Moreover, $\theta^{\boldsymbol{n}}_{\infty,b} \ge \theta^{\boldsymbol{n}}_{a,b}$ for all a and b such that $(a,b) \in \mathcal{C}(\boldsymbol{n})$ and, therefore, $\tau_{\text{h}}(\boldsymbol{n}) \le \varpi(\boldsymbol{n})$. Our goal is to demonstrate that $\tau_{\text{h}}(\boldsymbol{n}) = \varpi(\boldsymbol{n})$ and that the half-space surface tension $\tau_{\text{h}}(\boldsymbol{n})$ equals the full space surface tension $\tau(\boldsymbol{n})$.

Let $\epsilon > 0$. Then there is a $b^\star$ such that $\theta^{\boldsymbol{n}}_{\infty,b^\star} \ge e^{-b^\star(\tau_{\text{h}}(\boldsymbol{n})+\epsilon)}$. However, since $\theta^{\boldsymbol{n}}_{\infty,b^\star}$ simply *equals* the limit of $\theta^{\boldsymbol{n}}_{a,b^\star}$ as $a \to \infty$, there is an $a^\star$ such that $\theta^{\boldsymbol{n}}_{a^\star,b^\star} \ge e^{-b^\star(\tau_{\text{h}}(\boldsymbol{n})+2\epsilon)}$. Thence $\varpi(\boldsymbol{n}) \le \tau_{\text{h}}(\boldsymbol{n})$ and the equality of $\tau_{\text{h}}(\boldsymbol{n})$ and $\varpi(\boldsymbol{n})$ follows. To remove the half-space constraint, consider the analogue of the previously defined events. Let x and y be related to $R^{\boldsymbol{n}}_{a,b}$ as in the definition of event $\mathcal{B}^{\boldsymbol{n}}_{a,b}$ and let $D^{\boldsymbol{n}}_{a,b}$ denote the union of $R^{\boldsymbol{n}}_{a,b}$ and its reflection through the line joining x and y. Let

$$\rho^{\boldsymbol{n}}_{a,b} = P^{\text{w},\beta}_{\text{FK}}\bigl(\{x \underset{D^{\boldsymbol{n}}_{a,b}}{\longleftrightarrow} y\}\bigr). \tag{3.22}$$

Reasoning identical to that employed thus far yields

$$e^{-\tau(\boldsymbol{n})} = \lim_{b\to\infty}\lim_{a\to\infty}(\rho^{\boldsymbol{n}}_{a,b})^{1/b} = \lim_{a\to\infty}\lim_{b\to\infty}(\rho^{\boldsymbol{n}}_{a,b})^{1/b}, \tag{3.23}$$

where we tacitly assume $(a,b) \in \mathcal{C}(\boldsymbol{n})$ for the production of both limits. Now, obviously, $\rho^{\boldsymbol{n}}_{a,b} \ge \theta^{\boldsymbol{n}}_{a,b}$ and hence $\tau(\boldsymbol{n}) \le \tau_{\text{h}}(\boldsymbol{n})$. To derive the opposite inequality, we note that for each $a \in \mathcal{A}(\boldsymbol{n})$, there is a $g(a) > 0$ such that

$$\theta^{\boldsymbol{n}}_{2a,b} \ge g(a)\rho^{\boldsymbol{n}}_{a,b}, \qquad (a,b) \in \mathcal{C}(\boldsymbol{n}). \tag{3.24}$$

Indeed, the event giving rise to $\theta^{\boldsymbol{n}}_{2a,b}$ can certainly be achieved by connecting the bottom corners of $R^{\boldsymbol{n}}_{2a,b}$ directly to the middle points and then connecting the middle points on the opposite a-sides of $R^{\boldsymbol{n}}_{2a,b}$. Then (3.24) follows by FKG. (To get that $g(a) > 0$, we also used that $\beta > 0$.) Taking the $1/b$-th power of both sides of (3.24) and letting $b \to \infty$ followed by $a \to \infty$ we arrive at $\varpi(\boldsymbol{n}) = \tau_{\text{h}}(\boldsymbol{n}) = \tau(\boldsymbol{n})$ as promised.

To finish the proof, we must account for the effects of finite volume. Consider the event $\mathcal{F}^{\boldsymbol{n}}_{a,b} = \{\partial R^{\boldsymbol{n}}_{a,b} \leftrightarrow \partial\Lambda_L\}$. Should $\mathcal{F}^{\boldsymbol{n}}_{a,b}$ *not* occur, a vacant ring separates $R^{\boldsymbol{n}}_{a,b}$ from $\partial\Lambda_L$ and, using fairly standard arguments, we have

$$P^{\text{free},\beta}_{L,\text{FK}}(\mathcal{B}^{\boldsymbol{n}}_{a,b}) \ge P^{\text{w},\beta}_{\text{FK}}\bigl(\mathcal{B}^{\boldsymbol{n}}_{a,b}\big|(\mathcal{F}^{\boldsymbol{n}}_{a,b})^{\text{c}}\bigr). \tag{3.25}$$

On the other hand, by Lemma 3.3, we have

$$P_{\text{FK}}^{\text{w},\beta}(\mathcal{F}_{a,b}^{\boldsymbol{n}}) \le P_{L,\text{FK}}^{\text{w},\beta}(\mathcal{F}_{a,b}^{\boldsymbol{n}}) \le 8L(a+b)\, e^{-\operatorname{dist}(\partial R_{a,b}^{\boldsymbol{n}},\partial\Lambda_L)/\xi}. \tag{3.26}$$

Thus if the distance between $\partial R_{a,b}^{\boldsymbol{n}}$ and $\partial\Lambda_L$ exceeds a large multiple of $b_L+\log L$, the dominant contribution to $P_{\text{FK}}^{\text{w},\beta}(\mathcal{B}_{a,b}^{\boldsymbol{n}})$ comes from $P_{\text{FK}}^{\text{w},\beta}(\mathcal{B}_{a,b}^{\boldsymbol{n}}\big|(\mathcal{F}_{a,b}^{\boldsymbol{n}})^{\text{c}})$. Using (3.25), the claim follows. $\square$

3.2.3 Corona estimates. We recall the "corona" regions $\mathbb{K}_{t,r}^{\text{I}}$–$\mathbb{K}_{t,r}^{\text{III}}$ associated with some given polygon P. In addition, we will also need to consider the collection of dual sites $\mathbb{K}_{t,r}^{*\text{II}} = K_{t,r}^{\text{II}} \cap (\mathbb{Z}^2)^*$, where $(\mathbb{Z}^2)^*$ is the lattice dual to $\mathbb{Z}^2$. (This differs slightly from the graph dual to $\mathbb{K}_{t,r}^{\text{II}}$ by some boundary sites.) In the context of the random cluster model (and its dual) we will consider three events: The first event, to be denoted $\mathcal{E}_{t,r}^{\text{I}}$, takes place in $\mathbb{K}_{t,r}^{\text{I}}$ and is defined by

$$\mathcal{E}_{t,r}^{\text{I}} = \bigl\{\omega\in\Omega\colon \textit{ there is a circuit of occupied bonds in } \mathbb{K}_{t,r}^{\text{I}} \textit{ surrounding the origin}\bigr\}. \tag{3.27}$$

The event $\mathcal{E}_{t,r}^{\text{III}}$ is defined similarly except that the circuit takes place in the region $\mathbb{K}_{t,r}^{\text{III}}$. Finally, one more circuit, this time a dual circuit in the region $\mathbb{K}_{t,r}^{\text{II}*}$. We define

$$\mathcal{E}_{t,r}^{\text{II}*} = \bigl\{\omega\in\Omega\colon \textit{ there is a dual circuit of vacant bonds in } \mathbb{K}_{t,r}^{*\text{II}} \textit{ surrounding the origin}\bigr\}. \tag{3.28}$$

As we will see in the proof of Lemma 3.2, the event $\mathcal{E}_{t,r}^{\text{I}}\cap\mathcal{E}_{t,r}^{\text{II}*}\cap\mathcal{E}_{t,r}^{\text{III}}$ more or less implies the desired event $\mathcal{E}_{t,r}$. The desired lower bound will then be an immediate consequence of the following lemma:

Lemma 3.5 *Let $\beta>\beta_{\text{c}}$ and let P be as in Lemma 3.2. For any sequences (t_L) and (r_L) satisfying (3.2), there is a sequence (ϵ_L'') such that $\epsilon_L''\to 0$ and, for all L,*

$$P_{L,\text{FK}}^{\text{w},\beta}\bigl(\mathcal{E}_{t_L,r_L}^{\text{I}}\cap\mathcal{E}_{t_L,r_L}^{\text{II}*}\cap\mathcal{E}_{t_L,r_L}^{\text{III}}\bigr) \ge \exp\bigl\{-t_L\mathcal{W}_\beta(\mathsf{P})(1+\epsilon_L'')\bigr\}. \tag{3.29}$$

Proof. In the course of this proof, let us abbreviate $\mathcal{E}_L^{\text{I}}=\mathcal{E}_{t_L,r_L}^{\text{I}}$, and similarly for $\mathcal{E}_L^{\text{II}*}$ and $\mathcal{E}_L^{\text{III}}$, as well as $\mathbb{K}_L^{\text{I}}$, $\mathbb{K}_L^{*\text{II}}$, and $\mathbb{K}_L^{\text{III}}$. We will start with an estimate for $P_{L,\text{FK}}^{\text{w},\beta}(\mathcal{E}_L^{\text{II}*})$, which is in any case the central ingredient of this lemma. Let T be the smallest integer $T\ge 2$ such that the polygon P magnified by T has all vertices on $\mathbb{Z}^2$. Let $u_L = T\lfloor (t_L+r_L)/T\rfloor + T$ and let $x_1,\dots,x_N$ be the vertices of the polygon P magnified by u_L. Let $x_1^*,\dots,x_N^*$ be the corresponding vertices of the polygon P magnified by u_L and translated by $(-\frac12,-\frac12)$. Notice that (once t_L and r_L are large enough) the sites $x_1^*,\dots,x_N^*$ lie inside the "corona" $\mathbb{K}_L^{*\text{II}}$. We use $\boldsymbol{n}_i$ to denote the unit vector constituting the outer normal to the side between x_{i+1}^* and x_i^* (where x_{N+1}^* is identified with x_1^*). By our construction, $x_1,\dots,x_N\in\mathbb{Z}^2$, $x_1^*,\dots,x_N^*\in(\mathbb{Z}^2)^*$ and $\boldsymbol{n}_i$ have rationally related components.

For $i=1,\dots,N$, let us consider the rectangles $R_{a_i,b_i}^{\boldsymbol{n}_i}$ with the base coinciding with the line between x_i^* and x_{i+1}^*. Here a_i is the largest possible number such that $(a_i,b_i)\in\mathcal{C}(\boldsymbol{n}_i)$ and $R_{a_i,b_i}^{\boldsymbol{n}_i}\subset\mathbb{K}_L^{*\text{II}}$. We remark that all (a_i) and (b_i) have L-dependence which is notationally suppressed and that these tend to infinity as $L\to\infty$. In particular, the b_i's scale with u_L. Let us denote

$$\mathfrak{b}_i = \lim_{L\to\infty} \frac{b_i}{t_L}, \qquad i = 1, \dots, N, \tag{3.30}$$

where the limit exists by the construction of b_i's and where we noted that $t_L/u_L \to 1$ as $L \to \infty$.

Let $\mathcal{B}_i^*$ be the event that there is a dual vacant connection $x_i^* \longleftrightarrow x_{i+1}^*$ in the box $R^{\boldsymbol{n}_i}_{a_i,b_i}$ and let $\mathcal{B}_i$ be the corresponding "direct" event that there is a direct occupied path $x_i \longleftrightarrow x_{i+1}$ contained in $(\frac12,\frac12)$-translate of $R^{\boldsymbol{n}_i}_{a_i,b_i}$. It is clear that the intersection $\bigcap_{i=1}^N \mathcal{B}_i^*$ produces the event $\mathcal{E}_L^{\text{II}*}$ and that these events are FKG-correlated. Moreover, by duality, we have

$$P_{L,\text{FK}}^{\text{w},\beta}(\mathcal{B}_i^*) = P_{L-1,\text{FK}}^{\text{free},\beta^*}(\mathcal{B}_i) \tag{3.31}$$

(c.f., the paragraph before (3.15)). Now we are perfectly positioned to apply Lemma 3.4: Using FKG, the scaling relation (3.30), and the fact that also the a_j's tend to infinity by our construction, we have as a consequence of the above-mentioned lemma that

$$\lim_{L\to\infty} P_{L,\text{FK}}^{\text{w},\beta}\bigl(\mathcal{E}_L^{\text{II}*}\bigr)^{1/t_L} = \exp\Bigl\{-\sum_{j=1}^N \mathfrak{b}_j \tau_\beta(\boldsymbol{n}_j)\Bigr\}. \tag{3.32}$$

The remainder of the proof concerns the estimate of the probability $P_{L,\text{FK}}^{\text{w},\beta}(\mathcal{E}_L^{\text{I}} \cap \mathcal{E}_L^{\text{III}} | \mathcal{E}_L^{\text{II}*})$. We claim that this conditional probability tends to one as $L \to \infty$. First, as a worst-case scenario, consider the event $V_L^{\text{II}*}$ that all bonds in $\mathbb{K}_L^{*\text{II}}$ are vacant. By monotonicity in boundary conditions and the strong FKG property of $P_{L,\text{FK}}^{\text{w},\beta}$ it is seen that

$$P_{L,\text{FK}}^{\text{w},\beta}\bigl(\mathcal{E}_L^{\text{I}} \cap \mathcal{E}_L^{\text{III}} \big| \mathcal{E}_L^{\text{II}*}\bigr) \ge P_{L,\text{FK}}^{\text{w},\beta}\bigl(\mathcal{E}_L^{\text{I}} \cap \mathcal{E}_L^{\text{III}} \big| V_L^{\text{II}*}\bigr). \tag{3.33}$$

Under the condition that $V_L^{\text{II}*}$ occurs, $\mathcal{E}_L^{\text{I}}$ and $\mathcal{E}_L^{\text{III}}$ are independent and we may treat them separately. The arguments are virtually identical for both events, so we need only be explicit about $P_{L,\text{FK}}^{\text{w},\beta}(\mathcal{E}_L^{\text{I}} | V_L^{\text{II}*})$.

Let ℓ_L be a maximal integer such that there is a circuit of dual cites, $z_1^*, \dots, z_m^*$, separating the boundaries of $\mathbb{K}_L^{\text{I}}$ with the property that, if $\Lambda_{\ell_L}^*(z_j^*)$ is the translate of $\Lambda_{\ell_L}^*$ by (the vector) z_j^*, then $\Lambda_{\ell_L}^*(z_j^*) \subset \mathbb{K}_L^{\text{I}}$. Note that $\liminf_{L\to\infty} \ell_L/r_L > 1/3$. Now, for the event $\mathcal{E}_L^{\text{I}}$ *not* to occur, there must be a dual occupied path connecting some dual site on the outer boundary of $\mathbb{K}_L^{\text{I}}$ to another on the inner boundary and hence at least one z_j^* has to be connected to the boundary of its $\Lambda_{\ell_L}^*(z_j^*)$ by a path of dual occupied bonds. Using subadditivity of the probability measure, we find

$$1 - P_{L,\text{FK}}^{\text{w},\beta}\bigl(\mathcal{E}_L^{\text{I}} \big| V_L^{\text{II}*}\bigr) \le \sum_{j=1}^m P_{L,\text{FK}}^{\text{w},\beta}\bigl(z_j^* \longleftrightarrow \partial\Lambda_{\ell_L}^*(z_j^*) \big| V_L^{\text{II}*}\bigr). \tag{3.34}$$

Now, again invoking monotonicity in the boundary conditions, the probability of the above connection events may be estimated from above by placing dual wired (i.e., direct free) boundary conditions on $\Lambda_{\ell_L}^*(z_j^*)$. But then, by duality, we have exactly the event which is the subject of Lemma 3.3. Explicitly,

$$P_{L,\text{FK}}^{\text{w},\beta}\bigl(z_j^* \longleftrightarrow \partial\Lambda_{\ell_L}^*(z_j^*) \big| V_L^{\text{II}*}\bigr) \le P_{\ell_L,\text{FK}}^{\text{w},\beta^*}\bigl(0 \longleftrightarrow \partial\Lambda_{\ell_L}\bigr) \tag{3.35}$$

holds for all $j = 1, \dots, m$, and the bound in (3.16) can be applied. Now the number of sites z_j^* which comprise the circuit does not exceed a multiple of t_L. Thus, for some constant C independent of L we have

$$P_{L,\mathrm{FK}}^{\mathrm{w},\beta}\bigl(\mathcal{E}_L^{\mathrm{I}}\big|V_L^{\mathrm{II}*}\bigr) \ge 1 - C\ell_L t_L e^{-\ell_L/\xi}. \tag{3.36}$$

By the condition stated in (3.2), the fact that $r_L \ge \ell_L \ge r_L/3$ for sufficiently large L, and the observation that $\xi^{-1} = \tau_{\min}$, the desired result for $\mathcal{E}_L^{\mathrm{I}}$ follows. Similarly for $\mathcal{E}_L^{\mathrm{III}}$. □

Proof of Lemma 3.2. We make liberal use of the correspondence between the graphical configurations ω and (sets of) spin configurations as described, e.g., in [2, 10, 27]. Each connected cluster in ω represents the spin configurations in which all sites of the cluster have spins of the same type. Thus, if $\mathcal{E}_L^{\mathrm{I}} \cap \mathcal{E}_L^{\mathrm{II}*} \cap \mathcal{E}_L^{\mathrm{III}}$ occurs, then the inner circuit of occupied bonds in $\mathbb{K}_L^{\mathrm{I}}$ forces the spins on these sites to be of the same type. Since these are disconnected from the boundary of Λ_L by the dual vacant circuit in $\mathbb{K}_L^{*\mathrm{II}}$, with probability one-half, all spins on the circuit are minus. Similarly, the outer circuit of bonds in $\mathbb{K}_L^{\mathrm{III}}$ is plus-type with probability one if it is connected to $\partial\Lambda_L$ and with probability $1/2$ otherwise. Thus, $P_L^{+,\beta}(\mathcal{E}_{t_L,r_L}|\mathcal{E}_L^{\mathrm{I}} \cap \mathcal{E}_L^{\mathrm{II}*} \cap \mathcal{E}_L^{\mathrm{III}})$ is certainly bigger than $1/4$, and the claim follows using Lemma 3.5. □

4. ABSENCE OF INTERMEDIATE CONTOUR SIZES

4.1 Statement and outline.

The goal of this section is to prove that, with probability tending to one as $L \to \infty$, there will be no contours with a diameter between the scales of $\log L$ and $\sqrt{v_L}$ in the "canonical" ensemble of the Ising model in volume Λ_L. This result is by far the most difficult part of the proof of our main results stated in Section 1.3.

We start with a standard notion from contour theory. Let $\Gamma(\sigma)$ denote the set of all contours of a configuration σ in Λ_L with plus boundary condition. Applying the rounding rule, contours are self-avoiding simple curves in $\mathbb{R}^2$. Recall that $\Gamma_s(\sigma)$ is the set of contours of σ that have a non-trivial s-skeleton. We say that $\gamma \in \Gamma(\sigma)$ is an *external* contour, if it is not surrounded by any other contour from Γ. We will use $\Gamma_s^{\mathrm{ext}}(\sigma)$ to denote the set of external contours of $\Gamma_s(\sigma)$. (We remark that $\Gamma_s^{\mathrm{ext}}(\sigma)$, namely the external contours of $\Gamma(\sigma)$ which are big enough to have an s-skeleton, coincides exactly with the set of external contours of the collection $\Gamma_s(\sigma)$.)

Using this notation, the event $\mathcal{A}_{\varkappa,s,L}$ from Theorem 1.2 is best described via its complement:

$$\mathcal{A}_{\varkappa,s,L}^{\mathrm{c}} = \bigl\{\sigma\colon \exists\gamma \in \Gamma_s^{\mathrm{ext}}(\sigma),\, \operatorname{diam}\gamma \le \varkappa\sqrt{v_L}\bigr\}. \tag{4.1}$$

The relevant claim is then restated as follows:

Theorem 4.1 *Let $\beta > \beta_{\mathrm{c}}$ and let (v_L) be a sequence of positive numbers that make $m^\star|\Lambda_L| - 2m^\star v_L$ an allowed value of M_L for all L. Suppose the limit Δ in (1.10) obeys $\Delta \in (0, \infty)$. For each $c_0 > 0$ there exist $\varkappa > 0$, $K_0 < \infty$ and $L_0 < \infty$ such that if $K \ge K_0$, $L \ge L_0$ and*

$s = K \log L$, *then*

$$P_L^{+,\beta}\big(\mathcal{A}^{\mathrm{c}}_{\varkappa,s,L}\big|M_L = m^\star|\Lambda_L| - 2m^\star\, v_L\big) \le L^{-c_0} \tag{4.2}$$

Let $s = K \log L$ be a scale function and recall that a contour γ is s-*large* if $\gamma \in \Gamma_s(\sigma)$. For $\varkappa > 0$, a contour γ large enough to be an s-large contour but satisfying $\operatorname{diam}\gamma \le \varkappa\sqrt{v_L}$ will be called a $\varkappa$-*intermediate* contour. Thus, Theorem 4.1 shows that, in the canonical ensemble with the magnetization fixed to $m^\star|\Lambda_L| - 2m^\star\, v_L$, there are no $\varkappa$-intermediate contours with probability tending to one as L tends to infinity. This statement, which is of interest in its own right, reduces the proof of our main result to a straightforward application of isoperimetric inequalities for the Wulff functional as formulated in Lemma 2.8.

Remark 9. The reason why a *power* of L appears on the right-hand side is because we only demand the absence of contours with sizes over $K \log L$. Indeed, for a general s, the right-hand side of (4.2) could be replaced by $e^{-\alpha s}$ for some constant $\alpha > 0$. In particular, the decay can be made substantially faster by easing the lower limit of what we chose to call an intermediate size contour. Finally, we note that L_0 in Theorem 4.1 depends not only on β, Δ, and c_0, but also on how fast the limit $v_L^{3/2}/|\Lambda_L|$ is achieved.

The proof of Theorem 4.1 will require some preparations. In particular, we will need to estimate the (conditional) probability of five highly unprobable events that we would like to exclude explicitly from the further considerations. All five events are defined with reference to a positive number $\varkappa$ which, more or less, is the same $\varkappa$ that appears in Theorem 4.1.

The first event, $\mathcal{R}^1_{\varkappa,s,L}$, collects the configurations for which the combined length of all s-large contours in Λ_L exceeds $\varkappa^{-1}s\sqrt{v_L}$. These configurations need to be *a priori* excluded because all of the crucial Gaussian estimates from Section 2.3 can only be applied to regions with a moderate surface-to-volume ratio. Next, we show that one can ignore configurations whose large contours occupy too big volume. This is the basis of the event $\mathcal{R}^2_{\varkappa,s,L}$. The remaining three events concern the magnetization deficit in two random subsets of Λ_L: A set $\mathrm{Int}^\circ \subset \mathbb{V}(\Gamma_s^{\mathrm{ext}}(\sigma))$ of sites enclosed by an s-large contour and a set Ext° of sites outside all s-large contours. The precise definition of these sets follows in Section 4.2. The respective events are:

(3) The event $\mathcal{R}^3_{\varkappa,s,L}$ that $M_{\mathrm{Int}^\circ} \le -m^\star|\mathrm{Int}^\circ| - \varkappa^{-1}sv_L^{3/4}$.
(4) The event $\mathcal{R}^4_{\varkappa,s,L}$ that $M_{\mathrm{Ext}^\circ} \ge m^\star|\mathrm{Ext}^\circ| - 2\varkappa m^\star v_L$.
(5) The event $\mathcal{R}^5_{\varkappa,s,L}$ that $M_{\mathrm{Ext}^\circ} \le m^\star|\mathrm{Ext}^\circ| - 2(1+\varkappa^{-1})m^\star v_L$.

By choosing $\varkappa$ sufficiently small, the events $\mathcal{R}^1, \dots, \mathcal{R}^5$ will be shown to have a probability vanishing exponentially fast with $\sqrt{v_L}$. These estimates are the content of Lemma 4.2 and Lemmas 4.6-4.8.

Once the preparatory statements have been proven, we consider a rather extreme version of the restricted contour ensemble, namely, one in which no contour that is larger than $\varkappa$-intermediate is allowed to appear. We show, in a rather difficult Lemma 4.9, that despite this restriction, bounds similar to those of (4.2) still hold. The final step—the proof of Theorem 4.1—is now achieved by conditioning on the location(s) of the large contour(s), which by the "$\mathcal{R}$-lemmas" are typically not *too* big and not *too* rough. By definition, the exterior region is now in the restricted ensemble featured in Lemma 4.9 and the result derived therein allows a relatively easy endgame.

Throughout Sections 4.2-4.4 we will let $\beta > \beta_c$ be fixed and let (v_L) be a sequence of positive numbers such that $m^\star|\Lambda_L| - 2m^\star v_L$ is an allowed value of M_L for all L. Moreover, we will assume that (v_L) is such that the limit Δ in (1.10) exists with $\Delta \in (0,\infty)$.

4.2 Contour length and volume.

In this section we will prepare the grounds for the proof of Theorem 4.1. In particular, we derive rather crude estimates on the total length of large contours and the volume inside and outside large external contours. These results come as Lemmas 4.2 and 4.4 below.

4.2.1 Total contour length. We begin by estimating the combined length of large contours. Let s be a scale function and, for any $\varkappa > 0$, let $\mathcal{R}^1_{\varkappa,s,L}$ be the event

$$\mathcal{R}^1_{\varkappa,s,L} = \Bigl\{ \sigma\colon \sum_{\gamma\in\Gamma_s(\sigma)} |\gamma| \ge \varkappa^{-1} s\sqrt{v_L}\Bigr\}. \tag{4.3}$$

The probability of event $\mathcal{R}^1_{\varkappa,s,L}$ is then estimated as follows:

Lemma 4.2 *For each $c_1 > 0$ there exist $\varkappa_0 > 0$, $K_0 < \infty$ and $L_0 < \infty$ such that*

$$P_L^{+,\beta}\bigl(\mathcal{R}^1_{\varkappa,s,L}\big|M_L = m^\star\,|\Lambda_L| - 2m^\star\, v_L\bigr) \le e^{-c_1\sqrt{v_L}} \tag{4.4}$$

holds for all $\varkappa \le \varkappa_0$, $K \ge K_0$, $L \ge L_0$, and $s = K\log L$.

Proof. Let K_0 be the quantity $K_0(\frac12,\beta)$ from Lemma 2.5 and let us recall that τ_{min} denotes the minimal value of the surface tension. We claim that it suffices to show that, for all $c_1' > 0$ and an appropriate choice of $\varkappa$, the bound

$$P_L^{+,\beta}(\mathcal{R}^1_{\varkappa,s,L}) \le e^{-c_1'\sqrt{v_L}} \tag{4.5}$$

holds true once L is sufficiently large. Indeed, if (4.5) is established, we just choose c_1' so large that the difference $c_1' - c_1$ exceeds the rate constant from the lower bound in Theorem 3.1 and the estimate (4.4) immediately follows.

In order to prove (4.5), fix $c_1' > 0$ and let $\varkappa_0^{-1} = 2g_1c_1'/\tau_{\text{min}}$, where g_1 is as in (2.9). Let $K \ge K_0$, $\varkappa \le \varkappa_0$ and $s = K\log L$. We claim that if $\sigma \in \mathcal{R}^1_{\varkappa,s,L}$ and $\mathfrak{S}$ is a collection of s-skeletons such that $\mathfrak{S} \sim \sigma$, then (2.9) and (2.11) force

$$\varkappa^{-1}s\sqrt{v_L} \le \sum_{\gamma\in\Gamma_s(\sigma)} |\gamma| \le g_1 s\sum_{S\in\mathfrak{S}} \bigl|\mathsf{P}(S)\bigr| \le g_1 s\tau_{\text{min}}^{-1}\mathcal{W}_\beta(\mathfrak{S}). \tag{4.6}$$

Hence, for each $\sigma \in \mathcal{R}^1_{\varkappa,s,L}$ there is at least one $\mathfrak{S}$ such that $\mathfrak{S} \sim \sigma$ and $\mathcal{W}_\beta(\mathfrak{S}) \ge 2c_1'\sqrt{v_L}$. By Corollary 2.6 with $\kappa = 2c_1'\sqrt{v_L}$ and $\alpha = \frac12$, and our choice of K_0, (4.5) follows. □

4.2.2 Interiors and exteriors. Given a scale function s and a configuration σ, let $\Gamma_s^{\text{ext}}(\sigma)$ be the set of external contours in $\Gamma_s(\sigma)$. (Note that these contours will also be external in the set of all contours of σ.) Define $\text{Int} = \text{Int}_{s,L}(\sigma)$ to be the set of all sites in Λ_L enclosed by some $\gamma \in \Gamma_s^{\text{ext}}(\sigma)$ and let $\text{Ext} = \text{Ext}_{s,L}(\sigma)$ be the complement of Int, i.e., $\text{Ext} = \Lambda_L \setminus \text{Int}$.

Given a set of external contours Γ, we claim that under the condition that $\Gamma_s^{\text{ext}}(\sigma) = \Gamma$, the measure $P_L^{+,\beta}$ is a product of independent measures on Ext and Int. A coarse look might suggest a product of plus-boundary condition measure on Ext and the minus measure on Int. Indeed, all spins in Ext up against a piece of Γ are necessarily pluses and similarly all spins on the Int sides of these contours are minuses. But this is not quite the end of the story, two small points are in order: First, we have invoked a rounding rule. Thus, for example, certain spins in Ext (at some corners but not up against the contours) are *forced* to be plus otherwise the rounding rule would have drawn the contour differently. On the other hand, some corner spins *are* permitted either sign because the rounding rule would separate any such resulting contour. Fortunately, the upshot of these "rounding anomalies" is only to force a few additional *minus* spins in Int and *plus* spins in Ext than would appear from a naive look at Γ.

To make the aforementioned observations notationally apparent, we define $\text{Int}^\circ \subset \text{Int}$ to be the set of sites that can be flipped without changing Γ and similarly for Ext. We thus have $\sigma_x = -1$ for all $x \in \text{Int}\setminus\text{Int}^\circ$ and $\sigma_x = +1$ for all $x \in \text{Ext}\setminus\text{Ext}^\circ$. Explicitly, there are a few more boundary spins than one might have thought, but they are always of the correct type. Thus, clearly, although rather trivially, the measure $P_L^{+,\beta}(\cdot|\Gamma_s^{\text{ext}}(\sigma) = \Gamma)$ restricted to Int is simply the measure in Int with minus boundary conditions. The same measure on Ext is not quite the corresponding plus-measure due to the condition that Γ constitutes *all* the external contours visible on the scale s. Thus, beyond the scale s in Ext, we must see ... no contours. But this is precisely the definition of the restricted ensemble.

We conclude that the conditional measure splits on Int and Ext into independent measures that are well understood. Explicitly, if $\mathcal{A}$ is an event depending only on the spins in Int° and $\mathcal{B}$ is an event depending only on the spins in Ext°, then

$$P_L^{+,\beta}\bigl(\mathcal{A}\cap\mathcal{B}\big|\Gamma_s^{\text{ext}}(\sigma) = \Gamma\bigr) = P_{\text{Int}^\circ}^{-,\beta}(\mathcal{A})P_{\text{Ext}^\circ}^{+,\beta,s}(\mathcal{B}). \tag{4.7}$$

This observation will be crucial for our estimates in the next section.

Next we will notice that the number of sites associated with the contours can be easily bounded in terms of the total length of Γ:

Lemma 4.3 *There exists a geometrical constant $g_4 < \infty$ such that the following is true: If Γ is a set of external contours and* Int° *and* Ext° *are as defined above, then*

$$|\Lambda_L \setminus (\text{Int}^\circ \cup \text{Ext}^\circ)| \le g_4 \sum_{\gamma\in\Gamma} |\gamma|. \tag{4.8}$$

Proof. Each site from $\Lambda_L \setminus (\text{Int}^\circ \cup \text{Ext}^\circ)$ is within some (Euclidean) distance from a dual lattice site $x^* \in (\mathbb{Z}^2)^*$ such that some contour $\gamma \in \Gamma$ passes through x^*. On the other hand, the number of dual lattice sites x^* visited by contours from Γ does not exceed twice the total length of all contours in Γ. From here the existence of a g_4 satisfying (4.8) follows. $\square$

The definition of the event $\mathcal{R}^1_{\varkappa,s,L}$ gives us the following easy bounds:

Lemma 4.4 *Let g_4 be as in Lemma 4.3. Let $\sigma \notin \mathcal{R}^1_{\varkappa,s,L}$ and let the sets* $\text{Int} = \text{Int}_{s,L}(\sigma)$ *and* $\text{Int}^\circ = \text{Int}^\circ_{s,L}(\sigma)$ *be as above. Then we have the bounds*

$$|\partial\text{Int}^\circ| \le g_4\varkappa^{-1}s\sqrt{v_L} \quad \textit{and} \quad |\partial\text{Ext}^\circ| \le g_4\varkappa^{-1}s\sqrt{v_L} \tag{4.9}$$

and

$$|\text{Int}^\circ| \le |\text{Int}| \le g_4^2\varkappa^{-2}s^2v_L. \tag{4.10}$$

Proof. The bound (4.9) is an immediate consequence of the estimate $|\partial\text{Int}^\circ| \le g_4\sum_{\gamma\in\Gamma_s(\sigma)}|\gamma|$ and the fact that $\sigma \notin \mathcal{R}^1_{\varkappa,s,L}$. The other bound, (4.10), then follows by the inclusion $\partial\text{Int} \in \text{Ext}\setminus\text{Ext}^\circ$, the inequality $|\text{Ext}\setminus\text{Ext}^\circ| \le g_4\sum_{\gamma\in\Gamma_s(\sigma)}|\gamma|$ and the isoperimetric inequality $|\Lambda| \le \frac{1}{16}|\partial\Lambda|^2$ valid for any $\Lambda\subset\mathbb{R}^2$ that is a finite union of closed unit squares (see, e.g., Lemma A.1 in [14]). □

4.2.3 Volume of large contours. The preceding lemma asserts that, for typical configurations, the interior of large contours is not too big. Actually, one can be a bit more precise. Namely, introducing

$$\mathcal{R}^2_{\varkappa,s,L} = \bigl\{\sigma\colon |V(\Gamma_s^{\text{ext}}(\sigma))| \ge (1-\varkappa)v_L\bigr\}, \tag{4.11}$$

we will show in the next lemma that, whenever $\varkappa$ is sufficiently small, the conditional probability of $\mathcal{R}^2_{\varkappa,s,L}$ given the M_L's of interest is still exponentially small in $\sqrt{v_L}$. However, unlike in Lemma 4.2 (and Lemma 4.6 below), here the constant multiplying $\sqrt{v_L}$ in the exponent can no longer be made arbitrarily large.

Lemma 4.5 *There exist constants $c_2>0$, $\varkappa_0>0$, $K_0<\infty$, and $L_0<\infty$ such that*

$$P_L^{+,\beta}\bigl(\mathcal{R}^2_{\varkappa,s,L}\big|M_L = m^\star|\Lambda_L| - 2m^\star v_L\bigr) \le e^{-c_2\sqrt{v_L}} \tag{4.12}$$

holds for all $K\ge K_0$, $\varkappa\in(0,\varkappa_0]$, $L\ge L_0$, and $s = K\log L$.

Proof. Let $\Phi^\star_\Delta$ be as defined in in (2.2). Clearly, it suffices to prove the statement for *some* $\varkappa>0$, so let $\varkappa\in(0,1)$ be such that

$$c_2 = w_1\bigl[(1-\varkappa)^2 - (\Phi^\star_\Delta + 2\varkappa)\bigr] > 0. \tag{4.13}$$

(This is possible because $\Phi^\star_\Delta < 1$ for all $\Delta<\infty$.) Let L_0 be so large that ϵ_L from Theorem 3.1 satisfies $\epsilon_L \le \varkappa$ for all $L\ge L_0$. Let K_0 be chosen to exceed the quantity $K_0(\varkappa,\beta)$ from Lemma 2.5.

Fix $K\ge K_0$, $L\ge L_0$, and $s = K\log L$. Let now $\sigma\in\mathcal{R}^2_{\varkappa,s,L}$ and let us temporarily abbreviate $\Gamma = \Gamma_s(\sigma)$ and $\Gamma' = \Gamma_s^{\text{ext}}(\sigma)$. Let $\mathfrak{S}$ be any s-skeleton such that $\mathfrak{S}\sim\Gamma$, and let $\mathfrak{S}'$ be the set of skeletons in $\mathfrak{S}$ corresponding to Γ'. First we note that we may as well assume that, for some fixed $B>0$ to be specified later

$$\sum_{S\in\mathfrak{S}'}\bigl|\mathsf{P}(S)\bigr| \le \frac{B}{\tau_{\min}}\sqrt{v_L}. \tag{4.14}$$

Indeed, the contribution of the configurations violating this bound can be directly estimated, combining Corollary 2.6 with $\alpha = \varkappa$ and (2.11), by $e^{-(1-\varkappa)B\sqrt{v_L}}$. For configurations satisfying (4.14), Lemma 2.3 in turn implies

$$\bigl|V(\mathfrak{S}')\bigr| \geq \bigl|V(\Gamma')\bigr| - g_3 s \sum_{S\in\mathfrak{S}'} \bigl|\mathsf{P}(S)\bigr| \geq (1-\varkappa)^2 v_L, \tag{4.15}$$

provided L is sufficiently large to ensure that $g_3 K \frac{\log L}{\sqrt{v_L}} \frac{B}{\tau_{\text{min}}} \ll 1$. As a consequence of this and the Wulff variational problem, $\mathcal{W}_\beta(\mathfrak{S}') \geq (1-\varkappa)\sqrt{v_L} w_1$. Since $\mathfrak{S} \supset \mathfrak{S}'$, we have $\mathcal{W}_\beta(\mathfrak{S}) \geq \mathcal{W}_\beta(\mathfrak{S}')$ and thus for every $\sigma \in \mathcal{R}^2_{\varkappa,s,L}$ satisfying (4.14) there is a collection $\mathfrak{S}$ of s-skeletons such that $\mathfrak{S} \sim \sigma$ and $\mathcal{W}_\beta(\mathfrak{S}) \geq (1-\varkappa)\sqrt{v_L} w_1$. Using, once more, Corollary 2.6 with $\alpha = \varkappa$ and our choice of K_0, we have

$$P_L^{+,\beta}(\mathcal{R}^2_{\varkappa,s,L}) \leq e^{-(1-\varkappa)^2 w_1\sqrt{v_L}} + e^{-(1-\varkappa)B\sqrt{v_L}}. \tag{4.16}$$

Letting $B = (1-\varkappa)w_1$, the right-hand side beats the lower bound $P_L^{+,\beta}(M_L = m^\star|\Lambda_L| - 2m^\star v_L) \geq \exp\{-w_1\sqrt{v_L}(\Phi^\star_\Delta + \varkappa)\}$ from Theorem 3.1 and our choice of L_0 and $\varkappa$ by exactly $2e^{-(c_2+\varkappa w_1)\sqrt{v_L}}$. Using the leeway in the exponent to absorb the extra factor of 2 (which may require that we further increase L_0), the estimate (4.12) follows. $\square$

4.3 Magnetization deficit due to large contours.

In this section we will provide the necessary control over the magnetization deficit inside and outside large contours. The relevant statements come as Lemmas 4.6-4.8.

4.3.1 Magnetization inside. Our next claim concerns the total magnetization inside the large contours in Λ_L. Recalling the definition of Int°, we reintroduce the event

$$\mathcal{R}^3_{\varkappa,s,L} = \bigl\{\sigma\colon M_{\text{Int}^\circ} \leq -m^\star|\text{Int}^\circ| - \varkappa^{-1} s v_L^{3/4}\bigr\}. \tag{4.17}$$

For the probability of $\mathcal{R}^3_{\varkappa,s,L}$ we have the following bound:

Lemma 4.6 *For each $c_3 > 0$ there exist $\varkappa_0 > 0$, $K_0 < \infty$ and $L_0 < \infty$ such that*

$$P_L^{+,\beta}\bigl(\mathcal{R}^3_{\varkappa,s,L}\big|M_L = m^\star|\Lambda_L| - 2m^\star v_L\bigr) \leq e^{-c_3\sqrt{v_L}} \tag{4.18}$$

for any $\varkappa \leq \varkappa_0$, $K \geq K_0$, $L \geq L_0$, and $s = K\log L$.

Proof. Fix a $c_3 > 0$. By Lemma 4.2, there are $\vartheta < \infty$, $K_0 < \infty$ and $L_0 < \infty$ such that $P_L^{+,\beta}(\mathcal{R}^1_{\vartheta,s,L}|M_L = m^\star|\Lambda_L| - 2m^\star v_L) \leq e^{-2c_3\sqrt{v_L}}$ whenever $s = K\log L$ and $L \geq L_0$. Let $\boldsymbol{\Gamma} = \{\Gamma_s^{\text{ext}}(\sigma)\colon \sigma \notin \mathcal{R}^1_{\vartheta,s,L}\}$. Recalling the lower bound in Theorem 3.1, it is clearly sufficient to prove that for some $c_3' > 0$ large enough,

$$P_L^{+,\beta}\bigl(\mathcal{R}^3_{\varkappa,s,L}\big|\Gamma_s^{\text{ext}}(\sigma) = \Gamma\bigr) \leq 2e^{-c_3'\sqrt{v_L}} \tag{4.19}$$

holds for all $\Gamma \in \boldsymbol{\Gamma}$ and all L sufficiently large provided $\varkappa$ is sufficiently small and that the K in $s = K\log L$ is sufficiently large. (Note that, for (4.19) to imply (4.18), c_3' will have to exceed c_3 by a β-dependent factor. The factor of "2" was put in for later convenience.)

Pick a $\Gamma \in \mathbf{\Gamma}$. Since $\mathcal{R}^3_{\varkappa,s,L}$ depends only on the configuration in Int°, (4.7) implies

$$P_L^{+,\beta}\big(\mathcal{R}^3_{\varkappa,s,L}\big|\Gamma_s^{\text{ext}}(\sigma)=\Gamma\big) = P_{\text{Int}^\circ}^{-,\beta}\big(\mathcal{R}^3_{\varkappa,s,L}\big). \tag{4.20}$$

In order to apply Lemma 2.10, we need to compare $-m^\star|\text{Int}^\circ|$ with the actual average magnetization of the Ising model in volume Int° with minus boundary condition. By (4.10) and (4.9), we have $|\text{Int}^\circ| \le g_4^2\vartheta^{-2}s^2v_L$ and $|\partial\text{Int}^\circ| \le g_4\vartheta^{-1}s\sqrt{v_L}$. Then Lemma 2.9 and (2.37) imply the existence of constants $\alpha_1 = \alpha_1(\beta) < \infty$ and $\alpha_2 = \alpha_2(\beta) > 0$ such that

$$\big|\langle M_{\text{Int}^\circ}\rangle_{\text{Int}^\circ}^{-,\beta} + m^\star|\text{Int}^\circ|\big| \le \alpha_1\big(g_4\vartheta^{-1}s\sqrt{v_L} + g_4^2s^2\vartheta^{-2}v_Le^{-\alpha_2 s}\big). \tag{4.21}$$

Now, since $s = K\log L$, for K sufficiently large the right-hand side does not exceed $2\alpha_1g_4\vartheta^{-1}s\sqrt{v_L}$. Thus, if L is so large that the latter does not exceed $\frac{1}{2}\varkappa^{-1}sv_L^{3/4}$ (i.e., if $4\alpha_1g_4\vartheta^{-1}s\sqrt{v_L} \le \varkappa^{-1}sv_L^{3/4}$), then $\sigma \in \mathcal{R}^3_{\varkappa,s,L}$ and $\Gamma_s^{\text{ext}}(\sigma) = \Gamma$ imply

$$M_{\text{Int}^\circ} \le \langle M_{\text{Int}^\circ}\rangle_{\text{Int}^\circ}^{-,\beta,s} - \frac{1}{2}\varkappa^{-1}sv_L^{3/4}. \tag{4.22}$$

Let now $\varkappa_0 > 0$ be such that $c_3' \le \vartheta^2(8\varkappa_0^2\chi)^{-1}$, where $\chi = \chi(\beta)$ is the susceptibility, and let $\varkappa \le \varkappa_0$. By Lemma 2.10, equation (2.41), and the fact that $|\text{Int}^\circ| \le g_4^2\vartheta^{-2}s^2v_L$, the right-hand side of (4.20) is bounded by $2e^{-c_3'\sqrt{v_L}}$. The bound (4.19) is thus proved. $\square$

4.3.2 Magnetization outside. Recall the definition of Ext°. Our first concern here is an upper bound on the total magnetization in Ext°. Let $\mathcal{R}^4_{\varkappa,s,L}$ be the event

$$\mathcal{R}^4_{\varkappa,s,L} = \big\{\sigma\colon M_{\text{Ext}^\circ} \ge m^\star\,|\text{Ext}^\circ| - 2\varkappa m^\star\,v_L\big\}. \tag{4.23}$$

To bound the conditional probability of this event is easy; we will actually show that it can be included into the preceding ones for configurations contained in $\mathcal{M}_L = \{\sigma\colon M_L = m^\star|\Lambda_L| - 2m^\star\,v_L\}$.

Lemma 4.7 *For any $\varkappa > 0$ and $s < \infty$, we have*

$$\mathcal{R}^4_{\varkappa/2,s,L} \cap \mathcal{M}_L \subset \big(\mathcal{R}^1_{\varkappa,s,L} \cup \mathcal{R}^2_{\varkappa,s,L} \cup \mathcal{R}^3_{\varkappa,s,L}\big) \cap \mathcal{M}_L \tag{4.24}$$

for any L large enough.

Proof. Let $\varkappa$ and K be fixed. Let us abbreviate $\text{Int}^\circ = \text{Int}^\circ(\sigma)$ and $\text{Ext}^\circ = \text{Ext}^\circ(\sigma)$ for a configuration σ which we will take to be in $(\mathcal{R}^1_{\varkappa,s,L})^{\text{c}} \cap (\mathcal{R}^2_{\varkappa,s,L})^{\text{c}} \cap (\mathcal{R}^3_{\varkappa,s,L})^{\text{c}}$. First, we note that if $\sigma \notin \mathcal{R}^1_{\varkappa,s,L}$, we can use Lemmas 4.3 and 4.4 to get

$$|\Lambda_L| - \big(|\text{Ext}^\circ| + |\text{Int}^\circ|\big) \le g_4\varkappa^{-1}s\sqrt{v_L} \tag{4.25}$$

and hence

$$|M_L - M_{\text{Ext}^\circ} - M_{\text{Int}^\circ}| \le g_4\varkappa^{-1}s\sqrt{v_L}. \tag{4.26}$$

Now, since the total magnetization is held fixed, i.e., $\sigma \in \mathcal{M}_L$, we have $M_L = m^\star |\Lambda_L| - 2m^\star v_L$ and by a simple calculation we get

$$\begin{aligned} M_{\text{Ext}^\circ} &\le M_L - M_{\text{Int}^\circ} + g_4 \varkappa^{-1} s\sqrt{v_L} \\ &= m^\star (|\Lambda_L| - |\text{Int}^\circ|) - M_{\text{Int}^\circ} + m^\star |\text{Int}^\circ| - 2m^\star v_L + g_4 \varkappa^{-1} s\sqrt{v_L}. \end{aligned} \tag{4.27}$$

At the expense of another factor of $g_4\varkappa^{-1}s\sqrt{v_L}$, we can replace $|\Lambda_L| - |\text{Int}^\circ|$ with $|\text{Ext}^\circ|$. Finally, for $\sigma \notin \mathcal{R}^2_{\varkappa,s,L} \cup \mathcal{R}^3_{\varkappa,s,L}$ we will use the bounds

$$|\text{Int}^\circ| \le |V(\Gamma^{\text{ext}}_s(\sigma))| \le (1-\varkappa)v_L \tag{4.28}$$

and

$$M_{\text{Int}^\circ} \ge -m^\star |\text{Int}^\circ| - \varkappa^{-1} s v_L^{3/4} \tag{4.29}$$

in succession to arrive at

$$M_{\text{Ext}^\circ} \le m^\star |\text{Ext}^\circ| - 2m^\star \varkappa v_L + 2g_4\varkappa^{-1} s\sqrt{v_L} + \varkappa^{-1} s v_L^{3/4}. \tag{4.30}$$

From here we see that $\sigma \notin \mathcal{R}^4_{\varkappa/2,s,L}$ once L is so large that the remaining terms on the right-hand side are swamped by $-m^\star \varkappa v_L$. $\square$

Our second task concerning the magnetization outside the large external contours is to show that $M_{\text{Ext}^\circ} - m^\star|\text{Ext}^\circ|$ will not get substantially below the deficit value forced in by the condition on overall magnetization. (Note, however, that we have to allow for the possibility that $\text{Ext}^\circ = \Lambda_L$ in which case the exterior takes the entire deficit.) Let $\varkappa > 0$ and consider the event

$$\mathcal{R}^5_{\varkappa,s,L} = \bigl\{\sigma\colon M_{\text{Ext}^\circ} \le m^\star |\text{Ext}^\circ| - 2m^\star (1+\varkappa^{-1}) v_L\bigr\}. \tag{4.31}$$

The probability of $\mathcal{R}^5_{\varkappa,s,L}$ is bounded as follows:

Lemma 4.8 *For any $c_5 > 0$ there exist constants $\varkappa_0 > 0$, $K_0 < \infty$ and $L_0 < \infty$ such that*

$$P_L^{+,\beta}\bigl(\mathcal{R}^5_{\varkappa,s,L} \big| M_L = m^\star |\Lambda_L| - 2m^\star v_L\bigr) \le e^{-c_5\sqrt{v_L}} \tag{4.32}$$

for all $K \ge K_0$, $\varkappa \le \varkappa_0$ and $L \ge L_0$, and $s = K\log L$.

Proof. With $\Phi^\star_\Delta$ as in (2.2) and c_5 fixed, choose $\varkappa_0$ so that

$$c_5 \le \frac{w_1}{2}\Bigl[\Delta + \frac{\Delta}{3\varkappa_0} - \Phi^\star_\Delta\Bigr]. \tag{4.33}$$

For this $\varkappa_0 > 0$, let L_0 be so large that for all $L \ge L_0$, the finite-L expression on the right-hand side of (1.10) exceeds $\Delta(1+\frac{1}{2\varkappa_0})^{-1}$ and, at the same time, ϵ_L from Theorem 3.1 is bounded by $\frac{\Delta}{6\varkappa_0}$.

First, we can restrict ourselves to the complement of $\mathcal{R}^1_{\vartheta,s,L}$ with ϑ so small that the corresponding c_1 exceeds $2c_5$. Once again using Lemma 2.9, we get

$$\bigl|\langle M_{\text{Ext}^\circ}\rangle^{+,\beta}_{\text{Ext}^\circ} - m^\star|\text{Ext}^\circ|\bigr| \le \alpha_1\bigl(g_4\vartheta^{-1}s\sqrt{v_L} + 4L + L^4 e^{-\alpha_2 s}\bigr). \tag{4.34}$$

Now, since $s = K\log L$ and $v_L \sim L^{4/3}$, for K sufficiently large the right-hand side does not exceed $8\alpha_1 L$. Thus, if L is so large that the latter does not exceed $m^\star v_L \varkappa_0^{-1}$, it suffices to prove the corresponding bound for the event

$$\overline{\mathcal{R}} = \bigl\{\sigma\colon M_{\text{Ext}^\circ} \le \langle M_{\text{Ext}^\circ}\rangle_{\text{Ext}^\circ}^{+,\beta} - m^\star\,(2+\varkappa_0^{-1})v_L\bigr\}. \tag{4.35}$$

Clearly, $\overline{\mathcal{R}}$ depends only on the configuration in Ext°, and thus (4.7) makes the estimates in Lemma 2.11 available. We get

$$\begin{aligned} P_L^{+,\beta}\bigl(\overline{\mathcal{R}}\big|\Gamma_s^{\text{ext}}(\sigma)=\Gamma\bigr) &\le C\exp\Bigl\{-2\frac{(m^\star v_L)^2}{\chi|\text{Ext}^\circ|}\Bigl(1+\frac{1}{2\varkappa_0}\Bigr)^2\Bigr\} \\ &\le C\exp\Bigl\{-w_1\Delta\Bigl(1+\frac{1}{2\varkappa_0}\Bigr)\sqrt{v_L}\Bigr\}. \end{aligned} \tag{4.36}$$

Here $C = C(\beta) < \infty$ is independent of Γ and the second inequality follows from our assumption about L_0. Now, using (4.33) and the fact that $\epsilon_L \le \frac{\Delta}{6\varkappa_0}$, we derive the bound

$$P_L^{+,\beta}\bigl(\overline{\mathcal{R}}\big|\Gamma_s^{\text{ext}}(\sigma)=\Gamma\bigr) \le Ce^{-w_1\sqrt{v_L}(\Phi_\Delta^\star+\epsilon_L)-2c_5\sqrt{v_L}}. \tag{4.37}$$

The claim then follows by comparing the right-hand side with the lower bound in Theorem 3.1 and summing over all Γ with the above properties. $\square$

4.4 Proof of Theorem 4.1.

The ultimate goal of this section is to rule out the occurrence of intermediate contours. As a first step we derive an upper bound on the probability of the occurrence of contours of intermediate sizes in a contour ensemble constrained to not contain contours with diameters larger than $\varkappa\sqrt{v_L}$. The relevant statement comes as Lemma 4.9. Once this lemma is established, we will give a proof of Theorem 4.1.

4.4.1 A lemma for the restricted ensemble. Recall our notation $P_\Lambda^{+,\beta,s'}$ for the probability measure in volume $\Lambda \subset \Lambda_L$ conditioned on the event that the contour diameters do not exceed s'. We will show that the occurrence of intermediate contours is improbable in $P_\Lambda^{+,\beta,s'}$ with $s' = \varkappa\sqrt{v_L}$ and magnetization restricted to "reasonable" values. For any $\Lambda \subset \Lambda_L$ and any $s > 0$ and $\varkappa > 0$, let

$$\mathcal{A}_{\varkappa,s,\Lambda}^{\text{c}} = \bigl\{\sigma\colon \text{there exists } \gamma \text{ in } \Lambda \text{ such that } K\log L \le \operatorname{diam}\gamma \le \varkappa\sqrt{v_L}\bigr\}. \tag{4.38}$$

Then we have the following estimates:

Lemma 4.9 *For any $c_6 > 0$, $\varphi_0 > 1$, and $\vartheta > 1$, there exist $\varkappa_0 \in (0,1)$, $K_0 < \infty$, and $L_0 < \infty$, such that for $s = K\log L$, all $\varkappa \in (0,\varkappa_0]$, $K \ge K_0$, $L \ge L_0$, all $\Lambda \subset \Lambda_L$ satisfying the bounds*

$$|\Lambda| \ge \vartheta^{-1}L^2 \quad \textit{and} \quad |\partial\Lambda| \le \vartheta L, \tag{4.39}$$

and all $\varphi \in [\varkappa_0,\varphi_0]$ that make $m^\star\,|\Lambda| - 2\varphi m^\star\, v_L$ an allowed value of M_Λ, we have

$$P_\Lambda^{+,\beta,\varkappa\sqrt{v_L}}\bigl(\mathcal{A}_{\varkappa,s,\Lambda}^{\text{c}}\big|M_\Lambda = m^\star\,|\Lambda| - 2\varphi m^\star\, v_L\bigr) \le L^{-c_6}. \tag{4.40}$$

Proof. Notice that the event $\mathcal{A}^{\mathrm{c}}_{\varkappa,s,\Lambda}$ is monotone in $s = K\log L$ and thus it is sufficient to prove the claim for only a fixed K (chosen suitably large). Fix a set $\Lambda\subset\mathbb{Z}^2$ satisfying (4.39) and let

$$\mathcal{M}_\Lambda(\varphi) = \bigl\{\sigma\colon M_\Lambda = m^\star\,|\Lambda| - 2\varphi m^\star\, v_L\bigr\}. \tag{4.41}$$

Let us define

$$\delta_\Lambda = \langle M_\Lambda\rangle_\Lambda^{+,\beta,s} - m^\star|\Lambda| \tag{4.42}$$

and note that, on $\mathcal{M}_\Lambda(\varphi)$, we have $M_\Lambda = \langle M_\Lambda\rangle_\Lambda^{+,\beta,s} - \delta_\Lambda - 2\varphi m^\star v_L$.

The proof of (4.40) will be performed by writing the conditional probability as a quotient of two probabilities with unconstrained contour sizes, and estimating separately the numerator and the denominator. Let

$$\mathcal{E} = \bigl\{\sigma\colon \forall\gamma\in\Gamma_s(\sigma),\ \operatorname{diam}\gamma\le\varkappa\sqrt{v_L}\bigr\} \tag{4.43}$$

and, using the shorthand $\mathcal{A}=\mathcal{A}_{\varkappa,s,\Lambda}$, write

$$P_\Lambda^{+,\beta,\varkappa\sqrt{v_L}}\bigl(\mathcal{A}^{\mathrm{c}}\big|\mathcal{M}_\Lambda(\varphi)\bigr) = \frac{P_\Lambda^{+,\beta}(\mathcal{A}^{\mathrm{c}}\cap\mathcal{M}_\Lambda(\varphi)\cap\mathcal{E})}{P_\Lambda^{+,\beta}(\mathcal{M}_\Lambda(\varphi)\cap\mathcal{E})}. \tag{4.44}$$

As to the bound on the denominator, we restrict the contour sizes in Λ to $s=K\log L$ as in (3.5) and apply Lemmas 2.11 and 2.7 with the result

$$P_\Lambda^{+,\beta}(\mathcal{M}_\Lambda(\varphi)\cap\mathcal{E}) \ge \frac{C_1}{L^2}\exp\Bigl\{-2\frac{(m^\star\,v_L)^2}{\chi|\Lambda|}\varphi^2 + 2\frac{m^\star\,\varphi\, v_L}{\chi|\Lambda|}\delta_\Lambda\Bigr\}, \tag{4.45}$$

where $C_1 = C_1(\beta,\vartheta,\varphi_0)>0$. Here, we note that two distinct terms were incorporated into the constant C_1: First, a term proportional to δ_Λ^2 since, by Lemma 2.9 and (4.39), $|\delta_\Lambda|\le 2\alpha_1\vartheta L$ once K is sufficiently large and thus $|\delta_\Lambda|^2/|\Lambda|$ is bounded by a constant independent of L. Second, a term that comes from the bound (2.47) yielding $|\Omega_\Lambda^s(\varphi v_L+\frac{\delta_\Lambda}{2m^\star})|\le C_2\bigl(K\frac{\log L}{L^{1/3}}\vee 1\bigr)$ with some $C_2=C_2(\beta,\vartheta,\varphi_0)<\infty$. (Notice that, to get a constant C_1 independent of L, we have to choose L_0 after a choice of K is done.) Although the second term on the right-hand side of (4.45) is negligible compared to the first one, its exact form will be needed to cancel an inconvenient contribution of the complement of intermediate contours.

In order to estimate the numerator, let $\mathbf{\Gamma} = \{\Gamma_s(\sigma)\colon \sigma\in\mathcal{E},\ \Gamma_s(\sigma)\ne\emptyset\}$ be the set of all collections of s-large contours that can possibly contribute to $\mathcal{E}$. (We also demand that $\Gamma_s(\sigma)\ne\emptyset$, because on $\mathcal{A}^{\mathrm{c}}$ there will be at least one s-large contour.) Then we have

$$P_\Lambda^{+,\beta}\bigl(\mathcal{A}^{\mathrm{c}}\cap\mathcal{M}_\Lambda(\varphi)\cap\mathcal{E}\bigr) \le \sum_{\Gamma\in\mathbf{\Gamma}} P_\Lambda^{+,\beta}\bigl(\mathcal{M}_\Lambda(\varphi)\big|\Gamma_s(\sigma)=\Gamma\bigr)P_\Lambda^{+,\beta}\bigl(\Gamma_s(\sigma)=\Gamma\bigr). \tag{4.46}$$

Our strategy is to derive a bound on $P_\Lambda^{+,\beta}(\mathcal{M}_\Lambda(\varphi)|\Gamma_s(\sigma)=\Gamma)$ which is uniform in $\Gamma\in\mathbf{\Gamma}$ and to estimate $P_\Lambda^{+,\beta}(\Gamma_s(\sigma)=\Gamma)$ using the skeleton upper bound.

Let $\Gamma\in\mathbf{\Gamma}$ and let $\mathfrak{S}$ be an s-skeleton such that $\mathfrak{S}\sim\Gamma$. We claim that, for some $C'=C'(\beta,\vartheta)<\infty$ and some $\eta_0=\eta_0(\beta,\vartheta)<\infty$, independent of Γ, $\mathfrak{S}$ and L,

$$\frac{P_\Lambda^{+,\beta}(\mathcal{M}_\Lambda(\varphi)|\Gamma_s(\sigma)=\Gamma)}{P_\Lambda^{+,\beta}(\mathcal{M}_\Lambda(\varphi)\cap\mathcal{E})} \le C' L^2 e^{\eta_0\sqrt{\varkappa}\mathcal{W}_\beta(\mathfrak{S})} \tag{4.47}$$

holds true. Indeed, let Γ' be the abbreviation for the set of external contours in Γ and let $\mathfrak{S}'$ be the set of skeletons in $\mathfrak{S}$ corresponding to Γ'. Recall the definition of Int and Int$^\circ$ and note that $\mathbb{V}(\Gamma') = \text{Int}$ and $\mathcal{W}_\beta(\mathfrak{S}) \geq \mathcal{W}_\beta(\mathfrak{S}')$, since $\mathfrak{S} \supset \mathfrak{S}'$. Also note that, by (2.10) and (2.11) and the fact that $\operatorname{diam}\gamma \leq \varkappa\sqrt{v_L}$ for all $\gamma \in \Gamma'$, we have

$$|\text{Int}| \leq g_2 \varkappa \sqrt{v_L} \sum_{S \in \mathfrak{S}'} \bigl|\mathsf{P}(S)\bigr| \leq g_2 \varkappa \tau_{\text{min}}^{-1} \sqrt{v_L}\, \mathcal{W}_\beta(\mathfrak{S}). \tag{4.48}$$

This bound tells us that we might as well assume that $|\text{Int}| \leq \sqrt{\varkappa} v_L$. Indeed, in the opposite case, the bound (4.47) would directly follow by noting that (4.45) implies $P_L^{+,\beta}(\mathcal{M}_\Lambda(\varphi) \cap \mathcal{E}) \geq CL^{-2} e^{-\eta_1 \sqrt{\varkappa} \mathcal{W}_\beta(\mathfrak{S})}$ with η_1 given by

$$\eta_1 = 2g_2 \frac{(m^\star \varphi)^2}{\chi \tau_{\text{min}}} \frac{v_L^{3/2}}{|\Lambda|}. \tag{4.49}$$

Notice that η_1 is bounded uniformly in L by (4.39) and the fact that $\Delta < \infty$. A similar bound, using (2.9) instead of (2.10), shows that also $|\partial \text{Int}| \leq s\sqrt{v_L}$.

Thus, let us assume that $|\text{Int}| \leq \sqrt{\varkappa} v_L$ and $|\partial\text{Int}| \leq s\sqrt{v_L}$ hold true. In order for $\mathcal{M}_\Lambda(\varphi)$ to occur, the total magnetization in Λ should deviate from $m^\star|\Lambda|$ by $-2\varphi m^\star v_L$, while the volume Int can help the bulk only by at most $-|\text{Int}|$. More precisely, M_{Ext° is forced to deviate from its mean value $\langle M_{\text{Ext}^\circ}\rangle_{\text{Ext}^\circ}^{+,\beta,s}$ by at least $-2m^\star u$ where u is defined by

$$-2m^\star u = -2\varphi m^\star v_L - \delta_{\text{Ext}^\circ} + 2|\text{Int}|, \tag{4.50}$$

with $\delta_{\text{Ext}^\circ}$ as in (4.42). By the estimates $|\text{Int}| \leq \sqrt{\varkappa} v_L$, $|\text{Ext}^\circ| \geq \frac{1}{2}\vartheta^{-1}L^2$, $|\partial\text{Ext}^\circ| \leq 2\vartheta L$, and $u \leq C_3 L^{4/3} \ll L^2/\log L$, with $C_3 = C_3(\beta, \vartheta, \varphi_0)$ (all these bounds hold for L sufficiently large—in particular, to ensure that $K\sqrt{v_L}\log L \leq \vartheta L$), we now have, once more, Lemma 2.11 at our disposal. Thus,

$$P_\Lambda^{+,\beta}\bigl(\mathcal{M}_\Lambda(\varphi)\big|\Gamma_s(\sigma) = \Gamma\bigr) \leq C_4 \exp\Bigl\{-2\frac{(m^\star v_L)^2}{\chi|\Lambda|}\varphi^2 + 2\frac{m^\star \varphi v_L}{\chi|\Lambda|}\bigl(\delta_{\text{Ext}^\circ} - 2|\text{Int}|\bigr)\Bigr\}, \tag{4.51}$$

where $C_4 = C_4(\beta, \vartheta, \varphi_0) < \infty$. Similarly as in (4.45), the constant C_4 incorporates also the error term $\Omega_{\text{Ext}^\circ}^s(u)$. To compare the right-hand side of (4.51) and (4.45), we invoke the second part of Lemma 2.9 to note that, for K sufficiently large and some $\alpha_1 = \alpha_1(\beta) < \infty$,

$$\delta_{\text{Ext}^\circ} - \delta_\Lambda \leq \alpha_1 |\Lambda \setminus \text{Ext}^\circ|. \tag{4.52}$$

Using (4.48) again, $|\text{Int}|$ is bounded by a constant times $\varkappa\mathcal{W}_\beta(\mathfrak{S})$ and the same holds for $|\Lambda \setminus \text{Ext}^\circ|$. Therefore, there is a constant $\eta_2 = \eta_2(\beta, \vartheta) < \infty$ such that

$$2\frac{m^\star \varphi v_L}{\chi|\Lambda|}\bigl(\delta_{\text{Ext}^\circ} - \delta_\Lambda - 2|\text{Int}|\bigr) \leq \eta_2 \varkappa \mathcal{W}_\beta(\mathfrak{S}), \tag{4.53}$$

holds true for all $\Gamma \in \mathbf{\Gamma}$ and their associated skeletons $\mathfrak{S}$. By combining this with (4.51) and (4.45), the bound (4.47) is established with $\eta_0 = \eta_1 \vee \eta_2$, taking into account that $\varkappa \leq 1$.

With (4.47), the proof is easily concluded. Indeed, a straightforward application of the skeleton bound to the second term on the right-hand side of (4.46) then shows that

$$P_\Lambda^{+,\beta,\varkappa\sqrt{v_L}}\bigl(\mathcal{A}^{\rm c}\big|\mathcal{M}_\Lambda(\varphi)\bigr)\le\sum_{\mathfrak{S}\ne\emptyset}C'L^2e^{-(1-\eta_0\sqrt{\varkappa})\mathcal{W}_\beta(\mathfrak{S})}. \tag{4.54}$$

Now, for $\varkappa$ sufficiently small, we have $1-\eta_0\sqrt{\varkappa}>2/3$. Then we can extract the term $C'e^{-\frac13\mathcal{W}_\beta(\mathfrak{S})}$ which, choosing the K in $s=K\log L$ sufficiently large, can be made less than L^{-2-c_6}, for any c_6 initially prescribed. Invoking Lemma 2.5, the remaining sum is then estimated by one. □

4.4.2 Absence of intermediate contours. Lemmas 4.2 and 4.5-4.9 finally put us in the position to rule out the intermediate contours altogether.

Proof of Theorem 4.1. Recall that our goal is to prove (4.2), i.e., $P_L^{+,\beta}(\mathcal{A}^{\rm c}|\mathcal{M}_L)\le L^{-c_0}$. Pick any $c_0>0$ and $\varkappa_0<1$. Let K_0 and L_0 be chosen so that Lemmas 4.2, 4.5, 4.6, and 4.8 hold with *some* $c_1,c_2,c_3,c_5>0$ for all $\varkappa\le2\varkappa_0$, $K\ge K_0$ and $L\ge L_0$. We also assume that L_0 is chosen so that Lemma 4.7 is valid for $\varkappa=2\varkappa_0$. We wish to restrict attention to configuration outside the sets $\mathcal{R}^1_{\varkappa_0,s,L}$, $\mathcal{R}^4_{\varkappa_0,s,L}$ and $\mathcal{R}^5_{\varkappa_0,s,L}$, but since $\mathcal{R}^4_{\varkappa_0,s,L}$ is essentially included in $\mathcal{R}^2_{\varkappa_0,s,L}$ and $\mathcal{R}^3_{\varkappa_0,s,L}$, we might as well focus on the event $\mathcal{R}^{\rm c}$, where $\mathcal{R}=\bigcup_{\ell=1}^5\mathcal{R}^\ell_{\varkappa_0,s,L}$. Fix any $\varkappa\le\varkappa_0$, let $s=K\log L$ and let us introduce the shorthand $\mathcal{A}=\mathcal{A}_{\varkappa,s,L}$. Appealing to the aforementioned lemmas, our goal will be achieved if we establish the bound $P_L^{+,\beta}(\mathcal{A}^{\rm c}\cap\mathcal{R}^{\rm c}|\mathcal{M}_L)\le L^{-2c_0}$.

Let $q=\varkappa\sqrt{v_L}$ and let $\boldsymbol{\Gamma}=\{\Gamma_q^{\rm ext}(\sigma)\colon\sigma\in\mathcal{R}^{\rm c}\}$ be the set of all collections of external contours that can possibly arise from $\mathcal{R}^{\rm c}$. Fix $\Gamma\in\boldsymbol{\Gamma}$ and recall our notation Ext$^\circ$ for the exterior component of Λ_L induced by the contours in Γ. To prove (4.2), it suffices to show that, for all $\Gamma\in\boldsymbol{\Gamma}$,

$$P_L^{+,\beta}\bigl(\mathcal{A}^{\rm c}\cap\mathcal{R}^{\rm c}\cap\mathcal{M}_L\big|\Gamma_q^{\rm ext}(\sigma)=\Gamma\bigr)\le L^{-2c_0}P_L^{+,\beta}\bigl(\mathcal{M}_L\big|\Gamma_q^{\rm ext}(\sigma)=\Gamma\bigr). \tag{4.55}$$

Indeed, multiplying (4.55) by $P_L^{+,\beta}(\Gamma_q(\sigma)=\Gamma)$ and summing over all $\Gamma\in\boldsymbol{\Gamma}$, we derive that

$$P_L^{+,\beta}\bigl(\mathcal{A}^{\rm c}\cap\mathcal{R}^{\rm c}\cap\mathcal{M}_L\bigr)\le L^{-2c_0}P_L^{+,\beta}(\mathcal{M}_L). \tag{4.56}$$

Thence, $P_L^{+,\beta}(\mathcal{A}^{\rm c}\cap\mathcal{R}^{\rm c}|\mathcal{M}_L)\le L^{-2c_0}$ which, in the light of the bound $P_L^{+,\beta}(\mathcal{R}|\mathcal{M}_L)\le4e^{-c\sqrt{v_L}}$ where $c=\min\{c_1,c_2,c_3,c_5\}$, implies (4.2) once L is sufficiently large.

It remains to prove (4.55) for all $\Gamma\in\boldsymbol{\Gamma}$. Let $\varphi\ge0$ be such that $m^\star|\text{Ext}^\circ|-2\varphi m^\star v_L$ is an allowed value of M_{Ext° and consider the corresponding event $\mathcal{M}_{\text{Ext}^\circ}(\varphi)$ (cf. (4.41)). Note that, by the restriction to the complements of $\mathcal{R}^4_{\varkappa_0,s,L}$ and $\mathcal{R}^5_{\varkappa_0,s,L}$, we only need to consider $\varphi\in[\varkappa_0,1+\varkappa_0^{-1}]$. We claim that, for all such allowed values of φ, we have

$$P_L^{+,\beta}\bigl(\mathcal{A}^{\rm c}\big|\{\Gamma_q^{\rm ext}(\sigma)=\Gamma\}\cap\mathcal{M}_L\cap\mathcal{M}_{\text{Ext}^\circ}(\varphi)\bigr)=P_{\text{Ext}^\circ}^{+,\beta,\varkappa\sqrt{v_L}}\bigl(\mathcal{A}^{\rm c}\big|\mathcal{M}_{\text{Ext}^\circ}(\varphi)\bigr). \tag{4.57}$$

Indeed, given that $\Gamma_q^{\rm ext}(\sigma)=\Gamma$, the event $\mathcal{A}$ depends only on the configurations in Ext$^\circ$. Moreover, $\mathcal{M}_L\cap\mathcal{M}_{\text{Ext}^\circ}(\varphi)$ can be written as an intersection of $\mathcal{M}_{\text{Ext}^\circ}(\varphi)$, which also depend only on σ in Ext$^\circ$, and the event $\{\sigma\colon M_{\Lambda_L\setminus\text{Ext}^\circ}=m^\star(|\Lambda_L|-|\text{Ext}^\circ|)-2m^\star(1-\varphi)v_L\}$, which depends only on the configuration in Int$^\circ$. Thus, (4.57) follows from (4.7) and some elementary manipulations.

By the restriction to the complement of $\mathcal{R}^1_{\varkappa_0,s,L}$, we have $|\text{Ext}^\circ| \ge L^2/2$ and $|\partial\text{Ext}^\circ| \le 8L$ for all $\Gamma \in \boldsymbol{\Gamma}$. Choosing now $c_6 = 2c_0$ and then K_0 and L_0 (if necessary, even bigger than before) so that Lemma 4.9 can be applied, the right-hand side of (4.57) can be bounded by $L^{-c_6} = L^{-2c_0}$ uniformly in $\Gamma \in \boldsymbol{\Gamma}$, provided $\varkappa$ is sufficiently small and $L \ge L_0$. Using (4.57), we thus have

$$\begin{aligned} P_L^{+,\beta}\bigl(\mathcal{A}^{\text{c}} \cap \mathcal{R}^{\text{c}} \cap \mathcal{M}_L \cap \mathcal{M}_{\text{Ext}^\circ}(\varphi)\big|\Gamma_q(\sigma) = \Gamma\bigr) \qquad\qquad\qquad\qquad \\ \le P_L^{+,\beta}\bigl(\mathcal{A}^{\text{c}}\big|\{\Gamma_q^{\text{ext}}(\sigma) = \Gamma\} \cap \mathcal{M}_L \cap \mathcal{M}_{\text{Ext}^\circ}(\varphi)\bigr) \\ \times P_L^{+,\beta}\bigl(\mathcal{M}_L \cap \mathcal{M}_{\text{Ext}^\circ}(\varphi)\big|\Gamma_q(\sigma) = \Gamma\bigr) \\ \le L^{-2c_0} P_L^{+,\beta}\bigl(\mathcal{M}_L \cap \mathcal{M}_{\text{Ext}^\circ}(\varphi)\big|\Gamma_q(\sigma) = \Gamma\bigr), \end{aligned} \tag{4.58}$$

for *all* φ for which $m^\star\,|\text{Ext}^\circ| - 2\varphi m^\star\, v_L$ is an allowed value of M_{Ext°. (In the cases when $\varphi \notin [\varkappa_0, 1 + \varkappa_0^{-1}]$ we have $\mathcal{R}^{\text{c}} \cap \mathcal{M}_{\text{Ext}^\circ}(\varphi) = \emptyset$ and the left-hand side vanishes.) This implies (4.55) by summing over all allowed values of φ. $\square$

5. Proof of main results

Having established the absence of intermediate-size contours, we are now in the position to prove our main results.

Proof of Theorem 1.2. Fix a $\zeta > 0$ and recall our notation $\mathcal{M}_L = \{\sigma\colon M_L = m^\star|\Lambda_L| - 2m^\star\, v_L\}$. Our goal is to estimate the conditional probability $P_L^{+,\beta}(\mathcal{A}^{\text{c}}_{\varkappa,s,L} \cup \mathcal{B}^{\text{c}}_{\epsilon,s,L}|\mathcal{M}_L)$ by $L^{-\zeta}$. Let $c_0 > \zeta$ and note that, by Theorem 4.1, we have

$$P_L^{+,\beta}(\mathcal{A}^{\text{c}}_{\varkappa,s,L}|\mathcal{M}_L) \le L^{-c_0}, \tag{5.1}$$

provided $\varkappa$ is sufficiently small and L sufficiently large. This means we can restrict our attention to the event $\mathcal{B}^{\text{c}}_{\epsilon,s,L} \setminus \mathcal{A}^{\text{c}}_{\varkappa,s,L}$. Furthermore, we can use Lemmas 4.2, 4.5, 4.6, and 4.7 to exclude the events $\mathcal{R}^1_{\vartheta,s,L}$, $\mathcal{R}^2_{\vartheta,s,L}$, $\mathcal{R}^3_{\vartheta,s,L}$, and $\mathcal{R}^4_{\vartheta,s,L}$, provided ϑ is sufficiently small. We therefore introduce the event $\mathcal{E}_{\epsilon,\varkappa,\vartheta}$ defined by

$$\mathcal{E}_{\epsilon,\varkappa,\vartheta} = \mathcal{B}^{\text{c}}_{\epsilon,s,L} \setminus (\mathcal{A}^{\text{c}}_{\varkappa,s,L} \cup \mathcal{R}^1_{\vartheta,s,L} \cup \mathcal{R}^2_{\vartheta,s,L} \cup \mathcal{R}^3_{\vartheta,s,L} \cup \mathcal{R}^4_{\vartheta,s,L}), \tag{5.2}$$

where we have suppressed $s = K \log L$ and L from the notation.

On the basis of the aforementioned Lemmas, the proof of Theorem 1.2 will follow if we can establish that for each $\varkappa > 0$ and each $\epsilon > 0$ there are $K_0 < \infty$, $\vartheta > 0$ and $c_7 > 0$ such that

$$P_L^{+,\beta}(\mathcal{E}_{\epsilon,\varkappa,\vartheta}|\mathcal{M}_L) \le e^{-c_7\sqrt{v_L}} \tag{5.3}$$

whenever L is sufficiently large. The proof of (5.3) will be performed by conditioning on the set of s-large exterior contours and applying separately the Gaussian estimates and the skeleton upper bound. The argument will be split into several cases, depending on which of the bounds (1.14–1.16) constituting the event $\mathcal{B}_{\epsilon,s,L}$ fail to hold.

Let us write $\mathcal{E}_{\epsilon,\varkappa,\vartheta}$ as the disjoint union $\mathcal{E}^1_{\epsilon,\varkappa,\vartheta} \cup \mathcal{E}^2_{\epsilon,\varkappa,\vartheta}$, where $\mathcal{E}^1_{\epsilon,\varkappa,\vartheta}$ is the set of all configurations on which one of (1.14) or (1.15) fail and where $\mathcal{E}^2_{\epsilon,\varkappa,\vartheta} = \mathcal{E}_{\epsilon,\varkappa,\vartheta} \setminus \mathcal{E}^1_{\epsilon,\varkappa,\vartheta}$. Let $\boldsymbol{\Gamma} = \{\Gamma_s^{\text{ext}}(\sigma)\colon \sigma \in \mathcal{E}_{\epsilon,\varkappa,\vartheta}\}$ be the set of all collections of exterior contours allowed by $\mathcal{E}_{\epsilon,\varkappa,\vartheta}$. (Here

$s = K \log L$.) Since $\Gamma_s(\sigma)$ is non-empty for all σ contributing to $\mathcal{B}^{\mathrm{c}}_{\epsilon,s,L}$, we have $\Gamma \neq \emptyset$ for all $\Gamma \in \boldsymbol{\Gamma}$. Let

$$\lambda_\Gamma = v_L^{-1} |V(\Gamma)|. \tag{5.4}$$

To apply the Gaussian estimate, we need the following *upper* bound on the magnetization in Ext°.

Lemma 5.1 *Let $\epsilon > 0$, $\varkappa > 0$ and $\vartheta > 0$ and let the K in $s = K \log L$ be sufficiently large. Then there exists a sequence (κ_L) with $\lim_{L\to\infty} \kappa_L = 0$ such that for both $i = 1, 2$, all $\Gamma \in \boldsymbol{\Gamma}$ and all $\sigma \in \mathcal{M}_L \cap \mathcal{E}^i_{\epsilon,\varkappa,\vartheta} \cap \{\Gamma^{\mathrm{ext}}_s(\sigma) = \Gamma\}$, the magnetization $M_{\mathrm{Ext}^\circ} = M_{\mathrm{Ext}^\circ_{s,L}(\sigma)}(\sigma)$ obeys the bound*

$$M_{\mathrm{Ext}^\circ} \le \langle M_{\mathrm{Ext}^\circ} \rangle^{+,\beta,s}_{\mathrm{Ext}^\circ} - 2m^\star\, v_L (1 - \lambda_\Gamma + \epsilon_i - \kappa_L). \tag{5.5}$$

Here $\epsilon_1 = 0$ and $\epsilon_2 = \epsilon/(2m^\star)$.

Proof. Recall the exact definition of Ext°. The proof is similar in spirit to the reasoning (4.28–4.30). First we will address the case of configurations in $\mathcal{E}^1_{\epsilon,\varkappa,\vartheta}$. Using the equality $M_L = m^\star |\Lambda_L| - 2m^\star\, v_L$ and our restriction to the complement of $\mathcal{R}^1_{\vartheta,s,L}$, we have

$$M_L \le m^\star |\mathrm{Ext}^\circ| + m^\star |V(\Gamma)| - 2m^\star v_L + g_4 \vartheta^{-1} s \sqrt{v_L}, \tag{5.6}$$

where $g_4 \vartheta^{-1} s \sqrt{v_L}$ bounds the volume of $\mathrm{Ext} \setminus \mathrm{Ext}^\circ$ according to Lemma 4.3. Next, in view of the restriction to $(\mathcal{R}^3_{\vartheta,s,L})^{\mathrm{c}}$, we have

$$M_{\mathbb{V}(\Gamma)} \ge -m^\star |V(\Gamma)| - \vartheta^{-1} s v_L^{3/4} - g_4 \vartheta^{-1} s \sqrt{v_L}. \tag{5.7}$$

Finally, since $M_{\mathrm{Ext}^\circ} \le M_L - M_{\mathbb{V}(\Gamma)} + g_4 \vartheta^{-1} s \sqrt{v_L}$ and since (4.34) implies that $m^\star |\mathrm{Ext}^\circ| - \langle M_{\mathrm{Ext}^\circ} \rangle^{+,\beta,s}_{\mathrm{Ext}^\circ}$ can be bounded by $8\alpha_1 L$ once K is sufficiently large, we have (5.5) with κ_L given by

$$2m^\star \kappa_L = \vartheta^{-1} s v_L^{-1/4} + 3 g_4 \vartheta^{-1} s v_L^{-1/2} + 8\alpha_1 L v_L^{-1}. \tag{5.8}$$

Since $v_L \sim L^{4/3}$, we have $\lim_{L\to\infty} \kappa_L = 0$ as claimed.

Next we will attend to the case of configurations from $\mathcal{E}^2_{\epsilon,\varkappa,\vartheta}$, for which the bound (1.16) must fail. Since $\mathcal{E}^2_{\epsilon,\varkappa,\vartheta}$ is still a subset of $(\mathcal{R}^3_{\vartheta,s,L})^{\mathrm{c}}$, we still have the bound (5.7) at our disposal implying that $M_{\mathbb{V}(\Gamma)} \ge -m^\star |V(\Gamma)| - \epsilon v_L$ once L is sufficiently large. However, this means that the only way (1.16) can fail is that, in fact, the lower bound

$$M_{\mathbb{V}(\Gamma)} \ge -m^\star |V(\Gamma)| + \epsilon v_L \tag{5.9}$$

holds. Substituting this stronger bound in the above derivation in the place of (5.7), the desired estimate follows. $\square$

With Lemma 5.1 in the hand, we are ready to start proving the bound (5.3). We begin with the Gaussian estimate. By the restriction to the complement of $\mathcal{R}^2_{\vartheta,s,L}$, we have the bound $\lambda_\Gamma \le 1 - \vartheta$ and thus $1 - \lambda_\Gamma + \epsilon_i - \kappa_L \ge 0$ once L is sufficiently large. Moreover, since we also discarded $\mathcal{R}^1_{\vartheta,s,L}$, Lemma 2.11 for $A = \text{Ext}^\circ$ applies. Combining this with the observation (4.7) and the bound (5.5), there exists a constant $C < \infty$ such that

$$P_L^{+,\beta}\bigl(\mathcal{M}_L \cap \mathcal{E}^i_{\epsilon,\varkappa,\vartheta} \big| \Gamma_s^{\text{ext}}(\sigma) = \Gamma\bigr) \le C \exp\biggl\{-2\frac{(m^\star\, v_L)^2}{\chi|\Lambda_L|}(1-\lambda_\Gamma+\epsilon_i-\kappa_L)^2\biggr\} \tag{5.10}$$

holds for all $\Gamma \in \mathbf{\Gamma}$. Next we will estimate the probability that $\Gamma_s^{\text{ext}}(\sigma) = \Gamma$. Let $\mathfrak{S}$ be a collection of skeletons corresponding to Γ. The skeleton upper bound in Lemma 2.4 along with the estimates featured in Lemma 2.5 then yields

$$P_L^{+,\beta}\bigl(\Gamma_s^{\text{ext}}(\sigma) = \Gamma\bigr) \le \sum_{\mathfrak{S}' \supseteq \mathfrak{S}} e^{-\mathcal{W}_\beta(\mathfrak{S}')} \le C' e^{-\mathcal{W}_\beta(\mathfrak{S})}, \tag{5.11}$$

where $C' < \infty$ and where $\mathfrak{S}'$ corresponds to the skeleton of a full set $\Gamma_s(\sigma)$ with $\Gamma_s^{\text{ext}}(\sigma) = \Gamma$.

To estimate the probability of $\mathcal{M}_L \cap \mathcal{E}^i_{\epsilon,\varkappa,\vartheta} \cap \{\Gamma_s^{\text{ext}}(\sigma) = \Gamma\}$, we will write $\mathbf{\Gamma}$ as the union of two disjoint sets, $\mathbf{\Gamma} = \mathbf{\Gamma}_1 \cup \mathbf{\Gamma}_2$. Here

$$\mathbf{\Gamma}_1 = \bigl\{\Gamma \in \mathbf{\Gamma} \colon \exists \mathfrak{S} \sim \Gamma,\, \mathcal{W}_\beta(\mathfrak{S}) \le w_1\sqrt{\lambda_\Gamma v_L}(1+\epsilon c^{-2})\bigr\}, \tag{5.12}$$

where c is the constant from Lemma 2.8, and $\mathbf{\Gamma}_2 = \mathbf{\Gamma} \setminus \mathbf{\Gamma}_1$. First we will study the cases when $\Gamma \in \mathbf{\Gamma}_1$. By the restriction to the event $\mathcal{A}_{\varkappa,s,L}$, we know that $\operatorname{diam}\gamma \ge \varkappa\sqrt{v_L}$ for all $\gamma \in \Gamma$. Using that $\lambda_\Gamma \le 1-\vartheta$—recall that we are in the complement of $\mathcal{R}^2_{\vartheta,s,L}$—we have $\operatorname{diam}\gamma \ge c(\epsilon c^{-2})\sqrt{|V(\Gamma)|}$ whenever $\varkappa \ge \epsilon/c$. Moreover, the upper bound on $\mathcal{W}_\beta(\mathfrak{S})$ from (5.12) along with the estimate $\mathcal{W}_\beta(\mathfrak{S}) \ge \tau_{\min}\varkappa\sqrt{v_L}$ imply that λ_Γ is bounded away from zero and thus $\epsilon\sqrt{|V(\Gamma)|} = \epsilon\sqrt{\lambda_\Gamma v_L} \ge s$ for L sufficiently large. This verifies the assumptions of Lemma 2.8 with ϵ replaced by ϵc^{-2}, which then guarantees that Γ is a singleton, $\Gamma = \{\gamma_0\}$, and that

$$\inf_{z\in\mathbb{R}^2} d_{\mathrm{H}}\bigl(V(\gamma_0), \sqrt{|V(\gamma_0)|}W + z\bigr) \le \sqrt{\epsilon}\sqrt{|V(\gamma_0)|}. \tag{5.13}$$

Now, $|V(\gamma_0)| = \lambda_\Gamma v_L \le v_L$ (because, as before, $\lambda_\Gamma \le 1$), which means that the right-hand side is less than $\sqrt{\epsilon v_L}$ and (1.14) holds. But on $\mathcal{E}^i_{\epsilon,\varkappa,\vartheta}$ the event $\mathcal{B}_{\epsilon,s,L}$ must fail, so we must have either that $\Phi_\Delta(\lambda_\Gamma) > \Phi_\Delta^\star + \epsilon$, which only applies when $i=1$, or that (1.16) fails, which only applies when $i=2$.

We claim that, in both cases, there exists an $\epsilon' > 0$ and an $\alpha > 0$—both proportional to ϵ—such that for some $\mathfrak{S} \sim \Gamma$, we have

$$(1-\alpha)\mathcal{W}_\beta(\mathfrak{S}) + 2\frac{(m^\star\, v_L)^2}{\chi|\Lambda_L|}(1-\lambda_\Gamma+\epsilon_i-\kappa_L)^2 \ge w_1\sqrt{v_L}\bigl(\Phi_\Delta^\star + \epsilon'\bigr). \tag{5.14}$$

Indeed, the Wulff variational problem in conjunction with Lemma 2.3, the restriction to $(\mathcal{R}^1_{\vartheta,s,L})^{\mathrm{c}}$ and the bound $(1-x)^{1/2} \ge 1-x$ for $x \in [0,1]$ imply that

$$\begin{aligned}\mathcal{W}_\beta(\mathfrak{S}) &\ge w_1|\mathbb{V}(\mathfrak{S})|^{1/2} \ge w_1\Bigl(|V(\gamma_0)| - g_3\vartheta^{-1}s^2\sqrt{v_L}\Bigr)^{1/2}\\ &\ge w_1\sqrt{\lambda_\Gamma v_L} - g_3 w_1\bigl(\vartheta\sqrt{\lambda_\Gamma}\bigr)^{-1}s^2.\end{aligned} \tag{5.15}$$

Observing also that the difference $(m^\star)^2/(\chi|\Lambda_L|)v_L^{3/2} - w_1\Delta \to 0$ as $L \to \infty$, the left hand side of (5.14) can be bounded from below by

$$w_1\sqrt{v_L}\Phi_\Delta(\lambda_\Gamma) - \alpha w_1\sqrt{\lambda_\Gamma v_L} - \delta_L\sqrt{v_L} + 2w_1\Delta\sqrt{v_L}(\epsilon_i - \kappa_L)\vartheta, \tag{5.16}$$

where $\delta_L \to 0$ (as well as $\kappa_L \to 0$) with $L \to \infty$. (Here we again used that $1 - \lambda_\Gamma \ge \vartheta$.) Now, for $i = 1$ we have $\Phi_\Delta(\lambda_\Gamma) > \Phi_\Delta^\star + \epsilon$ from which (5.14) follows once $\alpha < \epsilon$ and L is sufficiently large. For $i = 2$, we use $\Phi_\Delta(\lambda_\Gamma) \ge \Phi_\Delta^\star$ and get the same conclusion since (5.16) now contains the positive term $2w_1\Delta\epsilon_2\sqrt{v_L} \propto \epsilon\sqrt{v_L}$.

By putting (5.10) and (5.11) together, applying (5.14), choosing $K \ge K_0(\alpha,\beta)$ and invoking Lemma 2.5 to bound the sum over all skeletons $\mathfrak{S}$, we find that

$$P_L^{+,\beta}\bigl(\mathcal{M}_L \cap \mathcal{E}_{\epsilon,\varkappa,\vartheta} \cap \{\Gamma_s^{\text{ext}}(\sigma) \in \boldsymbol{\Gamma}_1\}\bigr) \le 2CC' \exp\bigl\{-w_1\sqrt{v_L}\bigl(\Phi_\Delta^\star + \epsilon'\bigr)\bigr\}. \tag{5.17}$$

whenever $\varkappa \le \sqrt{\epsilon}$ and L is sufficiently large. (Here the embarrassing factor "2" comes from combining the corresponding estimates for $i = 1$ and $i = 2$.)

Thus, we are down to the cases $\Gamma \in \boldsymbol{\Gamma}_2$, which means that for every skeleton $\mathfrak{S} \sim \Gamma$, we have $\mathcal{W}_\beta(\mathfrak{S}) > w_1\sqrt{\lambda_\Gamma v_L}(1 + \epsilon c^{-2})$. Moreover, since $\mathcal{E}_{\epsilon,\vartheta,\varkappa} \subset \mathcal{A}_{\varkappa,s,L}$, all s-large contours that we have to consider actually satisfy that $\operatorname{diam}\gamma \ge \varkappa\sqrt{v_L}$. In particular, we also have that $\mathcal{W}_\beta(\mathfrak{S}) \ge \tau_{\min}\varkappa\sqrt{v_L}$. Combining these bounds we derive that, for some $c' > 0$ and regardless of the value of λ_Γ,

$$\mathcal{W}_\beta(\mathfrak{S}) \ge w_1\bigl(\sqrt{\lambda_\Gamma} + c'\bigr)\sqrt{v_L}. \tag{5.18}$$

Disregarding the factor ϵ_i in (5.10) and performing similar estimates as in the derivation of (5.17), we find that (5.14) holds again for some $\alpha > 0$. Hence an analogue of (5.17) is valid also for all $\Gamma \in \boldsymbol{\Gamma}_2$. A combination of these estimates in conjunction with Theorem 3.1 show that, indeed, (5.3) is true with a c_7 proportional to ϵ. This finishes the proof. $\square$

The previous proof immediately provides us with the proof of the other main results:

Proof of Theorem 1.1. In light of Theorem 3.1, we need to prove an appropriate upper bound on $P_L^{+,\beta}(\mathcal{M}_L)$, where $\mathcal{M}_L = \{\sigma\colon M_L = m^\star|\Lambda_L| - 2m^\star v_L\}$. First we note that for L sufficiently large, the probability $P_L^{+,\beta}(\mathcal{M}_L)$ is comparable with $P_L^{+,\beta}(\mathcal{F}_L)$, where $\mathcal{F}_L$ is the event

$$\mathcal{F}_L = \mathcal{M}_L \cap \mathcal{A}_{\varkappa,s,L} \cap \mathcal{B}_{\epsilon,s,L} \cap \bigl(\mathcal{R}^1_{\vartheta,s,L} \cup \mathcal{R}^3_{\vartheta,s,L} \cup \mathcal{R}^4_{\vartheta,s,L}\bigr)^{\mathrm{c}} \tag{5.19}$$

with ϵ, $\varkappa$, ϑ as in the proof of Theorem 1.2. But on $\mathcal{F}_L$, we have at most one large contour and the skeleton and Gaussian upper bounds readily give us that

$$P_L^{+,\beta}(\mathcal{F}_L) \le Ce^{-w_1\sqrt{v_L}(\Phi_\Delta^\star - \epsilon')}. \tag{5.20}$$

for some $C < \infty$ and some $\epsilon' > 0$ proportional to ϵ. From here and Theorem 3.1, the claim (1.11) follows by letting $L \to \infty$ and $\epsilon \downarrow 0$. $\square$

Our last task is to prove Corollary 1.3.

Proof of Corollary 1.3. By Proposition 2.1, if $\Delta < \Delta_{\mathrm{c}}$, the unique minimizer of $\Phi_\Delta(\lambda)$ is $\lambda = 0$. Thus, for $\epsilon > 0$ sufficiently small and L large enough, the contour volumes are restricted to a small number times v_L. Since (1.14) says that the contour volume is proportional to the square of its diameter, this (eventually) forces $\operatorname{diam}\gamma < \varkappa\sqrt{v_L}$ for any fixed $\varkappa > 0$. But that contradicts the fact that $\mathcal{A}_{\varkappa,s,L}$ holds for a $\varkappa$ sufficiently small. Hence, no such intermediate γ exists and all contours have a diameter smaller than $K \log L$.

In the cases $\Delta > \Delta_c$, the function $\Phi_\Delta(\lambda)$ is minimized by a non-zero λ (which is, in fact, larger than $2/3$). Since, again, $\operatorname{diam}\gamma \geq \varkappa\sqrt{v_L}$ for all potential contours, Theorem 1.2 guarantees that there is only one such contour and it obeys the bounds (1.14) and (1.15). All the other contours have diameter less than $K \log L$. $\square$

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