Overcoming Nonrenormalizability

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Abstract

A suitable counterterm for a Euclidean space lattice version of φ_n^4 theories, $n \geq 4$, is combined with several additional procedures so that in the continuum limit the resultant quantum field theory is nontrivial by design.

Introduction

Perturbatively nonrenormalizable or renormalizable but not asymptotically free quantized fields arise in theories of physical interest, such as in quantum gravity or in the Higgs field of the standard model. As representatives of such fields we consider φ_n^4 models for spacetime dimensions $n \geq 4$.

For $n \geq 5$, it is known that the quartically coupled relativistic scalar quantum field φ_n^4 is nonrenormalizable when formulated perturbatively, and trivial (equivalent to a free or generalized free field) when constructed as the continuum limit of a Euclidean space lattice theory with a conventional lattice action [1]. For $n = 4$, the φ_4^4 theory is perturbatively renormalizable, not asymptotically free, and nontrivial despite strong evidence from lattice space computer studies that imply the theory is again trivial. Finally, we observe that infinite-order perturbation theory also points toward triviality due to the presence of Landau poles. Conventional approaches lead to triviality;

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in the present paper, we follow an unconventional path in our studies of φ^4_n theories.

Triviality may be readily characterized by the behavior of selected lattice space correlation functions when the lattice spacing a is very small. We assume units are chosen so that a is a dimensionless variable. Mean field theory, generally regarded as valid when $n \geq 5$, leads to [2] (T denotes truncated)

$$\begin{aligned} \Sigma_k \langle\varphi_0\varphi_k\rangle &\propto a^{-2} \ , \\ \Sigma_{k_2,\dots,k_{2r}} \langle\varphi_0\varphi_{k_2}\cdots\varphi_{k_{2r}}\rangle^T &\propto a^{-2-6(r-1)} \ , \\ \Sigma_k k^2 \langle\varphi_0\varphi_k\rangle &\propto a^{-4} \ . \end{aligned}$$

Here, φ_k denotes the field value at the lattice site $k = (k_1, \dots, k_n)$, $k_j \in \{0, \pm 1, \pm 2, \dots\}$, $1 \leq j \leq n$, while φ_0 is the field at the origin. To remove all dimensional and rescaling aspects, as well as to allow for a continuum limit, one focuses on the quotients

$$g_r \equiv -\frac{\Sigma_{k_2,\dots,k_{2r}} \langle\varphi_0\varphi_{k_2}\cdots\varphi_{k_{2r}}\rangle^T}{[\Sigma_k \langle\varphi_0\varphi_k\rangle]^r \, [\Sigma_k k^2 \langle\varphi_0\varphi_k\rangle / 6\Sigma_k \langle\varphi_0\varphi_k\rangle]^{n(r-1)/2}} \propto a^{(r-1)(n-4)} \ .$$

When $n \geq 5$, $g_r \to 0$ as $a \to 0$ for all $r \geq 2$ indicative for such examples of a strictly trivial result. For $n = 4$, mean field theory has logarithmic corrections and it follows that

$$g_r \propto |\ln(a)|^{-(r-1)} \ ,$$

which still has the property that $g_r \to 0$ as $a \to 0$ for all $r \geq 2$ leading again to the conclusion that the continuum theory is trivial.

The dependence on a that leads to the conclusion of triviality arises from the long range order that develops close to a second-order phase transition. If, by some procedure, we could simultaneously arrange that the *magnitude* of all correlation functions was *uniformly rescaled for all* $r \geq 1$ so that

$$\begin{aligned} \langle\varphi_{k_1}\varphi_{k_2}\cdots\varphi_{k_{2r}}\rangle &\propto a^{n-4}, \qquad && n \geq 5 \ , \\ \langle\varphi_{k_1}\varphi_{k_2}\cdots\varphi_{k_{2r}}\rangle &\propto |\ln(a)|^{-1}, \qquad && n = 4 \ , \end{aligned}$$

then for all $n \geq 4$ we learn that $g_r \propto a^0 = 1$ for all $r \geq 1$, and the door to nontriviality is open. What follows is a conservative procedure to achieve the required uniform rescaling that also promises to produce a genuine covariant quantum field theory (after Wick rotation).

Building to Specifications

Our basic goal is to present a formula for the Euclidean space generating functional $S\{h\}$. To understand this formula, it is helpful to construct it in a step-by-step procedure from several different elements – indeed, rather like a modern day "powerpoint presentation". To that end, we express our proposal for the Euclidean space generating functional in the schematic form

$$S\{h\} = L_6\, P_5\, F_3\, N_2^{-1} \int e^{\Sigma h_k \varphi_k\, a^n - \mathfrak{A}_1 - \mathfrak{P}_4}\, \Pi\, d\varphi_k \;.$$

In this expression:

Element 1: $\mathfrak{A}_1$ denotes the conventional lattice action,

$$\mathfrak{A}_1 \equiv \tfrac{1}{2}\Sigma(\varphi_{k^*} - \varphi_k)^2\, a^{n-2} + \tfrac{1}{2} m_o^2(a)\,\Sigma \varphi_k^2\, a^n + \lambda(a)\,\Sigma \varphi_k^4\, a^n \;,$$

where k^* denotes each of the n positive nearest neighbors to k, the coupling constant $\lambda(a) \geq 0$, and the mass term $m_o^2(a)$ may even be negative when $\lambda(a) > 0$. Sums extend over k and k^* as needed, and we suppose that the lattice is a large but finite hypercube with periodic boundary conditions.

Element 2: N_2 denotes a normalization factor,

$$N_2 \equiv \int e^{-\mathfrak{A}_1}\, \Pi\, d\varphi_k \;,$$

which implies that $D \equiv N_2^{-1} e^{-\mathfrak{A}_1}$ is a probability density distribution for the conventional lattice φ_n^4 theory.

Element 3: F_3 is a dimensionless factor designed to uniformly rescale the conventional correlation functions, and is given by

$$\begin{aligned} F_3 &\equiv K\, a^{n-4}, && n \geq 5 \;, \\ F_3 &\equiv K\, |\ln(a)|^{-1}, && n = 4 \;, \end{aligned}$$

where K is a fixed positive constant. We focus on $F_3 \leq 1$. Whenever $F_3 < 1$, the positive distribution $F_3 N_2^{-1} e^{-\mathfrak{A}_1}$ is no longer normalized. To fix the normalization we introduce an additional term to the lattice action.

Element 4: $\mathfrak{P}_4$ is an auxiliary and nonclassical ($\propto \hbar^2$) factor added to the action – a kind of designer counterterm – which is given by

$$\mathfrak{P}_4 \equiv \tfrac{1}{2}\, A(a)\, \Sigma\, [\varphi_k^2 - B(a)]/[\varphi_k^2 + B(a)]^2\, a^n \ ;$$

why this form is chosen instead of a more familiar counterterm is discussed below. At this point we only note that both $A(a)$ and $B(a)$ are positive, and that $B(a) \to 0$ as $a \to 0$. Therefore, the terms in $\mathfrak{P}_4$ are significant for small field values ($\varphi_k^2 \simeq B(a)$) which is where the terms in $\mathfrak{A}_1$ are small; in contrast, the terms in $\mathfrak{P}_4$ behave like $\frac{1}{2} A(a)\, \varphi_k^{-2}$ for large field values ($\varphi_k^2 \simeq a^{-(n-2)}$) which is where the terms in $\mathfrak{A}_1$ are important. The choice of φ_k^{-2} only involves the dimensional factor $\hbar^2$, and requires no new dimensional coefficient for any n. As discussed below, there is a wide latitude available in which to *choose* the functional form of $B(a)$. Once that is done, however, the amplitude factor $A(a)$ in $\mathfrak{P}_4$ is then *determined* [in relation to the choice made for $B(a)$] by requiring that the distribution

$$D' \equiv F_3\, N_2^{-1}\, e^{-\mathfrak{A}_1 - \mathfrak{P}_4}$$

is *normalized* and hence is a probability density; an approximation for $A(a)$ [in relation to a given $B(a)$] is derived below. Furthermore, observe that the correlation functions of D' are similar to those of $F_3 D$ since $\mathfrak{P}_4$ has introduced changes primarily for small φ values, which would tend to contribute relatively little to the correlation functions in the first place.

We next need to restore the various correlation functions to macroscopic values. To that end we introduce multiple replicas of the present system.

Element 5: P_5 denotes a product over N_R independent, identical distributions, i.e., $P_5 = \Pi_l$, $D' \to D'_l$, $1 \le l \le N_R$, all for the same test sequence $\{h_k\}$ coupled to each factor. Here $N_R = [\![a^{-(n-4)}]\!]$ for $n \ge 5$, while $N_R = [\![|\ln(a)|]\!]$ for $n = 4$, where $[\![\cdot]\!]$ denotes the integer part of its argument. The resultant product has given the correlation functions macroscopic values as desired.

With the specific construction as described above, the stage is set for:

Element 6: L_6 is the continuum limit $a \to 0$ including, for convenience, a subsequent increase of the lattice volume in a natural way so that all of $\mathbb{R}^n$ is covered.

The resultant expression takes the form

$$S\{h\} = \exp\left\{K \int [e^{\int h(x)\varphi(x)\,d^nx} - 1]\,d\sigma(\varphi)\right\},$$

where σ is a positive measure on fields that fulfills $\int d\sigma(\varphi) = \infty$. As such we recognize $S\{h\}$ as determined by a generalized Poisson process [3]. The parameter K may be chosen, e.g., to satisfy some normalization condition on the two-point function.

* * *

Elements 1 through 6 represent our recipe for resolving the ramifications of triviality.[1]

Frequently Asked Questions

Why have we chosen $\mathfrak{P}_4$ as we have?

Elsewhere (see Chapter 8 in [5]) we have argued that, for any positive value of the coupling constant, nonrenormalizable interactions act as *hard cores in field space* projecting out certain fields that would otherwise have been allowed by the noninteracting theory alone. As such, the interacting theories are not even continuously connected to the noninteracting theory as the coupling constant passes to zero! As a consequence, perturbation theory is a highly unreliable guide for what the counterterms to nonrenormalizable theories should be. The choice made by $\mathfrak{P}_4$ represents a local self interaction, without derivatives, that is suggested by the form taken for certain idealized,

[1]The author has long been concerned with nonrenormalizable theories, and, in particular, counterterms that are regularized versions of local inverse square field powers have been previously proposed [4, 5]. The version of $\mathfrak{P}_4$ suggested in the present paper involves only a single unknown function of the lattice spacing, $A(a)$, and it is recognized here for the first time that normalization of D' offers a clear requirement to uniquely determine $A(a)$, once we have made a choice for $B(a)$. In previous work of the author it was hoped to restore the correlation functions to macroscopic values by an appropriate choice of the field strength renormalization factor. In this paper it is recognized that this suggestion is incorrect and it is replaced with the use of replicas. Of all the author's many attempts to study nonrenormalizable theories, the scenario offered in this paper seems to be the least arbitrary and most compelling. It is also possible that similar procedures may have a wider range of applicability.

but fully soluble, nonrenormalizable quantum field models (see Chapter 10 in [5]).

It is not too surprising that the choice of $\mathfrak{P}_4$ has led to the fact that $S\{h\}$ is expressed as a generalized Poisson process. What is novel is that Elements 1 through 6 offer a relatively direct recipe to construct the underlying measure $\sigma(\varphi)$.

How can modifying just the small field behavior change things?

While we have stressed that the introduction of $\mathfrak{P}_4$ makes relatively small quantitative impact on the large-field behavior of the lattice action, it must also be stressed that the small-field changes made by $\mathfrak{P}_4$ bring about a profound modification of the overall theory. This claim is already evident from the vast redistribution of the field probability in passing from the probability density distribution D to the probability density distribution D'.

What does it mean to introduce replicas?

The introduction of replicas implies that the underlying sharp-time field operator algebra is *reducible*. A familiar example pertains to a so-called generalized free field. It is clear that exploiting reducible representations opens up a new direction that is normally not pursued.

Does our procedure lead to a quantum theory?

We first note that after Elements 1 and 2 one is dealing with the conventional Euclidean lattice space formulation of the φ^4_n problem. For a choice of parameters which leads to a positive mass theory, it is natural to assume that the lattice expression will support a continuum limit that respects n-dimensional Euclidean invariance, reflection positivity, and clustering. Moreover, the factoring of the generating functional implicit in clustering should in no way be effected by the addition of Elements 3 and 4, namely by rescaling the whole expression by F_3 and the introduction of another local potential to restore normalization as represented by the term $\mathfrak{P}_4$; in short, the long range correlations and the associated decay of truncated correlation functions as two spacetime regions become asymptotically separated should in no significant way be altered by the introduction of $\mathfrak{P}_4$. The product of identical systems called for in Element 5 does not interfere with invariance, reflection positivity, or clustering. Consequently, assuming a uniform lower bound on a positive mass, these important properties still ought to hold for the final result of our construction after the continuum limit. The relative growth of the resultant correlation functions should also not be affected by modifications at small field values. Therefore, although these arguments by no means constitute a proof, we do not anticipate any conflict with requirements that our final theory admits a covariant quantum field theory under Wick rotation [6]. (Indeed, if coincident point singularities of the correlation functions are not integrable, then our expressions for $S\{h\}$ should be understood as standing for a corresponding set of noncoincident point correlation functions.)

Can one calculate something?

Assuming that the continuum limit is relatively smooth, it is plausible that certain aspects of the present model may well be studied by the use of numerical simulation and Monte Carlo techniques. In particular, calculation of a nonzero value for g_2 as a is made as small as possible would support the expectation of a nontrivial continuum limit.

Where should one begin?

Observe that the suggested construction of $S\{h\}$ also permits us to consider the limit in which the coupling constant of the quartic interaction goes to zero. Note well that the result of this limit is *not* the traditional free theory but is what is called the *pseudofree theory*. The pseudofree theory is worthy of examination in its own right. For one thing, it is the theory to which the interacting theories are continuously connected (as opposed to the free theory). In addition, from a Monte Carlo point of view the pseudofree theory is a natural place to begin since it deals with one parameter less than the full theory. Clarification of the pseudofree theory would undoubtedly facilitate elucidation of the interacting theory.

Clearly the entire construction rests heavily on choosing a suitable function $B(a)$, and then on finding the right function $A(a)$ to go with that choice. Let us next address that issue (in an approximate way, at least).

Choosing $B(a)$ and Approximating $A(a)$

In this section we discuss how to choose $B(a)$ and develop an approximation for the all-important function $A(a)$ that appears in $\mathfrak{P}_4$, and in so doing we focus only on Elements 1-4. As a preliminary to that analysis, however, we discuss a simpler, one-dimensional quantum problem as motivation.

The inverse square potential of interest to us can be illustrated in a simple quantum mechanical problem. In the Schrödinger representation, an eigenfunction of the form $\psi(x) = x^{-\gamma} f(x)$, with f smooth, $\gamma > 0$, and $x \neq 0$, satisfies a Schrödinger equation of the form

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = 0 \ ,$$

where the potential $V(x)$ takes the form

$$V(x) \equiv \frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} = \frac{\hbar^2}{2m}\Big[\frac{\gamma(\gamma+1)}{x^2} - 2\frac{\gamma}{x}\frac{f'(x)}{f(x)} + \frac{f''(x)}{f(x)}\Big] \ .$$

If $f'/f \propto x$ near $x = 0$, then the only singular term is $\hbar^2\gamma(\gamma+1)/(2mx^2)$. Observe that this singular potential is invariant under $\gamma \rightarrow -(\gamma+1)$; we only consider the form $x^{-\gamma} f(x)$ [rather than $x^{\gamma+1} f(x)$] since we are interested in eigenfunctions that have *enhanced* probability near $x = 0$ due to the singular potential.

Focusing on a small interval near $x = 0$, let us assume for simplicity that $f = 1$ in that region and then introduce a regularization for the singular eigenfunction that remains. To that end we change $\psi(x)$ to read $(x^2 + \epsilon)^{-\gamma/2}$, $\epsilon > 0$, and learn that

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{\hbar^2}{2m}\frac{\gamma(\gamma+1)\,[x^2 - \epsilon(\gamma+1)^{-1}]}{(x^2+\epsilon)^2}\,\psi(x) = 0 \ ,$$

which applies for all x near to and including zero. For the regularized potential, it is noteworthy that the invariance under $\gamma \rightarrow -(\gamma+1)$ is lost; in particular, the attractive well close to the origin evident in the regularized potential above, becomes repulsive when $\gamma \rightarrow -(\gamma+1)$. The similarity of the one-dimensional regularized potential with the expression adopted in $\mathfrak{P}_4$ is clear, and we next turn our attention to the field theory case. We predominantly treat the case $n \geq 5$; comments regarding $n = 4$ are made when appropriate.

The lattice action $\mathfrak{A}_1 + \mathfrak{P}_4$ corresponds to a certain Schrödinger representation, lattice-space Hamiltonian operator as well. To develop a comparison with the one-dimensional example, let us first focus on the contribution of just the kinetic energy and the regularized, auxiliary potential in the lattice (Euclidean) action, which reads as (k' refers to the eventual time direction)

$$\tfrac{1}{2}\Sigma\,(\varphi_{k'} - \varphi_k)^2 a^{n-2} + \tfrac{1}{2}A(a)\,\Sigma\,[\varphi_k^2 - cB(a)]\,/\,[\varphi_k^2 + B(a)]^2 a^n \ ,$$

where we have made one generalization to the form of the potential previously given. In particular, we have introduced a new (positive) parameter c; this change will let us see how $c = 1$ is selected as the analysis proceeds. In the classical lattice Hamiltonian these two terms appear as

$$\tfrac{1}{2}a^{-(n-1)}\Sigma\,\pi_k^2 + \tfrac{1}{2}A(a)\,\Sigma\,[\varphi_k^2 - c\,B(a)]/[\varphi_k^2 + B(a)]^2\,a^{n-1}\ ,$$

where π_k denotes the classical lattice momentum conjugate to the field φ_k at the site k. Correspondingly, this part of the Hamiltonian operator becomes

$$-\tfrac{1}{2}\hbar^2\,a^{-(n-1)}\,\Sigma\,\partial^2/\partial\varphi_k^2 + \tfrac{1}{2}A(a)\,a^{n-1}\,\Sigma\,[\varphi_k^2 - c\,B(a)]/[\varphi_k^2 + B(a)]^2\ .$$

Now, in order that the auxiliary potential arises in the manner previously indicated for the one-dimensional example, it follows that

$$\begin{aligned} A(a) &= \hbar^2\,a^{-(2n-2)}\,\gamma(\gamma+1)\ , \\ c &= (\gamma+1)^{-1}\ , \end{aligned}$$

where $\gamma > 0$ remains undetermined. If γ is a constant, this result

suggests that $A(a)$ should diverge like $a^{-(2n-2)}$ as $a \to 0$, for any $n \geq 4$. Below, however, we present arguments which suggest that although $A(a)$ may diverge as $a \to 0$, there is no compelling reason for $A(a)$ to diverge as fast as $a^{-(2n-2)}$. Accepting this argument implies that γ is *not* a constant, but instead that $\gamma = \gamma(a)$ and moreover that $\gamma(a) \to 0$ as $a \to 0$. In fact, the property that $\gamma(a) \to 0$ is not too surprising when it is recognized that the normalization change introduced by F_3 is just $K\,a^{n-4}$, and that we have many field variables to contribute toward this relatively small change. If $\gamma(a) \to 0$ as $a \to 0$, then to leading order we may already set $\gamma + 1 = 1$ and thus let $c = 1$. (This possibility has already been anticipated earlier; see [5], p. 304.)

We now take up the question of estimating $\gamma(a)$, at least in a rough sort of fashion. For the normalization integral that enforces D' to be a probability density, let us assume that the entire lattice volume $\mathcal{V} \equiv (La)^n$ breaks up into an integral number M of cells each of volume $v \equiv (\xi a)^n$, and that within each cell all the fields are completely correlated. On the other hand, we assume that fields in different cells are completely uncorrelated. With these strong assumptions, the normalization integral for D' (for $n \geq 5$, $\lambda(a) = 0$,

$c = 1$, and reverting to $\hbar = 1$) takes the form given by

$$(K\,a^{n-4})^{-1/M} = \frac{\int \exp\{-\frac{1}{2}v m_o^2 \varphi^2 - \frac{1}{2}v A(a)\,[\varphi^2 - B(a)]/[\varphi^2 + B(a)]^2\}\;d\varphi}{\int \exp\{-\frac{1}{2}v m_o^2 \varphi^2\}\;d\varphi}\;.$$

A change of variables from φ to $\varphi\sqrt{B(a)}$, along with the introduction of $E(a) \equiv v\,A(a)/2B(a)$, leads us to the relation

$$\begin{aligned}(K\,a^{n-4})^{-1/M} &= 2\sqrt{v m_o^2 B(a)/2\pi}\\ &\times \int_0^\infty \exp\{-\tfrac{1}{2}v m_o^2 \varphi^2 B(a) - E(a)\,[\varphi^2 - 1]/[\varphi^2 + 1]^2\}\;d\varphi\;,\end{aligned}$$

which is an equation that implicitly defines $E(a)$ as a function of a for sufficiently small a where our real interest lies. For $n = 4$, the left-hand side in this relation should be replaced by $(K/|\ln(a)|)^{-1/M}$. In such equations observe that we have isolated $v = (\xi a)^n$ – the correlation volume – which, near a second-order phase transition, is effectively a constant independent of a.

On the basis of our approximate normalization condition, $E(a)$ is determined as a function of a, as well as a function of the other (constant) parameters, i.e., K, v, m_o^2, and M, along with the assumed choice for $B(a)$. An explicit expression for $E(a)$ is not possible, but we can at least seek an *approximate* expression for $E(a)$ which captures its *leading functional form for very small* a. This leading functional form is designed to capture the primary term $a^{-(n-4)/M}$ on the left-hand side as well as the form of $B(a)$, but it will not account for the constant factors: K, v, or m_0^2. Nevertheless, this leading dependence will be enough to suggest which form of $B(a)$ would be suitable. With that choice made we can then complete the estimate of $E(a)$ and hence determine the leading behavior of $A(a)$ in order to characterize our approximate model problem. All in all, this analysis will suggest a specific choice for $B(a)$ and offer an approximate form for $A(a)$, both of which can serve as starting approximations in a more careful analysis of the true normalization condition satisfied by the original expression D'.

Since $B(a)$ becomes small when a becomes small, it follows that $E(a)$ must become large to maintain the normalization condition. This property may be seen most easily if we rewrite the former integral in the form

$$(K\,a^{n-4})^{-1/M}$$

$$= 2\sqrt{vm_o^2 B(a)/2\pi}\int_0^1 \exp\{-\tfrac{1}{2}vm_o^2\varphi^2 B(a) + E(a)\,[1-\varphi^2]/[1+\varphi^2]^2\}\,d\varphi$$
$$+2\sqrt{vm_o^2 B(a)/2\pi}\int_1^\infty \exp\{-\tfrac{1}{2}vm_o^2\varphi^2 B(a) - E(a)\,[\varphi^2-1]/[\varphi^2+1]^2\}\,d\varphi\ .$$

Observe that the latter integral (from 1 to ∞) is bounded by one, so, for very small a, the former integral (from 0 to 1) must become large to accompany the large left-hand side. A steepest descent evaluation of the integral from 0 to 1 establishes that the leading behavior of this expression is given by

$$(K\,a^{n-4})^{-1/M} = \sqrt{vm_o^2 B(a)/2\pi}\,\sqrt{\pi/3E(a)}\,e^{E(a)}\ .$$

Taking the logarithm of both sides leads to

$$E(a) = [(n-4)/M]\,|\ln(a)| + \tfrac{1}{2}\ln(E(a)) + \tfrac{1}{2}|\ln(B(a))| + O(1)\ ,$$

where the latter term contains the parameters K, v, m_o^2, and other factors.

At this point in the analysis we need to choose $B(a)$. From a *mathematical* point of view, we could imagine choosing $B(a)$ as we like, e.g. $B(a) = a^2$, or even $B(a) = e^{-(1/a)}$. However, from a *computational* point of view, such choices of $B(a)$ are inappropriate since they tend to overwhelm the primary term of interest, namely $[(n-4)/M]|\ln(a)|$, especially when M is large. Therefore, to ensure that the primary term remains dominant we select (for $n \geq 5$), e.g.,

$$B(a) = |\ln(a)|^{-2}\ .$$

With this choice, the equation for $E(a)$ becomes

$$E(a) = [(n-4)/M]|\ln(a)| + \tfrac{1}{2}\ln(E(a)) + |\ln(|\ln(a)|)| + O(1)\ ,$$

which has a leading order solution given by

$$E(a) = [(n-4)/M]|\ln(a)| + (3/2)\,|\ln(|\ln(a)|)|\ , \qquad n \geq 5\ .$$

From $E(a)$ we are led to

$$\begin{aligned} A(a) &= (2/v)B(a)\,E(a) \\ &= 2(n-4)\,(Mv)^{-1}|\ln(a)|^{-1} + \text{l.o.t.}\ , \\ &= 2(n-4)\,(La)^{-n}|\ln(a)|^{-1} + \text{l.o.t.}\ , \qquad n \geq 5\ , \end{aligned}$$

where "l.o.t." denotes "lower order terms". There are several ways to interpret this result. On the one hand, we can imagine a limit (to be called limit I) in which $a \rightarrow 0$ and where L is held fixed which is rather the way a computer study might take place. On the other hand, we can imagine another limit (to be called limit II) in which, as $a \rightarrow 0$, we simultaneously increase L so that the overall lattice volume $\mathcal{V} = (La)^n$ remains constant. Other limits are possible, but these two examples illustrate the issue. If we choose limit I, then $A(a)$ indeed diverges as $a \rightarrow 0$ but not as fast, for $n \geq 5$, as $a^{-(2n-2)}$. If we choose limit II, it follows as $a \rightarrow 0$ that $A(a)$ does *not* diverge, but instead goes to zero. In either of these two limits, it follows, as described previously, that $\gamma(a) \rightarrow 0$ as $a \rightarrow 0$; specifically,

$$\begin{aligned}\gamma(a) &= a^{(2n-2)}\, A(a) \\ &= 2(n-4)L^{-n}\, a^{(n-2)}\, |\ln(a)|^{-1} + \text{l.o.t.}\,, \qquad n \geq 5\,.\end{aligned}$$

The reader can readily determine for themselves the vanishing behavior of $\gamma(a)$ as $a \rightarrow 0$ in either limit I or II.

For $n = 4$ a slightly different argument applies. When $n = 4$, the leading behavior of the basic equation becomes

$$(K/|\ln(a)|)^{-1/M} = \sqrt{v m_o^2 B(a)/2\pi}\, \sqrt{\pi/3E(a)}\, e^{E(a)}\,,$$

which leads to

$$E(a) = M^{-1}|\ln(|\ln(a)|)| + \tfrac{1}{2}\ln(E(a)) + \tfrac{1}{2}|\ln(B(a))| + O(1)\,.$$

This time we choose

$$B(a) = |\ln(|\ln(a)|)|^{-2}$$

in order not to compete with the principal term. Therefore, we find that

$$E(a) = M^{-1}|\ln(|\ln(a)|) + \tfrac{1}{2}\ln(E(a)) + |\ln(|\ln(|\ln(a)|)|)| + O(1)\,,$$

which has a leading order solution given by

$$E(a) = M^{-1}|\ln(|\ln(a)|)| + (3/2)|\ln(|\ln(|\ln(a)|)|)|\,, \qquad n = 4\,.$$

In turn, we find that

$$\begin{aligned}A(a) &= (2/v)B(a)\,E(a) \\ &= 2\,(La)^{-4}|\ln(|\ln(a)|)|^{-1} + \text{l.o.t.}\,, \qquad n = 4\,,\end{aligned}$$

and

$$\begin{aligned}\gamma(a) &= a^6 A(a)\\ &= 2L^{-4}a^2\,|\ln(|\ln(a)|)|^{-1} + \text{l.o.t.}\ , \qquad n = 4\ .\end{aligned}$$

Once again, depending on the chosen limiting procedure (i.e., type I or II limits), it follows that $A(a)$ may diverge or vanish, but even if $A(a)$ diverges that divergence is sufficiently slow to ensure that $\gamma(a)$ always vanishes as $a \to 0$.

The equations above tagged with $n \geq 5$ and $n = 4$ indicate, under the strong approximations made in this section, an acceptable choice of functions $B(a)$ and $A(a)$ [and thereby of $\gamma(a)$] whenever $n \geq 5$ and $n = 4$, respectively. In all cases, $\gamma(a) \to 0$ as $a \to 0$, and as a consequence, *the change in the lattice action introduced, at each field point, actually vanishes as* $a \to 0$, relative to the kinetic energy contribution. Over the whole lattice, however, the *cumulative effect* of those soon-to-vanish individual contributions accounts for the desired change in the distribution. While the results obtained in this section surely depend on the strong assumptions that were introduced, it is nevertheless plausible that the results are sufficiently robust so that certain qualitative features of the solution survive even within a full calculation. We have in mind, that if we were to use the particular $B(a)$ developed in this section in the correct calculation of the normalization condition for D', the resulting correct expression for $A(a)$ would still imply a correct form for $\gamma(a)$ that also vanished in the continuum limit. This conjecture is based on the fact that $\gamma(a)$ as derived in this section does not go to zero marginally, but, for all the cases discussed in this section and even in the least favorable limit (type I), $\gamma(a)$ goes to zero faster than a^2 as $a \to 0$, a fact which suggests that even if the functional form of $\gamma(a)$ changes, its asymptotic behavior, as $a \to 0$, may well be the same. Based on this assumption, we have already set $c = 1$ in the original version of $\mathfrak{P}_4$.

In summary, the results of the analysis in this section suggest that suitable choices for $B(a)$ are given by

$$\begin{aligned}B(a) &= |\ln(a)|^{-2}\ , \qquad && n \geq 5\ ,\\ B(a) &= |\ln(|\ln(a)|)|^{-2}\ , \qquad && n = 4\ .\end{aligned}$$

It is also expected that these choices would prove satisfactory in a full analysis to determine the correct form of the remaining function $A(a)$ that enters the nonclassical, auxiliary potential $\mathfrak{P}_4$, and which should then render our version of the φ^4_n theory nontrivial for any $n \geq 4$. As a place to begin to look for the proper function $A(a)$ we can only recommend to start with the expressions found in the study in this section, namely,

$$A(a) = 2(n-4)(La)^{-n} |\ln(a)|^{-1} \,, \qquad n \geq 5 \,,$$
$$A(a) = 2(La)^{-4} |\ln(|\ln(a)|)|^{-1} \,, \qquad n = 4 \,.$$

In these expressions, L denotes the number of sites on each edge of the lattice, while $(La)^n$ denotes the lattice volume.

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References

[1] R. Fernández, J. Fröhlich, and A. Sokal, *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory*, (Springer-Verlag, New York, 1992).

[2] M.E. Fisher, The Theory of Equilibrium Critical Phenomena, *Rep. Prog. Phys.* **30**, 615-730 (1967).

[3] B. DeFinetti, *Theory of Probability, Vol. 2*, (John Wiley & Sons, London, 1975).

[4] J. R. Klauder, "Covariant Diastrophic Quantum Field Theory", Phys. Rev. Lett. **28**, 769-772 (1972); "Remarks on Nonrenormalizable Interactions", Phys. Lett. B **47**, 523-525 (1973); "Field Structure Through Model Studies: Aspects of Nonrenormalizable Theories", Acta Physica Austriaca, Suppl. **XI**, 341-387 (1973); "New Measures for Nonrenormalizable Quantum Field Theory", Annals of Physics **117**, 19-55 (1979); "Poisson Distributions and Nontriviality of φ^4 Theory," Phys. Rev. Lett. **73**, 3051-3054 (1994); "Nonrenormalizability and Nontriviality", hep-th/9811181.

[5] J.R. Klauder, *Beyond Conventional Quantization*, (Cambridge University Press, Cambridge, 2000).

[6] J. Glimm and A. Jaffe, *Quantum Physics*, (Springer-Verlag, New York, Second Edition, 1987).