EXISTENCE OF RESONANCES FOR MATRIX SCHRÖDINGER OPERATORS.

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Abstract. We study the resonances of matrix Schrödinger operators, in the semi-classical limit, motivated by the Born-Oppenheimer approximation. Precisely, we are interested in a lower bound on the number of resonances near a point. We obtain the lower bound $Ch^{-1}|\ln h|^{-\frac{3}{2}}$ for $n = 3$.

1. Introduction

In this paper we study the spectral properties of matrix Schrödinger operators. Such operators appear in the Born-Oppenheimer approximation. The justification of this approximation and a classification of matrix Schrödinger operators can be found in the work of Hagedorn [5], Klein, Martinez, Seiler, and Wang [10], and Combes, Duclos, and Seiler [2] .

More precisely, we are interested in resonances, which in physics are associated with quasi-stable states. For the definition of resonances, see the work of Aguilar and Combes [1], Helffer and Martinez [6], and Sjöstrand and Zworski [17].

We find a link between the properties of the potential and the number of resonances for the self-adjoint Schrödinger operator

$$P_0 = -h^2\Delta + V_0$$

on $\mathbb{R}^n$, where V_0 is a Hermitian smooth function and n is odd.

When V_0 is scalar, Sjöstrand proved [14] that there exist $\mathcal{O}(h^{-n})$ resonances near any point of the analytic singular support of the distribution ν defined as $\nu(E) = \int_{x, V_0(x)\geq E} dx$ (if $V_0(x) \to 0$ when $|x| \to \infty$ and $E > 0$). (There could also exist many resonances near regular points of the distribution ν.) This result is an application of the local trace formula of Sjöstrand, see [15] and [16].

In the paper [11], we proved a similar trace formula for matrix operators and that there exist $\mathcal{O}(h^{-n})$ resonances near any point of the analytic singular support of a distribution ν when the dimension n is even.

We are interested in the number of resonances near a point that is not in the singular support of the distribution ν. The considered case is the generic crossing of two eigenvalues of a Hermitian matrix V_0

when $n = 3$. With this paper, we continue the classification which was started in [11].

The proof of the lower bound on the number of resonances uses a computation of the trace included in Theorem 2.1, using mainly the residue theorem. The estimate on the number of resonances is in Theorem 2.2. The number of resonances, near the point studied, is at least $Ch^{-1}|\ln h|^{-\frac{3}{2}}$. For related results on the same model, see [4], [5] and [9].

2. NOTATION AND RESULTS

Let $n > 0$ be an odd integer. Denote by $\mathcal{M}_r(\mathbb{C})$ the set of Hermitian matrices of size (r, r), and let $\|M\| = \sup_{\|u\|=1} \|Mu\|$. There exists $C > 0$ such that $|\mathrm{Tr}M| \leq C\|M\|$. Let $x \mapsto V_0(x)$ be a bounded smooth function of $\mathbb{R}^n$ with values in $\mathcal{M}_r(\mathbb{C})$. Let $x = r\hat{x}$ with $r = |x|$, $\hat{x} \in \mathbb{S}^{n-1}$.

We make the following assumptions on V_0 :

(H1): There exist $\theta_0 \in [0, \frac{\pi}{2}[$, $\varepsilon > 0$, $R > 0$, $\tilde{R} > R > 0$ such that for all $\hat{x} \in \mathbb{S}^{n-1}$, $r \mapsto V_0(r\hat{x})$ extends to an analytic function on $\mathrm{S} \subset \mathbb{C}$, where

$$\mathrm{S} = \left\{r \in \mathbb{C}; r \in \mathbb{C}, |r| > R, \arg r \in [-\varepsilon, \theta_0 + \varepsilon[\,\right\} \cup \left\{r; r \in \mathbb{C}, |r| < \tilde{R}\right\}.$$

(H2): There exist $V_\infty \in \mathcal{M}_r(\mathbb{C})$, $\tilde{n} \in \mathbb{R}$ $\tilde{n} > n$ and $C > 0$ such that :

$$\forall r \in \mathrm{S},\ \|V_0(x) - V_\infty\| \leq C\langle x\rangle^{-\tilde{n}}.$$

Let I_r be the identity matrix of size (r, r). Let P_0 be the operator $-h^2\Delta \otimes I_r + V_0(x)$ with domain $\mathcal{D} = H^2(\mathbb{R}^n)$. Set $P_\infty = -h^2\Delta \otimes I_r + V_\infty$. Let $\lambda_{1,\infty} \leq \lambda_{2,\infty} \leq \cdots \leq \lambda_{r,\infty}$ be the eigenvalues of V_∞. For all $x \in \mathbb{R}^n$, denote by $\lambda_{1,0}(x),\ \lambda_{2,0}(x), \cdots, \lambda_{r,0}(x)$ the eigenvalues of $V_0(x)$. Set $\xi = \rho\hat{\xi}$ with $\hat{\xi} \in \mathbb{S}^{n-1}$.

Remark 1. *Under these assumptions, the operators P_0 and P_∞ are self-adjoint. For all $\hat{x} \in \mathbb{S}^{n-1}$, the functions $r \mapsto \lambda_{j,0}(r\hat{x})$ can be labeled so that they are holomorphic functions in a neighbourhood of* 0 *in* $\mathbb{C}$.

(H3): Let $E_0 \in \mathbb{R} \setminus \{0\}$ and $E_0 \notin \{\lambda_{j,\infty}\}_{j=1}^n$. Assume that

- $\forall j,\ \lambda_{j,0}(0) = E_0$.
- There exists a neighborhood $S_0 \subset S$ of 0 such that for all $r \in S_0 \setminus \{0\}$, $\lambda_{j,0}(r\hat{x})$ is a simple eigenvalue of $V_0(r\hat{x})$.
- For all r such that $\lambda_{j,0}(r\hat{x}) = 0$ we have $\nabla_r \lambda_{j,0}(r\hat{x}) \neq 0$.

In what follows, the bracket $[L_\bullet]_0^\infty$ will denote the difference $L_\infty - L_0$. Denote $\langle w\rangle = (1 + w^2)^{\frac{1}{2}}$. Let λ be a positive constant (that will later depend on h). Let $a > 0$, $b > 0$, $b_0 > 0$,

$$\Omega = [-a, a] + [-ib, ib_0] \subset \mathbb{C}.$$

such that $\lambda_{j,\infty} \notin \Omega + E_0$. Let $(z,\lambda) \mapsto f(z,\lambda)$ be a holomorphic function in z defined on $\Omega + E_0$, satisfying the following four conditions :

(H4): For all $0 < \check{b} < b$, there exist $C_{\check{b}} > 0$ and $\tilde{C}_{\check{b}} > 0$ such that

$$|f(z+E_0,\lambda)| \leq \tilde{C}_{\check{b}} e^{-C_{\check{b}}\lambda}.$$

uniformly in $\lambda \geq 0$ and for all $z \in [-a,a] + [-i\check{b}, -ib]$.

(H5): There exist $0 < \tilde{a} < a$, $b_0 > 0$, $C_0 > 0$, and $\tilde{C}_0 > 0$ such that

$$|f(z+E_0,\lambda)| \leq \tilde{C}_0 e^{-C_0\lambda}$$

uniformly in $\lambda \geq 0$ and for all $z \in \Omega_0$,

$$\Omega_0 = ([-a,-\tilde{a}] + [-ib, ib_0]) \cup ([\tilde{a}, a] + [-ib, ib_0]).$$

(H6): There exists $\tilde{C} > 0$ and $C_1 > 0$ such that

$$|f(z+E_0,\lambda)| \leq \tilde{C} e^{C_1\lambda}.$$

uniformly in $\lambda \geq 0$ for all $z \in \Omega$.

(H7): $f(E_0) \neq 0$.

An example of such a function is $f(z,\lambda) = e^{-i\lambda(z-E_0)} e^{-\lambda(z-E_0)^2}$ for Ω a small enough neighborhood of 0. We will choose $0 < b < a$, and b_0 arbitrary small. Let $\tilde{a} < \hat{a} < a$. Let $\chi \in \mathcal{C}_0^\infty(\,]-\hat{a}+E_0, \hat{a}+E_0[\,)$ be a function such that $\chi(x+E_0) = 1$ for all $x \in [-\tilde{a}, \tilde{a}]$. Let $\tilde{b}_0 < \hat{b}_0 < b_0$, and $\tilde{b} < \hat{b} < b$. Let $\tilde{\chi}$ be an almost-analytic extension of χ of support strictly included in Ω, satisfying

$$\overline{\partial}\tilde{\chi}(z) = \mathcal{O}(|\mathrm{Im}(z)|^\infty),$$

$$\tilde{\chi}(z+E_0) = 1 \quad \text{for} \quad z \in [-\tilde{a}, \tilde{a}] + [-i\tilde{b}, i\tilde{b}_0],$$

$$\tilde{\chi}(z+E_0) = 0 \quad \text{for} \quad \mathrm{Im}(z) \geq \hat{b}_0 \quad \text{or} \quad \mathrm{Im}(z) \leq -\hat{b} \quad \text{or} \quad |\mathrm{Re(z)}| \geq \hat{\mathrm{a}}.$$

Let $m \in \mathbb{N}$ be a constant such that $m - 1 > \frac{n}{2}$. Let z_0 be an element of $\mathbb{C} \setminus \Omega$ such that $\mathrm{Im}(z_0) > 0$. Let

$$\tilde{f}(z,\lambda) = f(z,\lambda)(z-z_0)^m. \tag{2.1}$$

Denote by

$$\|f\|_\infty = \sup_{x\in\Omega} \|f(x+E_0,\lambda)\|.$$

Denote $r_{0,0}(x,\xi,z) = (\xi^2 + V_0(x) - z)^{-1}$ and $D = \frac{\partial}{i}$. Recall that $x = r\hat{x}, r = |x|, \hat{x} = x/|x| \in \mathrm{S}^{n-1}, \xi = \rho\hat{\xi}, \rho = |\xi|, \hat{\xi} = \xi/|\xi| \in \mathrm{S}^{n-1}$. Denote $L(d\hat{x})$ the measure given by $dx = r^{n-1}dr L(d\hat{x})$, and $L(d\hat{\xi})$ given by $d\xi=\rho^{n-1}d\rho L(d\hat{\xi})$, and $\|A\|_{\mathrm{Tr}}$ the trace norm of A. Let

$$\mu(x,\xi,z)=\sum_i\Big\{\mathrm{Tr}\Big(-r_{0,0}^3(x,\xi,z)(D_{x_i}V_0(x))r_{0,0}(x,\xi,z)(D_{x_i}V_0(x))$$
$$+\frac{1}{2}r_{0,0}^2(x,\xi,z)(D_{x_i}V_0(x))r_{0,0}^2(x,\xi,z)(D_{x_i}V_0(x))\Big)\Big\}$$
$$+\sum_{i=1}^n\Big\{2\xi_i^2\mathrm{Tr}\Big(-r_{0,0}^2(x,\xi,z)(D_{x_i}V_0(x))r_{0,0}^3(x,\xi,z)(D_{x_i}V_0(x))$$
$$+r_{0,0}^4(x,\xi,z)(D_{x_i}V_0(x))r_{0,0}(x,\xi,z)(D_{x_i}V_0(x))\Big)\Big\}.$$

Let $\partial D(z,\varepsilon)=\{z'\in\mathbb{C},|z-z'|=\varepsilon\}$, and $D(z,\varepsilon)=\{z'\in\mathbb{C},|z-z'|\le\varepsilon\}$.

Theorem 2.1. *The operator $\chi f(P_\infty,\lambda)-\chi f(P_0,\lambda)$ is of trace class and there exists $C_2>0$ and $<\nu,f>\in\mathbb{C}$ such that :*

(2.2)
$$\mathrm{Tr}[(\chi f)(P_\bullet,\lambda)]_0^\infty=<\nu,f>h^{2-n}+\|f\|_\infty\mathcal{O}(h^{4-n})+\mathcal{O}(e^{-C_2\lambda}h^{-n}).$$

Furthermore, there exist ρ_0, $\tilde{r}_0$ such that for all $r_1\le\tilde{r}_0$ there exist $\mu_1(r_1)$ such that

$$<\nu,f>=-\frac{(2\pi)^{-n}}{16i\pi}\sum_{j=1}^n\mathrm{sgn}(\partial_r\lambda_{j,0}(0))\iint_{\hat{x},\hat{\xi}\in S^{n-1}}\int_{\rho\in e^{-i\frac{\pi}{4}}\mathbb{R},|\rho|\le\rho_0}$$
$$\iint_{\substack{r\in\partial D(0,r_1)\\ z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)}}f(z,\lambda)\mu(x,\xi,z)\ r^{n-1}dr L(d\hat{x})L(d\hat{\xi})\rho^{n-1}d\rho dz.$$

for all $\mu_1\le\mu_1(r_1)$

A similar formula to 2.2 was obtained in [11] with the main term h^{-n} instead of h^{2-n} for the case n even.

For the special case $r=2$, V_0 is Hermitian so there exist $(a_j(x))_{1\le j\le 3}$, $b(x)$ such that

$$V_0(x)=b(x)I_2+\begin{pmatrix}a_1(x) & a_2(x)-ia_3(x)\\ a_2(x)+ia_3(x) & -a_1(x)\end{pmatrix}.$$

We denote $\mathrm{Vol}(\mathrm{S}^2)$ the volume of S^2 with respect to Lebesgue measure.

Theorem 2.2. *Let $n=3$ and $r=2$. Suppose V_0 satisfy $a_1(0)=a_2(0)=a_3(0)=0$, $b(0)=E_0$, $\{\nabla a_j(0)\}_{j=1}^3$ are linearly independent. Let (a_1,a_2,a_3) be local coordinates and denote by $\nabla_{a_1,a_2,a_3}b$ the gradient of b for these coordinates. Suppose $|\nabla_{a_1,a_2,a_3}b(0)|<1$ and suppose that $r\mapsto(\lambda_{1,0}-\lambda_{2,0})(r\hat{x})$ has a zero of order one at $r=0$.*

Then there exist $C(V_0)>0$, and $\rho_0>0$ such that :

$$<\nu,f>=\frac{i}{2^5\pi^2}\mathrm{Vol}(\mathrm{S}^2)C(V_0)\int_{|\rho|\le\rho_0}f(E_0+\rho^2,\lambda)\ \rho^2d\rho,$$

with

$$C(V_0) = \frac{\pi}{3}\det(\nabla a_1(0), \nabla a_2(0), \nabla a_3(0))^{-1} \sum_{j=1}^{3} |\nabla a_j(0)|^2.$$

Denote $\mathrm{Res}(P_0)$ the set of resonances of P_0. We then have the following

Theorem 2.3. *With the same hypotheses as in Theorem 2.2, for any neighborhood W of E_0, there exist $h_0 > 0$ and $C > 0$ such that*

$$\forall h \in]0, h_0[, \qquad \mathrm{card}\{\mathrm{Res}(P_0) \cap W\} \geq Ch^{-1}|\ln h|^{-\frac{3}{2}}.$$

Throughout the proofs, we will denote by C any constants.

3. Reduction using functional calculus

In this section we use the functional calculus to get an expression for $[(\chi f)(P_\infty, \lambda) - (\chi f)(P_0, \lambda)]$, using only the operator $(P_\bullet - z)^{-1}$ for z such that $\mathrm{Im}(z) = \delta$, up to a remainder. The main result of this section is Lemma 3.1. A similar reduction can be found in Dimassi-Sjöstrand [3]. For the functional calculus, we use the Helffer-Sjöstrand formula, see [7].

Let $b > \check{b} > \hat{b}$. Let $\hat{a} \in]\tilde{a}, \check{a}[$. Denote by $L(dz) = d(\mathrm{Re}(z))d(\mathrm{Im}(z))$. Let δ be a constant such that $0 < \delta < \mathrm{Min}\{\tilde{b}_0, \tilde{b}, 1\}$. By the functional calculus we get (for $\tilde{f}$ see (2.1))

(3.1)

$$[(\chi f)(P_\bullet, \lambda)]_0^\infty = \frac{1}{\pi}\int_{z\in\Omega} \tilde{f}(z,\lambda)\overline{\partial}\tilde{\chi}(z)\left[(P_\bullet - z)^{-1}(P_\bullet - z_0)^{-m}\right]_0^\infty L(dz).$$

We have the following lemma

Lemma 3.1. *With $\mathcal{Z} = \{z \in \Omega; \mathrm{Im}(z) = \delta\}$*

$$\begin{aligned}[(\chi f)(P_\bullet, \lambda)]_0^\infty = &-\frac{1}{2i\pi}\int_{\mathcal{Z}} \tilde{f}(z,\lambda)\tilde{\chi}(z)\left[(P_\bullet - z)^{-1}(P_\bullet - z_0)^{-m}\right]_0^\infty dz \\ &+ \mathcal{O}(h^{-n})e^{-\min\{C_0, C_{\tilde{b}}\}\lambda}.\end{aligned}$$

The remainder is estimated with the trace norm.

Proof : Denote $\varpi(z) = \tilde{f}(z,\lambda)\overline{\partial}\tilde{\chi}(z)\left[(P_\bullet - z)^{-1}(P_\bullet - z_0)^{-m}\right]_0^\infty$. Using Stokes formula and (3.1), we get

$$\begin{aligned}[(\chi f)(P_\bullet, \lambda)]_0^\infty = &\frac{1}{\pi}\int_{\substack{z\in\Omega \\ \mathrm{Im}(z)\geq\delta}} \varpi(z)L(dz) \\ &+ \frac{1}{\pi}\int_{z\in\Omega;|\mathrm{Im}(z)|\leq\delta} \varpi(z)L(dz) + \frac{1}{\pi}\int_{z\in\Omega;\mathrm{Im}(z)\leq-\delta} \varpi(z)L(dz) \\ =:&A_0 + A_1 + A_2.\end{aligned}$$

Stokes theorem again gives

$$A_0 = -\frac{1}{2i\pi}\int_{\mathcal{Z}} \tilde{f}(z,\lambda)\tilde{\chi}(z)[(P_\bullet - z)^{-1}(P_\bullet - z_0)^{-m}]_0^\infty dz.$$

Proposition A.3 implies

$$\|A_1\|_{\mathrm{Tr}} = \Big\|\frac{1}{\pi}\int_{z\in\Omega;|\mathrm{Im}(z)|\le\delta} \varpi(z)L(dz)\Big\|_{\mathrm{Tr}} \le \mathcal{O}(h^{-n})e^{-C_0\lambda}.$$

and

$$\|A_2\|_{\mathrm{Tr}} \le \Big\|\frac{1}{\pi}\int_{z\in\Omega;\mathrm{Im}(z)\le-\delta} \varpi(z)L(dz)\Big\|_{\mathrm{Tr}} \le \mathcal{O}(h^{-n})e^{-C_{\tilde{\delta}}\lambda}.$$

This finishes the proofs of Lemma 3.1 ■

4. Symbolic calculus

In this section we use the pseudo-differential calculus to simplify the right hand side in Lemma 3.1. Some notation and results are recalled in the appendix. To be able to compute the trace of $[(\chi f)(P_\bullet,\lambda)]_0^\infty$ we reduce the operator $(P_\bullet - z)^{-1}(P_\bullet - z_0)^{-m}$ to a pseudo-differential operator with a symbol of finite expansion in h. The following lemma is the main result of this section. Denote by $\mathrm{Op}^w(r_\bullet^N)$ an approximation of $(P_\bullet - z)^{-1}$ given by $r_\bullet^N(x,\xi,z) = \sum_{j<N} h^j r_{j,\bullet}(x,\xi,z)$ with

$$\begin{aligned} r_{j+1,\bullet}(x,\xi,z) =& - r_{0,\bullet}(x,\xi,z) \times \sum_{0\le l\le j\ ,l+|\alpha|+|\beta|=j+1} \Gamma(\alpha,\beta) \\ & \times (\partial_\xi^\alpha D_x^\beta (p_\bullet - z\mathrm{I}_r)(x,\xi))(\partial_\xi^\beta D_x^\alpha r_{l,\bullet}(x,\xi,z)), \\ \Gamma(\alpha,\beta) =& \frac{(-1)^{|\beta|}}{\alpha!\beta!2^{|\alpha|+|\beta|}}, \quad r_{0,\bullet}(x,\xi,z) = (p_\bullet(x,\xi) - z\mathrm{I}_r)^{-1}. \end{aligned}$$

Lemma 4.1. *Let $b_\bullet^N$ be the sum of the $(N-1)$ first terms of the expansion in powers of h of the symbol of $\mathrm{Op}^w(r_\bullet^N)(P_\bullet - z_0)^{-m}$. Then*

$$\begin{aligned}\mathrm{Tr}[(\chi f)(P_\bullet,\lambda)]_0^\infty =& -\mathrm{Tr}\left(\frac{1}{2i\pi}\int_{\mathcal{Z}} \tilde{\chi}(z)\tilde{f}(z,h)\mathrm{Op}^w\left(\left[b_\bullet^N(x,\xi,z)\right]_0^\infty\right)dz\right) \\ & + \|f\|_\infty \mathcal{O}_\delta(h^{N-n}) + \mathcal{O}(h^{-n})e^{-\mathrm{Min}\{C_0,C_{\tilde{\delta}}\}\lambda}.\end{aligned}$$

Remark 2. *The remainder could be improved to obtain, uniformly in δ, for all M,*

$$\|f\|_\infty \mathcal{O}(h^{N-n})\delta^{-3-2N} + \mathcal{O}(h^{-n})\delta^M e^{-\mathrm{Min}\{C_0,C_{\tilde{\delta}}\}\lambda}.$$

Proof : The demonstration is quite usual, unless we care about the integration in x of any remainder (for that we use the difference $V_0 - V_\infty$), and we get estimates uniformly in z for $z\in\mathcal{Z}$. Let us first recall some properties of the approximation of $(P_\bullet - z\mathrm{I}_r)^{-1}$ for $z\in\mathcal{Z}$. Let $p_\bullet$

be the symbol of $P_\bullet$, then $p_\bullet - z \in \mathrm{S}(\langle\xi\rangle^2, 1, 1)$ (the definition is in the appendix). For all $\alpha, \beta \in \mathbb{N}$ and $z \in \mathcal{Z}$ we get

$$\|\partial_x^\alpha \partial_\xi^\beta r_{j,\bullet}(x,\xi,z)\| \le C\langle\xi\rangle^{-2},$$

$$\|\partial_x^\alpha \partial_\xi^\beta (r_{j,\infty} - r_{j,0})(x,\xi,z)\| \le C\langle\xi\rangle^{-2}\langle x\rangle^{-\tilde{n}}.$$

Hence

$$r_{j,\bullet}(x,\xi,z) \in \mathrm{S}(\langle\xi\rangle^{-2}, 1, 1)$$

and

$$[r_\bullet^N]_0^\infty(x,\xi,z) \in \mathrm{S}(\langle\xi\rangle^{-2}\langle x\rangle^{-\tilde{n}}, 1, 1),$$

uniformly in z for fixed δ. The symbol of $(P_\bullet - z_0)^{-l}$, denoted by $(p_\bullet - z_0)^{-\#l}(x,\xi)$, belongs to $\mathrm{S}(\langle\xi\rangle^{-2l}, 1, 1)$. We have choosen $r_\bullet^N$ such that

$$r_\bullet^N(x,\xi,z)\#(p_\bullet(x,\xi) - z) - 1 =: s_\bullet^N(x,\xi,z) \in h^N\mathrm{S}(1,1,1),$$

uniformly in z.

We now use the above mentioned propreties to simplify the right hand side in Lemma 3.1.

For $1 \le k \le m$, let us denote

$$\begin{aligned} A_k = &- \frac{1}{2i\pi}\int_{\mathcal{Z}} \tilde{f}(z,\lambda)\tilde{\chi}(z) \\ \times & \quad \big(P_\infty - z\big)^{-1}(P_\infty - z_0)^{-k}(V_0 - V_\infty)(P_0 - z_0)^{-m-1+k}dz, \end{aligned}$$

$$B = -\frac{1}{2i\pi}\int_{\mathcal{Z}} \tilde{f}(z,\lambda)\tilde{\chi}(z)[(P_\bullet - z)^{-1}]_0^\infty(P_0 - z_0)^{-m}dz.$$

Using the expression

$$\begin{aligned} [(P_\bullet - z)^{-1}(P_\bullet - z_0)^{-m}]_0^\infty = &\sum_{k=1}^m (P_\infty - z)^{-1}(P_\infty - z_0)^{-k}(V_0 - V_\infty) \\ &\times (P_0 - z_0)^{-m-1+k} + [(P_\bullet - z)^{-1}]_0^\infty(P_0 - z_0)^{-m}, \end{aligned}$$

we obtain, in trace norm, that

$$[(\chi f)(P_\bullet,\lambda)]_0^\infty = \sum_{k=1}^m A_k + B + \mathcal{O}(h^{-n})e^{-\mathrm{Min}\{C_0, C_{\tilde{\delta}}\}\lambda}. \tag{4.1}$$

We first simplify the expression for A_k. Using Proposition A.1 and

$$[s_\bullet^N]_0^\infty = [r_\bullet^N]_0^\infty\#(p_\infty - z) + r_0^N\#(p_\infty - p_0),$$

we get a control in x :

$$[s_\bullet^N(x,\xi,z)]_0^\infty \in h^N\mathrm{S}(\langle x\rangle^{-\tilde{n}}, 1, 1). \tag{4.2}$$

The equation

$$(\mathrm{Op}^w(p_\bullet - z))^{-1} = \mathrm{Op}^w(r_\bullet^N) - (\mathrm{Op}^w(p_\bullet - z))^{-1}\,\mathrm{Op}^w(s_\bullet^N)$$

gives

$$
\begin{aligned}
A_k = &- \frac{1}{2i\pi} \int_{\mathcal{Z}} \tilde{f}(z,\lambda)\tilde{\chi}(z)\mathrm{Op}^w(r_\infty^N)\mathrm{Op}^w((p_\infty - z_0)^{-\#k}) \\
&\times (V_0 - V_\infty)\mathrm{Op}^w((p_0 - z_0)^{-\#m+1-k})dz \\
&+ \frac{1}{2i\pi} \int_{\mathcal{Z}} \tilde{f}(z,\lambda)\tilde{\chi}(z) \left(\mathrm{Op}^w(p_\infty - z)\right)^{-1} \mathrm{Op}^w(s_\infty^N) \\
&\times \mathrm{Op}^w\left((p_\infty - z_0)^{-\#k}\right)(V_0 - V_\infty)\mathrm{Op}^w((p_0 - z_0)^{-\#m+1-k})dz \\
=:&\hat{A}_k + \tilde{A}_k.
\end{aligned}
$$

We get

$$
\| \left(\mathrm{Op}^w(p_\infty - z)\right)^{-1} \|_{L^2 \to L^2} \leq C|\mathrm{Im}(z)|^{-1} = C\delta^{-1}.
$$

and the operator

$$
\mathrm{Op}^w(s_\infty^N)\mathrm{Op}^w\left((p_\infty - z_0)^{-\#k}\right)(V_\infty - V_0)\mathrm{Op}^w\left((p_0 - z_0)^{-\#(m+1-k)}\right)
$$

therefore belongs to $\Sigma(h^N\langle x\rangle^{-\tilde{n}}\langle\xi\rangle^{-2m-2}, 1, 1)$, so $\tilde{A}_k$ is a trace class operator, and we get

$$
\|\tilde{A}_k\|_{\mathrm{Tr}} \leq \|f\|_\infty \mathcal{O}(h^{-n+N}).
$$

We get from this inequality the simplification for A_k :

$$
\begin{aligned}
\sum_{k=1}^m A_k = &-\frac{1}{2i\pi} \int_{\mathcal{Z}} \tilde{f}(z,\lambda)\tilde{\chi}(z)\mathrm{Op}^w\left(r_\infty^N \# [(p_\bullet - z_0)^{-\#m}]_0^\infty\right)dz \\
&+ \|f\|_\infty \mathcal{O}(h^{-n+N}).
\end{aligned} \tag{4.3}
$$

We return to the term B of the equation (4.1). We get, using (4),

$$
\begin{aligned}
B = &- \frac{1}{2i\pi} \int_{\mathcal{Z}} \tilde{f}(z,\lambda)\tilde{\chi}(z)[\mathrm{Op}^w(r_\bullet^N]_0^\infty (P_0 - z_0)^{-m} dz \\
&+ \frac{1}{2i\pi} \int_{\mathcal{Z}} \tilde{f}(z,\lambda)\tilde{\chi}(z) \\
&\quad \times [(\mathrm{Op}^w(p_\bullet - z))^{-1}]_0^\infty \mathrm{Op}^w(s_\infty^N)\mathrm{Op}^w(p_0 - z_0)^{-\#m} \\
&+ \frac{1}{2i\pi} \int_{\mathcal{Z}} \tilde{f}(z,\lambda)\tilde{\chi}(z) \\
&\quad \times \left(\mathrm{Op}^w(p_0 - z)\right)^{-1} [\mathrm{Op}^w(s_\bullet^N)]_0^\infty \mathrm{Op}^w\left((p_0 - z_0)^{-\#m}\right) dz \\
=:&F_0 + F_1 + F_2.
\end{aligned}
$$

A basic calculus gives

$$
\begin{aligned}
F_1 =&\frac{1}{2i\pi} \int_{\mathcal{Z}} \tilde{f}(z,\lambda)\tilde{\chi}(z) \left(\mathrm{Op}^w(p_\infty - z)\right)^{-1} (V_0 - V_\infty) \\
&\left(\mathrm{Op}^w(p_0 - z)\right)^{-1} \mathrm{Op}^w(s_\infty^N \# (p_0 - z_0)^{-\#m})dz.
\end{aligned}
$$

Using the cyclicity of the trace and the fact that

$$
s_\infty^N \# (p_0 - z_0)^{-\#m} \# (V_0 - V_\infty) \in \mathrm{S}(h^N\langle x\rangle^{-\tilde{n}}\langle\xi\rangle^{-2m}, 1, 1),
$$

we get :

$$
\begin{aligned}
\|\mathrm{Tr} F_1\| &\leq \|f\|_\infty \int_{\mathcal{Z}} \left\| \mathrm{Op}^w (s_\infty^N \# (p_0 - z_0)^{-\# m} \# (V_0 - V_\infty) \right\|_{\mathrm{Tr}} |dz| \\
&\leq \|f\|_\infty \mathcal{O}(h^{-n+N}).
\end{aligned} \tag{4.4}
$$

The relation (4.2) proves that

$$
\begin{aligned}
\|F_2\| \leq & \Big\| \frac{1}{2i\pi} \int_{\mathcal{Z}} \tilde{f}(z,\lambda)\tilde{\chi}(z) \\
& \quad \times \left(\mathrm{Op}^w(p_0 - z)\right)^{-1} \mathrm{Op}^w \left([s_\bullet^N]_0^\infty \# (p_0 - z_0)^{-\# m} \right) dz \Big\|_{\mathrm{Tr}} \\
\leq & \|f\|_\infty \mathcal{O}(h^{-n+N})
\end{aligned} \tag{4.5}
$$

We finally obtain form (4.4), (4.5), (4.3) and (4.1) :

$$
\begin{aligned}
\mathrm{Tr}[(\chi f)(P_\bullet, \lambda)]_0^\infty = & -\frac{1}{2i\pi} \mathrm{Tr} \Big(\int_{\mathcal{Z}} \tilde{f}(z,\lambda)\tilde{\chi}(z) \\
& \times \mathrm{Op}^w [r_\bullet^N \# (p_\bullet - z_0)^{-\# m}]_0^\infty dz \Big) \\
& + \mathcal{O}(h^{-n}) e^{-\mathrm{Min}\{C_0, C_{\tilde{\delta}}\}\lambda} + \|f\|_\infty \mathcal{O}(h^{N-n}).
\end{aligned}
$$

We still do not have a symbol with finite expansion in h, so we compare the symbol $[r_\bullet^N \# (p_\bullet - z_0)^{-\# m}]_0^\infty$ with the first terms of its expansion. Denote

$$
\begin{aligned}
(p_\bullet - z_0)^{-\# m}(x,\xi) =: & \sum_{k=0}^{N-1} h^k l_{k\bullet}(x,\xi) + h^N \mathrm{S}(\langle \xi \rangle^{-2m}, 1, 1) \\
=: & \, l_\bullet^N(x,\xi) + h^N \mathrm{S}(\langle \xi \rangle^{-2m}, 1, 1).
\end{aligned}
$$

Proposition A.1 proves that :

$$
\begin{aligned}
[r_\bullet^N(x,\xi,z) \# (p_\bullet - z_0)^{-\# m}(x,\xi)]_0^\infty = & [b_\bullet^N(x,\xi,z)]_0^\infty \\
& + h^N \mathrm{S}(\langle x \rangle^{-\tilde{n}} \langle \xi \rangle^{-m-1}, 1, 1),
\end{aligned}
$$

with

$$
\begin{aligned}
b_\bullet^N(x,\xi,z) = & \sum_{k=0}^{N-1} \frac{1}{k!} \big((\frac{ih}{2} \sigma(D_x, D_\xi; D_y, D_\eta))^k r_\bullet^N(x,\xi,z) l_\bullet^N(y,\eta) \big)|_{y=x,\eta=\xi} \\
=: & \sum_{j=0}^{N-1} h^j b_{j\bullet}(x,\xi,z),
\end{aligned}
$$

where

$$
\begin{aligned}
b_{j\bullet}(x,\xi,z) = & \sum_{k+p+m=j} \frac{i^k}{k!} 2^{-k} (\sigma(D_x, D_\xi; D_y, D_\eta))^k \\
& \big(r_{p\bullet}(x,\xi,z) l_{\bullet m}(y,\eta) \big)|_{y=x,\eta=\xi}.
\end{aligned}
$$

This gives Lemma 4.1. ∎

5. SIMPLIFICATION OF THE TRACE

In this section we compute the trace of the right hand side in Lemma 4.1. Using the residue theorem in z we prove Lemma 5.2 below. In this lemma the symbol of $(P_\bullet - z_0)^{-m}$ is used for technical reasons. Lemma 5.1 is the similar result with the symbol of $(P_\bullet - z_0)^{-m}(P_\bullet - z)^{-1}$ replaced by the symbol of $(P_\bullet - z)^{-1}$, and is the main result of this section. Denote by $d\mathcal{L} = r^{n-1}drL(d\hat{x})\rho^{n-1}d\rho L(d\hat{\xi})$ and $\mathcal{D} = \{\rho \in e^{-i\frac{\pi}{4}}\mathbb{R}, |\rho| \leq \rho_0, \hat{x}, \hat{\xi} \in \mathrm{S}^{n-1}\}$.

Lemma 5.1. *There exist ρ_0 and $\tilde{r}_0$ such that for all $r_1 \leq \tilde{r}_0$ there exists $\mu_1(r_1) > 0$ such that*

$$\begin{aligned}\mathrm{Tr}[(\chi f)(P_\bullet, \lambda)]_0^\infty = &-\frac{(2\pi h)^{-n}}{16i\pi} \int_{\mathcal{D}} \int_{r\in\partial D(0,r_1)} \sum_{j=1}^{n} \mathrm{sgn}(\partial_r \lambda_{j,0}(0)) \\ &\times \int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)} f(z,\lambda) \mathrm{Tr}\left(r_0^N(x,\xi,z)\right) dz d\mathcal{L} \\ &+ \mathcal{O}(h^{-n})e^{-C_2\lambda} + \|f\|_\infty \mathcal{O}(h^{N-n}).\end{aligned}$$

for all $\mu_1 \leq \mu_1(r_1)$, with $C_2 = \inf\{C_0, C_{\tilde{b}}, C_{\tilde{b}-\mu_0}\} > 0$.

Lemma 5.2. *There exist ρ_0 and $\tilde{r}_0$ such that for all $r_1 \leq \tilde{r}_0$ there exists $\mu_1(r_1) > 0$ such that*

$$\begin{aligned}\mathrm{Tr}[(\chi f)(P_\bullet, \lambda)]_0^\infty = &-\frac{(2\pi h)^{-n}}{16i\pi} \int_{\mathcal{D}} \int_{r\in\partial D(0,r_1)} \sum_{j=1}^{n} \mathrm{sgn}(\partial_r \lambda_{j,0}(0)) \\ &\int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)} \tilde{f}(z,\lambda) \mathrm{Tr}(b_0^N(x,\xi,z)) dz d\mathcal{L} \\ &+ \mathcal{O}(h^{-n})e^{-C_2\lambda} + \|f\|_\infty \mathcal{O}(h^{N-n}),\end{aligned}$$

for all $\mu_1 \leq \mu_1(r_1)$, with $C_2 = \inf\{C_0, C_{\tilde{b}}, C_{\tilde{b}-\mu_0}\} > 0$.

Proof : Denote

$$B_\bullet = -\frac{1}{2i\pi} \int_{\mathcal{Z}} \tilde{\chi}(z)\tilde{f}(z,\lambda)\mathrm{Op}^w(b_\bullet^N(x,\xi,z))dz.$$

From Lemma 4.1 and the standard expression ([12]) for the trace of a pseudo-differential operator, we get

$$\mathrm{Tr}[B_\bullet]_0^\infty = \frac{-1}{2i\pi(2\pi h)^n} \iiint_{x,\xi\in\mathbb{R}^n,\mathcal{Z}} \tilde{\chi}(z)\tilde{f}(z,\lambda)\mathrm{Tr}[b_\bullet^N(x,\xi,z)]_0^\infty \, dz\, dx\, d\xi.$$

Let $x = r\hat{x}$, $\xi = \hat{\xi}\rho$. For the first time we use the fact that n is odd:

$$\begin{aligned}\mathrm{Tr}[B_\bullet]_0^\infty = -\frac{(2\pi h)^{-n}}{8i\pi} &\int_{r\in\mathbb{R}} \iint_{\hat{x},\hat{\xi}\in\mathrm{S}^{n-1}} \int_{\rho\in\mathbb{R}} \\ &\int_{\mathcal{Z}} \tilde{\chi}(z)\tilde{f}(z,\lambda)\mathrm{Tr}[b_\bullet^N(x,\xi,z)]_0^\infty \, d\mathcal{L}\, dz.\end{aligned}$$

The function

$$(r,\rho)\mapsto \int_{\mathcal{Z}} \tilde{\chi}(z)\tilde{f}(z,\lambda)\mathrm{Tr}[b_\bullet^N(r\hat{x},\rho\hat{\xi},z)]_0^\infty r^{n-1}\rho^{n-1}\,dz$$

is analytic in every convex domain in $(r,\rho)\in\mathbb{C}^2$ where $\mathrm{Im}(\lambda_{j,0}(r\hat{x})+\rho^2)\neq\delta$ for all j and $r\in S_0$ (we use **(H3)**). Studying $b_{j,\bullet}$ we obtain uniformly in ι in the domain in x,ξ,z such that $\forall k,\ d(\lambda_{k,\bullet}(x)+\xi^2,z)\geq\iota$ and $\forall k,\ d(\lambda_{k,\bullet}(x)+\xi^2,z_0)\geq C\langle\xi\rangle^2$, $x\in \mathrm{S}$,

$$\left\|\partial_x^\alpha\partial_\xi^\beta[b_{j,\bullet}(x,\xi,z)]_0^\infty\right\|\leq C\,\mathrm{Min}\{\iota,\iota^{r(2j+\alpha+\beta+1)}\}^{-1}\langle x\rangle^{-\tilde{n}}\langle\xi\rangle^{-2m-2}. \tag{5.1}$$

There exists $r_0>0$ such that for all $s\in[0,r_0]$ we get

$$(\mathbf{C1})\ \forall j,r\in\mathbb{R},\ \ |\mathrm{Im}(\lambda_{j,\bullet}((r+is)\hat{x}))|\leq\frac{\delta}{2}.$$

Using the analyticity in r and ρ to change the integration path, and (5.1), we get

$$\mathrm{Tr}[B_\bullet]_0^\infty=-\frac{(2\pi h)^{-n}}{16i\pi}\int_{r\in\{\mathbb{R}+ir_0\}\cup\{\mathbb{R}-ir_0\}}\iint_{\hat{x},\hat{\xi}\in\mathrm{S}^{n-1}}\int_{\rho\in e^{-i\frac{\pi}{4}}\mathbb{R}}\int_{\mathcal{Z}}\tilde{\chi}(z)\tilde{f}(z,\lambda)\mathrm{Tr}[b_\bullet^N(x,\xi,z)]_0^\infty d\mathcal{L}dz$$

Denote $\mathrm{Tr}[B_\bullet]_0^\infty=:A_1+A_2$ with

$$A_1=-\frac{(2\pi h)^{-n}}{16i\pi}\int_{r\in\{\mathbb{R}+ir_0\}\cup\{\mathbb{R}-ir_0\}}\iint_{\hat{x},\hat{\xi}\in\mathrm{S}^{n-1}}\int_{\rho\in e^{-i\frac{\pi}{4}}\mathbb{R}}\int_{z\in\Omega;\mathrm{Im}(z)=\delta+\frac{1}{2}\mathrm{Im}(\rho^2)}\tilde{\chi}(z)\tilde{f}(z,\lambda)\mathrm{Tr}[b_\bullet^N(x,\xi,z)]_0^\infty d\mathcal{L}dz,$$

$$A_2(r)=-\frac{(2\pi h)^{-n}}{16i\pi}\int_{r\in\{\mathbb{R}+ir_0\}\cup\{\mathbb{R}-ir_0\}}\iint_{\hat{x},\hat{\xi}\in\mathrm{S}^{n-1}}\int_{\rho\in e^{-i\frac{\pi}{4}}\mathbb{R}}\int_{z\in\Omega;\delta+\frac{1}{2}\mathrm{Im}(\rho^2)\leq\mathrm{Im}(z)\leq\delta}\overline{\partial}_z\tilde{\chi}(z)\tilde{f}(z,\lambda)\mathrm{Tr}[b_\bullet^N(x,\xi,z)]_0^\infty d\mathcal{L}dz.$$

We obtain, using (5.1),

$$A_2=\mathcal{O}(h^{-n})e^{-\mathrm{Min}\{C_0,C_{\tilde{b}}\}\lambda}.$$

Let ρ_0 be a positive constant such that

$$(\mathbf{C2})\quad \tilde{b}\leq\frac{1}{2}\rho_0^2-\delta.$$

We cut A_1 in two

$$
\begin{aligned}
A_1 = & -\frac{(2\pi h)^{-n}}{16i\pi}\int_{r\in\{\mathbb{R}+ir_0\}\cup\{\mathbb{R}-ir_0\}}\int_{\mathcal{D}} \\
& \int_{z\in\Omega;\mathrm{Im}(z)=\delta+\frac{1}{2}\mathrm{Im}(\rho^2)} \tilde{\chi}(z)\tilde{f}(z,\lambda)\mathrm{Tr}[b_\bullet^N(x,\xi,z)]_0^\infty d\mathcal{L}dz \\
& -\frac{(2\pi h)^{-n}}{16i\pi}\int_{r\in\{\mathbb{R}+ir_0\}\cup\{\mathbb{R}-ir_0\}}\int_{\rho\in e^{-i\frac{\pi}{4}}\mathbb{R},|\rho|\geq\rho_0}\iint_{\hat{x},\hat{\xi}\in\mathrm{S}^{n-1}} \\
& \int_{z\in\Omega;\mathrm{Im}(z)=\delta+\frac{1}{2}\mathrm{Im}(\rho^2)} \tilde{\chi}(z)\tilde{f}(z,\lambda)\mathrm{Tr}[b_\bullet^N(x,\xi,z)]_0^\infty d\mathcal{L}dz \\
=: & A_{11}+A_{12}.
\end{aligned}
\tag{5.2}
$$

We observe, using **(C2)**, that

$$
A_{12}=\mathcal{O}(h^{-n})e^{-C_{\tilde{\delta}}\lambda}.
$$

Let $\tilde{r}_0\leq r_0$. Let γ be a path in z included in $\tilde{\chi}=0$ and $\{z\in\mathbb{C},\ \mathrm{Im}(z)\leq\delta\}$, starting in the region $\{z\in\mathbb{C},\ \mathrm{Im}(z)=\delta+\frac{1}{2}\mathrm{Im}(\rho^2),\mathrm{Re}(z)=-a\}$, and ending in the region $\{z\in\mathbb{C},\ \mathrm{Im}(z)=\delta+\frac{1}{2}\mathrm{Im}(\rho^2),\mathrm{Re}(z)=a\}$. Let $\mu_0>0$ be a constant small enough such that the following hypotheses are satisfied :

- **(C3)** $d(\gamma,\mathrm{Supp}\tilde{\chi})>2\mu_0$,
- **(C4)** $\mu_0\leq\frac{\delta}{2}$,
- **(C5)** $\inf\limits_{\mathbb{R}\pm ir_0,i\neq j} d(\lambda_{j,0}(x),\lambda_{i,0}(x))\geq 2\mu_0$,
- **(C6)** $\mathrm{Supp}(1_{\mathrm{Supp}\tilde{\chi}}-\tilde{\chi})+D(0,\mu_0)\in\Omega+E_0$,
- **(C7)** $\mathrm{Supp}\tilde{\chi}+D(0,2\mu_0)\in\Omega+E_0$,
- **(C8)** $D(\lambda_{j\infty},\mu_0)+i\mathbb{R}\notin\mathrm{Supp}\tilde{\chi}$,
  and
- **(C9)** $\inf_{r\notin D(0,\tilde{r}_0)} d(\lambda_{j,0}(r\hat{x}),\lambda_{i,0}(r\hat{x}))\geq 2\mu_0$.

Mostly these conditions are necessary to define the future integration that we will use. The main point we use is that the discs $\partial D(\lambda_{0,j}(x),\mu_0)$ for $|\mathrm{Im}(x)|=r_0$ are disjoint (**(C5)**) and included in $\{z,\mathrm{Im}(z)\leq\delta\}$ (**(C1)**, **(C4)**). Changing the integration path $\mathrm{Im}(z)=\delta+\frac{1}{2}\mathrm{Im}(\rho^2)$

into the path γ in the formula (5.2), using **(C7)** and **(C3)**, gives :

$$\begin{aligned}\mathrm{Tr}[B_\bullet]_0^\infty = &-\frac{(2\pi h)^{-n}}{16i\pi}\int_{r\in\{\mathbb{R}\pm ir_0\}}\int_{\mathcal{D}} d\mathcal{L}\\ &[\int_{\substack{\mathrm{Im}(z)\leq\delta+\frac{1}{2}\mathrm{Im}(\rho^2)\\ z\notin\cup_j D(\lambda_{j,\bullet}(x)+\xi^2,\mu_0)}}\overline{\partial}_z\tilde{\chi}(z)\tilde{f}(z,\lambda)\mathrm{Tr}(b_\bullet^N(x,\xi,z))L(dz)]_0^\infty\\ &-\frac{(2\pi h)^{-n}}{16i\pi}\int_{r\in\{\mathbb{R}\pm ir_0\}}\int_{\mathcal{D}} d\mathcal{L}\\ &[\int_{z\in(\cup_j\partial D(\lambda_{j,\bullet}(x)+\xi^2,\mu_0))}\tilde{\chi}(z)\tilde{f}(z,\lambda)\mathrm{Tr}(b_\bullet^N(x,\xi,z))dz]_0^\infty\\ &+\mathcal{O}(h^{-n})e^{-C_2\lambda}.\end{aligned}$$

We get from (5.1) that

$$\begin{aligned}&-\frac{(2\pi h)^{-n}}{16i\pi}\int_{r\in\{\mathbb{R}\pm ir_0\}}\int_{\mathcal{D}} d\mathcal{L}\\ &\quad[\int_{\substack{\mathrm{Im}(z)\leq\delta+\frac{1}{2}\mathrm{Im}(\rho^2)\\ z\notin\cup_j D(\lambda_{j,\bullet}(x)+\xi^2,\mu_0)}}\overline{\partial}_z\tilde{\chi}(z)\tilde{f}(z,\lambda)\mathrm{Tr}(b_\bullet^N(x,\xi,z))]_0^\infty L(dz)\\ &\quad=\mathcal{O}(h^{-n})e^{-C_2\lambda}.\end{aligned}$$

The condition **(C9)** implies that

$$\begin{aligned}\mathrm{Tr}[B_\bullet]_0^\infty = &\frac{(2\pi h)^{-n}}{16i\pi}\iint_{\substack{r\in\{\mathbb{R}\pm ir_0\}\\ z\in(\cup_j\partial D(\lambda_{j,0}(x)+\xi^2,\mu_0))}}\int_{\mathcal{D}} d\mathcal{L}\\ &\tilde{\chi}(z)\tilde{f}(z,\lambda)\mathrm{Tr}(b_0^N(x,\xi,z))dz+\mathcal{O}(h^{-n})e^{-C_2\lambda}.\end{aligned}$$

Since $\mathrm{Supp}(1_{\mathrm{Supp}\tilde{\chi}}-\tilde{\chi})+D(0,\mu_0)$ is included in a region where f is defined **(C6)**, and included in a region where f is bounded by $e^{-C_2\lambda}$, we get from (5) and **(C5)** that

$$\begin{aligned}\mathrm{Tr}[B_\bullet]_0^\infty = &\frac{(2\pi h)^{-n}}{16i\pi}\int_{\mathcal{D}}\sum_{j=1}^n\iint_{\substack{r\in\{\mathbb{R}\pm ir_0\}\\ z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_0)}} 1_{\mathrm{Supp}\tilde{\chi}}(\lambda_{j,0}(x)+\xi^2)\\ &\tilde{\chi}(z)\tilde{f}(z,\lambda)\mathrm{Tr}(b_0^N(x,\xi,z))dzd\mathcal{L}+\mathcal{O}(h^{-n})e^{-C_2\lambda}\\ = &\frac{(2\pi h)^{-n}}{16i\pi}\sum_{j=1}^n\iint_{\substack{r\in\{\mathbb{R}\pm ir_0\}\\ z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_0)}}\int_{\mathcal{D}}\tilde{\chi}(\lambda_{j,0}(r\hat{x})+\xi^2)\\ &\tilde{f}(z,\lambda)\mathrm{Tr}(b_0^N(x,\xi,z))dzd\mathcal{L}+\mathcal{O}(h^{-n})e^{-C_2\lambda}.\end{aligned}$$

Let, for $\rho\in e^{-i\frac{\pi}{4}}\mathbb{R}$,

$$g(r)=\sum_{j=1}^n\tilde{\chi}(\lambda_{j,0}(r\hat{x})+\xi^2)\int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_0)}\tilde{f}(z,\lambda)\mathrm{Tr}(b_0^N(x,\xi,z))dz.$$

We get $\lim_{|r|\to\infty, r\in S}\lambda_{j,0}(r\hat{x}) = \lambda_{j\infty}$ and **(C8)** so g is of compact support. **(H3)** implies that there exist $\tilde{r}_1$ and C_{V_0} (dependent of V_0) such that for all $r_1 < \tilde{r}_1$, $r \in S_0$ and all j satisfying $\partial_r\lambda_j(0) > 0$ we have $\mathrm{Im}(\lambda_{j,0}((r - ir_1)\hat{x})) \leq -C_{V_0}r_1$ (resp. for all j satisfying $\partial_r\lambda_{j,0}(0) < 0$ we have $\mathrm{Im}(\lambda_{j,0}((r + ir_1)\hat{x})) \leq -C_{V_0}r_1$.) Let us add the last condition on μ_0 and $\hat{b}$:

$$(\mathbf{C10}) \quad C_{V_0}\tilde{r}_1 \geq \tilde{b}, C_{V_0}\tilde{r}_1 - \mu_0 \geq \tilde{b}.$$

There exist paths $r \mapsto \gamma_j(r)$ such that $\gamma_j(r) = r$ for large r, $\gamma_j(r) = r \pm i\tilde{r}_1$ for r near 0 and

$$\mathrm{Im}(\lambda_{j,0}(\gamma_j(r)\hat{x}) \leq -\tilde{b} - \mu_0$$

for r such that $\lambda_{j,0}(r\hat{x}) + \xi^2 \in \mathrm{Supp}\tilde{\chi}$. Changing one of the paths $\mathbb{R} \pm ir_0$ into γ_j, using **(C9)**, gives :

$$\begin{aligned}
\mathrm{Tr}[B_\bullet]_0^\infty = & \frac{(2\pi h)^{-n}}{8i\pi}\sum_{j=1}^{n}\int_{r\in\gamma_j}\int_{\mathcal{D}}\tilde{\chi}(\lambda_{j,0}(r\hat{x}) + \rho^2) \\
& \int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_0)}\tilde{f}(z,\lambda)\mathrm{Tr}(b_0^N(x,\xi,z))dzd\mathcal{L} \\
& - \frac{(2\pi h)^{-n}}{16i\pi}\sum_{j=1}^{n}\mathrm{sgn}(\partial_r\lambda_{j,0}(0)) \\
& \times \int_{\mathcal{D}}\iint_{\substack{r\in\partial D(0,\tilde{r}_0)\\ z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_0)}}\tilde{\chi}(\lambda_{j,0}(r\hat{x}) + \rho^2) \\
& \times \int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_0)}\tilde{f}(z,\lambda)\mathrm{Tr}(b_0^N(x,\xi,z))dzd\mathcal{L} + \mathcal{O}(h^{-n})e^{-C_2\lambda}.
\end{aligned}$$

Assumption **(H4)** gives

$$\begin{aligned}
\mathrm{Tr}[B_\bullet]_0^\infty = & -\frac{(2\pi h)^{-n}}{16i\pi}\sum_{j=1}^{n}\mathrm{sgn}(\partial_r\lambda_{j,0}(0)) \\
& \int_{\mathcal{D}}\int_{r\in\partial D(0,\tilde{r}_0)}\int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_0)}\tilde{f}(z,\lambda)\mathrm{Tr}(b_0^N(x,\xi,z))dzd\mathcal{L} \\
& + \mathcal{O}(h^{-n})e^{-C_2\lambda}.
\end{aligned}$$

The function $z \mapsto \tilde{f}(z,\lambda)\mathrm{Tr}(b_0^N(x,\xi,z))$ is holomorphic in z in a neighborhood of $\lambda_{j,0}(x) + \xi^2$ except at the points $\lambda_{j,0}(x) + \xi^2$. Therefore for all $\mu_1 \leq \mu_0$

$$\begin{aligned}
& \int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_0)}\tilde{f}(z,\lambda)\mathrm{Tr}(b_0^N(x,\xi,z))dz \\
& \qquad = \int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)}\tilde{f}(z,\lambda)\mathrm{Tr}(b_0^N(x,\xi,z))dz.
\end{aligned}$$

Due to analyticity in r, we get for all $r_1 \leq \tilde{r}_0$ and

$$\mu_1 \leq \frac{1}{2} \inf_{r \in \partial D(0,r_1), i \neq j} d(\lambda_{j,0}(x), \lambda_{i,0}(x))$$

$$\begin{aligned}\mathrm{Tr}[B_\bullet]_0^\infty = & -\frac{(2\pi h)^{-n}}{16 i\pi} \sum_{j=1}^{n} \mathrm{sgn}(\partial_r \lambda_{j,0}(0)) \\ & \int_{\mathcal{D}} \iint_{\substack{r \in \partial D(0,r_1) \\ z \in \partial D(\lambda_{j,0}(x)+\xi^2, \mu_1)}} \tilde{f}(z,\lambda) \mathrm{Tr}(b_0^N(x,\xi,z)) dz d\mathcal{L} \\ & + \mathcal{O}(h^{-n}) e^{-C_2 \lambda}.\end{aligned}$$

This gives Lemma 5.2. ■

Proof of Lemma 5.1 : An iteration will prove

(5.3)

$$\begin{aligned}\int_{z \in \partial D(\lambda_{j,0}(x)+\xi^2, \mu_1)} & f(z,\lambda) \mathrm{Tr}(r_0^N(x,\xi,z))\, dz \\ & = \int_{z \in \partial D(\lambda_{j,0}(x)+\xi^2, \mu_1)} \tilde{f}(z,\lambda) \mathrm{Tr}(b_0^N(x,\xi,z))\, dz.\end{aligned}$$

Let $c^{N,m}(x,\xi,z)$ be the sum of the $N-1$ first terms of the expansion in h of the symbol of $(P_0 - z)^{-1}(P_0 - z_0)^{-m}$. Let $d^{N,m}(x,\xi)$ be the sum of the $N-1$ first terms of the expansion in h of the symbol of $(P_0 - z_0)^{-m}$. We get by the Weyl calculus : $c^{N,m} \# (p_0 - z_0) = c^{N,m-1}$, $c^{N,0} = r_0^N$ and $c^{N,m} = b_0^N$. Let $c^{N,m} = \sum_{j=0}^{j=N-1} h^j c_j^{N,m}$. The functions $c_j^{N,m}$ satisfy

$$\begin{aligned}c_j^{N,m-1}(x,\xi,z) = & \sum_{k+p=j} \frac{i^k}{k! 2^k} (\sigma(D_x, D_\xi; D_y, D_\eta))^k c_p^{N,m}(x,\xi,z) \\ & \times (p_0 - z_0)(y,\eta)|_{y=x,\eta=\xi} \\ = \sum_{k+p=j, k>0} & \frac{i^k}{k! 2^k} (\sigma(D_x, D_\xi; D_y, D_\eta))^k c_p^{N,m}(x,\xi,z)(p_0 - z_0)(y,\eta)|_{y=x,\eta=\xi} \\ & + c_j^{N,m}(x,\xi,z)(p_0 - z_0)(x,\xi).\end{aligned}$$

For $k > 0$ we have

$$\begin{aligned}\sigma(D_x, D_\xi; D_y, D_\eta)^k c_p^{N,m}(x,\xi,z)(p_0 - z_0)(y,\eta)|_{y=x,\eta=\xi} = \\ \sigma(D_x, D_\xi; D_y, D_\eta)^k c_p^{N,m}(x,\xi,z)(p_0 - z)(y,\eta)|_{y=x,\eta=\xi}.\end{aligned}$$

Therefore, the difference

$$c_j^{N,m-1}(x,\xi,z) - c_j^{N,m}(x,\xi,z)(z - z_0)$$

is a function independent of z. This proves that

$$\int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)} (z-z_0)^{m-1} f(z,\lambda)\mathrm{Tr}(c_j^{N,m-1}(x,\xi,z))\, dz r^{n-1}\, dr\, L(d\hat{x})$$
$$= \int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)} (z-z_0)^{m} f(z,\lambda)\mathrm{Tr}(c_j^{N,m}(x,\xi,z))\, dz r^{n-1}\, dr\, L(d\hat{x}).$$

The iteration proves the relation (5.3). ■

6. CANCELLATIONS

We compute the first terms of the expansion in powers of h of $\mathrm{Tr}[\chi f(P_\bullet,\lambda)]_0^\infty$ up to errors of order $\mathcal{O}(h^{-n})e^{-C_2\lambda}$.

Define the numbers T_s by the equation

$$\mathrm{Tr}[(\chi f)(P_\bullet,\lambda)]_0^\infty = \sum_{s\leq N} h^{-n+s} T_s + \mathcal{O}(h^{-n})e^{-C_2\lambda} + \mathcal{O}(h^{N+1-s})\|f\|_\infty.$$

We prove that $T_0 = 0$:

$$T_0 = -\frac{(2\pi)^{-n}}{16 i\pi}\sum_{j=1}^{n}\mathrm{sgn}(\partial_r\lambda_{j,0}(0))\int_{r\in\partial D(0,r_1)}\int_{\mathcal{D}}$$
$$\int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)} f(z,\lambda)\mathrm{Tr}(r_{0,0}(x,\xi,z))\, dz\, d\mathcal{L}.$$

Since $r_{0,\bullet}(x,\xi,z) = (\xi^2+V_\bullet(x)-z)^{-1}$ we have that

$$\frac{1}{2i\pi}\int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)} f(z,\lambda)\mathrm{Tr}(r_{0,0}(x,\xi,z))\, dz$$
$$=\frac{1}{2i\pi}\int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)} f(z,\lambda)\sum_{k=1}^{r}(\xi^2+\lambda_{k,0}(x)-z)^{-1}\, dz$$
$$=-\, f(\xi^2+\lambda_{j,0}(x),\lambda).$$

Then

$$T_0 =\frac{1}{8(2\pi)^{-n}}\sum_{j,\partial_r\lambda_{j,0}(0)>0}\int_{\mathcal{D}}\int_{r\in\partial D(0,r_1)} f(\xi^2+\lambda_{j,0}(x),\lambda)\, d\mathcal{L}$$
$$-\frac{1}{8}(2\pi)^{-n}\sum_{j,\partial_r\lambda_{j,0}(0)<0}\int_{\mathcal{D}}\int_{r\in\partial D(0,r_1)} f(\xi^2+\lambda_{j,0}(x),\lambda)\, d\mathcal{L}.$$

The function $r\mapsto f(\xi^2+\lambda_{j,0}(r\hat{x}),\lambda)$ is analytic so, $T_0 = 0$. ■

Next, we prove $T_1 = 0$. A computation gives that

$$r_{1,0} = \sum_{i=1}^{n}\xi_i(r_{0,0}^2(D_{x_i}V)r_{0,0} - r_{0,0}(D_{x_i}V)r_{0,0}^2), \tag{6.1}$$

so $\mathrm{Tr}\,(r_{1,0}) = 0$. ■

Let us compute the term T_2 :

$$T_2 = -\frac{(2\pi)^{-n}}{16i\pi}\sum_{j=1}^{n}\operatorname{sgn}(\partial_r\lambda_{j,0}(0))\int_{\mathcal{D}}\int_{r\in\partial D(0,r_1)}$$
$$\int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)} f(z,\lambda)\operatorname{Tr}(r_{2,0}(x,\xi,z))\,dz\,d\mathcal{L}.$$

We get :

$$\begin{aligned} r_{2,0}(x,\xi,z) = &- r_{0,0}(x,\xi,z)\times\frac{1}{8}\Big(\sum_i 2(D^2_{x_i}r_{0,0})(x,\xi,z)\\ &+\sum_{i,j}(D^2_{x_i,x_j}V)(x)\partial_{\xi_i}\partial_{\xi_j}r_{0,0}(x,\xi,z)\Big)\\ &- r_{0,0}(x,\xi,z)\frac{1}{2}\Big(\sum_i 2\xi_i(D_{x_i}r_{1,0})(x,\xi,z)\\ &-(D_{x_i}V_0)\partial_{\xi_i}r_{1,0}(x,\xi,z)\Big). \end{aligned}\tag{6.2}$$

Let

$$\mathcal{A}_1 = -r_{0,0}\frac{1}{8}\left[\sum_i 2(D^2_{x_i}r_{0,0})+\sum_{i,j}(D^2_{x_i,x_j}V_0)\partial_{\xi_i}\partial_{\xi_j}r_{0,0}\right](x,\xi,z).$$

Let $\mathcal{A}_2 = r_{2,0}(x,\xi,z)-A_1$. Then

$$\begin{aligned}\mathcal{A}_1 =&\frac{1}{4}\sum_i r^2_{0,0}(x,\xi,z)(D^2_{x_i}V_0)r_{0,0}(x,\xi,z)\\ &-\frac{1}{2}\sum_i r^2_{0,0}(x,\xi,z)(D_{x_i}V_0)r_{0,0}(x,\xi,z)(D_{x_i}V_0)r_{0,0}(x,\xi,z)\\ &-\sum_{i,j}\xi_i\xi_j r_{0,0}(x,\xi,z)(D^2_{x_i,x_j}V_0)r^3_{0,0}(x,\xi,z)\\ &+\frac{1}{4}\sum_i r_{0,0}(x,\xi,z)(D^2_{x_i}V_0)r^2_{0,0}(x,\xi,z).\end{aligned}$$

So

$$\begin{aligned}\operatorname{Tr}\mathcal{A}_\infty =&\operatorname{Tr}\left(\frac{1}{2}\sum_i r^3_{0,0}(x,\xi,z)(D^2_{x_i}V_0)\right)\\ &-\operatorname{Tr}\left(\frac{1}{2}\sum_i r^3_{0,0}(x,\xi,z)(D_{x_i}V_0)r_{0,0}(x,\xi,z)D_{x_i}V_0\right)\\ &-\operatorname{Tr}\left(\sum_{i,j}\xi_i\xi_j(D^2_{x_i,x_j}V_0)r^4_{0,0}(x,\xi,z)\right).\end{aligned}$$

Now we need the following remark:

Remark 3. *If $D(x,\xi)$ is a matrix holomorphic in r, then there exist $d_j(x,\xi)$, holomorphic in $r \in S_0 \setminus \{0\}$, such that for all p :*

$$\operatorname{Tr}\big(D(x,\xi) r_{0,0}^p(x,\xi,z)\big) = \sum_j d_j(x,\xi)(\xi^2 + \lambda_{j,0}(x) - z)^{-p},$$

and the function

$$r \mapsto \frac{1}{2i\pi} \int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)} f(z,\lambda) \operatorname{Tr}\big(D(x,\xi) r_{0,0}^p(x,\xi,z)\big) dz$$

is holomorphic, for all j.

This remark proves that

$$\begin{aligned}\int_{r\in\partial D(0,r_1)} \int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)} & f(z,\lambda)\operatorname{Tr}\big(A_1(x,\xi,z)\big) dz r^{n-1} dr \\ &= \iint_{\substack{r\in\partial D(0,r_1)\\ z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)}} f(z,\lambda) \\ \times \operatorname{Tr}\Big(-\frac{1}{2}\sum_i r_{0,0}^3(x,\xi,z)&(D_{x_i}V_0) r_{0,0}(x,\xi,z)(D_{x_i}V_0)\Big) r^{n-1} dr dz.\end{aligned}$$

Next, let us compute the next term in (6.2):

(6.3)
$$\mathcal{A}_2 = -\frac{1}{4} r_{0,0}(x,\xi,z) \sum_i \big(2\xi_i(D_{x_i} r_{1,0}) - (D_{x_i}V_0)(\partial_{\xi_i} r_{1,0})\big)(x,\xi,z).$$

Recall that $r_{1,0}$ is given in (6.1). Therefore :

$$\begin{aligned} D_{x_i} r_{1,0}(x,\xi,z) = \sum_{j=1}^n \xi_j \Big(& - r_{0,0}(D_{x_i}V_0) r_{0,0}^2 (D_{x_j}V_0) r_{0,0} \\ & - r_{0,0}^2(D_{x_i}V_0) r_{0,0}(D_{x_j}V_0) r_{0,0} - r_{0,0}^2 (D_{x_j}V_0) r_{0,0}(D_{x_i}V_0) r_{0,0} \\ & + r_{0,0}^2 (D^2_{x_j,x_i}V_0) r_{0,0} + r_{0,0}(D_{x_i}V_0) r_{0,0}(D_{x_j}V_0) r_{0,0}^2 \\ & - r_{0,0}(D^2_{x_j,x_i}V_0) r_{0,0}^2 + r_{0,0}(D_{x_j}V_0) r_{0,0}(D_{x_i}V_0) r_{0,0}^2 \\ & + r_{0,0}(D_{x_j}V_0) r_{0,0}^2 (D_{x_i}V_0) r_{0,0}\Big), \end{aligned}$$

and

$$\begin{aligned} \partial_{\xi_i} r_{1,0}(x,\xi,z) = &\big(r_{0,0}^2 (D_{x_i}V_0) r_{0,0} - r_{0,0}(D_{x_i}V_0) r_{0,0}^2\big) \\ & + \sum_{j=1}^n 2\xi_j\xi_i \left(-2r_{0,0}^3 (D_{x_j}V_0) r_{0,0} + 2 r_{0,0}(D_{x_j}V_0) r_{0,0}^3\right). \end{aligned}$$

Plugging this into (6.3), we get

$$\mathrm{Tr}\mathcal{A}_2 = \frac{1}{2}\sum_{i=1}^{n} \mathrm{Tr}\Big(r_{0,0}^2(D_{x_i}V_0)r_{0,0}^2(D_{x_i}V_0) - r_{0,0}^3(D_{x_i}V_0)r_{0,0}(D_{x_i}V_0)\Big)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n} 2\xi_j\xi_i \mathrm{Tr}\Big(- r_{0,0}^2(D_{x_i}V_0)r_{0,0}^3(D_{x_j}V_0) + r_{0,0}^4(D_{x_i}V_0)r_{0,0}(D_{x_j}V_0)\Big).$$

So we obtain

$$\iint_{\substack{r\in\partial D(0,r_1)\\ z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)}} f(z,\lambda)\mathrm{Tr}\Big(r_{2,0}(x,\xi,z)\Big)r^{n-1}dr dz$$

$$= \iint_{\substack{r\in\partial D(0,r_1)\\ z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)}} \sum_i \mathrm{Tr}\Big(\frac{1}{2}r_{0,0}^2(x,\xi,z)(D_{x_i}V_0)r_{0,0}^2(x,\xi,z)(D_{x_i}V_0)$$

$$- r_{0,0}^3(D_{x_i}V_0)r_{0,0}(D_{x_i}V_0)\Big)f(z,\lambda)r^{n-1}dr dz$$

$$+ \iint_{\substack{r\in\partial D(0,r_1)\\ z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)}} \sum_{i=1}^{n}\sum_{j=1}^{n} 2\xi_j\xi_i\mathrm{Tr}\Big(- r_{0,0}^2(D_{x_i}V_0)r_{0,0}^3(D_{x_j}V_0)$$

$$+ r_{0,0}^4(D_{x_i}V_0)r_{0,0}(D_{x_j}V_0)\Big)f(z,\lambda)r^{n-1}dr dz.$$

We get by symmetry, for $i \neq j$,

$$\int_{\hat{\xi}}\int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)} f(z,\lambda)L(d\hat{\xi})dz 2\xi_j\xi_i$$

$$\mathrm{Tr}\Big(- r_{0,0}^2(D_{x_i}V_0)r_{0,0}^3(D_{x_j}V_0) + r_{0,0}^4(D_{x_i}V_0)r_{0,0}(D_{x_j}V_0)\Big) = 0.$$

We then get the expression of T_2. Next, we prove, by induction, that for all j :

$$r_{j+1,\bullet}(x,-\xi,z) = (-1)^{j+1}r_{j+1,\bullet}(x,\xi,z).$$

The relation is true for $j=0$: $r_{0,\bullet}(x,\xi,z) = (p_\bullet(x,\xi) - z\mathrm{I}_r)^{-1}$. Then the iteration is obtained by the relation (A.3).

For s odd,

$$T_s = -\frac{(2\pi)^{-n}}{16i\pi}\int_{\mathcal{D}}\int_{r\in\partial D(0,r_1)}\sum_{j=1}^{n}\mathrm{sgn}(\partial_r\lambda_{j,0}(0))$$

$$\int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)} f(z,\lambda)\mathrm{Tr}(r_{s,0}(x,\xi,z))\,dz\,d\mathcal{L}$$

$$= -\frac{(2\pi)^{-n}}{16i\pi}\int_{\mathcal{D}}\int_{r\in\partial D(0,r_1)}\sum_{j=1}^{n}\mathrm{sgn}(\partial_r\lambda_{j,0}(0))$$

$$\int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)} f(z,\lambda)\mathrm{Tr}(r_{s,0}(x,-\xi,z))\,dz\,d\mathcal{L}$$

$$= -T_s$$

So $T_s = 0$ for s odd. This finishes the proof of Theorem 2.1.

7. Resonances due to the matrix potential

In this section we prove that the potential V_0 generate resonances, using techniques similar to the ones used in [14].
Proof of Theorem 2.2 : We improve the computation obtained above in Theorem 2.1 in the case $r = 2$ and $n = 3$. This is done using the residue theorem in r.

Remark 4. *Let $B_i(x,\xi)$ and $B_j(x,\xi)$ be two matrices independent of z. Let $\alpha > 0$ and $\beta > 0$. For fixed x, and for P such that $P^{-1}V_0P$ is diagonal, denote*

$$P^{-1}B_iP = \begin{pmatrix} b_{1,i} & b_{2,i} \\ b_{3,i} & b_{4,i} \end{pmatrix},$$

$$\begin{aligned} \mathrm{Tr}\ r_{0,0}^{\alpha}B_ir_{0,0}^{\beta}B_j =& b_{1,i}b_{1,j}(\xi^2+\lambda_{1,0}-z)^{-\alpha-\beta} \\ &+ b_{2,i}b_{3,j}(\xi^2+\lambda_{1,0}-z)^{-\alpha}(\xi^2+\lambda_{2,0}-z)^{-\beta} \\ &+ b_{3,i}b_{2,j}(\xi^2+\lambda_{2,0}-z)^{-\alpha}(\xi^2+\lambda_{1,0}-z)^{-\beta} \\ &+ b_{4,i}b_{4,j}(\xi^2+\lambda_{2,0}-z)^{-\alpha-\beta}. \end{aligned}$$

Then, for $\alpha \neq 1$ and $\beta \neq 1$,

$$\int_{z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)} f(z,\lambda)\mathrm{Tr}(r_{0,0}^{\alpha}B_ir_{0,0}^{\beta}B_j)dz = 0.$$

This remark proves :

$$\begin{aligned} &\iint_{\substack{r\in\partial D(0,r_1)\\ z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)}} f(z,\lambda)\mathrm{Tr}(r_{2,0}(x,\xi,z))\ r^2drdz = \\ &-\iint_{\substack{r\in\partial D(0,r_1)\\ z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)}} f(z,\lambda)\sum_{j=1}^{n}\mathrm{Tr}(r_{0,0}^{3}(D_{x_j}V_0)r_{0,0}(D_{x_j}V_0))\ r^2drdz \\ &+\iint_{\substack{r\in\partial D(0,r_1)\\ z\in\partial D(\lambda_{j,0}(x)+\xi^2,\mu_1)}} f(z,\lambda)\sum_{j=1}^{n}2\xi_j^2\mathrm{Tr}(r_{0,0}^{4}(D_{x_j}V_0)r_{0,0}(D_{x_j}V_0))\ r^2drdz. \end{aligned}$$

Define

$$P^{-1}(D_{x_i}V_0)P = \begin{pmatrix} a_i^0 & b_i^0 \\ c_i^0 & d_i^0 \end{pmatrix}.$$

Denote $d\mathcal{L}^3 = r^2 dr L(d\hat{x})\rho^2 d\rho L(d\hat{\xi})$, $\mathcal{D}^3 = \{x, \xi \in \mathrm{S}^2, \rho \in e^{-i\frac{\pi}{4}}\mathbb{R}, |\rho| \leq \rho_0\}$. Suppose $\partial_r \lambda_{1,0}(0) > 0$, then $\partial_r \lambda_{2,0}(0) < 0$ and so

$$\begin{aligned}\langle \nu, f\rangle =& \frac{1}{4^3\pi^3}\int_{\mathcal{D}^3}\int_{r\in\partial D(0,r_1)} \{(\sum_i d_i^0 b_i^0)(x) \\ & (f(\lambda_{2,0}(x)+\xi^2,\lambda) + f(\lambda_{1,0}(x)+\xi^2,\lambda))(\lambda_{1,0}-\lambda_{2,0})^{-3}\} d\mathcal{L}^3 \\ & + \frac{1}{4^3\pi^3}\int_{\mathcal{D}^3}\int_{r\in\partial D(0,r_1)} (f(\lambda_{1,0}(x)+\xi^2,\lambda) - f(\lambda_{2,0}(x)+\xi^2,\lambda)) \\ & \sum_{j=1}^n 2\xi_j^2 (d_i^0 b_i^0)(x)\big[(\lambda_{1,0}-\lambda_{2,0})(x)\big]^{-4} d\mathcal{L}^3.\end{aligned}$$

Then the residue theorem gives

$$\begin{aligned}\langle \nu, f\rangle &= \frac{i}{4\pi^2}\int_{\mathcal{D}^3} \{\partial_r(\lambda_{1,0}-\lambda_{2,0})(0,\hat{x}))\}^{-3} L(d\hat{x})L(d\hat{\xi})\rho^2 d\rho \\ \times & \Big[(\sum_i d_i^0 b_i^0)(0,\hat{x}) f(\rho^2+E_0,\lambda) + \sum_i \big((d_i^0 b_i^0)(0,\hat{x})\xi_i^2\big) f'(\rho^2+E_0,\lambda)\Big].\end{aligned}$$

A deformation of the contour in ρ gives

$$\begin{aligned}\langle \nu, f\rangle = \frac{i}{4\pi^2}\iint_{\hat{x},\hat{\xi}\in\mathrm{S}^2}\int_{\rho\in\mathbb{R},|\rho|\leq\sqrt{a}} &\{\partial_r(\lambda_{1,0}-\lambda_{2,0})(0,\hat{x}))\}^{-3} L(d\hat{x})L(d\hat{\xi}) \\ & \Big[(\sum_i d_i^0 b_i^0)(0,\hat{x}) f(\rho^2+E_0,\lambda) \\ & + \big(\sum_i (d_i^0 b_i^0)(0,\hat{x})\xi_i^2\big) f'(\rho^2+E_0,\lambda)\Big]\rho^2 d\rho.\end{aligned}$$

We get, for all $j \in \{1, n\}$, that

$$\int_{\hat{\xi}\in\mathrm{S}^2} \xi_j^2 \, L(d\hat{\xi}) = \frac{\mathrm{Vol}(\mathrm{S}^2)}{3}\rho^2,$$

$$\begin{aligned}\int_{\rho\in\mathbb{R},|\rho|\leq\sqrt{a}} f'(\rho^2+E_0,\lambda)\rho^4 d\rho = & -\frac{3}{2}\int_{\rho\in\mathbb{R},|\rho|\leq\sqrt{a}} f(\rho^2+E_0,\lambda)\rho^2 \, d\rho \\ & + \mathcal{O}(e^{-C_2\lambda}).\end{aligned}$$

Up to a remainder, included in (2.2), we get

$$\begin{aligned}\langle \nu, f\rangle =& \frac{i}{2(4\pi)^2}\mathrm{Vol}(\mathrm{S}^2)\int_{\hat{x}\in\mathrm{S}^2}\int_{\rho\in\mathbb{R},|\rho|\leq\sqrt{a}} f(\rho^2+E_0,\lambda) \\ & \times \Big[(\sum_i d_i^0 b_i^0)(0,\hat{x})\{\partial_r(\lambda_{1,0}-\lambda_{2,0})(0,\hat{x})\}^{-3}\Big] L(d\hat{x})\rho^2 d\rho,\end{aligned}$$

with

$$C(V_0) = \int_{\hat{x}\in\mathrm{S}^2} \big(\sum_i d_i^0 b_i^0\big)(0,\hat{x})\partial_r(\lambda_{1,0}-\lambda_{2,0})(0,\hat{x})^{-3} L(d\hat{x}).$$

Denote by $a_1, a_2, a_3 \in \mathbb{R}^3$ the vectors such that the first order Taylor expansion of $V_0(x) - b(x)I_3$ is

$$\tilde{V}_0(x) = \begin{pmatrix} a_1\dot{x} & a_2\dot{x} + ia_3\dot{x} \\ a_2\dot{x} - ia_3\dot{x} & -a_1\dot{x} \end{pmatrix}$$

We get $C(V_0) = C(\tilde{V}_0)$. Using the change of coordinates

$$y_1 = a_1\dot{x},\ y_2 = a_2\dot{x},\ y_3 = a_3\dot{x},\ y = s\hat{y},\ \hat{y} \in \mathrm{S}^2,$$

we obtain :

$$\begin{aligned} C(\tilde{V}_0) =& \det(a_1, a_2, a_3)^{-1} \\ &\times \int_{\hat{y}\in \mathrm{S}^2} (\sum_i d_i^0 b_i^0)(0,\hat{y})(\partial_r(\lambda_{1,0} - \lambda_{2,0})(0,\hat{y}))^{-3} L(d\hat{y}). \end{aligned}$$

A computation gives $\{\partial_s(\lambda_{1,0} - \lambda_{2,0})(0,\hat{y})\}^{-3} = \frac{1}{8}$, and

$$\int_{\hat{y}\in \mathrm{S}^2} (\sum_i d_i^0 b_i^0)(0,\hat{y})\, L(d\hat{y}) = \frac{8\pi}{3}\sum_{j=1}^{3} |a_j|^2.$$

So

$$C(V_0) = \frac{\pi}{3}(\det(a_1, a_2, a_3))^{-1}\sum_{j=1}^{3} |a_j|^2 > 0$$

This implies Theorem 2.2.

Proof of Theorem 2.3 : We compute a Fourier-Bros-Iagolnitzer transform of ν (see [13] for the definition).

Let H be the Heaviside function. Let f be the function $f(u,\lambda) = e^{-i\lambda(u-E_0)}e^{-\lambda(u-E_0)^2}$ and let $g(u) = f(u + E_0, \lambda)$. Denote by $\mathcal{F}$ the Fourier transform. Then there exists $a \in \mathbb{R}$ such that :

$$\begin{aligned} \langle \nu, f\rangle =& C\int_{-a^{\frac14}}^{a^{\frac14}} g(u)\sqrt{u}H(u)du = C\int_{-\infty}^{\infty} g(u)\sqrt{u}H(u)du + \mathcal{O}(e^{-C_0\lambda}) \\ =& C\int_{-\infty}^{\infty} \mathcal{F}g(\xi)\mathcal{F}(\sqrt{u}H(u))(\xi)d\xi + \mathcal{O}(e^{-C_0\lambda}). \end{aligned}$$

There exists $C \in \mathbb{R}$ such that $\mathcal{F}(\sqrt{u}H(u))(\xi) = C(\xi - i0)^{-\frac32}$. Using this and $\mathcal{F}g(\xi) = \frac{1}{\sqrt{\lambda}}e^{-2\lambda^{-1}(\xi-\lambda)^2}$, therefore

$$\begin{aligned} \langle \nu, f\rangle =& C\int_{-\infty}^{\infty} e^{-2\lambda^{-1}(\xi-\lambda)^2}\sqrt{\frac{2}{\lambda}}(\xi - i0)^{-\frac32}d\xi + \mathcal{O}(e^{-C_0\lambda}) \\ =& C\lambda^{-2}\int_{-\infty}^{\infty} e^{-2\lambda v^2}(v + 1 - i0)^{-\frac32}dv + \mathcal{O}(e^{-C_0\lambda}). \end{aligned}$$

The theorem of stationary phase gives, for large λ, that

$$\langle \nu, f\rangle = 2C\lambda^{-3/2} + \mathcal{O}(\lambda^{-5/2}) + \mathcal{O}(e^{-C_0\lambda}).$$

So there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$

$$|\langle \nu, f\rangle| \geq C\lambda^{-\frac{3}{2}}.$$

We next recall the local trace formula [11]:

Theorem 7.1. *Let $f = f(z, \lambda)$ be holomorphic in z on Ω, dependent on $h \in \mathbb{R}$, and such that for all $z \in \Omega \setminus W$ for some closed set W, $|f(z, \lambda)| \leq 1$. Then the operator $(\chi f)(P_0, \lambda) - (\chi f)(P_\infty, \lambda)$ is of trace class and*

$$\mathrm{Tr}\big[(\chi f)(P_\bullet, \lambda)\big]_0^\infty = -\sum_{z_j \in \mathrm{Res}(P_0)\cap W} f(z_j, \lambda) + \mathcal{O}(h^{-n})\|f\|_{\Omega\setminus W}.$$

We choose W such that $\Omega + E_0 \setminus W \subset \Omega_0 + E_0$. The function f gives $C_{\check{b}} = \check{b} + \mathrm{o}(\check{b})$, $C_1 = b_0 + \mathrm{o}(b_0)$ and $C_0 = \tilde{a}^2 - b_0 + \mathrm{o}(b_0)$. Recall that $C_2 = \inf\{C_0, C_{\tilde{b}}, C_{\tilde{b}-\mu_0}\}$. We can choose b_0 as small as we need to get $C_1 < C_2$. The function f satisfies all the hypotheses (**H4**) – (**H7**) so

$$\mathrm{Tr}[(\chi f)(P_\bullet, \lambda)]_0^\infty = -\sum_{z_j \in \mathrm{Res}(P_0)\cap W} f(z_j, \lambda) + e^{-C_2\lambda}\mathcal{O}(h^{-3}). \tag{7.1}$$

The result of the previous section gives for $\lambda \geq \lambda_0$:

$$|\mathrm{Tr}[(\chi f)(P_\bullet, \lambda)]_0^\infty| \geq C\lambda^{-\frac{3}{2}}h^{-1} + \mathcal{O}(h)e^{C_1\lambda} + \mathcal{O}(h^{-3})e^{-C_2\lambda}. \tag{7.2}$$

The comparison of (7.2) and (7.1) gives that

$$\sum_{z_j \in \mathrm{Res}(P_0)\cap W} |f(z_j, \lambda)| \geq Ch^{-1}\lambda^{-\frac{3}{2}} + \mathcal{O}(h)e^{C_1\lambda} + \mathcal{O}(h^{-3})e^{-C_2\lambda}.$$

Let us choose $\hat{C}$ such that $C_1 < 2\hat{C} < C_2$. Let h_1 be defined by $\ln h_1 = -\hat{C}\lambda_0$. Now we fix the relation between λ and h, we choose $\ln h = -\hat{C}\lambda$. There exist $h_0 \leq h_1$ such that for $h < h_0$ we obtain :

$$\begin{aligned} &Ch^{-1}\lambda^{-\frac{3}{2}} + \mathcal{O}(h)e^{C_1\lambda} + \mathcal{O}(h^{-3})e^{-C_2\lambda} \\ &\qquad = Ce^{\tilde{C}\lambda}\lambda^{-\frac{3}{2}} + \mathcal{O}(1)e^{(C_1-\tilde{C})\lambda} + \mathcal{O}(1)e^{(3\tilde{C}-C_2)\lambda} \\ &\qquad \sim Ce^{\tilde{C}\lambda}\lambda^{-\frac{3}{2}}. \end{aligned}$$

Hence

$$2Ch^{-1}\lambda^{-\frac{3}{2}} + \mathcal{O}(h)e^{C_1\lambda} + \mathcal{O}(h^{-3})e^{-C_2\lambda} \geq C|\ln h|^{-\frac{3}{2}}h^{-1}.$$

For f we see that $|f(z, \lambda)| \leq 1$ for $\mathrm{Im}(z) \leq 0$. We conclude that

$$\begin{aligned} \mathrm{card}\{\mathrm{Res}(P_0)\cap W\} &= \sum_{z_j \in \mathrm{Res}(P)\cap W} 1 \\ &\geq \sum_{z_j \in \mathrm{Res}(P)\cap W} |f(z_j, \lambda)| \geq Ch^{-1}|\ln h|^{-\frac{3}{2}}. \end{aligned}$$

This finishes the proof of Theorem 2.3.

Appendix A. Weyl calculus

We gather in this appendix some results on the Weyl calculus of h speudo differential operators needed various places in the paper. Let $m(x,\xi)$ be a tempered function on $\mathbb{R}^{2n}$ (i.e. there exists $N>0$ and $C>0$ such that :

$$\forall (x,\xi,u,v)\in\mathbb{R}^{4n}\times\mathbb{C},\ \ |m(x+u,\xi+v)|\leq C|m(x,\xi)|(1+u^2+v^2)^N).$$

Let Ω be a compact set. We denote by $\mathrm{S}(m(x,\xi),1,1)$ the set of functions $a\in\mathcal{C}^\infty(\mathbb{R}^{2n})$, depending on h and z, called symbols, with values in $\mathcal{M}_r(\mathbb{C})$ such that, for all $(\alpha,\beta)\in\mathbb{N}^{2n}$, there exist $C_{\alpha,\beta}$ such that :

$$\forall (x,\xi,h,z)\in\mathbb{R}^{2n}\times]0,h_0[\times\Omega\ :\ \|\partial_x^\alpha\partial_\xi^\beta a(x,\xi,h)\|\leq C_{\alpha,\beta}\, m(x,\xi)).$$

A symbol a is h-admissible if and only if there exists symbols $a_j\in\mathrm{S}(m(x,\xi),1,1)$ such that for all N :

$$a(x,\xi,h,z)=\sum_{j<N}h^ja_j(x,\xi,z)+h^Na'_N(x,\xi,h,z),$$

with $a'_N\in\mathrm{S}(m(x,\xi),1,1)$. To an h-admissible symbol a we associate an operator by the relation :

$$\big(\mathrm{Op}^w(a)u\big)(x,z)=(2\pi h)^{-n}\int_{\mathbb{R}^{2n}}e^{i\langle x-y,\xi\rangle/h}a(\frac{x+y}{2},\xi,h,z)u(y)\,dy\,d\xi.$$

An operator A belongs to $\Sigma(m,1,1)$ if and only if there exists a symbol $a\in\mathrm{S}(m,1,1)$ such that $A=\mathrm{Op}^w(a)$. Standard techniques of Weyl calculus, as described in [8] and [12], give the following proposition :

Proposition A.1. *For h-admissible symbols $a\in\mathrm{S}(m_1,1,1)$ and $b\in\mathrm{S}(m_2,1,1)$ and for all N, there exists symbols c and d such that*

$$\mathrm{Op}^w(a)\mathrm{Op}^w(b)=\mathrm{Op}^w(c)+h^N\mathrm{Op}^w(d)$$

where $c\in\mathrm{S}(m_1m_2,1,1)$ is a h-admissible symbol and d is an element of $\mathrm{S}(m_1m_2,1,1)$. The symbol c is given by :

$$c(x,\xi)=\sum_{j=0}^{N-1}h^jc_j(x,\xi)\tag{A.3}$$

with

$$c_j(x,\xi)=\frac{1}{2^j}\sum_{|\alpha|+|\beta|=j}\frac{(-1)^{|\beta|}}{\alpha!\beta!}(\partial_\xi^\alpha D_x^\beta a)(\partial_\xi^\beta D_x^\alpha b)(x,\xi),\ \ 0\leq j<N$$

We denote $c+h^Nd=a\#b$

To determine if an operator belongs to the trace class we have :

Proposition A.2. [12] *If $\int_{\mathbb{R}^{2n}}m(x,\xi)dxd\xi$ is finite, every operator $A\in\Sigma(m,1,1)$ is of trace class and satisfies $\|A\|_{\mathrm{Tr}}=\mathcal{O}(h^{-n})$.*

Let us recall the following results [11] for the operator P_0 :

Proposition A.3. *The operator* $[(P_\bullet - z)^{-1}(P_\bullet - z_0)^{-m}]_0^\infty$ *is of trace class and we have, uniformly in* $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\|[(P_\bullet - z)^{-1}(P_\bullet - z_0)^{-m}]_0^\infty\|_{\mathrm{Tr}} = \frac{\mathcal{O}(h^{-n})(1+z^2)^{\frac{1}{2}}}{|\mathrm{Im}(z)|^2}.$$

Proposition A.4. *For all* $g \in \mathcal{C}_0^\infty(\mathbb{R})$ *the operator* $(g(P_\infty) - g(P_0))$ *is of trace class and*

$$\mathrm{Tr}(g(P_\infty) - g(P_0)) = \mathcal{O}(h^{-n}) \int m(x,\xi) dx d\xi.$$

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