RANDOM SCHRÖDINGER OPERATORS ON MANIFOLDS

DANIEL LENZ, NORBERT PEYERIMHOFF, AND IVAN VESELIĆ

Abstract. We consider a random family of Schrödinger operators on a cover X of a compact Riemannian manifold $M = X/\Gamma$. We present several results on their spectral theory, in particular almost sure constancy of the spectral components and existence and non-randomness of an integrated density of states. We also sketch a groupoid based general framework which allows to treat basic features of random operators in different contexts in a unified way. Further topics of research are also discussed.

1. Introduction

This paper is devoted to the study of spectral properties of random operators on manifolds. Their counterparts in Euclidean geometry play a prominent part in solid state physics [BBEE$^+$84, ES84, Lif85, LGP88]. Homogeneous random Hamiltonians are used to describe the propagation of both quantum mechanical and classical waves in random media. The model which most of the mathematical physics literature is devoted to is a Schrödinger operator with random potential, see for instance [CFKS87, Kir89, CL90, PF92, Sto01]. Hereby the potentials, more precisely the level of randomness they describe, may take quite different forms. For instance, a random potential generated by a Poissonian stochastic field describes an amorphous medium, whereas a quasi-periodic potential models a medium whose structure is very close to a crystalline one. A model where the randomness lies somewhere between the two just mentioned extremes is the (discrete) Anderson model on $l^2(\mathbb{Z}^d)$ and its continuum counterpart on $L^2(\mathbb{R}^d)$, the alloy-type model.

Characteristic for the last mentioned model is, that while the members of the random family of operators act on an L^2-space over an continuum configuration space, the homogeneity is expressed via the action of the discrete group $\mathbb{Z}^d$. We study here analogues of such operators, which act on more general geometries, namely covering Riemannian manifolds with compact quotients. Moreover, we allow also that the randomness enters in the Laplace-Beltrami operator via the metric.

By the variety of the models which fit under the common roof of random (Schrödinger) operators it cannot be expected that they will share all spectral features in detail. In fact, almost-periodic Schrödinger operators alone illustrate that all measure-theoretic types of spectrum may occur (see e.g. Chapter 10 in [CFKS87]). However basic spectral features are shared by all models and can be traced back to rather mild, abstract conditions on the considered random family of operators. These features include the non-randomness of the spectrum and existence of a self-averaging integrated density of states (IDS).

These basic spectral properties are presented in this paper for random Schrödinger operators on manifolds and random Laplace-Beltrami operators. Furthermore we

---

Date: 18th December 2002.

2000 *Mathematics Subject Classification.* 35J10; 58J35; 82B44.

Key words and phrases. integrated density of states, random metrics, random operators, Schrödinger operators on manifolds, Von Neumann algebra, trace.

indicate their origin from the above mentioned abstract properties using the framework of groupoids and von Neumann algebras.

Let us say something about the physical picture one has in mind when studying these models. Random Schrödinger operators are used to study conductance properties of solids. In certain cases it is known that the boundary plays a important role for the wave transport properties. This suggest the study of spectral properties of the Laplace-Beltrami operator of the two-dimensional surface which forms the boundary of the solid. Since the surface is not perfectly planar, we want to account for its random holes by using a stochastic field as the metric.

If one describes wave propagation properties of thin films or layers — idealized by surfaces — it may be necessary to incorporate an additional random "effective potential", cf. for instance [EK]. However this zero order term of the operator is part of the kinetic energy and is a reminiscence of the fact that the operator on the surface is an idealization of an operator acting on a very thin layer.

Let us outline the structure of the paper. In the next section we introduce the random operators on manifolds we are dealing with. Sections 3 and 4 are devoted to basic spectral properties of those operators and the existence of an selfaveraging integrated density of states, while Section 5 and 6 provide insight in the groupoid/von Neumann algebra background of these results. In Section 7 we specialize to alloy-type models on manifolds. For those we discuss Wegner estimates. Finally the last section is devoted to some interesting questions for further study.

2. Our model

In this section we shortly discuss the geometric situation underlying our model and then introduce the random Schrödinger operators we are dealing with. The model is a generalization of the model introduced in [PV02] as has been studied in [LPVa, LPVb]. Our treatment here mainly follows [LPVa, LPVb] to which we refer the reader for further details.

Our model is based on the following geometric situation: Let X be a cover of a compact Riemannian manifold $M = X/\Gamma$ where Γ is a discrete, finitely generated subgroup of the isometries of X with $|\Gamma| = \infty$. Let g_0 be the fixed Γ-periodic smooth Riemannian metric on the cover X, inherited from M. Denote by $\mathcal{F}$ a precompact connected Γ-fundamental domain with piecewise smooth boundary. Furthermore, let $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ be a probability space on which Γ acts ergodically by measure preserving transformations $\gamma\colon \Omega \to \Omega, \gamma \in \Gamma$. The expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$.

We will consider two types of random objects over $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$. The first is a family of *random metrics* on X, the second is a family of *random potentials*. Put together, they will give rise to a family of random operators.

As for the random metrics, the manifold X is equiped with a family of metrics $\{g_\omega\}_{\omega\in\Omega}$ with corresponding volume forms vol_ω with the following properties: The map $(\omega, v) \mapsto g_\omega(v, v)$ is jointly measurable for all $(\omega, v) \in \Omega \times TX$. There are constants $C_g, C_\rho > 0$ such that

$$C_g^{-1} g_0(v, v) \leq g_\omega(v, v) \leq C_g g_0(v, v) \;\forall\, (\omega, v) \in \Omega \times TX \tag{1}$$

and

$$|\nabla_0 \rho_\omega(x)|_0 \leq C_\rho \;\forall\, (\omega, x) \in \Omega \times X. \tag{2}$$

Here ∇_0 denotes the gradient with respect to g_0, ρ_ω is the unique smooth density satisfying $d\mathrm{vol}_0 = \rho_\omega d\mathrm{vol}_\omega$, and $|v|_0^2 = g_0(v, v)$. The Ricci curvature of all metrics

g^ω is bounded below by a fixed constant $K \in \mathbb{R}$. The metrics are compatible in the sense that the deck transformations

$$\gamma \colon (X, g_\omega) \to (X, g_{\gamma\omega})$$

are isometries and that the induced maps

$$U_{(\omega,\gamma)} \colon L^2(X, \mathrm{vol}_{\gamma^{-1}\omega}) \to L^2(X, \mathrm{vol}_\omega), \quad (U_{(\omega,\gamma)} f)(x) = f(\gamma^{-1}x)$$

are unitary operators on the family of Hilbert spaces over the manifolds $\{(X, g_\omega)\}_{\omega\in\Omega}$.

As for the random potentials, let $V : \Omega \times X \longrightarrow [0, \infty)$ be jointly measurable with $V_\omega \equiv V(\omega, \cdot) \in L^1_{loc}(X, g_\omega)$ for all $\omega \in \Omega$. Assume furthermore that $V(\gamma\omega, x) = V(\omega, \gamma^{-1}x)$ for arbitrary $x \in X$, $\omega \in \Omega$ and $\gamma \in \Gamma$.

Given a random metric and a random potential, we can now introduce the corresponding random Schrödinger operators as $H_\omega := \Delta_\omega + V_\omega$ on the Hilbert spaces $L^2(X, \mathrm{vol}_\omega)$. In fact, these unbounded operators are defined by means of quadratic forms. Moreover, they satisfy the *equivariance condition*

$$H_\omega = U_{(\omega,\gamma)} H_{\gamma^{-1}\omega} U^*_{(\omega,\gamma)}, \tag{3}$$

for all $\gamma \in \Gamma$ and $\omega \in \Omega$. We will refer to $\{H_\omega\}$ as a *random (Schrödinger) operator on the manifold* X and denote this model by **(RSM)**.

3. General results

In this section we discuss measurability properties of our random operators, almost everywhere constancy of their spectral properties and we introduce the abstract density of states. Again, the results in this section are taken from [LPVa, LPVb].

We start by defining the notion of measurability in our setting.

Definition 3.1. Let D be an open subset of X. A family of selfadjoint operators $\{H_\omega\}_\omega$, where the domain of H_ω is a dense subspace $\mathcal{D}_\omega$ of $L^2(D, \mathrm{vol}_\omega)$, is called a *measurable family of operators* if

$$\omega \mapsto (f_\omega, F(H_\omega) f_\omega)_\omega \tag{4}$$

is measurable for all measurable $F \colon \mathbb{R} \to \mathbb{C}$ with $|F|$ bounded and all $f \colon \Omega \times D \to \mathbb{R}$ measurable with $f_\omega \in L^2(D, \mathrm{vol}_\omega)$, $f_\omega(x) = f(\omega, x)$, for every $\omega \in \Omega$. Here, $(\cdot,\cdot)_\omega$ denotes the inner product on $L^2(D, \mathrm{vol}_\omega)$.

Our random operators are measurable in this sense as can be seen from the next theorem.

Theorem 1. *A random operator $\{H_\omega\}_{\omega\in\Omega}$ on X as well as its restriction $\{H^D_\omega\}$ to an arbitrary open set $D \subset X$ with Dirichlet boundary condition is a measurable family of operators.*

The proof of the theorem is not too complicated but somewhat technical. In fact, for technical reasons it is needed that the σ-algebra of Ω is countably generated. In our model this can be established by changing to an equivalent version of the defining stochastic processes given by the random potential and the random metric. This has been done for the potential explicitly in Remark 2.8 in [LPVa]. Given the measurabilty one can use the results of [LPVa] to establish a result on the non-randomness of the spectrum.

Theorem 2. *Let $\{H_\omega\}_{\omega\in\Omega}$ be a random operator on X. Then there exist $\Omega' \subset \Omega$ of full measure and $\Sigma, \Sigma_\bullet \subset \mathbb{R}$, such that $\sigma(H_\omega) = \Sigma$, $\quad \sigma_\bullet(H_\omega) = \Sigma_\bullet \quad$ for all $\omega \in \Omega'$ where $\bullet \in \{disc, ess, ac, sc, pp\}$. Moreover, $\Sigma_{disc} = \emptyset$.*

Note that σ_{pp} denotes the closure of the set of eigenvalues.

Next, we introduce the *(abstract) density of states* for a random operator $\{H_\omega\}$ as the measure on $\mathbb{R}$, given by

$$\rho_H(f) := \frac{\mathbb{E}\left[\operatorname{tr}\left(\chi_{\mathcal{F}} f(H_\bullet)\right)\right]}{\mathbb{E}\left[\operatorname{vol}_\bullet(\mathcal{F})\right]}, \qquad \text{for bounded measurable } f. \tag{5}$$

The measure ρ_H completely determines the spectral theory of the direct integral $H = \int_\Omega^\oplus H_\omega$ as can be seen from the following proposition. Recall that a measure ϕ on $\mathbb{R}$ is a *spectral measure* for the selfadjoint operator H with spectral family E_H if, for Borel measurable $B \subset \mathbb{R}$, $\phi(B) = 0 \Leftrightarrow E_H(B) = 0$.

Theorem 3. *The measure ρ_H is a spectral measure for the direct integral operator*

$$H := \int_\Omega^\oplus H_\omega \, d\mathbb{P}(\omega).$$

In particular, the almost sure spectrum Σ coincides with the topological support $\{\lambda \in \mathbb{R} : \rho(\,]\lambda-\epsilon, \lambda+\epsilon[\,) > 0 \quad \text{for all } \epsilon > 0\}$ of ρ_H.

Remark 3.2. Of course, the spectral properties of H will in general be different and in, fact, "smoother" than the spectral properties of the single operators H_ω.

It will turn out that the measure ρ_H, its distribution function $E \mapsto \rho_H(-\infty, E)$ and a certain trace on a suitable von Neumann algebra are intimately connected. This will be studied in the next sections.

4. Šubin-Pastur trace formula

In this section we discuss how the distribution function of the abstract density of states can be calculated by an exhaustion procedure. As the distribution function is essentially given by a trace (see below), this gives effectively a way to calculate a trace by an exhaustion procedure. This type of formula is associated to the names of Pastur and Šubin after the seminal work [Pas71, Šub79, Šub82]. The material of this section is taken from [LPVb].

We consider the following normalized eigenvalue counting function for the restricted random operator $\{H_\omega^D\}$ on a given open set $D \subset X$:

$$N_\omega^D(\lambda) = \frac{\#\{i \mid \lambda_i(H_\omega^D) < \lambda\}}{\operatorname{vol}_\omega(D)}. \tag{6}$$

Obviously, N_ω^D is a distribution function and has countably many discontinuity points. Our aim is now to exhaust X by an increasing set $\{D^j\}$ of open domains. For well-definedness of the limit (at all continuity points) we need that the group Γ is *amenable*. A geometric description of amenability is the existence of a Følner sequence. Let us introduce the relevant notions in the next definition. The function ϕ associates to every finite subset $I \subset \Gamma$ a corresponding open set $\phi(I) = \operatorname{int}\left(\bigcup_{\gamma \in I} \gamma\overline{\mathcal{F}}\right)$ of X, with the help of the fixed fundamental domain $\mathcal{F}$ of Section 2.

Definition 4.1. (a) A sequence $\{I_j\}_j$ of finite subsets in Γ is called a *Følner sequence* if $\lim_{j\to\infty} \frac{|I_j \Delta I_j\gamma|}{|I_j|} = 0$ for all $\gamma \in \Gamma$.
(b) A Følner sequence $\{I_j\}_j$ is called a *tempered Følner sequence* if it is monotonously increasing and satisfies $\sup_{j\in\mathbb{N}} \frac{|I_{j+1}I_j^{-1}|}{|I_{j+1}|} < \infty$.
(c) A sequence $\{D^j\}_j$ of subsets of X is called *admissible* if there exists a tempered Følner sequence $\{I_j\}_j$ in Γ with $D^j = \phi(I_j)$, $j \in \mathbb{N}$.

Lindenstrauss shows in [Lin01] that every Følner sequence has a tempered subsequence, and uses this to prove an ergodic theorem for locally compact amenable groups.

Using Lindenstrauss results together with suitable heat kernel estimates, it is shown in [LPVa] that, for any admissible sequence $\{D^j\}$, and the corresponding normalized eigenvalue functions $N^j_\omega := N^{D^j}_\omega$, the limit

$$N_H(\lambda) \equiv \lim_{j\to\infty} N^j_\omega(\lambda)$$

exists for almost all $\omega \in \Omega$ and for all $\lambda \in \mathbb{R}$ with at most countably many exceptions. The limit is again a distribution function and carries the name *integrated density of states*. It agrees with the (nonrandom) distribution function of the abstract density of states. This result is stated in the following theorem.

Theorem 4. *Let $\{D^j\}_j$ be an admissible sequence and $\{H_\omega\}_\omega$ be as above. There exists a set Ω' of full measure such that*

$$\lim_{j\to\infty} N^j_\omega(\lambda) = N_H(\lambda) = \rho_H(\,]-\infty,\lambda[\,),$$

for every $\omega \in \Omega'$ and every point $\lambda \in \mathbb{R}$ with $\rho_H(\{\lambda\}) = 0$.

Using Dirichlet-Neumann bracketing, we obtain the following corollary as an immediate consequence of the preceeding theorem (and its proof).

Corollary 4.2. *For every $\lambda \in \mathbb{R}$ with $\rho_H(\{\lambda\}) = 0$ we have*

$$\frac{\mathbb{E}(\mathrm{tr}(\chi_{]-\infty,\lambda[}(H^{\mathcal{F},D}_\bullet)))}{\mathbb{E}(\mathrm{vol}_\bullet(\mathcal{F}))} \le N_H(\lambda) \le \frac{\mathbb{E}(\mathrm{tr}(\chi_{]-\infty,\lambda[}(H^{\mathcal{F},N}_\bullet)))}{\mathbb{E}(\mathrm{vol}_\bullet(\mathcal{F}))},$$

where $\chi_{]-\infty,\lambda[}(H^{\mathcal{F},\#}_\omega)$, $\# = D$ or $\# = N$ denotes the spectral projection of the restricted operator H_ω to $\mathcal{F}$ with Dirichlet, resp., Neumann boundary condition onto the interval $]-\infty,\lambda[$.

5. The density of states as a trace on a von Neumann algebra

The abstract density of states ρ_H is closely related to a trace τ of a suitable von Neumann algebra $\mathcal{N}$.

In order to describe the von Neumann algebra $\mathcal{N}$, we first introduce the concept of a bounded random operator:

Definition 5.1. A family $\{A_\omega\}_{\omega\in\Omega}$ of bounded operators $A_\omega\colon L^2(X,\mathrm{vol}_\omega) \to L^2(X,\mathrm{vol}_\omega)$ is called a *bounded random operator* if it satisfies:

(i) $\omega \mapsto \langle g_\omega, A_\omega f_\omega\rangle$ is measurable for arbitrary $f,g \in L^2(\Omega\times X, \mathbb{P}\circ\mathrm{vol})$.
(ii) There exists a $C \ge 0$ with $\|A_\omega\| \le C$ for almost all $\omega \in \Omega$.
(iii) For all $\omega \in \Omega, \gamma \in \Gamma$ the equivariance condition $A_\omega = U_{(\omega,\gamma)} A_{\gamma^{-1}\omega} U^*_{(\omega,\gamma)}$ is satisfied.

Two bounded random operators $\{A_\omega\}_\omega, \{B_\omega\}_\omega$ are called *equivalent*, $\{A_\omega\}_\omega \sim \{B_\omega\}_\omega$, if $A_\omega = B_\omega$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Each equivalence class of bounded random operators $\{A_\omega\}_\omega$ gives rise to a *single* bounded operator A on the larger Hilbert space $L^2(\Omega\times X, \mathbb{P}\circ\mathrm{vol}_\bullet)$ by $(Af)(\omega,x) := A_\omega f_\omega(x)$, see Appendix A in [LPVa]. This allows us to identify the equivalence class of $\{A_\omega\}_\omega$ with the bounded operator A.

Now, the set of bounded random operators form the von Neumann algebra $\mathcal{N}$ (see Theorem 3.1 in [LPVa]). This von Neumann algebra possesses a trace [LPVa]) which is given by the following procedure: Choose a measurable $u\colon \Omega\times X \to \mathbb{R}^+$ with $\sum_{\gamma\in\Gamma} u_{\gamma^{-1}\omega}(\gamma^{-1}x) \equiv 1$ on $\Omega \times X$ and define

$$\tau(A) := \mathbb{E}\left[\mathrm{tr}(u_\bullet A_\bullet)\right]$$

on the set of non-negative operators in $\mathcal{N}$. This τ is independent of u (chosen as above). Under an additional freeness condition (see Theorem 4.2(b) in [LPVa]), $\mathcal{N}$ is a factor of type II_∞ and its trace is uniquely determined (up to a constant multiple).

By (3) and the last section, the resolvents, spectral projections and the semigroup associated to $\{H_\omega\}_\omega$ are all bounded random operators. This can be used to show how ρ_H and τ are related. Namely, the family $\{E_\omega(\lambda)\}_\omega$ of spectral projections onto the interval $]-\infty,\lambda[$ of the random operator $\{H_\omega\}_\omega$ is an element of $\mathcal{N}$ and agrees with the spectral projection of $H := \int_\Omega^\oplus H_\omega \, d\mathbb{P}(\omega)$ onto $]-\infty,\lambda[$. Hence $\tau(E(\lambda))$ is well defined.

Choosing $u_\omega(x) = \chi_{\mathcal{F}}(x)$ we can summarize these considerations in the following result.

Corollary 5.2. *Set* $\mathcal{D} \equiv \mathbb{E}\,(\mathrm{vol}_\bullet \mathcal{F})$. *Then,* $\frac{1}{\mathcal{D}}\,\tau(E(\lambda)) = N_H(\lambda)$.

6. The general framework: Random operators and groupoids

The abstract considerations in the last section motivate naturally a *more general abstract framework* which covers also random operators arising in various other settings: random operators on abelian (or more general) amenable groups (see, e.g., [Bel86, Bel92, KX87]), random operators on Bethe lattices (see e.g. [Kle96]), on tilings and Delone sets as (see, e.g., [BHZ00, Kel95, LS02, LS]), on foliations and on manifolds (see above). This general framework allows a unified treatment of basic spectral features of random operators in these different contexts by the use of Connes non-commutative integration theory [Con79]. This has been done by the authors in [LPVa].

There also, as an illustration, the details have been worked out for random Schrödinger operators on manifolds. A corresponding treatment of quasicrystals has been done in [LS02, LS02] (see [Kel95, BHZ00] as well). For almost periodic operators it can be found in [Bel86, Bel92, KX87, Len99].

The abtract framework starts with a measurable groupoid $\mathcal{G}$ (in our model $\mathcal{G} = \Omega \times \Gamma$). The corresponding set of units, denoted by Ω, carries a measure μ (which is invariant w.r.t. $\mathcal{G}$) and can in our model (RSM) be canonically identified with the underlying probability space. The groupoid $\mathcal{G}$ acts on a suitable space $\mathcal{X}$. This space is a bundle over Ω with natural projection map $\pi : \mathcal{X} \to \Omega$ and equiped with a family (α^ω) of measures s.t. α^ω is supported in $\mathcal{X}^\omega \equiv \pi^{-1}(\omega)$.

Given this setting one can easily define random operators. A *bounded random operator* is a family of operators $\{A_\omega\}$ acting on the Hilbert spaces $L^2(\mathcal{X}^\omega, \alpha^\omega)$ of the fibers $\mathcal{X}^\omega = \pi^{-1}(\omega)$ which satisfies the following conditions analogous to the ones given in Definition 5.1

- the map $\omega \mapsto A_\omega$ is measurable and essentially bounded
- the family is equivariant, i.e. compatible with the action of $\mathcal{G}$.

Note that this setting very naturally produces the two main features of random operators: a family of operators indexed by a probability space and an equivariance condition.

As above, one associates to an equivalence class of a bounded random operator $\{A_\omega\}$ in one-one correspondence a single bounded operator A on the larger Hilbert space $\mathcal{H} := L^2(\mathcal{X}, \mu\circ\alpha)$, and the set of these operators on $\mathcal{H}$ defines a von Neumann algebra $\mathcal{N}$.

While this formalism may seem rather abstract and somewhat vague, we should like to stress that, firstly, it applies to a large number of models and secondly, the involved quantities can in concrete cases often easily be identified. We include the following table to make this transparent. Here, c.m. means counting measure on a discrete space and L.m. means the Lebesque measure:

Model	Groupoid	Units of $\mathcal{G}$	Typical fibre $\mathcal{X}_\omega$ of $\mathcal{X}$	measure α^ω on $\mathcal{X}_\omega$
Anderson model	$\Omega \times \mathbb{Z}^d$	$\otimes_{\mathbb{Z}^d}\mathbb{R}$	$\mathbb{Z}^d$	c.m.
Alloy-type model	$\Omega \times \mathbb{Z}^d$	$\otimes_{\mathbb{Z}^d}\mathbb{R}$	$\mathbb{R}^d$	L.m.
Lifschitz-Poisson model	$\Omega \times \mathbb{R}^d$	$\otimes_{\mathbb{R}^d}\mathbb{R}$	$\mathbb{R}^d$	L.m.
Random Gaussian potential	$\Omega \times \mathbb{R}^d$	$\otimes_{\mathbb{R}^d}\mathbb{R}$	$\mathbb{R}^d$	L.m.
Almost periodic potential	$\Omega \times \mathbb{R}^d$	$\overline{\{f(\cdot - t) \mid t \in \mathbb{R}^d\}}$	$\mathbb{R}^d$	L.m.
(RSM)	$\Omega \times \Gamma$	Ω	(X, g_ω)	vol_ω
Delone system	$\Omega \times \mathbb{R}^d$	Ω	ω	c.m.
Tiling $\mathcal{T}$	$\mathcal{G}(\mathcal{T})$	Tiling space	Tiling	c.m.
Foliation	$\mathcal{G}(X, \mathcal{F})$	X	$\mathcal{F}_\omega$	$\mathrm{vol}_{\mathcal{F}_\omega}$

As can be seen in the diagram, in many models the groupoid is given by the product of a group with the underlying probability space.

Now, given this framework as well as some additional technical conditions (see Definition 2.6, Theorem 4.2(b) and Lemma 5.6 in [LPVa]) the following properties of the von Neumann algebra $\mathcal{N}$ and an affiliated random family $\{H_\omega\}$ can be established [LPVa]:

(i) Almost sure constancy of the spectral components of $\{H_\omega\}$,
(ii) absence of discrete spectrum for almost every $\{H_\omega\}$,
(iii) existence of a faithful trace τ on $\mathcal{N}$, which gives rise to a spectral measure of the family $\{H_\omega\}$.

Under suitable additional conditions one can even provide a trace formula, or more precisely:

(iv) An explicit way to calculate the trace using an exhaustion procedure.

To infer such a formula, one has to have information not only on the groupoid $\mathcal{G}$, but also on the space $\mathcal{X}$ and on the family of operators (H_ω) on the Hilbert space $L^2(\mathcal{X}_\omega, m_\omega)$. As for the properties of $\mathcal{X}$, one has to be able to control boundary effects. More precisely, a exhausting sequence of subsets of $\mathcal{X}$ must exist, such that in the limit the "area" of the boundaries becomes negligible when compared to the volume of the subsets in the exhaustion sequence. Furthermore, an appropriate ergodic theorem (for $\mathcal{G}$) must be at disposal. It ensures that spectral quantities are averaged out as one proceeds along the exhaustion sequence. The operators H_ω acting on $L^2(\mathcal{X}_\omega, m_\omega)$ — more precisely, certain auxiliary functions of them — must satisfy a "finite range" condition or a "principle of not feeling the boundary". The discrete Laplacians on graphs and continuous Laplacians are examples of such operators.

7. Wegner estimates and continuity properties of the IDS

In this section we discuss Wegner estimates and continuity properties of the integrated density of states in our model. The material is taken from [LPPV].

Let $\{H_\omega\}$ be our random operator. Let $J \subset \Gamma$ be a finite subset of Γ, $D := \phi(J)$ and denote by P_ω^J the spectral family of the restricted operator H_ω^D with Dirichlet boundary condition. An estimate of the form

$$\mathbb{E}(\mathrm{tr}(P^J_\bullet([E-\epsilon, E+\epsilon]))) \leq C\epsilon^\alpha |J|^\beta \tag{7}$$

with $C, \alpha, \beta > 0$ and E in a suitable interval I in $\mathbb{R}$ is called a *Wegner estimate* after the work [Weg81] of Wegner. Wegner estimates play a crucial role in proofs of localization (see e.g. [FS83, vDK89, CH94, Sto01, GK01]). Moreover, they can be used to establish continuity properties of the integrated density of states. For example, if (7) holds with $\beta = 1$ and C and α independent of J for all energies in a certain energy interval I, then the results of the last section, in particular, Theorem 4 imply

$$N([E-\epsilon, E+\epsilon]) \leq C\epsilon^\alpha \tag{8}$$

for all $E \in I$. This means that N is Hölder continuous with exponent α.

To obtain Wegner estimates in our setting, we must further specialize our model. In the following we present two specific examples where it is possible to derive Wegner estimates. In both models one part of the energy, either the kinetic or the potential, is of alloy type. In the fist one the potential has this structure, while in the second it is the metric of the Laplace-Beltrami operator.

We call a random operator $\{H_\omega\}$ alloy potential model if the potential V_ω is relatively small with respect to Δ_ω, uniformly in ω and can be decomposed into two parts $V_\omega = V_{per} + V^\omega$. The first one V_{per} is Γ-periodic, while the second one is the generalization of the well known Euclidean alloy-type potential

$$V^\omega(x) = \sum_{\gamma\in\Gamma} q_\gamma(\omega) v(\gamma^{-1}x).$$

Here the *coupling constants* $q_\gamma, \gamma \in \Gamma$ are a collection of independent, identically distributed random variables. The distribution measure $\tilde{\mu}$ of q_e (which coincides with the distribution measure of q_γ for every γ) is assumed to have a density $f \in L^\infty_c$ with respect to the Lebesgue measure. The *single site potential* $v \in L^\infty_c$ is a function satisfying $v \geq \tilde{\kappa}\chi_{\mathcal{F}}$ for some $\tilde{\kappa} > 0$.

In this model we assume the metric to be non-random, i.e. let $g_\omega = g_0$ for all $\omega \in \Omega$.

The other model where a Wegner estimate can be derived is a random operator $\{H_\omega\}$ with alloy-type metric. Here we again have to assume that the potential V_ω is a relatively small perturbation with respect to Δ_ω, uniformly in ω.

The metric is random and resembles the alloy-type potential above. Namely, for a collection of independently identically distributed random variables $r_\gamma \colon \Omega \to \mathbb{R}, \gamma \in \Gamma$, whose distribution measure μ has a compactly supported density f in the Sobolev space $W^1_1(\mathbb{R})$ and a *single site deformation* $u \in C^\infty_c(X)$ with $u \geq \kappa\chi_{\mathcal{F}}, \kappa > 0$ define for each $\omega \in \Omega$

$$g_\omega(x) = \left(\sum_{\gamma\in\Gamma} e^{r_\gamma(\omega)}\, u(\gamma^{-1}x)\right) g_0(x).$$

In this model we assume that the stochastic process V_ω is independent of $\{r_\gamma\}_\gamma$.

In both cases, a Wegner estimate can be established in a suitable energy region. We refer the reader to [LPPV] for further details.

8. Outlook

The line of research started in [LPVa, LPVb] and (partly) summarized above can be pursued in various directions. Two of the main directions are

(α) the study of more specific spectral features for more specialized models on manifolds,

(β) to extend the investigation to other models and geometries.

In both of these directions, various questions are immediate. We summarize some of them shortly in the remainder of the section.

As for (α), the following topics deserve special mentioning:

($*$) Lifshitz tails for our models,

($*$) localization phenomena of Laplacians on manifolds with randomly perturbed metrics,

($*$) decay properties of eigenfunctions,

($*$) random Schrödinger operators with magnetic fields.

As for (β), the following directions are of particular interest to us:

($*$) Integrated density of states for foliations: existence and a Šubin trace formula,

($*$) a Šubin trace formula for random graphs,

($*$) integrated density of states for divergence type operators on manifolds.

Some of these topics are subject of our current research.

References

[BBEE+84] V.L. Bonch-Bruevich, R. Enderlein, B. Esser, R. Keiper, A.G. Mirnov, and I.P. Zyvagin. *Elektronentheorie ungeordneter Halbleiter.* VEB Deutscher Verlag der Wissenschaften, 1984. Russisches Original: Moskau, Nauka,1981.

[Bel86] J. Bellissard. K-theory of C^*-algebras in solid state physics. In *Statistical mechanics and field theory: mathematical aspects (Groningen, 1985)*, pages 99–156. Springer, Berlin, 1986.

[Bel92] J. Bellissard. Gap labelling theorems for Schrödinger operators. In *From number theory to physics (Les Houches, 1989)*, pages 538–630. Springer, Berlin, 1992.

[BHZ00] J. Bellissard, D. J. L. Herrmann, and M. Zarrouati. Hulls of aperiodic solids and gap labeling theorems. In *Directions in mathematical quasicrystals*, volume 13 of *CRM Monogr. Ser.*, pages 207–258. Amer. Math. Soc., Providence, RI, 2000.

[CFKS87] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon. *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry.* Text and Monographs in Physics. Springer, Berlin, 1987.

[CH94] J.-M. Combes and P.D. Hislop. Localization for some continuous, random Hamiltionians in d-dimensions. *J. Funct. Anal.*, 124:149–180, 1994.

[CL90] R. Carmona and J. Lacroix. *Spectral Theory of Random Schrödinger Operators.* Birkhäuser, Boston, 1990.

[Con79] A. Connes. Sur la théorie non commutative de l'intégration. In *Algèbres d'opérateurs (Sém., Les Plans-sur-Bex, 1978)*, pages 19–143. Springer, Berlin, 1979.

[EK] P. Exner and S. Kondej. Bound states due to a strong δ interaction supported by a curved surface. submitted to J. Phys. **A**, math-ph/0207025 at arXiv.org.

[ES84] A. L. Efros and B. I. Shklovski. *Electronic Properties of Doped Semi-conductors.* Springer, Berlin, 1984.

[FS83] J. Fröhlich and T. Spencer. Absence of diffusion in the Anderson tight binding model for large disorder or low energy. *Commun. Math. Phys.*, 88:151–184, 1983.

[GK01] F. Germinet and A. Klein. A characterization of the Anderson metal-insulator transport transition. Preprint, www.ma.utexas.edu/mp_arc/, 2001.

[Kel95] J. Kellendonk. Noncommutative geometry of tilings and gap labelling. *Rev. Math. Phys.*, 7(7):1133–1180, 1995.

[Kir89] W. Kirsch. Random Schrödinger operators. In H. Holden and A. Jensen, editors, *Schrödinger Operators*, Lecture Notes in Physics, **345**, Berlin, 1989. Springer.

[Kle96] A. Klein. Spreading of wave packets in the Anderson model on the Bethe lattice. *Comm. Math. Phys.*, 177(3):755–773, 1996.

[KX87] J. Kaminker and J. Xia. The spectrum of operators elliptic along the orbits of $\mathbf{r}^n$ actions. *Comm. Math. Phys.*, 110(3):427–438, 1987.

[Len99] D. H. Lenz. Random operators and crossed products. *Math. Phys. Anal. Geom.*, 2(2):197–220, 1999.

[LGP88] I. M. Lifshitz, S. A. Gredeskul, and L. A. Pastur. *Introduction to the Theory of Disordered Systems*. Wiley, New York, 1988. Russian original: Nauka, Moscow, 1982.

[Lif85] Lifschitz Memorial Issue. *J. Statist. Phys.*, 38(1-2), 1985.

[Lin01] E. Lindenstrauss. Pointwise theorems for amenable groups. *Invent. Math.*, 146(2):259–295, 2001.

[LPPV] D. Lenz, N. Peyerimhoff, O. Post, and I. Veselić. Continuity properties of the integrated density of states on manifolds. in preparation.

[LPVa] D. Lenz, N. Peyerimhoff, and I. Veselić. Integrated density of states random metrics on manifolds. submitted, September 2002.

[LPVb] D. Lenz, N. Peyerimhoff, and I. Veselić. Von Neumann algebras, groupoids and the integrated density of states. (math-ph/0203026 on arXiv.org), submitted, March 2002.

[LS] D. H. Lenz and P. Stollmann. An ergodic theorem for delone dynamical systems and existence of the density of states. in preparation.

[LS02] Daniel Lenz and Peter Stollmann. Quasicrystals, aperiodic order, and groupoid von Neumann algebras. *C. R. Math. Acad. Sci. Paris*, 334(12):1131–1136, 2002.

[Pas71] L. A. Pastur. Selfaverageability of the number of states of the Schrödinger equation with a random potential. *Mat. Fiz. i Funkcional. Anal.*, (Vyp. 2):111–116, 238, 1971.

[PF92] L. A. Pastur and A. L. Figotin. *Spectra of Random and Almost-Periodic Operators*. Springer Verlag, Berlin, 1992.

[PV02] N. Peyerimhoff and I. Veselić. Integrated density of states for ergodic random Schrödinger operators on manifolds. *Geom. Dedicata*, 91(1):117–135, 2002.

[Sto01] P. Stollmann. *Caught by disorder: A Course on Bound States in Random Media*, volume 20 of *Progress in Mathematical Physics*. Birkhäuser, 2001.

[Šub79] M. A. Šubin. Spectral theory and the index of elliptic operators with almost-periodic coefficients. *Uspekhi Mat. Nauk*, 34(2(206)):95–135, 1979.

[Šub82] M. A. Šubin. Density of states of self adjoint operators with almost periodic coefficients. *Amer. Math. Soc. Translations*, 118:307–339, 1982.

[vDK89] H. von Dreifus and A. Klein. A new proof of localization in the Anderson tight binding model. *Commun. Math. Phys.*, 124:285–299, 1989.

[Weg81] F. Wegner. Bounds on the DOS in disordered systems. *Z. Phys. B*, 44:9–15, 1981.

(D. Lenz) Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz, Germany
E-mail address: `dlenz@mathematik.tu-chemnitz.de`
URL: `www.tu-chemnitz.de/mathematik/analysis/dlenz`

(N. Peyerimhoff) Fakultät für Mathematik, Ruhr-Universität Bochum, Germany
E-mail address: `peyerim@math.ruhr-uni-bochum.de`
URL: `www.ruhr-uni-bochum.de/mathematik10/Norbert.html`

(I. Veselić) Postdoctoral research fellow of the Deutsche Forschungsgemeinschaft, visiting the Department of Mathematics 253-37, California Institute of Technology, CA 91125, USA
E-mail address: `veselic@caltech.edu`
URL: `www.its.caltech.edu/~veselic`