Perturbative Analysis of Dynamical Localisation

J. C. A. Barata[1] and D. A. Cortez[2]

Universidade de São Paulo
Instituto de Física
Caixa Postal 66318
São Paulo - 05315 970 - SP - Brasil

Abstract. In this paper we extend previous results on convergent perturbative solutions of the Schrödinger equation of a class of periodically time-dependent two-level systems. The situation treated here is particularly suited for the investigation of two-level systems exhibiting the phenomenon of (approximate) dynamical localisation. We also present a convergent perturbative expansion for the secular frequency and discuss in detail the particular case of monochromatic interactions (ac-dc fields), providing a complete perturbative solution for that case. Our method is based on a "renormalisation" procedure, which we develop in a more systematic way here. For being free of secular terms and uniformly convergent in time, our expansions allow a rigorous study of the long-time behaviour of such systems and are also well-suited for numerical computations, as we briefly discuss, leading to very accurate calculations of quantities like transition probabilities for very long times compared to the cycles of the external field.

1 General Description and Previous Results

The study of periodically or quasi-periodically time-dependent two-level systems is of basic importance for many physical applications, ranging from condensed matter physics to quantum optics, as in problems of the theory of spin resonance, in problems of quantum tunnelling or in the semi-classical theory of the laser. They can be used, for instance, to describe the behaviour of a spin $1/2$ system in a time-dependent magnetic field, in which case the corresponding Schrödinger equation takes the form (we adopt $\hbar = 1$)

$$i\partial_t \Psi \;=\; H(t)\Psi\,, \qquad \text{with} \qquad H(t) \;=\; -\frac{1}{2}\vec{B}(t)\cdot\vec{\sigma}\,, \tag{1.1}$$

where $\Psi(t) = \begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix} \in \mathbb{C}^2$, $\vec{B}(t) = (B_1(t),\, B_2(t),\, B_3(t))$ and $\vec{\sigma} = (\sigma_1,\, \sigma_2,\, \sigma_3)$ are the Pauli matrices.

Systems like this have been analysed by many authors in various approximations, as in the pioneering works of Rabi [8], of Bloch and Siegert [9] and of

[1] Partially supported by CNPq. E-mail: jbarata@if.usp.br
[2] Work supported by FAPESP. E-mail: dacortez@fma.if.usp.br

Autler and Townes [10] (see also [5, 6] for more recent discussions). We should remark, however, that the interest in the solutions of (1.1) is not restricted to the investigation of quantum systems. As first pointed by Feynman, Vernon and Hellwarth [11] (see also the recent discussion in [4]), the quantum system (1.1) is equivalent to the classical Hamiltonian system describing a classical gyromagnet precessing in a magnetic field: $\frac{d}{dt}\vec{\mathcal{S}} = -\vec{B}(t) \times \vec{\mathcal{S}}$, where $\vec{\mathcal{S}}$ is a unit vector.

Of particular interest is the situation where the Schrödinger equation takes the form

$$i\partial_t \Psi(t) \;=\; H_1(t)\Psi(t), \qquad \text{with} \qquad H_1(t) \;:=\; \epsilon\sigma_3 - f(t)\sigma_1, \tag{1.2}$$

where $f(t)$ is a function of time t and $\epsilon \in \mathbb{R}$ is constant. By a time-independent unitary transformation, representing a rotation of $\pi/2$ around the 2-axis, we get the equivalent system

$$i\partial_t \Phi(t) \;=\; H_2(t)\Phi(t), \qquad \text{with} \qquad H_2(t) \;:=\; \epsilon\sigma_1 + f(t)\sigma_3, \tag{1.3}$$

where $\Phi(t) := \exp\left(-i\pi\sigma_2/4\right)\Psi(t)$ and $H_2(t) := \exp\left(-i\pi\sigma_2/4\right)\, H_1(t)\, \exp\left(i\pi\sigma_2/4\right)$.

One can either interpret the system (1.2) as describing a spin 1/2 system as (1.1) under a magnetic field $\vec{B} = (2f(t),\, 0,\, -2\epsilon)$, or as a system with an unperturbed diagonal Hamiltonian $H_0 := \epsilon\sigma_3$, representing a two-level system with energy levels $\pm\epsilon$, subjected to a time-dependent perturbation $H_I(t) := -f(t)\sigma_1$, inducing a time-depending transition between the unperturbed eigenstates of H_0. The equivalent system (1.3), in turn, represents either a spin 1/2 system as (1.1) under a magnetic field $\vec{B} = (-2\epsilon,\, 0,\, -2f(t))$, or a two-level system composed by two uncoupled (for $\epsilon = 0$) orthogonal time-dependent states $\exp\left(-i\int_0^t f(\tau)d\tau\right)\binom{1}{0}$ and $\exp\left(+i\int_0^t f(\tau)d\tau\right)\binom{0}{1}$, subjected to a constant perturbation $\epsilon\sigma_1$ inducing a transition between them.

To explain the purpose of the present paper, we have to describe some of our previous results. In [1] and [2] we studied the system described by (1.2) or (1.3) in the situation where f is a periodic or quasi-periodic function of time and ϵ is "small". It is well know that the usual perturbative approach, based, f.i., on the Dyson series, leads to difficulties involving secular terms (i.e., polynomials in t that appear order by order in perturbation theory and spoil the uniform convergence (in t) of the perturbative series) and, for quasi-periodic interactions, small denominators. This last problem is typical of perturbative approximations for solutions of differential equations with quasi-periodic coefficients and is well-known as one of the main sources of problems in the mathematically precise treatment of such equations.

In [1] and [2], a special perturbative expansion (power series expansion in ϵ) was developed, whose main virtue is to be free of secular terms. The algorithm employed involves an inductive "renormalization" of a sort of effective field introduced through an exponential Ansatz (the function g, to be introduced below). For the sake of the reader we will shortly recall our method of elimination of secular terms in Section 2. In the general case where f is *quasi-periodic*, it was established in [1] that the coefficients of the expansion are also well-defined quasi-periodic functions of time but, due mainly to the presence of small denominators, we were not able to prove convergence of our ϵ-expansion. Actually, a convergent power expansion in ϵ is not expected without further assumptions (for a detailed analysis of these issues in related systems, see [12]).

Less problematic is the situation where f is a *periodic* function, when the obstacle represented by the small denominators is naturally absent. In [2], we showed how the difficulties analysed in [1] can be circumvented in the case of periodic f and we were able to establish the convergence of our perturbative ϵ-expansion uniformly in $t \in \mathbb{R}$.

As discussed in [2], our method not only recovers the Floquet form of the solution of the time-depending Schrödinger equation (see (1.8)-(1.9) below), but also allows the computation of the secular frequency and of the Fourier coefficients in terms of explicit *convergent* ϵ-expansions, what constitutes a feature of our algorithm, compared to other expansion methods.

Due to the technical difficulties involved, we restricted our analysis in [2] to two classes of periodic functions, namely those satisfying the conditions (I) or (II) presented below (see [2]). Our purpose in the present paper is to extend the results of [2] to an additional class of periodic functions. The inclusion of this additional class leads to a essentially complete perturbative solution for some simple periodic functions, as $f(t) = F_0 + \varphi \cos(\omega t)$, representing the important case of a monochromatic interaction (also known as ac-dc fields).

The situation we treat here is also relevant for the rigorous discussion of the phenomenon of dynamical localisation, also known (less properly) as coherent destruction of tunnelling. This phenomenon, first pointed in [14], indicates the possibility to (approximately) freeze the initial state of a quantum system through the action of a suitable external time-dependent interaction. This effect has been the object of various recent investigations. In [7], for instance, a rigorous general criterion for the occurrence of dynamical localisation was established and applied to interesting situations, like the ac-dc field and the bichromatic field. Some of the conclusions of [7] on the ac-dc field are indirectly reproduced in Section 5, below. We refer the reader to [3] and [7] for more references on this subject.

The main result of [2] can be captured in the next theorem, for whose statement we need a definition we will repeatedly use in this work: for an almost periodic function h we denote by $M(h)$ the "mean value" of h, defined as

$$M(h) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} h(t)\, dt \,. \tag{1.4}$$

We remark that the limit in (1.4) is always well defined for any quasi-periodic function h. The mean value $M(h)$ equals the constant term in the Fourier expansion of h. Details can be found in [15, 16].

Theorem 1.1 *Let f be a real T_ω-periodic function of time ($T_\omega := 2\pi/\omega$ with $\omega > 0$) whose Fourier decomposition $f(t) = \sum_{n\in\mathbb{Z}} F_n e^{in\omega t}$, contains only a finite number of terms, i.e., the set of integers $\{n \in \mathbb{Z} | \, F_n \neq 0\}$ is a finite set. Let*

$$\Phi(t) = \begin{pmatrix} \phi_+(t) \\ \phi_-(t) \end{pmatrix} = U(t)\Phi(0) = U(t,\,0)\Phi(0) \tag{1.5}$$

be the solution of the Schrödinger equation (1.3). Consider the two following distinct conditions on f:

(I) $M(\mathcal{Q}_0) \neq 0$.

(II) $M(\mathcal{Q}_0) = 0$ *but* $M(\mathcal{Q}_1) \neq 0$, *where*

$$q(t) := \exp\left(i\int_0^t f(\tau)d\tau\right), \quad \mathcal{Q}_0(t) := q(t)^2 \;=\; \exp\left(2i\int_0^t f(\tau)d\tau\right) \tag{1.6}$$

and

$$\mathcal{Q}_1(t) \;:=\; \mathcal{Q}_0(t)\int_0^t \left(\mathcal{Q}_0(\tau)^{-1} - M\left(\mathcal{Q}_0^{-1}\right)\right) d\tau. \tag{1.7}$$

Then, for each f as above, satisfying condition (I) or (II), there exists a constant $K > 0$ (depending on the Fourier coefficients $\{F_n,\; n \in \mathbb{Z}\;, n \neq 0\}$ and on ω) so that, for each ϵ with $|\epsilon| < K$, there are $\Omega \in \mathbb{R}$ and T_ω-periodic functions $u_{11}^{\pm}$ and $u_{12}^{\pm}$ such that the propagator $U(t)$ of (1.5) can be written as

$$U(t) \;=\; \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ & \\ U_{21}(t) & U_{22}(t) \end{pmatrix} \;=\; \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ & \\ -\overline{U_{12}(t)} & \overline{U_{11}(t)} \end{pmatrix},$$

with

$$U_{11}(t) \;=\; e^{-i\Omega t}\, u_{11}^{-}(t) + e^{i\Omega t}\, u_{11}^{+}(t), \quad U_{12}(t) \;=\; e^{-i\Omega t}\, u_{12}^{-}(t) + e^{i\Omega t}\, u_{12}^{+}(t). \tag{1.8}$$

The functions $u_{11}^{\pm}$ and $u_{12}^{\pm}$ have absolutely and uniformly converging Fourier expansions

$$u_{11}^{\pm}(t) \;=\; \sum_{n\in\mathbb{Z}} \mathcal{U}_{11}^{\pm}(n)e^{in\omega t}, \qquad u_{12}^{\pm}(t) \;=\; \sum_{n\in\mathbb{Z}} \mathcal{U}_{12}^{\pm}(n)e^{in\omega t}. \tag{1.9}$$

Moreover, under the same assumptions, Ω and the Fourier coefficients $\mathcal{U}_{11}^{\pm}(n)$ and $\mathcal{U}_{12}^{\pm}(n)$ can be expressed in terms of absolutely converging power series on ϵ. $\square$

Let us now discuss the conditions (I) and (II) of Theorem 1.1. Writing the Fourier decomposition of f as $f(t) = F_0 + \sum_{n=1}^{J}\left[\varphi_1^{(n)}\cos(n\omega t) + \varphi_2^{(n)}\sin(n\omega t)\right]$ the set $\mathfrak{F}_{J,\,\omega}$ of all possible functions f with a given J and ω can be identified with the parameter space $\mathbb{R}^{2J+1}$ of all real coefficients F_0, $\varphi_{1,2}^{(n)}$, $1 \leq n \leq J$. The (complex) condition $M(\mathcal{Q}_0) = 0$ determines a $(2J)$ or $(2J-1)$-dimensional subset of $\mathfrak{F}_{J,\omega}$, where condition (II) eventually applies. It is also on this subset that the more restrictive condition $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = 0$ should hold, restricting the parameter space of f to a $(2J-1)$, $(2J-2)$ or $(2J-3)$-dimensional subset, if it is non-trivial. One should, therefore, expect that successive conditions like (I) and (II) would eventually exhaust completely the set $\mathfrak{F}_{J,\omega}$.

To illustrate all this, let us consider the simplest example, when f represents a monochromatic interaction: $f(t) = \varphi_1 \cos(\omega t) + \varphi_2 \sin(\omega t)$, with $(\varphi_1, \varphi_2) \in \mathbb{R}^2$. A simple computation shows that $M(\mathcal{Q}_0) = e^{2i\gamma_f} J_0\left(\frac{2\varphi_0}{\omega}\right)$, where $\varphi_0 := \sqrt{\varphi_1^2 + \varphi_2^2}$, J_0 is the Bessel function of first kind and order zero and, in this case, $\gamma_f = \varphi_2/\omega$. Moreover, $M(\mathcal{Q}_1) = 0$ for <u>all</u> $(\varphi_1, \varphi_2) \in \mathbb{R}^2$. (See [2] for details). Hence, condition (I) is satisfied for all $(\varphi_1, \varphi_2) \in \mathbb{R}^2$, except in the circles defined by $\varphi_0 = \omega x_a/2$, $a = 1, 2, \ldots$, where x_a if the a-th zero of J_0 in $\mathbb{R}_+$. Condition (II), however, is never fulfilled in this case. To achieve a complete solution we have, therefore, to extend Theorem 1.1 to include further conditions beyond (I) and (II), holding on the circles $\varphi_0 = \omega x_a/2$.

The purposes of this paper are to identify the first condition following (I) and (II), which we call condition (III), to show that the method of elimination of secular terms holds in this case as well and, for periodic interactions, to show that the expansion (2.5) converges for $|\epsilon|$ sufficiently small, uniformly for $t \in \mathbb{R}$. As we will discuss, this leads to a complete perturbative solution for the monochromatic interaction. As we will see, the identification of condition (III) and the application of the method of elimination of secular terms to it are highly non-trivial tasks.

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This paper is organised as follows. In Section 2 we describe the general strategy employed to eliminate the secular terms and present our main theorems. In Section 3 we prove some interesting and useful mathematical results on the mean value of some quasi-periodic functions, introduce the "renormalisation" operation, important to organise our procedures, and introduce some notations we will use throughout the paper. In Section 4, which is the technically central piece or this work, we apply our strategy of elimination of the secular terms to the situation we wish to analyse. In Section 5 we apply our results to case of monochromatic interactions and discuss the issue of dynamical localisation in this case. In Section 6 we briefly describe some numerical calculations based on our results. Sections 5 and 6 contain some of the physical applications of our work. In Appendix A we treat several results used in Section 4, whose proofs unfortunately involve too many algebraic computations to be included in the main text. Appendix B is dedicated to the proof of convergence of our expansions for the periodic case. Finally, Appendix C contains some comments on the Fourier coefficients of the wave functions.

1.1 Comments on the Notation

In this paper, $\mathbb{Z}_+$ will denote the set of all non-negative integers (zero included) and $\mathbb{Z}_*$ the set of all integers, excluding zero. $\mathbb{Z}_{*+}$ is the set of all positive integers. These notations are also applied to $\mathbb{Z}^A$ and to the real line $\mathbb{R}$.

Vectors in $\mathbb{Z}^A$ (or $\mathbb{R}^A$) will be written as $\underline{v}$. The operation $\underline{v} \cdot \underline{u}$ will denote the scalar product in $\mathbb{Z}^A$ (or $\mathbb{R}^A$), defined as $\underline{v} \cdot \underline{u} := v_1 u_1 + \cdots + v_A u_A$.

For a quasi-periodic function $h : \mathbb{R} \to \mathbb{C}$, we write its Fourier decomposition as $h(t) = \sum_{\underline{m} \in \mathbb{Z}^A} H_{\underline{m}} e^{i\underline{m}\cdot\underline{\omega}_h t}$, where A is some positive integer and $\underline{\omega}_h \in \mathbb{R}_+^A$ (see, e.g., [15, 16]). The Fourier coefficient $H_{\underline{0}}$ will be denoted simply as H_0.

For $m \in \mathbb{Z}$ we denote by $\ll m \gg$ the following function:

$$\ll m \gg := \begin{cases} |m|, & \text{for } m \neq 0 \\ 1, & \text{for } m = 0 \end{cases} . \tag{1.10}$$

Beyond the functions $\mathcal{Q}_0$ and $\mathcal{Q}_1$ defined in (1.6) and (1.7) we will frequently use the following functions

$$\mathcal{Q}_2(t) \quad := \quad \mathcal{Q}_0(t) \int_0^t \left(\mathcal{Q}_0(\tau) - M\left(\mathcal{Q}_0\right)\right) d\tau, \tag{1.11}$$

$$\mathcal{Q}_3(t) \quad := \quad \mathcal{Q}_0(t) \int_0^t \left(\mathcal{Q}_1(\tau) - M\left(\mathcal{Q}_1\right)\right) d\tau. \tag{1.12}$$

Note that, by their definitions, the functions $\mathcal{Q}_i$, $i = 0, \ldots, 3$, are quasi-periodic if f is quasi-periodic. We will have more to say about their properties below.

2 Elimination of Secular Terms. The Main Results

We recall in this section some of the methods and techniques developed in the previous works [1] and [2]. A key result for our method is the theorem below, proven in [1], which presents the solution of the Schrödinger equation (1.3) in terms of particular solutions of a generalised Riccati equation.

Theorem 2.1 *Let $f : \mathbb{R} \to \mathbb{R}$, $f \in C^1(\mathbb{R})$ and $\epsilon \in \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{C}$, $g \in C^1(\mathbb{R})$, be a particular solution of the generalised Riccati equation*

$$\dot{G} - iG^2 - 2ifG + i\epsilon^2 \;=\; 0. \tag{2.1}$$

Then, the function $\Phi : \mathbb{R} \to \mathbb{C}^2$ given by $\Phi(t) = \left(\begin{smallmatrix} \phi_+(t) \\ \phi_-(t) \end{smallmatrix}\right) = U(t)\Phi(0) = U(t, 0)\Phi(0)$, where

$$U(t) \;:=\; \begin{pmatrix} R(t)\left(1 + ig(0)S(t)\right) & -i\epsilon R(t)S(t) \\ \\ -i\epsilon\overline{R(t)}\,\overline{S(t)} & \overline{R(t)}\left(1 - i\,\overline{g(0)}\,\overline{S(t)}\right) \end{pmatrix}, \tag{2.2}$$

with

$$R(t) \;:=\; \exp\left(-i\int_0^t (f(\tau)+g(\tau))\,d\tau\right) \qquad \textit{and} \qquad S(t) \;:=\; \int_0^t R(\tau)^{-2}\,d\tau,$$

is a solution of (1.3) with initial value $\Phi(0) = \left(\begin{smallmatrix}\phi_+(0)\\ \phi_-(0)\end{smallmatrix}\right) \in \mathbb{C}^2$. □

Let us briefly describe some of the ideas leading to Theorem 2.1 and to other results of [1]. As we saw in [1], the solutions of the Schrödinger equation (1.3) can be studied in terms of the solutions of a particular complex version of Hill's equation:

$$\ddot{\phi}(t) + \left(i\dot{f}(t) + \epsilon^2 + f(t)^2\right)\phi(t) \;=\; 0. \tag{2.3}$$

In fact, a simple computation shows that the components $\Phi(t)$ satisfy $\ddot{\phi}_{\pm} + (\pm i\dot{f} + \epsilon^2 + f^2)\phi_{+} = 0$. If we attempt to solve (2.3) using the Ansatz

$$\phi(t) = \exp\left(-i\int_0^t (f(\tau)+g(\tau))d\tau\right), \tag{2.4}$$

it follows that g has to satisfy the generalised Riccati equation (2.1). We then try to find solutions for g in terms of a power expansion in ϵ (vanishing for $\epsilon = 0$) like

$$g(t) \;=\; q(t)\sum_{n=1}^{\infty} v_n(t)\,\epsilon^n\,, \tag{2.5}$$

where the function q was defined in (1.6) and is of central importance in this work.

The heuristic idea behind the Ansätze (2.4) and (2.5) is the following. For $\epsilon \equiv 0$ a solution for (2.3) is given by $\exp\left(-i\int_0^t f(\tau)d\tau\right)$. Thus, in (2.4) and (2.5) we are searching for solutions in terms of an "effective external field" of the form $f+g$, with g given in terms of a convergent power series expansion in ϵ, vanishing for $\epsilon = 0$. A solution of the form (2.4) leads to one of the two independent solutions of (2.3). The full solution of (1.3) in terms of solutions of the generalised Riccati equation (2.1) is that described in Theorem 2.1 (see the discussion of [1]).

We then proceed inserting (2.5) into (2.1). The result is a set of recursive first order linear differential equations for the functions v_n which can be easily integrated. The solutions of these equations are

$$v_1(t) \;=\; \kappa_1 q(t)\,, \tag{2.6}$$

$$v_2(t) \;=\; q(t)\left[i\int_0^t \left(\kappa_1^2 \mathcal{Q}_0(\tau) - \mathcal{Q}_0(\tau)^{-1}\right)d\tau + \kappa_2\right], \tag{2.7}$$

$$v_n(t) \;=\; q(t)\left[i\left(\int_0^t \sum_{p=1}^{n-1} v_p(\tau)v_{n-p}(\tau)\,d\tau\right) + \kappa_n\right], \quad \text{for } n \geq 3\,, \tag{2.8}$$

where the κ_n's above, $n = 1, 2, \ldots$, are arbitrary integration constants. Defining

$$\mathcal{I}_2(t) \ := \ \kappa_1^2 \mathcal{Q}_0(t) - \mathcal{Q}_0(t)^{-1}, \qquad \mathcal{I}_n(t) \ := \sum_{p=1}^{n-1} v_p(t) v_{n-p}(t), \quad n \geq 3.$$

we can write (2.6)-(2.8) as

$$v_1(t) \ = \ \kappa_1 q(t), \qquad v_n(t) \ = \ iq(t) \int_0^t \mathcal{I}_n(\tau) d\tau + \kappa_n q(t), \quad n \geq 2.$$

Observe that, in particular, we could just set all the κ_n's equal to zero. However, this is not a clever choice, since it would result in polynomial terms on t (the so-called *Secular Terms*) for the series expansion (2.5) of g. This, of course, would restrict the convergence of the series just for small values of time. As noticed in [1], there is a choice of the constants κ_n for which one can eliminate completely all the polynomial terms on t that would eventually appear in g. This procedure, which we call the *Elimination of Secular Terms*, will be briefly described now.

First of all, assuming that the function $f : \mathbb{R} \to \mathbb{R}$ is quasi-periodic, it was proven in Appendix B of [1] that q, defined in (2.5), is also quasi-periodic. Hence, v_1 in (2.6) is quasi-periodic. The same is true for the integrand $\mathcal{I}_2$ which appears in v_2, equation (2.7). Recalling that $\mathcal{I}_2$ depends on the free integration constant κ_1, the key idea is to fix κ_1 in such a way that the mean value of $\mathcal{I}_2$ is equal to zero, that is $M(\mathcal{I}_2) = M(\kappa_1^2 \mathcal{Q}_0 - \mathcal{Q}_0^{-1}) = 0$. Since $\mathcal{Q}_0$ is a quasi-periodic function, it readily follows from this that

$$\kappa_1^2 \ = \ \frac{\overline{M(\mathcal{Q}_0)}}{M(\mathcal{Q}_0)} . \tag{2.9}$$

With this choice of κ_1 one guarantees the absence of a constant term in the Fourier expansion of $\mathcal{I}_2$. Since $\mathcal{I}_2$ is being integrated in time, this would imply the absence of a linear term on t in the final expression for v_1. An important remark is that (2.9) will only make sense if we assume $M(\mathcal{Q}_0) \neq 0$.

Under this assumption we can now proceed and fix recursively all integration constants κ_m's by imposing a zero mean value for the integrands $\mathcal{I}_n$'s, $n = 3, 4, \ldots$, which appear in (2.8). This procedure removes, order by order in ϵ, the presence of the secular terms in the series expansion (2.5) for g and recursively implies that all functions v_n are quasi-periodic. Once all secular terms have been removed, one can write the Fourier expansion for the functions v_n as

$$v_n(t) \ = \ \sum_{\underline{m} \in \mathbb{Z}^A} V_{\underline{m}}^{(n)} e^{i\underline{m}\cdot\underline{\omega}t} , \tag{2.10}$$

provided the sum converges absolutely. It was shown in [1] that this is indeed true. The proof of this fact was performed in the following way: first it was shown that the Fourier coefficients $Q_{\underline{m}}$ of the function q satisfies the bound $|Q_{\underline{m}}| \leq \mathcal{Q} e^{-\chi|\underline{m}|}$, for some $\chi > 0$. Then, the method of elimination of secular terms described above was applied to fix the integration constants κ_n leading to inductive bounds of the form $|V_{\underline{m}}^{(n)}| \leq \mathcal{K}_n e^{-(\chi-\delta_n)|\underline{m}|}$ for the Fourier coefficients of the function v_n, where $0 < \delta_n < \chi$, for all $n = 1, 2, \ldots$. This exponential decay is enough to proof the convergence of the sum in (2.10) and to establish by induction the quasi-periodicity of all the functions v_n. Unfortunately, due to the bad behaviour in n of the constants $\mathcal{K}_n$, it was not possible prove the convergence of the ϵ-expansion (2.5), hence (2.5) has to be seen as a formal quasi-periodic power series solution of the generalised Riccati equation (2.1).

The reason for the bad behaviour of $\mathcal{K}_n$ is related to the presence of convolutions and to the *small denominators* appearing in the recursive relations for the coefficients $V_{\underline{m}}^{(n)}$. A general discussion of these problems is found in [1]. However, in the situation where f is a *periodic* function, stronger results are possible. In [2], where this situation was studied, it was possible to prove the convergence of the power series (2.5) and uniform convergence of the Fourier series involved in the computation of the wave functions. Moreover, absolute convergence of the ϵ-expansions leading to the *secular frequency* and to the coefficients of the Fourier expansion of the wave functions was also proven.

All the work done in [1] and [2] was restricted to one of the mutually exclusive conditions (I) and (II) of Theorem 1.1. These conditions are consequences of the method of elimination of secular terms. Clearly, (I) is vital for (2.9). When (I) is not satisfied we have to apply condition (II). Both cases were studied in [1] and [2], where the method of elimination of secular terms has been applied and equivalent results concerning the solution g were obtained. In the present work, we apply the method of elimination of secular terms to study a more restrictive condition than those represented by (I) and (II). Namely, we are concerned here with the situation where $M(\mathcal{Q}_0) = 0$ and also $M(\mathcal{Q}_1) = 0$.

In this case, the complexity of the calculations involved to find the right choice of constants κ_n's grows enormously, in contrast with those needed in the cases (I) and (II), already studied. The reason for that is quite simple: due to the hypothesis $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = 0$ one needs to work explicitly with higher order terms involved in the expansion (2.5). This will become more clear in Section 4.

In Section 5 we will discuss an important example where conditions (I) and (II) are not satisfied. To solve it, we have to apply the solution (free of secular terms) obtained here. We present in Section 6 numerical calculations on this particular example and obtained some interesting results.

We are ready now to state two main theorems of this work.

Theorem 2.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a real quasi-periodic function satisfying*

$$\text{(III)} \quad M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = 0, \text{ but } M(\mathcal{Q}_3) \neq 0.$$

Then, there are constants κ_n, $n \geq 1$, such that all functions v_n given in (2.6)-(2.8) are quasi-periodic. The explicit recursive expressions for the constants κ_n are found in (4.36)-(4.38). □

The proof of this theorem is the main content of Section 4. It states that the procedure of elimination of secular terms outlined above also works under condition (III). When f is quasi-periodic this does not imply, however, that the formal solution (2.5) of the generalised Riccati equation (2.1) converges, since we have the same difficulties discussed in detail in [1].

For periodic f, the situation is different and stronger results can be proven. Let $f(t) = \sum_{m\in\mathbb{Z}} F_m e^{im\omega t}$ be a real periodic function with frequency ω. If $F_0 = M(f) = 0$, q and $\mathcal{Q}_0 = q^2$ are also periodic and their spectra of frequencies are subsets of $\{n\omega,\ n \in \mathbb{Z}\}$. Following the notation employed in [2], we write the Fourier expansions of q and $\mathcal{Q}_0$ as

$$q(t) \ = \ \sum_{m\in\mathbb{Z}} Q_m e^{im\omega t}, \qquad \mathcal{Q}_0(t) \ = \ q(t)^2 \ = \ \sum_{m\in\mathbb{Z}} Q_m^{(2)} e^{im\omega t}. \tag{2.11}$$

By relations (2.6)-(2.8) and with the choice of constants κ_n mentioned in Theorem 2.2 (see (4.36)-(4.38)), the functions v_n are also periodic and their spectra of frequencies are also subsets of $\{n\omega,\ n \in \mathbb{Z}\}$. We write their Fourier expansions as

$$v_n(t) \ = \ \sum_{m\in\mathbb{Z}} V_m^{(n)} e^{im\omega t}. \tag{2.12}$$

In Appendix B we prove the theorem below, which justifies our whole procedure for the case of periodic interactions and establishes convergence of (2.5).

Theorem 2.3 *Let $f(t)$ be as above with $F_0 = M(f) = 0$ and such that condition (III) of Theorem 2.2 is satisfied. Moreover, assume that the coefficients Q_m and $Q_m^{(2)}$ above satisfy the following: for any $\chi > 0$ there is a positive constant $\mathcal{Q} \equiv \mathcal{Q}(\chi)$ such that*

$$|Q_m| \ \leq \ \mathcal{Q}\frac{e^{-\chi|m|}}{\ll m \gg^2} \qquad \textit{and} \qquad \left|Q_m^{(2)}\right| \ \leq \ \mathcal{Q}\frac{e^{-\chi|m|}}{\ll m \gg^2}, \tag{2.13}$$

for all $m \in \mathbb{Z}$, where the symbol $\ll m \gg$ was defined in (1.10). Then, with the constants κ_n fixed as in Theorem 2.2, the Fourier coefficients of the functions v_n given in (2.6)-(2.8) satisfy

$$\left|V_m^{(n)}\right| \ \leq \ M_0\,(M_1)^n\,\frac{e^{-\chi|m|}}{\ll m \gg^2}$$

for all $m \in \mathbb{Z}$ and all $n \geq 1$, for some positive constants M_0 and M_1. As a consequence, the power series expansion (2.5), representing a solution of the generalised Riccati equation (2.1), converges uniformly for $t \in \mathbb{R}$, provided $|\epsilon| < 1/M_1$. □

Remark 2.4 *In [1], (2.13) was established for f is periodic and represented by a finite Fourier series. The condition $F_0 = M(f) = 0$ above is not crucial and can be eliminated following the procedure described in [2].*

It follows from this theorem that the main consequences of Theorem 1.1 are valid under condition (III) as well. In particular, the Floquet form (1.8) holds and the secular frequency Ω and the Fourier coefficients of (1.9) are analytic functions of ϵ for $|\epsilon|$ small enough. See Appendix C for a comment on this.

2.1 The Secular Frequency

A feature of our method is that it allows to present the complete ϵ-expansion for the secular frequency Ω (also known as Rabi frequency) associated to the solutions of (1.2)-(1.3) (see (1.8)). One has (see [1, 2, 3])

$$\Omega \ = \ M(f) + M(g) = F_0 + \sum_{n=1}^{\infty} \epsilon^n M\left(qv_n\right) \, .$$

By Theorem 2.3 above, this expansion is convergent for $|\epsilon|$ small enough. The knowledge of the complete expansion is particularly important for the qualitative investigation of the large-time behaviour of that solutions. After some simple calculations using (2.6)-(2.8) one gets

$$\begin{aligned} \Omega \ &= \ F_0 + \epsilon\kappa_1 M(\mathcal{Q}_0) + \epsilon^2 \left[i\kappa_1^2 M(\mathcal{Q}_2) - iM(\mathcal{Q}_1) + \kappa_2 M(\mathcal{Q}_0) \right] \\ &\quad + \epsilon^3 \left[2\kappa_1 M(\mathcal{Q}_3) + \kappa_3 M(\mathcal{Q}_0) \right] + O(\epsilon^4) \, . \end{aligned} \tag{2.14}$$

As we will show in Corollary 3.2 below, $M(\mathcal{Q}_0) = 0$ implies $M(\mathcal{Q}_2) = 0$. Hence, for case (II),

$$\Omega \ = \ F_0 - i\epsilon^2 M(\mathcal{Q}_1) + 2\kappa_1 \epsilon^3 M(\mathcal{Q}_3) + O(\epsilon^4) \, , \tag{2.15}$$

and for case (III),

$$\Omega \ = \ F_0 + 2\kappa_1 \epsilon^3 M(\mathcal{Q}_3) + O(\epsilon^4) \, . \tag{2.16}$$

Actually, after fixing κ_1 in Section 4.1, we will see that $\Omega = F_0 + 2\epsilon^3 \left| M(\mathcal{Q}_3) \right| + O(\epsilon^4)$.

In this case (III), if one additionally has $F_0 = 0$, then $\Omega = O(\epsilon^3)$, a fact first pointed in [3]. This implies long transition times for certain probability amplitudes, a phenomenon known as (approximate) dynamical localisation (see [7] and other references therein). In Sections 5 and 6 we discuss this situation for f describing a monochromatic interaction.

3 Properties of the Mean Value and the Renormalisation Operator

Let us introduce some notations that will be very useful. Since expressions like $f(\xi) \int_0^\xi d\xi' \, g(\xi')$ will often appear throughout our calculations, we define a shorthand notation

$$(f\mid g)_\xi \;:=\; f(\xi)\int_0^\xi d\xi'\,g(\xi')\,. \tag{3.1}$$

Moreover, if $(f\mid g)_t$ is quasi-periodic function on t, then $M(f\mid g)$ will denote its mean value. We also define

$$(f\mid g\mid h)_\xi \;:=\; \Big(f\,\Big|\,(g\mid h)_{\xi'}\Big)_\xi \;=\; f(\xi)\int_0^\xi d\xi'\,g(\xi')\int_0^{\xi'} d\xi''\,h(\xi'')\,. \tag{3.2}$$

Further compositions like $(f_1\,|f_2|\cdots|\,f_n)_\xi$ are defined in analogous way, so that, for $n>2$,

$$(f_1\,|f_2|\cdots|\,f_n)_\xi \;:=\; \Big(f_1\,\Big|\,(f_2|\cdots|\,f_n)_{\xi'}\Big)_\xi\,.$$

Note, finally, the trivial fact that

$$f_0(\xi)(f_1\,|f_2|\cdots|\,f_n)_\xi \;=\; (f_0f_1\,|f_2|\cdots|\,f_n)_\xi\,. \tag{3.3}$$

3.1 Properties of the Mean Value

The following general results on the mean value of some quasi-periodic functions (see definition (1.4)) will be used for many purposes in the present work.

Proposition 3.1 *Let $a:\mathbb{R}\to\mathbb{C}$, $b:\mathbb{R}\to\mathbb{C}$ and $c:\mathbb{R}\to\mathbb{C}$ be quasi-periodic functions with Fourier components denoted by $A_{\underline{m}}$, $B_{\underline{m}}$ and $C_{\underline{m}}$, $\underline{m}\in\mathbb{Z}^A$, respectively. We have the following statements:*

1. *If $M(c)=0$, then the Fourier components of the function $h(t):=(b\mid c)_t$ are given by*

$$H_{\underline{n}} \;=\; \sum_{\underline{m}\in\mathbb{Z}^A_*}\frac{i\,C_{\underline{m}}\left(B_{\underline{n}}-B_{\underline{n}-\underline{m}}\right)}{\underline{m}\cdot\underline{\omega}}\,, \tag{3.4}$$

   for all $\underline{n}\in\mathbb{Z}^A$. Moreover, if $M(b)=0$ and $C_{\underline{m}}B_{-\underline{m}}=C_{-\underline{m}}B_{\underline{m}}$ for all $\underline{m}\in\mathbb{Z}^A_*$, then $M(h)=0$.

2. *If $M(b)=0$ and $M(c)=0$, then $M(b\mid c)=-M(c\mid b)$.*

3. *If $M(c)=0$, then $M(c\mid c)=0$.*

4. *If $M(a)=M(c)=M(b\mid c)=0$, then*

$$M(a\mid b\mid c) \;=\; -M\left[b(t)\left(\int_0^t a(\tau)\,d\tau\right)\left(\int_0^t c(\tau)\,d\tau\right)\right]. \tag{3.5}$$

   Moreover, if also $M(b\mid a)=0$, then $M(a\mid b\mid c)=M(c\mid b\mid a)$.

5. *If $M(a) = M(b) = M(c) = 0$ and $M(b \mid c) = M(c \mid a) = M(a \mid b) = 0$, then*

$$M(a \mid b \mid c) + M(b \mid c \mid a) + M(c \mid a \mid b) \;=\; 0\,. \tag{3.6}$$

6. *If $M(a) = 0$, it follows from (3) and (5) that $M(a \mid a \mid a) = 0$.* □

The identity (3.6) is very remarkable. We have noticed that some non-trivial relations follow from it, specially if one write (3.6) in terms of the Fourier coefficients of a, b and c.

Proof. We can demonstrate (1) by explicitly computing the Fourier decomposition of $h(t)$. Thus,

$$\begin{aligned} h(t) &= \sum_{\underline{n}\in\mathbb{Z}^A} B_{\underline{n}} e^{i\underline{n}\cdot\underline{\omega} t}\left[\sum_{\underline{m}\in\mathbb{Z}^A_*} \frac{C_{\underline{m}}}{i\underline{m}\cdot\underline{\omega}}\left(e^{i\underline{m}\cdot\underline{\omega} t}-1\right)\right] \\ &= \sum_{\underline{n}\in\mathbb{Z}^A}\left[\sum_{\underline{m}\in\mathbb{Z}^A_*} \frac{i\,C_{\underline{m}}\left(B_{\underline{n}}-B_{\underline{n}-\underline{m}}\right)}{\underline{m}\cdot\underline{\omega}}\right] e^{i\underline{n}\cdot\underline{\omega} t}\,, \end{aligned}$$

proving the first statement of (1). Now, if $M(b) = B_0 = 0$ and $C_{\underline{m}}B_{-\underline{m}} = C_{-\underline{m}}B_{\underline{m}}$ for all $\underline{m} \in \mathbb{Z}^A_{*+}$, then

$$H_0 \;=\; \sum_{\underline{m}\in\mathbb{Z}^A_*} \frac{-i\,C_{\underline{m}}B_{-\underline{m}}}{\underline{m}\cdot\underline{\omega}} \;=\; \frac{-i}{2}\sum_{\underline{m}\in\mathbb{Z}^A_*} \frac{\left(C_{\underline{m}}B_{-\underline{m}} - C_{-\underline{m}}B_{\underline{m}}\right)}{\underline{m}\cdot\underline{\omega}} \;=\; 0\,. \tag{3.7}$$

Since $M(h) = H_0$, we completed the proof of (1). To demonstrate (2) we simply use the first equality of (3.7) to write

$$M(b \mid c) \;=\; \sum_{\underline{m}\in\mathbb{Z}^A_*} \frac{-i\,C_{\underline{m}}B_{-\underline{m}}}{\underline{m}\cdot\underline{\omega}} \qquad \text{and} \qquad M(c \mid b) \;=\; \sum_{\underline{m}\in\mathbb{Z}^A_*} \frac{-i\,B_{\underline{m}}C_{-\underline{m}}}{\underline{m}\cdot\underline{\omega}}\,.$$

Changing $\underline{m} \to -\underline{m}$ in the second equation above, we get the desired claim.

Statement (3) is a mere consequence of (2) when we take $b = c$. Statement (4) can be proven using (2). Indeed, let $h(t) := (b \mid c)_t$. Since $M(h) = M(b \mid c) = 0$ and $M(a) = 0$, by (2), we can write

$$\begin{aligned} M(a \mid h) \;=\; -M(h \mid a) &= -M\left(h(t)\int_0^t a(\tau)\,d\tau\right) \\ &= -M\left[b(t)\int_0^t c(\tau)\,d\tau\left(\int_0^t a(\tau)\,d\tau\right)\right]\,, \end{aligned}$$

which proves the first claim. To prove the second one, all we need to do is to interchange the the roles of c and a in the last equality (note that since $M(b \mid a) =$

0, the mean value of $(c \mid b \mid a\,)_t$ is well defined). Finally, statement (5) can be easily obtained as follows. Using (3.5), writing $b(t) = \frac{d}{dt}\int_0^t b(\tau)d\tau$, using definition (1.4) and integration by parts, one gets

$$
\begin{aligned}
M\,(a \mid b \mid c\,) \quad &= \quad -\lim_{T\to\infty} \frac{1}{2T} \left(\int_0^t b(\tau)\, d\tau\right) \left(\int_0^t a(\tau)\, d\tau\right) \left(\int_0^t c(\tau)\, d\tau\right)\Big|_{-T}^{T} \\
&\quad + M\left(\left(\int_0^t b(\tau)\, d\tau\right) a(t) \left(\int_0^t c(\tau)\, d\tau\right)\right) \\
&\quad + M\left(\left(\int_0^t b(\tau)\, d\tau\right) \left(\int_0^t a(\tau)\, d\tau\right) c(t)\right) .
\end{aligned}
$$

Now, since $M(a) = M(b) = M(c) = 0$, the integrals in the first line of (3.8) are all bounded. Thus, the limit $T \to \infty$ is zero because of the division by $2T$. Applying (3.5) to the remaining terms we obtain (5). ■

The following trivial corollary is of crucial importance for some of our calculations:

Corollary 3.2 *For $M(\mathcal{Q}_0) = 0$ one always has $M(\mathcal{Q}_2) = 0$.* □

Proof. If $M(\mathcal{Q}_0) = 0$ then, by (1.11), $\mathcal{Q}_2(t) = (\mathcal{Q}_0 \mid \mathcal{Q}_0\,)_t$. Hence, from statement (3) of Proposition 3.1, it follows that $M(\mathcal{Q}_2) = 0$. ■

3.2 The Renormalisation Operator

For general quasi-periodic functions $a_1, \ldots, a_n$, the function $(a_1 \mid \cdots \mid a_n)_t$, defined above, is not generally quasi-periodic, since an integration performed on a quasi-periodic function with a non-zero mean value would produce a (linear in t) secular term, which would eventually become a higher degree polynomial after further integrations. We will here describe an operation designed to produce a quasi-periodic function out of $(a_1 \mid \cdots \mid a_n)_t$ through interactive subtractions of the mean value of the functions being integrated, a procedure we call "renormalisation" due to the analogy to the procedure of perturbative renormalization in quantum field theory. We will use this procedure of renormalisation in the following sections and here we present its definition and basic properties.

Let $a_1, \ldots, a_n$ be quasi-periodic functions. We define inductively the *renormalisation operator* $\mathfrak{R}_n$ acting on $(a_1 \mid \cdots \mid a_n)$ by

$$
\begin{aligned}
\mathfrak{R}_1 a_1(t) \quad &:= \quad a_1(t), \\
\mathfrak{R}_2\,(a_1 \mid a_2)_t \quad &:= \quad (a_1 \mid \mathfrak{R}_1(a_2) - M(\mathfrak{R}_1(a_2)))_t \; = \; (a_1 \mid a_2 - M(a_2))_t\,, \\
\mathfrak{R}_n(a_1 \mid \cdots \mid a_n)_t \quad &:= \quad (a_1 \mid \mathfrak{R}_{n-1}(a_2 \mid \cdots \mid a_n) - M(\mathfrak{R}_{n-1}(a_2 \mid \cdots \mid a_n))\,)_t\,,
\end{aligned}
$$

for $n > 2$. We will now prove some elementary facts on $\mathfrak{R}_n$ which will be used below. The first important observation is that if $a_1, \ldots, a_n$ are quasi-periodic functions, then $\mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_n)$ is also quasi-periodic. This can be easily seen by induction, through the obvious remark that the mean value of $\mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_n) - M(\mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_n))$ is zero. Note also that, trivially

$$a_0\, \mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_n) \;=\; \mathfrak{R}_n(a_0 a_1 \,|\, \cdots \,|\, a_n)\,. \tag{3.8}$$

The following proposition is a trivial but useful restatement of the definition of the $\mathfrak{R}_n$'s:

Proposition 3.3 *For all $n \geq 2$ the following statement holds: if $a_1, \ldots, a_n$ are quasi-periodic functions, then*

$$\mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_n) \;=\; \mathfrak{R}_2\left(a_1 \;|\; \mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_n)\right).$$

Consequently, for $n > 2$,

$$\mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_n) \;=\; \mathfrak{R}_2\left(a_1 \;|\; \mathfrak{R}_2\left(a_2 \;|\; \mathfrak{R}_2(\cdots \,|\, \mathfrak{R}_2\left(a_{n-1} \;|\, a_n\right)) \cdots)\right)\right). \tag{3.9}$$

□

Proof. By the definition of $\mathfrak{R}_2$,

$$\begin{aligned}
&\mathfrak{R}_2(a_1 \,|\, \mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_n)) = \\
&\qquad = \left(a_1 \;|\; \mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_n) - M(\mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_n))\right) \\
&\qquad = \mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_n).
\end{aligned}$$

■

Relation (3.9) shows that the operation $\mathfrak{R}_n$ can be obtained by iteration of the operation $\mathfrak{R}_2$. One also has the following useful

Proposition 3.4 *For all $n \geq 1$ the following statement holds: if $a_1, \ldots, a_n$ are quasi-periodic functions and $a_n = \mathfrak{R}_2\,(b \,|\, c)$ for quasi-periodic functions b, c, then*

$$\mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_{n-1} \,|\, a_n) \;=\; \mathfrak{R}_{n+1}(a_1 \,|\, \cdots \,|\, a_{n-1} \,|\, b \,|\, c).$$

□

Proof. For $n = 1$, let $a_1 = \mathfrak{R}_2\,(b \,|\, c)$. Then $\mathfrak{R}_1 a_1 = a_1 = \mathfrak{R}_2\,(b \,|\, c)$, trivially. For $n = 2$, let $a_2 = \mathfrak{R}_2\,(b \,|\, c)$. Then, $\mathfrak{R}_2\,(a_1 \,|\, a_2) = (a_1 \,|\, a_2 - M(a_2)) = (a_1 \;|\; \mathfrak{R}_2\,(b \,|\, c) - M(\mathfrak{R}_2\,(b \,|\, c)))$, but, by definition,

$$\mathfrak{R}_3\,(a_1 \;|\, b \,|\, c\,) = (a_1 \;|\; \mathfrak{R}_2\,(b \,|\, c) - M(\mathfrak{R}_2\,(b \,|\, c)))$$

and the statement holds again. For $n > 2$, let $a_n = \mathfrak{R}_2\,(b \,|\, c)$. Then, by induction,

$$\begin{aligned}
&\mathfrak{R}_{n+1}(a_1 \,|\, \cdots \,|\, a_{n-1} \,|\, b \,|\, c) = \\
&\qquad = \left(a_1 \;|\; \mathfrak{R}_n(a_2 \,|\, \cdots \,|\, a_{n-1} \,|\, b \,|\, c) - M(\mathfrak{R}_n(a_2 \,|\, \cdots \,|\, a_{n-1} \,|\, b \,|\, c))\right) \\
&\qquad = \left(a_1 \;|\; \mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_{n-1} \,|\, a_n) - M(\mathfrak{R}_{n-1}(a_2 \,|\, \cdots \,|\, a_{n-1} \,|\, a_n)\right) \\
&\qquad = \mathfrak{R}_n(a_1 \,|\, \cdots \,|\, a_{n-1} \,|\, a_n).
\end{aligned}$$

■

If a and $b_1, \ldots, b_m$ are quasi-periodic, a function like $\sum_{k=1}^{m} (a \mid b_k)$ may not be a sum of quasi-periodic functions, even when $(a \mid \sum_{k=1}^{m} b_k) = \sum_{k=1}^{m} (a \mid b_k)$ is quasi-periodic, since we are not assuming that $M(b_k) = 0$ for each individual k. This fact notwithstanding, the following simple statement holds and will be repeatedly used:

Proposition 3.5 *Let a and $b_1, \ldots, b_m$ be quasi-periodic functions. Then*

$$\mathfrak{R}_2 \left(a \,\middle|\, \sum_{k=1}^{m} b_k \right) = \sum_{k=1}^{m} \mathfrak{R}_2 \left(a \mid b_k \right). \tag{3.10}$$

Consequently,

$$\mathfrak{R}_n \left(a_1 \,\middle|\, \cdots \,\middle|\, a_{n-1} \,\middle|\, \sum_{k=1}^{m} b_k \right) = \sum_{k=1}^{m} \mathfrak{R}_n (a_1 \mid \cdots \mid a_{n-1} \mid b_k) \tag{3.11}$$

for quasi-periodic functions $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_m$. □

Proof. We have

$$\begin{aligned} \mathfrak{R}_2 \left(a \,\middle|\, \sum_{k=1}^{m} b_k \right) &= \left(a \,\middle|\, \sum_{k=1}^{m} b_k - M\left(\sum_{k=1}^{m} b_k \right) \right) = \left(a \,\middle|\, \sum_{k=1}^{m} \left(b_k - M\left(b_k \right) \right) \right) \\ &= \sum_{k=1}^{m} \left(a \mid b_k - M(b_k) \right) = \sum_{k=1}^{m} \mathfrak{R}_2 \left(a \mid b_k \right). \end{aligned}$$

Note, in the third equality, that the mean value of $b_k - M(b_k)$ is zero and, hence, $(a \mid b_k - M(b_k))$ are quasi-periodic. Relation (3.11) follows from (3.9)-(3.10). ■

The following corollary follows from Propositions 3.4 and 3.5.

Corollary 3.6 *Let $a_1, \ldots, a_{n-1}, b_1, \ldots, b_m$ and $c_1, \ldots, c_m$ be quasi-periodic functions. Then,*

$$\mathfrak{R}_n \left(a_1 \,\middle|\, \cdots \,\middle|\, a_{n-1} \,\middle|\, \sum_{k=1}^{m} \mathfrak{R}_2 \left(b_k \mid c_k \right) \right) = \sum_{k=1}^{m} \mathfrak{R}_{n+1} (a_1 \mid \cdots \mid a_{n-1} \mid b_k \mid c_k). \quad \square$$

Remark 3.7 *The reader should be warned of the following fact: If $a_1, \ldots, a_n$, $b_1, \ldots, b_m$ are quasi-periodic functions and $(a_1 \mid \cdots \mid a_n) + (b_1 \mid \cdots \mid b_m)$ is also quasi-periodic, it is not always true that $(a_1 \mid \cdots \mid a_n) + (b_1 \mid \cdots \mid b_m)$ equals*

$$\mathfrak{R}_n (a_1 \mid \cdots \mid a_n) + \mathfrak{R}_m (b_1 \mid \cdots \mid b_m).$$

For a counter-example, take $a_1(t) = 1$, $a_2(t) = e^{it}$, $a_3(t) = 1$ $b_1(t) = ie^{it}$ and $b_2(t) = 1$. One has $(a_1 \mid a_2 \mid a_3) + (b_1 \mid b_2) = e^{it} - 1$, but $\mathfrak{R}_3 (a_1 \mid a_2 \mid a_3) + \mathfrak{R}_2 (b_1 \mid b_2) = 0$. Hence, the renormalisation operations are not additive in this sense.

3.3 More on the Notation and Some Definitions

With the shorthand notation introduced in (3.1) and the definition of $\mathcal{Q}_0$ in (1.6), we see from (1.7), (1.11) and (1.12) that

$$\mathcal{Q}_1(t) \;=\; \mathfrak{R}_2\left(\mathcal{Q}_0 \,\middle|\, \overline{\mathcal{Q}_0}\right)_t, \quad \mathcal{Q}_2(t) \;=\; \mathfrak{R}_2\left(\mathcal{Q}_0 \mid \mathcal{Q}_0\right)_t, \quad \mathcal{Q}_3(t) \;=\; \mathfrak{R}_2\left(\mathcal{Q}_0 \mid \mathcal{Q}_1\right)_t .$$

Below, we will often use the following compact notation

$$(i \mid j)_t := (\mathcal{Q}_i \mid \mathcal{Q}_j)_t, \quad \left(i \,\middle|\, \overline{j}\right)_t := \left(\mathcal{Q}_i \,\middle|\, \overline{\mathcal{Q}_j}\right)_t, \quad (i \mid j \mid k)_t := (\mathcal{Q}_i \mid \mathcal{Q}_j \mid \mathcal{Q}_k)_t,$$

etc, for $i,\ j,\ k = 0, \ldots, 3$. In other words, we simply use the index n of $\mathcal{Q}_n$ to denote $\mathcal{Q}_n$ itself. Moreover, by $M\,(i \mid j)$ we will denote the mean value of $(i \mid j)_t$, etc.

Note that for $M(\mathcal{Q}_0) = 0$ one has with this notation $\mathcal{Q}_1(t) = \left(0 \,\middle|\, \overline{0}\right)_t$ and $\mathcal{Q}_2(t) = (0 \mid 0)_t$ and for $M(\mathcal{Q}_1) = 0$ one has

$$\mathcal{Q}_3(t) \;=\; (0 \mid 1)_t .$$

We will often write $\mathcal{Q}_3$ this way.

For $M(\mathcal{Q}_0) = 0$, other identities will also be at hand. For instance, one has

$$(0 \mid 0 \mid 0)_t \;=\; (0 \mid (0 \mid 0)_{t'})_t \;=\; (0 \mid 2)_t \quad \text{and} \quad (1 \mid 0)_t \;=\; \left(2 \,\middle|\, \overline{0}\right)_t, \qquad (3.12)$$

since $M\,(0 \mid 0) = 0$, by item (3) of Proposition 3.1. Relations like these will be often employed.

4 The Case $M(\mathcal{Q}_0) = 0$ and $M(\mathcal{Q}_1) = 0$

In this Section we will prove the Theorem 2.2. Our interest is to study the situation complementary to cases (I) and (II), i.e., the situation where one has condition

$$(\mathrm{III}_0)\;\; M(\mathcal{Q}_0) = 0 \text{ and } M(\mathcal{Q}_1) = 0.$$

To remove the secular terms from g, applying the method described in the previous section, we will be forced to add a further restriction to (III_0), namely the condition $M\,(0 \mid 1) \neq 0$.

Recall that the functions q and $\mathcal{Q}_1$ depend primordially on the interaction f (see the definitions given in (1.6) and (1.7)), so conditions (I), (II) or (III_0) apply upon the properties of f. As we already saw in Section 1, the function $f(t) = \varphi_1 \cos(\omega t) + \varphi_2 \sin(\omega t)$ only satisfies condition (I) or (III_0), depending on the particular choice of the parameters φ_1, φ_2. We will have more to say about this example latter on in Section 5. Now, let us work with the expansion for g in order to remove all of its secular terms.

Again, our Ansatz to solve the generalised Riccati equation (2.1) is (2.5). The explicit solutions for the coefficients v_n are given in (2.6), (2.7) and (2.8). One sees immediately from condition (III_0) that v_1 and v_2 do not suffer from secular terms. Indeed, q in quasi-periodic and since $M(\mathcal{Q}_0^{-1}) = \overline{M(\mathcal{Q}_0)} = 0$, we conclude that the mean value of the integrand $\mathcal{I}_2$ occurring in (2.7) is zero. Therefore, the integration occurring in the definition of v_2 in (2.7) does not produce a linear term in t. These fats imply that v_1 and v_2 are quasi-periodic under (III_0). From these considerations we see that the condition $M(\mathcal{I}_n) = 0$, $n \geq 3$, becomes recursively identical to $\sum_{p=1}^{n-1} M(v_p v_{n-p})$, $n \geq 3$, since the v_n's become successively quasi-periodic when the interactive procedure is run.

If this is achieved, i.e., if we succeed in fixing $M(\mathcal{I}_n) = 0$ for all n, we can rewrite (2.6)-(2.8) in a "renormalised" form:

$$
\begin{aligned}
v_1(t) &= \kappa_1 q(t), && (4.1)\\
v_2(t) &= q(t)^{-1}\left(i\kappa_1^2\mathcal{Q}_2(t) - i\mathcal{Q}_1(t) + \kappa_2\mathcal{Q}_0(t)\right), && (4.2)\\
v_n(t) &= q(t)^{-1}\left\{ i\sum_{p=1}^{n-1} \mathfrak{R}_2\left(0\,|\,v_p v_{n-p}\right)_t + \kappa_n\mathcal{Q}_0(t)\right\}, \quad n \geq 3, && (4.3)
\end{aligned}
$$

where $\mathcal{Q}_0$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ were defined in (1.6), (1.7) and (1.11), respectively.

Let us move on and analyse the third order term. According to (2.8) the integrand $\mathcal{I}_3$ which appears in the definition of v_3 is given by $2v_1v_2$. Using (4.1) and (4.2) we have

$$
v_1 v_2 \;=\; i\kappa_1^3\mathcal{Q}_2 - i\kappa_1\mathcal{Q}_1 + \kappa_1\kappa_2\mathcal{Q}_0\,. \tag{4.4}
$$

From Corollary 3.2, we readily see that $M(\mathcal{I}_3) = 2M(v_1v_2) = 0$. This means that the integrand which appears in v_3 does not have a constant term in its Fourier expansion. Hence, v_3 is quasi-periodic.

Until now we have verified the absence of secular terms in the series expansion of g up to order three in ϵ. As we shall see next, for the same to be true up to order four, we have to make a especial choice for the value of the constant κ_1.

4.1 The Absence of Secular Terms in v_4. Fixing κ_1

As one sees from (2.8), the integrand in v_4 is $\mathcal{I}_4 := 2v_1v_3 + v_2^2$. Since v_1, v_2, v_3 are quasi-periodic, the mean value of $\mathcal{I}_4$ is well defined. Let us explicitly evaluate $\mathcal{I}_4$ using (2.6)–(2.8):

$$
\begin{aligned}
\mathcal{I}_4 &= 2\kappa_1\mathcal{Q}_0(t)\left(i\int_0^t 2v_1(\tau)v_2(\tau)\,d\tau + \kappa_3\right)\\
&\quad + \mathcal{Q}_0(t)\left(i\int_0^t \left(\kappa_1^2\mathcal{Q}_0(\tau) - \mathcal{Q}_0(\tau)^{-1}\right)\,d\tau + \kappa_2\right)^2\\
&= -4\kappa_1^4\left(0\,|\,2\right) + 4\kappa_1^2\left(0\,|\,1\right) + 6i\kappa_1^2\kappa_2\mathcal{Q}_2 + (2\kappa_1\kappa_3 + \kappa_2^2)\mathcal{Q}_0\\
&\quad -2i\kappa_2\mathcal{Q}_1 - \kappa_1^4\left(2\,|\,0\right) + 2\kappa_1^2\left(2\,\middle|\,\overline{0}\right) - \left(1\,\middle|\,\overline{0}\right)\,.
\end{aligned}\tag{4.5}
$$

The functions appearing in the right-hand side of (4.5) are all quasi-periodic, since $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = M(\mathcal{Q}_2) = 0$. Therefore, we are allowed to take the mean value of each individual term above. The result is

$$M(\mathcal{I}_4) \; = \; -4\kappa_1^4 M(\,0\,|\,2\,) + 4\kappa_1^2 M(\,0\,|\,1\,) - \kappa_1^4 M(\,2\,|\,0\,) + 2\kappa_1^2 M(\,2|\,\overline{0}\,) - M(\,1\,|\,\overline{0}\,).$$

By statements (2) and (6) of Proposition 3.1 and by (3.12) we have

$$M\,(2\,|\,0\,) \;\; = \;\; -M\,(0\,|\,2\,) \; = \; -M\,(0\,|\,0\,|\,0\,) = 0\,, \tag{4.6}$$

$$M\left(2\,\middle|\,\overline{0}\right) \;\; = \;\; M\,(1\,|\,0\,) \; = \; -M\,(0\,|\,1\,)\;. \tag{4.7}$$

Hence,

$$M(\mathcal{I}_4) \; = \; 2\kappa_1^2 M\,(0\,|\,1\,) - M\left(1\,\middle|\,\overline{0}\right)\;. \tag{4.8}$$

We have the following

Proposition 4.1 *Under $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = 0$, one has $M(\,1\,|\,\overline{0}\,) = 2\overline{M(\,0\,|\,1\,)}$.* □

Proof. By the definition of $\mathcal{Q}_1$,

$$M\,(0\,|\,1\,) \; = \; M\left(0\,\middle|\,0\,\middle|\,\overline{0}\,\right) \tag{4.9}$$

and

$$M\left(1\,\middle|\,\overline{0}\right) = M\left(\mathcal{Q}_0(t)\left(\int_0^t \mathcal{Q}_0(\tau)^{-1}\,d\tau\right)\left(\int_0^t \mathcal{Q}_0(\tau)^{-1}\,d\tau\right)\right) = -M\left(\overline{0}\,\middle|\,0\,\middle|\,\overline{0}\,\right)\,, \tag{4.10}$$

where, in the last equality, we have used statement (4) of Proposition 3.1. Now, by statement (5) of the same proposition, we can write

$$M\left(0\,\middle|\,0\,\middle|\,\overline{0}\,\right) + M\left(0\,\middle|\,\overline{0}\,\middle|\,0\,\right) + M\left(\overline{0}\,|\,0\,|\,0\,\right) \; = \; 0 \tag{4.11}$$

(recall that $M\left(0\,\middle|\,\overline{0}\right) = M\left(\overline{0}\,|\,0\right) = M\,(0\,|\,0\,) = 0$). Once again, by statement (4) of Proposition 3.1, $M\left(0\,\middle|\,0\,\middle|\,\overline{0}\,\right) = M\left(\overline{0}\,|\,0\,|\,0\,\right)$. Thus, (4.11) reads $2M\left(0\,\middle|\,0\,\middle|\,\overline{0}\,\right) + M\left(0\,\middle|\,\overline{0}\,\middle|\,0\,\right) = 0$. Taking the complex conjugate, yields

$$2\overline{M\left(0\,\middle|\,0\,\middle|\,\overline{0}\,\right)} + M\left(\overline{0}\,\middle|\,0\,\middle|\,\overline{0}\,\right) = 0.$$

Finally, using (4.9) and (4.10), we get $M\left(1\,\middle|\,\overline{0}\right) = 2\overline{M\,(0\,|\,1\,)}$. ■

We have just proven that $M(\mathcal{I}_4) \; = \; 2\left(\kappa_1^2\, M\,(0\,|\,1\,) - \overline{M\,(0\,|\,1\,)}\right)$. Now we impose $M(\mathcal{I}_4) = 0$. Of course, this will be the case if $M\,(0\,|\,1\,) = 0$ but, in the situation where $M\,(0\,|\,1\,) \neq 0$ this can be achieved by fixing κ_1 as (see (4.8))

$$\kappa_1 \; = \; \left(\frac{\overline{M\,(0\,|\,1\,)}}{M\,(0\,|\,1\,)}\right)^{1/2} \; = \; \left(\frac{\overline{M(\mathcal{Q}_3)}}{M(\mathcal{Q}_3)}\right)^{1/2}. \tag{4.12}$$

Thus, κ_1 is a phase: $|\kappa_1| = 1$. It will be henceforth assumed that $M\,(0\,|\,1\,) \neq 0$. If $M\,(0\,|\,1\,) = 0$, κ_1 has to be fixed by $M(\mathcal{I}_5) = 0$. We shall not treat this more restrictive case here.

So far, we have verified the absence of secular terms in the series expansion (2.5) for g up to order three in ϵ and we have eliminated them from v_4 by making a especial choice for the value of κ_1 (given by (4.12)). At this point we would like to proceed recursively by imposing $M(\mathcal{I}_n) = 0$, for all $n \geq 5$. This would give the correct values for the constants κ_p, $p \geq 2$, and guarantee the absence of secular terms in all v_n, $n \geq 5$. This recursive procedure was used in [1] to eliminate the secular terms from g in cases (I) and (II). Here, we still have to determine the constants κ_2 and κ_3 explicitly (not recursively) before running the recursive procedure.

4.2 The Absence of Secular Terms in v_5. Fixing κ_2

Let us begin by calculating the integrand $\mathcal{I}_5$ which appears in v_5. Taking $n = 5$ in (2.8) we get $\mathcal{I}_5 = 2v_1v_4 + 2v_2v_3$. We first evaluate v_1v_4 explicitly and then v_2v_3.

For v_1v_4, a lengthy computation (see Appendix A.1) shows that

$$\begin{aligned} v_1v_4 \quad = \quad & -6\kappa_1^3\kappa_2\;\mathfrak{R}_2\,(0\,|\,2\,) + 2\kappa_1\kappa_2\;\mathfrak{R}_2\,(0\,|\,1\,) + i\kappa_1(2\kappa_1\kappa_3 + \kappa_2^2)\,\mathcal{Q}_2 \\ & +\kappa_1\kappa_4\mathcal{Q}_0 + i\kappa_1\mathcal{A}_1\,, \end{aligned} \tag{4.13}$$

where $\mathcal{A}_1$ is the quasi-periodic function defined in (A.7) and depends only on the constant κ_1. Since we are working under the condition $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = M(\mathcal{Q}_2) = 0$, we can drop the symbol $\mathfrak{R}_2$ above. From this fact and from relation (4.6), we conclude that $M(v_1v_4) = 2\kappa_1\kappa_2\,M\,(0\,|\,1\,) + i\kappa_1 M(\mathcal{A}_1)$.

Let us now calculate the second term in $\mathcal{I}_5$, namely, v_2v_3. Another lengthy computation (see Appendix A.2) gives

$$\begin{aligned} v_2v_3 \quad = \quad & -2i\kappa_1^5\,\mathfrak{R}_2\,(2\,|\,2\,) + 2i\kappa_1^3\,\mathfrak{R}_2\,(2\,|\,1\,) - 2\kappa_1^3\kappa_2\,\mathfrak{R}_2\,(2\,|\,0\,) + 2i\kappa_1^3\,\mathfrak{R}_2\,(1\,|\,2\,) \\ & -2i\kappa_1\,\mathfrak{R}_2\,(1\,|\,1\,) + 2\kappa_1\kappa_2\,\mathfrak{R}_2\,(1\,|\,0\,) - 2\kappa_1^3\kappa_2\,\mathfrak{R}_2\,(0\,|\,2\,) \\ & +2\kappa_1\kappa_2\,\mathfrak{R}_2\,(0\,|\,1\,) + i(2\kappa_1\kappa_2^2 + \kappa_1^2\kappa_3)\mathcal{Q}_2 - i\kappa_3\mathcal{Q}_1 + \kappa_2\kappa_3\mathcal{Q}_0\,. \end{aligned} \tag{4.14}$$

Clearly the right hand side of equation (4.14) is quasi-periodic. Since we are working under the condition $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = M(\mathcal{Q}_2) = 0$, we can drop the symbol $\mathfrak{R}_2$ above and reorder (4.14) in the form

$$\begin{aligned} v_2v_3 \quad = \quad & i(2\kappa_1\kappa_2^2 + \kappa_1^2\kappa_3)\mathcal{Q}_2 - i\kappa_3\mathcal{Q}_1 + \kappa_2\kappa_3\mathcal{Q}_0 - 2i\kappa_1^5\,(2\,|\,2\,) - 2i\kappa_1\,(1\,|\,1\,) \\ & +2i\kappa_1^3\,[(2\,|\,1\,) + (1\,|\,2\,)] - 2\kappa_1^3\kappa_2\,[(2\,|\,0\,) + (0\,|\,2\,)] \\ & +2\kappa_1\kappa_2\,[(1\,|\,0\,) + (0\,|\,1\,)]\,. \end{aligned} \tag{4.15}$$

Above we also used $\mathfrak{R}_2\,(0\,|\,0\,) = \mathcal{Q}_2$. From (4.15), from the fact that $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = M(\mathcal{Q}_2) = 0$ and from statements (2) and (3) of Proposition 3.1, we see immediately that $M(v_2v_3) = 0$.

We are now ready to find the value of κ_2 by imposing the condition $M(\mathcal{I}_5) = 0$, which guarantees the absence of secular terms in v_5. Since $M(\mathcal{I}_5) = 2M(v_1v_4) + 2M(v_2v_3) = 4\kappa_1\kappa_2\, M\,(0\,|\,1\,) + 2i\kappa_1 M(\mathcal{A}_1)$, we conclude that

$$\kappa_2 \;=\; -\frac{iM(\mathcal{A}_1)}{2M\,(0\,|\,1\,)}\,. \tag{4.16}$$

Note that the right-hand side of (4.16) depends on the previously fixed κ_1.

4.3 The Absence of Secular Terms in v_6. Fixing κ_3

We still have to find κ_3 in order to fix recursively all κ_n's for $n \geq 4$. κ_3 will be fixed by eliminating the secular terms from v_6, that is, by imposing $M(\mathcal{I}_6) = 0$. First of all, we need to write $\mathcal{I}_6$. Using relation (2.8) for $n = 6$ we find that $\mathcal{I}_6 = 2v_1v_5 + 2v_2v_4 + v_3^2$. Let us calculate $2v_1v_5$. Another lengthy computation (see Appendix A.3) gives

$$\begin{aligned} 2v_1v_5 \;&=\; \mathcal{A}_2 - 12\kappa_1^3\kappa_3\,\mathfrak{R}_2\,(0\,|\,2\,) + 4\kappa_1\kappa_3\,\mathfrak{R}_2\,(0\,|\,1\,) \\ &\quad +4i\kappa_1(\kappa_2\kappa_3 + \kappa_1\kappa_4)\mathcal{Q}_2 + 2\kappa_1\kappa_5\mathcal{Q}_0\,, \end{aligned} \tag{4.17}$$

where $\mathcal{A}_2$ is the quasi-periodic function defined in (A.11) and depends only on the constants κ_1 and κ_2. Since $2v_1v_5$ given above is a sum of quasi-periodic functions we can take the mean value of each individual term which appears in the right hand side of (4.17) and write

$$2M(v_1v_5) \;=\; M(\mathcal{A}_2) + 4\kappa_3\kappa_1 M\,(0\,|\,1\,)\;, \tag{4.18}$$

where, once again, we have used $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = M(\mathcal{Q}_2) = 0$ and the identity (4.6).

We will now calculate the second term in $\mathcal{I}_5$, namely, $2v_2v_4$. We have (see Appendix A.4)

$$\begin{aligned} 2v_2v_4 \;&=\; \mathcal{A}_3 - 4\kappa_1^3\kappa_3\,\mathfrak{R}_2\,(2\,|\,0\,) + 4\kappa_1\kappa_3\,\mathfrak{R}_2\,(1\,|\,0\,) \\ &\quad +2i(\kappa_1^2\kappa_4 + 2\kappa_1\kappa_2\kappa_3)\mathcal{Q}_2 - 2i\kappa_4\mathcal{Q}_1 + 2\kappa_2\kappa_4\mathcal{Q}_0\,, \end{aligned} \tag{4.19}$$

where $\mathcal{A}_3$ is defined in (A.12) and depends only on the constants κ_1 and κ_2.

We can now proceed and take the mean value of $2v_2v_4$ from (4.19). Using $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = M(\mathcal{Q}_2) = 0$ and (4.6), we get

$$2M(v_2v_4) \;=\; M(\mathcal{A}_3) + 4\kappa_1\kappa_3 M\,(0\,|\,1\,)\;. \tag{4.20}$$

We are almost through with the calculation of $\mathcal{I}_6$. We still need v_3^2. Using relations (4.1)–(4.4),

$$\begin{aligned} v_3 \;&=\; q^{-1}\left\{2i\,\mathfrak{R}_2\,(0\,|\,v_1v_2\,) + \kappa_3\mathcal{Q}_0\right\} \\ &=\; q^{-1}\left\{-2\kappa_1^3\,\mathfrak{R}_2\,(0\,|\,2\,) + 2\kappa_1\,\mathfrak{R}_2\,(0\,|\,1\,) + 2i\kappa_1\kappa_2\mathcal{Q}_2 + \kappa_3\mathcal{Q}_0\right\}. \end{aligned} \tag{4.21}$$

Hence,

$$v_3^2 \;=\; \kappa_3^2\mathcal{Q}_0 + 4\kappa_1\kappa_3\left[-\kappa_1^2\,\mathfrak{R}_2\,(0\,|\,2) + \mathfrak{R}_2\,(0\,|\,1) + i\kappa_2\mathcal{Q}_2\right] - 4\mathcal{A}_4\,,$$

where

$$\mathcal{A}_4 \;:=\; \mathcal{Q}_0^{-1}\left[i\kappa_1^3\,\mathfrak{R}_2\,(0\,|\,2) - i\kappa_1\,\mathfrak{R}_2\,(0\,|\,1) + \kappa_1\kappa_2\mathcal{Q}_2\right]^2\,.$$

Again, $\mathcal{A}_4$ is quasi-periodic and depends only on the already fixed constants κ_1 and κ_2. Therefore, using once more $M(\mathcal{Q}_0) = M(\mathcal{Q}_2) = 0$ and (4.6), we get

$$M(v_3^2) \;=\; 4\kappa_3\kappa_1 M\,(0\,|\,1) - 4M(\mathcal{A}_4)\,. \tag{4.22}$$

Finally, imposing $M(\mathcal{I}_6) = 0$, i.e., $2M(v_1v_5) + 2M(v_2v_4) + M(v_3^2) = 0$, we obtain from our previous calculations (relations (4.18), (4.20) and (4.22)),

$$\kappa_3 \;=\; \frac{4M(\mathcal{A}_4) - M(\mathcal{A}_3) - M(\mathcal{A}_2)}{4\kappa_1 M\,(0\,|\,1)}\,. \tag{4.23}$$

Note that the right-hand side of (4.23) depends on the previously fixed κ_1 and κ_2.

4.4 The Absence of Secular Terms in v_n, $n \geq 7$. Fixing κ_{n-3} Recursively

So far, we have fixed the constants κ_1, κ_2 and κ_3 individually. Now we proceed to fix recursively all other κ_{n-3} for all $n \geq 7$. We have to impose

$$M(\mathcal{I}_n) \;=\; 0 \quad\Longrightarrow\quad M\left(\sum_{p=1}^{n-1} v_p v_{n-p}\right) \;=\; 0 \tag{4.24}$$

for all $n \geq 7$. Condition (4.24) guarantees the absence of secular terms in all v_n, $n \geq 7$.

The idea now is to use (4.24) to calculate recursively the constants κ_{n-3}, for all $n \geq 7$, that is, $\kappa_4,\ \kappa_5,\ \ldots$. Of course, we already have κ_1, κ_2 and κ_3 and, hence, we completely know v_1, v_2 and v_3. We also know that all functions, from v_1 to v_6, are quasi-periodic. From now on we will work inductively. Thus, it will be supposed for each $n \geq 7$ that we fixed $\kappa_1,\ \ldots,\ \kappa_{n-4}$ by imposing $M(\mathcal{I}_m) = 0$ for all $m = 2,\ \ldots,\ n-1$ and that, as a consequence, all functions $v_1,\ \ldots,\ v_{n-1}$ are quasi-periodic. Note that it is indeed true for $n = 7$.

By our inductive hypothesis, we are allowed to take the summation out of the mean value M in (4.24) and write

$$2\left[M(v_1v_{n-1}) + M(v_2v_{n-2}) + M(v_3v_{n-3})\right] + \sum_{p=4}^{n-4} M(v_pv_{n-p}) \;=\; 0\,, \tag{4.25}$$

where, by convention, $\sum_{p=4}^{n-4} M(v_p v_{n-p}) = 0$ for $n = 7$. Let us introduce now the following definition:

$$l_m(t) \ := \ q(t)(v_m(t) - \kappa_m q(t)) \,, \tag{4.26}$$

for all $m \geq 4$. Note that, by relation (2.8), the functions l_m's above can also be written as

$$l_m(t) \ = \ i\, q(t)^2 \left(\int_0^t \sum_{p=1}^{m-1} v_p(\tau) v_{m-p}(\tau) \, d\tau \right) \, . \tag{4.27}$$

For $m < n$ we are allowed to write

$$\begin{aligned} l_m(t) \ &= \ i\, q(t)^2 \left(\int_0^t \sum_{p=1}^{m-1} \left[v_p(\tau) v_{m-p}(\tau) - M\left(v_p v_{m-p}\right)\right] \, d\tau \right) \\ &= \ i \sum_{p=1}^{m-1} \mathfrak{R}_2 \left(0 \mid v_p v_{m-p}\right)_t \,, \end{aligned} \tag{4.28}$$

since we assumed $M(\sum_{p=1}^{m-1} v_p v_{m-p}) = M(\mathcal{I}_m) = 0$ for all $m < n$, by the inductive hypothesis. Hence, l_m are quasi-periodic for all $m < n$, by the inductive hypothesis.

Let us use the definition given in (4.26) and evaluate the first three terms which appear in (4.25). Beginning with the first one, we have

$$M(v_1 v_{n-1}) \ = \ M\left(v_1(q^{-1} l_{n-1} + \kappa_{n-1} q)\right) \ = \ \kappa_1 M(l_{n-1}) \,, \tag{4.29}$$

where we have used (2.6) and the fact that $M(\mathcal{Q}_0) = 0$. Using (4.26), the second term of (4.25) can be evaluated as

$$M(v_2 v_{n-2}) \ = \ M\left(v_2(q^{-1} l_{n-2} + \kappa_{n-2} q)\right) \ = \ M(q^{-1} v_2 l_{n-2}) \,, \tag{4.30}$$

where we have used (4.2) to express qv_2 and the fact that $M(\mathcal{Q}_0) = M(\mathcal{Q}_1) = M(\mathcal{Q}_2) = 0$. Finally, for the third term of (4.25), we have

$$M(v_3 v_{n-3}) \ = \ M\left(v_3(q^{-1} l_{n-3} + \kappa_{n-3} q)\right) \ = \ M(q^{-1} v_3 l_{n-3}) + \kappa_{n-3} M(q v_3) \, . \tag{4.31}$$

The product qv_3 can be obtained from (4.21), from which we conclude that $M(qv_3) = 2\kappa_1 M\left(0 \mid 1\right)$. Inserting this into (4.31), gives

$$M(v_3 v_{n-3}) \ = \ M(q^{-1} v_3 l_{n-3}) + 2\kappa_1 \kappa_{n-3} M\left(0 \mid 1\right) \, . \tag{4.32}$$

The substitution of (4.29), (4.30) and (4.32) into (4.25), gives us

$$2\left\{\underbrace{\kappa_1 M(l_{n-1})}_{\text{(i)}}+\underbrace{M(q^{-1}v_2 l_{n-2})}_{\text{(ii)}}+\underbrace{M(q^{-1}v_3 l_{n-3})}_{\text{(iii)}}+\underbrace{2\kappa_{n-3}\kappa_1 M\left(0\mid 1\right)}_{\text{(iv)}}\right\}$$

$$+\underbrace{\sum_{p=4}^{n-4} M(v_p v_{n-p})}_{\text{(v)}} \;=\; 0\,. \qquad (4.33)$$

Before we proceed, let us make some comments on our strategy. Equation (4.33) is a direct consequence of (4.24) and, thus, is being imposed for each $n \geq 7$, leading to the values of κ_4, κ_5 and so on. By our induction hypothesis, we have fixed the constants $\kappa_1, \ldots, \kappa_{n-4}$ and, hence, we completely know $v_1, \ldots, v_{n-4}$. For this reason the terms (iii) and (v) are known by assumption (by (4.28), the evaluation of l_{n-3} requires the knowledge of $v_1, \ldots, v_{n-4}$). Our aim is to use (4.33) as a condition to fix κ_{n-3} and we, therefore, have to isolate the dependence of (4.33) on κ_{n-3}. The function l_{n-2}, in term (ii), depends implicitly on v_{n-3} and, hence, on κ_{n-3} (see, again, relation (4.28)). The term (i), however, depends implicitly on κ_{n-2} and κ_{n-3}. This dependence on κ_{n-2} could be a problem, since we are still working to fix κ_{n-3}. Nevertheless, as will be shown, the conditions $M(\mathcal{Q}_1) = 0$ and $M(\mathcal{Q}_2) = 0$ fortunately eliminate κ_{n-2} from the final expression, and we will be led to a condition expressing κ_{n-3} in terms of known quantities.

Let us now compute terms (i) and (ii). After a long computation, found in Appendix A.5, we get

$$\kappa_1 M(l_{n-1}) \;=\; 2\kappa_1\kappa_{n-3} M\left(0\mid 1\right) + \mathcal{R}_n^{(1)}\,, \qquad (4.34)$$

where $\mathcal{R}_n^{(1)}$ is defined in (A.18) and depends on constants $\kappa_1, \ldots, \kappa_{n-4}$ only. For term (ii) of (4.33) we get, after another long computation presented in Appendix A.6,

$$M(q^{-1}v_2 l_{n-2}) \;=\; -2\kappa_1\kappa_{n-3} M\left(0\mid 1\right) + \mathcal{R}_n^{(2)}\,, \qquad (4.35)$$

where $\mathcal{R}_n^{(2)}$ is defined in (A.19). We again stress that $\mathcal{R}_n^{(2)}$ depends on $\kappa_1, \ldots, \kappa_{n-4}$ only.

We are now ready to give the precise value of κ_{n-3} in order to satisfy (4.24). Collecting (4.34) and (4.35) and inserting them into (4.33), we obtain

$$\kappa_{n-3} \;=\; \frac{-1}{4\kappa_1 M\left(0\mid 1\right)}\left\{\sum_{p=4}^{n-4} M(v_p v_{n-p}) + 2\mathcal{R}_n^{(1)} + 2\mathcal{R}_n^{(2)} + 2M(q^{-1}v_3 l_{n-3})\right\}\,,$$

for all $n \geq 7$. Note that $\mathcal{R}_n^{(1)}$ and $\mathcal{R}_n^{(2)}$ can be completely computed in any order of recursion (see equations (A.18) and (A.19)). The function l_{n-3} can also be computed in any order of recursion if one uses relation (4.28).

Summarising our conclusions, under conditions (III_0) and $M\,(0\,|\,1) \neq 0$, i.e., under condition (III) of Theorem 2.2, and with the constants κ_n recursively chosen as

$$\kappa_1 \;=\; \left(\frac{\overline{M\,(0\,|\,1)}}{M\,(0\,|\,1)}\right)^{1/2} , \tag{4.36}$$

$$\kappa_2 \;=\; -\frac{iM(\mathcal{A}_1)}{2M\,(0\,|\,1)} , \qquad \kappa_3 \;=\; \frac{4M(\mathcal{A}_4) - M(\mathcal{A}_3) - M(\mathcal{A}_2)}{4\kappa_1 M\,(0\,|\,1)} , \tag{4.37}$$

$$\kappa_{n-3} \;=\; \frac{-1}{4\kappa_1 M\,(0\,|\,1)} \left\{ \sum_{p=4}^{n-4} M\,(v_p v_{n-p}) + 2\mathcal{R}_n^{(1)} + 2\mathcal{R}_n^{(2)} + 2M\left(q^{-1} v_3 l_{n-3}\right) \right\} , \tag{4.38}$$

for $n \geq 7$, all secular terms are eliminated from the formal solution (2.5) of the generalised Riccati equation (2.1). Notice that hypothesis $M\,(0\,|\,1) \neq 0$ is the only additional restriction needed to (4.36)-(4.38). The proof of Theorem 2.2 is thus complete. ■

5 The Case of the Monochromatic Field

We illustrate our method and our results considering the simplest case of monochromatic interactions (ac-dc field)

$$f(t) \;=\; F_0 + \varphi\cos(\omega t), \tag{5.1}$$

which are of particular importance in physical applications. We want to show that with conditions (I)-(III) we obtain with our method convergent perturbative solutions of this problems for all parameters F_0 and φ, except perhaps for some spurious situations. For (5.1) one has

$$\mathcal{Q}_0(t) \;=\; \sum_{n\in\mathbb{Z}} J_n\,(\chi_1)\, e^{i(n+\chi_2)\omega t} ,$$

where J_n is the Bessel function of first kind and order n and where we defined $\chi_1 := 2\varphi/\omega$, $\chi_2 := 2F_0/\omega$. Hence, condition (I) (treated in detail in [2]) holds provided $\chi_2 = -m$, with m integer, and provided χ_1 is not a zero of the Bessel function J_m. Here, we have by (2.14),

$$\Omega \;=\; -\frac{m\omega}{2} + \epsilon J_m(\chi_1) + O(\epsilon^2) .$$

See also the discussion in [7]. Let us consider the complementary situations.

i. Consider the case where $\chi_2 = -m$, a non-zero integer, and χ_1 is a zero of J_m. One has $M(\mathcal{Q}_0) = 0$ and we have to look at $M(\mathcal{Q}_1)$. We get,

$$M(\mathcal{Q}_1) \;=\; \sum_{k\in\mathbb{Z}_*} \frac{J_{k+m}(\chi_1)^2}{ik\omega} .$$

Since χ_1 is bound to be a zero of J_m, the r.h.s. does not vanish, except, perhaps, for spurious points. Hence we have now condition (II). Moreover, by (2.15),

$$\Omega \;=\; -\frac{m\omega}{2} - \epsilon^2 \sum_{k\in\mathbb{Z}_*} \frac{J_{k+m}(\chi_1)^2}{k\omega} + O(\epsilon^3)\,.$$

ii. Consider the case where χ_2 is non-integer (see also the discussion in [7]). One has $M(\mathcal{Q}_0) = 0$ and we have to look at $M(\mathcal{Q}_1)$. We get,

$$M(\mathcal{Q}_1) \;=\; \frac{i}{\omega\chi_2}\left[J_0(\chi_1)^2 + 2\chi_2^2 \sum_{k=1}^{\infty} \frac{J_k(\chi_1)^2}{\chi_2^2 - k^2}\right]\,.$$

Generally the r.h.s is non-zero and we have condition (II). Hence, by (2.15),

$$\Omega \;=\; F_0 + \frac{\epsilon^2}{\omega\chi_2}\left[J_0(\chi_1)^2 + 2\chi_2^2 \sum_{k=1}^{\infty} \frac{J_k(\chi_1)^2}{\chi_2^2 - k^2}\right] + O(\epsilon^3)\,. \tag{5.2}$$

Note, however, that on each interval $\chi_2 \in (k,\; k+1)$, $k = 1, 2, \ldots$, the terms $\frac{J_k(\chi_1)^2}{\chi_2^2-k^2} + \frac{J_{k+1}(\chi_1)^2}{\chi_2^2-(k+1)^2}$ vary continuously from $+\infty$ to $-\infty$. Hence, there is on each interval $(k,\; k+1)$, $k = 0, 1, 2, \ldots$, a special value χ_2^s of χ_2 (depending on χ_1) for which $M(\mathcal{Q}_1) = 0$, and we would be out of case (II). But when $2\chi_2$ is non-integer, one has $M(\mathcal{Q}_3) = 0$. Hence, except for the very unlikely case where $2\chi_2^s$ is an integer, we would be out of condition (III) as well, and $\Omega = F_0 + O(\epsilon^4)$. This special case may not be very interesting, because χ_2 has to be chosen with precision. That could be difficult to fix it in some experimental setting.

Another special situation would occur when χ_2 is chosen to satisfy $F_0 - i\epsilon^2 M(\mathcal{Q}_1) = 0$. By the argument above, this is possible, but χ_2 will depend on ϵ. It is therefore unclear if Ω will be just $O(\epsilon^4)$ or "small" (eventually leading to an even stronger dynamical localisation than we have in case (III)). It is not even clear that we will be in a situation where our series converge, and we left this other special situation without more comments.

It is interesting to compare the expressions for the secular frequency Ω in the three situations above (for $F_0 \neq 0$) with the situation where $\varphi = 0$, where the secular frequency Ω_0 is $\Omega_0 := F_0\sqrt{1 + (\frac{\epsilon}{F_0})^2} = F_0 + \frac{\epsilon^2}{2F_0} + O(\epsilon^3)$. This reveals the effect of the ac-field $\varphi\cos(\omega t)$ on the secular frequency. Taking $\chi_1 \to 0$ in (5.2) we recover Ω_0.

iii. Consider the case where $\chi_2 = 0$, i.e., $F_0 = 0$, and $\chi_1 = x_a$, the a-th zero zero of the Bessel function J_0 on the positive real axis. This case is interesting in connection with the issue of dynamical localisation, as discussed by many authors ([14]. For more references, see [3, 7]). Here $M(\mathcal{Q}_0) = 0$ and,

$$M(\mathcal{Q}_1) \;=\; \frac{i}{\omega}\sum_{m=1}^{\infty}\left(\frac{J_m(\chi_1)^2 - J_{-m}(\chi_1)^2}{m}\right) \;=\; 0\,,$$

since $J_k(x) = (-1)^k J_{-k}(x)$. Thus, condition (II) does not apply and we have to look at $M(0 \mid 1)$. We obtain $M(0 \mid 1) = \frac{1}{\omega^2}\mathcal{T}(x_a)$, with

$$\mathcal{T}(x_a) := -\sum_{n,m\in\mathbb{Z}_*} \frac{J_n(x_a)\, J_{n-m}(x_a)\, J_m(x_a)}{n\, m} .$$

We conclude that condition (III) will be valid, except perhaps for spurious zeros of J_0 for which $\mathcal{T}(x_a)$ eventually vanishes. Numerical computations, though, indicate that such zeros may not exist. We abstain from giving an analytical proof that $\mathcal{T}(x_a) \neq 0$ for all zeros of the J_0 function. Figure 1 shows the values of $|\mathcal{T}(x_a)|$ calculated numerically on the first fifteen zeros x_a of J_0. We note an exponential decay of $|\mathcal{T}(x_a)|$ as x_a increases, suggesting that $|\mathcal{T}(x_a)| \to 0$ only for $a \to \infty$.

We conclude that condition (III) is suitable for studying the monochromatic field when χ_1 lies over the "resonant" points x_a, leading, together with condition (I), to a complete solution of (5.1) except, perhaps, for some rather spurious situations.

Note finally that in this case **iii** we have $\Omega = O(\epsilon^3)$ (see (2.16)). In fact, the first contribution to Ω is $2\frac{\epsilon^3}{\omega^2}\mathcal{T}(x_a)$. This weak dependence on ϵ implies long transition times for certain probability amplitudes (see below), an issue known as dynamical localisation (see [7]).

In order to test our algorithm and to extract more information from our solutions, we wrote a computer program to compute the matrix elements of the propagator $U(t)$ given in (1.8) for the case **iii** described above, where one has approximate dynamical localisation. The results are excellent and are briefly reported in the next section.

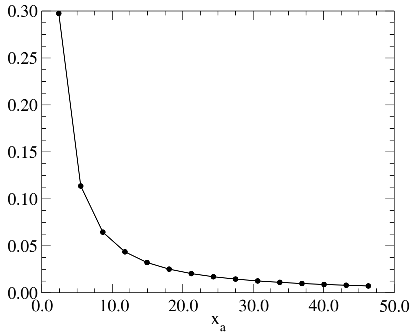

Figure 1: The filled circles represent the values of $|\mathcal{T}(x_a)|$ as a function of the a-th zero x_a of J_0 in $\mathbb{R}_+$. We infer an exponential decay of $|M(0 \mid 1)|$ as a increases.

Remark 5.1 *We obtained for case* **iii**, $\Omega = 2\frac{\epsilon^3}{\omega^2}\mathcal{T}(x_a) + O(\epsilon^4)$. *Following another* *path, we obtained in [3],*

$$\Omega = -2\frac{\epsilon^3}{\omega^2} \sum_{n_1,\, n_2 \in \mathbb{Z}} \frac{J_{2n_1+1}(x_a) J_{2n_2+1}(x_a) J_{-2(n_1+n_2+1)}(x_a)}{(2n_1+1)(2n_2+1)]} + O(\epsilon^4).$$

Thus, we must have

$$\sum_{n,m \in \mathbb{Z}_*} \frac{J_n(x_a)\, J_{n-m}(x_a)\, J_m(x_a)}{n\, m} = \sum_{n_1,\, n_2 \in \mathbb{Z}} \frac{J_{2n_1+1}(x_a) J_{2n_2+1}(x_a) J_{-2(n_1+n_2+1)}(x_a)}{(2n_1+1)(2n_2+1)}.$$

Curiously, such an identity is not easy to prove by direct means. It can be established, however, with the use of statement (6) of Proposition 3.1, namely, from $M(0\,|\,0\,|\,0) = 0$*. Actually, such an identity is not specific for Bessel functions, but is valid for sequences satisfying certain properties.*

6 Numerical Results

In this final section, we will briefly show some numerical results that were obtained using condition (III) to study the monochromatic interaction over the resonant points (see the discussion in the last section). Our goal here is to show how one can numerically use the results of Theorems 2.1, 2.2 and 2.3 to produce results of physical interest.

The computation of the wave function $\Phi(t)$ associated with the Schrödinger equation (1.3) can be performed by means of the propagator $U(t)$, given in (2.2), in terms of the solution of the generalised Riccati equation (2.1) given in Theorem 2.2. Due to the uniform convergence of $g(t)$ for all $t \in \mathbb{R}$ (see Theorem 2.3), one can use the proper expressions given in [2] to compute the elements of $U(t)$ via uniformly convergent power series in ϵ. Although this task may not be so trivial, it can be implemented numerically with great success, as we are about to show now.

The first step toward the computation of $U(t)$ is to evaluate the Fourier coefficients of $g(t)$ using the recursive relations given in (B.1)-(B.2). Of course, we need first to compute the constants κ_n. To this end, we have to use expressions (4.36) and (4.38). Those equations involves the mean value of some previously defined functions, which can be obtained trough the zero order terms of their Fourier decomposition. Equation (3.4) of Proposition 3.1 can be used to this end. Once $V_m^{(n)}$ are all computed, the m-th Fourier coefficient of order n of $g(t)$ can be obtained via the convolution $G_m^{(n)} = \sum_{p\in\mathbb{Z}} Q_{m-p} V_p^{(n)}$ (see equation (2.5)). As a second and final step towards the computation of $U(t)$, we have to use the expressions given in [2]. Basically, these expressions convert the integral form of the elements of the propagator $U(t)$ (see Theorem 2.1) into convergent power series in ϵ using the Fourier coefficients $G_m^{(n)}$ of $g(t)$. They also lead to the expression of $U(t)$ in terms of its *Floquet form* (see Theorem 1.1). A more detailed description of our numerical study will be postponed for a future publication [13]. For now, let us briefly show some of our results.

We have followed steps one and two above to numerically compute the propagator $U(t)$ associated with (1.3) when $f(t) = \varphi \cos(\omega t)$ in the situation where $M(\mathcal{Q}_0) = 0$, that is, over the "resonant" points defined by the condition $\varphi = \omega x_a/2$, x_a being the a-th positive zero of J_0.

Let $\Phi_+ = \binom{1}{0}$ and $\Phi_- = \binom{0}{1}$ be two orthogonal states of a system described by (1.3). The probability for the transition from the initial state Φ_+ to the final state Φ_- at time t is $P(t) := |\langle \Phi_+, U(t)\Phi_- \rangle|^2 = |U_{12}(t)|^2$. We computed $P(t)$ numerically using our expansions. To estimate the accuracy of our calculations, we tested the unitarity of the time evolution and considered the quantity $N(t) := |U_{11}(t)|^2 + |U_{12}(t)|^2$, which should be identically equal to 1 for unitary $U(t)$.

In Figure 2**(a)**, we show $P(t)$ for $\varphi = \omega x_1/2$, x_1 being the first positive zero of J_0. We took $\omega = 10.0$, $\epsilon = 0.1$ and worked with a sixth order expansion in ϵ. The time interval considered corresponds to 1.6×10^6 times the basic cycle $2\pi/\omega$ of f. We recall that in this case we have condition (III) and the *secular frequency*, which dominates the quasi-periodic evolution of the system, is of order ϵ^3/ω^2 (see [3] or the discussion in Section 2.1). This explains the long time ($\sim 10^6$ of the basic cycle of f) needed for the system to transit from Φ_+ to Φ_-.

In Figure 2**(b)**, we show the quantity $N(t) - 1$ as a function of time obtained from the same calculations leading to Figure 2**(a)**. By looking at the deviations of $N(t)$ from 1, we can infer that our perturbative solution produces errors of the order of only 0.3%. This excellent numerical precision for long times is a consequence of the uniform convergence in time of our expansions, i.e., of the elimination of the secular terms.

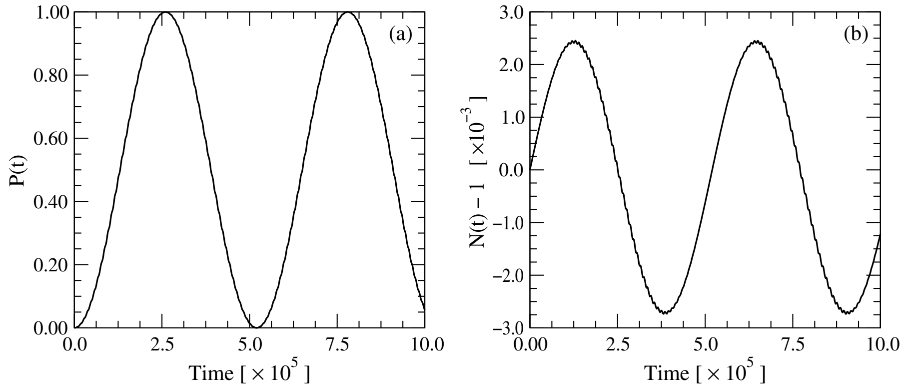

Figure 2: **(a)** The transition probability $P(t)$ as function of time. Here $\epsilon = 0.1$, $\omega = 10$ and $2\varphi_1/\omega = x_1$, the first zero of J_0. **(b)** The quantity $N(t) - 1$ that measures deviation from unitarity.

In Figure 3 we took $\omega = 1.0$, $\epsilon = 0.01$ and $\varphi_1 = \omega x_1/2$. From Figure 3**(b)** we infer errors of the order of only 0.05%. The time interval considered corresponds to 1.6×10^6 times the basic cycle $2\pi/\omega$ of f. In Figure 4 we took $\omega = 10.0$, $\epsilon = 0.01$ and again $\varphi_1 = \omega x_1/2$. From Figure 4**(b)** we infer errors of the order of only 0.003%. The time interval considered corresponds to 1.6×10^9 times the basic cycle $2\pi/\omega$ of f.

The numerical computations above involved the evaluation of the Fourier expansions of several functions, like q, v_n's etc. We typically computed about 20 terms of that expansions, but since the coefficients decay very fast, even less terms are needed. More details about the numerical computations will be presented in a future publication [13].

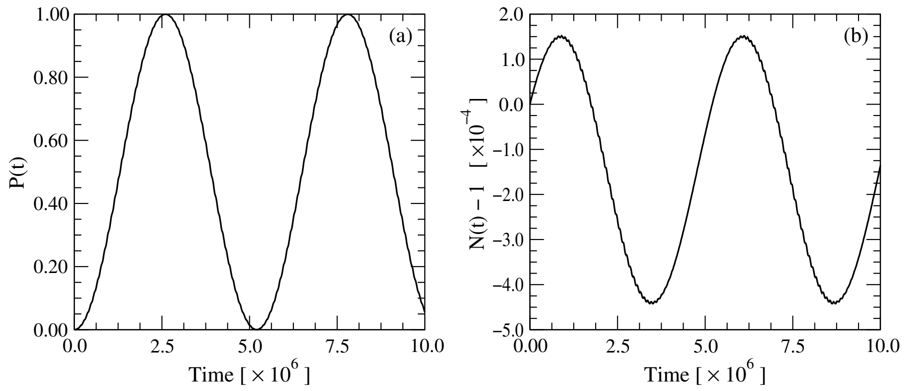

Figure 3: **(a)** The transition probability $P(t)$ as function of time. Here $\epsilon = 0.01$, $\omega = 1.0$ and $2\varphi_1/\omega = x_1$, the first zero of J_0. **(b)** The quantity $N(t) - 1$ that measures deviation from unitarity.

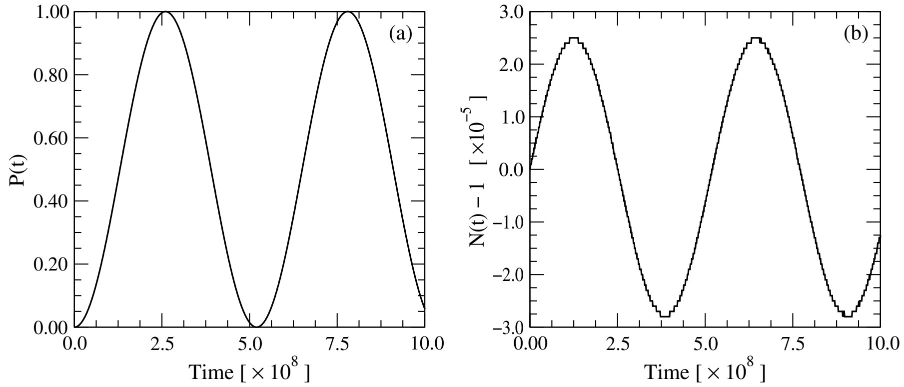

Figure 4: **(a)** The transition probability $P(t)$ as function of time. Here $\epsilon = 0.01$, $\omega = 10.0$ and $2\varphi_1/\omega = x_1$, the first zero of J_0. **(b)** The quantity $N(t) - 1$ that measures deviation from unitarity.

Appendices

A Some Special Relations

This appendix presents the proofs of some relations used in Section 4. Since they involve a somewhat large amount of algebraic manipulations we prefer to separate them from the main text. In the following we will (often without explicit mention) make repeated use of the propositions and corollaries proven in Sections 3.1 and 3.2.

A.1 Obtaining Relation (4.13)

Since we have $M(\mathcal{I}_3) = 0$ and since with the choice of κ_1 in (4.12) we imposed $M(\mathcal{I}_4) = 0$, we have by the recursive relations (4.1)-(4.3)

$$\begin{aligned} v_1 v_3 &= 2i\kappa_1 \, \mathfrak{R}_2 \left(0 \mid v_1 v_2\right) + \kappa_1 \kappa_3 \mathcal{Q}_0 \,, && \text{(A.1)} \\ v_1 v_4 &= 2i\kappa_1 \, \mathfrak{R}_2 \left(0 \mid v_1 v_3\right) + i\kappa_1 \, \mathfrak{R}_2 \left(0 \,\middle|\, v_2^2\right) + \kappa_1 \kappa_4 \mathcal{Q}_0 \,. && \text{(A.2)} \end{aligned}$$

Inserting (4.4) into the r.h.s. of (A.1) we get

$$v_1 v_3 \;=\; -2\kappa_1^4 \, \mathfrak{R}_2 \left(0 \mid 2\right) + 2\kappa_1^2 \, \mathfrak{R}_2 \left(0 \mid 1\right) + 2i\kappa_1^2 \kappa_2 \, \mathfrak{R}_2 \left(0 \mid 0\right) + \kappa_1 \kappa_3 \, \mathcal{Q}_0 \,. \quad \text{(A.3)}$$

Inserting this into the r.h.s. of (A.2) gives

$$\begin{aligned} v_1 v_4 \;=\;& -4i\kappa_1^5 \, \mathfrak{R}_3 \left(0 \mid 0 \mid 2\right) + 4i\kappa_1^3 \, \mathfrak{R}_3 \left(0 \mid 0 \mid 1\right) - 4\kappa_1^3 \kappa_2 \, \mathfrak{R}_3 \left(0 \mid 0 \mid 0\right) \\ & + 2i\kappa_1^2 \kappa_3 \, \mathfrak{R}_2 \left(0 \mid 0\right) + i\kappa_1 \, \mathfrak{R}_2 \left(0 \,\middle|\, v_2^2\right) + \kappa_1 \kappa_4 \mathcal{Q}_0 \,. && \text{(A.4)} \end{aligned}$$

Let us now compute v_2^2. Since $v_2 = \kappa_2 q + iq^{-1}(\kappa_1^2 \mathcal{Q}_2 - \mathcal{Q}_1)$, we have

$$v_2^2 \;=\; \kappa_2^2 \mathcal{Q}_0 + 2i\kappa_1^2\kappa_2 \mathcal{Q}_2 - 2i\kappa_2 \mathcal{Q}_1 + s\,, \tag{A.5}$$

where $s := -\mathcal{Q}_0^{-1}(\kappa_1 \mathcal{Q}_2 - \mathcal{Q}_1)^2$. Note that s contains κ_1 alone.

Now, we insert this into (A.4) and get

$$\begin{aligned} v_1 v_4 \;=\;& -4i\kappa_1^5\, \mathfrak{R}_3\,(0\,|\,0\,|\,2\,) + 4i\kappa_1^3\, \mathfrak{R}_3\,(0\,|\,0\,|\,1\,) - 6\kappa_1^3\kappa_2\, \mathfrak{R}_2\,(0\,|\,2) \\ & +2i\kappa_1^2\kappa_3\, \mathfrak{R}_2\,(0\,|\,0\,) + i\kappa_1\kappa_2^2\, \mathfrak{R}_2\,(0\,|\,0\,) + 2\kappa_1\kappa_2\, \mathfrak{R}_2\,(0\,|\,1\,) \\ & +i\kappa_1\, \mathfrak{R}_2\,(0\,|\,s\,) + \kappa_1\kappa_4 \mathcal{Q}_0\,, \end{aligned} \tag{A.6}$$

where we also used the fact that $\mathfrak{R}_3\,(0\,|\,0\,|\,0\,) = \mathfrak{R}_2\,(0\,|\,2\,)$.

Collecting in (A.6) the terms depending only on κ_1, we define

$$\mathcal{A}_1 \;:=\; -4\kappa_1^4\, \mathfrak{R}_3\,(0\,|\,0\,|\,2\,) + 4\kappa_1^2\, \mathfrak{R}_3\,(0\,|\,0\,|\,1\,) + \mathfrak{R}_2\,(0\,|\,s\,) \tag{A.7}$$

and rewrite (A.6) as

$$v_1 v_4 = -6\kappa_1^3\kappa_2\, \mathfrak{R}_2\,(0\,|\,2) + i\kappa_1(2\kappa_1\kappa_3 + \kappa_2^2)\, \mathcal{Q}_2 + 2\kappa_1\kappa_2\, \mathfrak{R}_2\,(0\,|\,1\,) + \kappa_1\kappa_4 \mathcal{Q}_0 + i\kappa_1 \mathcal{A}_1\,,$$

which is expression (4.13), as desired. From definition (A.7), it is evident that $\mathcal{A}_1$ is quasi-periodic, since it is a sum of quasi-periodic functions.

A.2 Obtaining Relation (4.14)

From relations (4.2)–(4.3)

$$v_2 v_3 \;=\; \mathcal{Q}_0^{-1}\left(i\kappa_1^2 \mathcal{Q}_2 - i\,\mathcal{Q}_1 + \kappa_2 \mathcal{Q}_0\right)\left(2i\, \mathfrak{R}_2\,(0\,|\,v_1 v_2\,) + \kappa_3 \mathcal{Q}_0\right)\,. \tag{A.8}$$

Now, by (3.8), we have $\mathcal{Q}_0^{-1} \mathcal{Q}_i\, \mathfrak{R}_2\,(0\,|\,v_1 v_2\,) = \mathfrak{R}_2\,(i\,|\,v_1 v_2\,)$. Hence, (A.8) becomes

$$\begin{aligned} v_2 v_3 \;=\;& -2\kappa_1^2\, \mathfrak{R}_2\,(2\,|\,v_1 v_2\,) + 2\, \mathfrak{R}_2\,(1\,|\,v_1 v_2\,) + 2i\kappa_2\, \mathfrak{R}_2\,(0\,|\,v_1 v_2\,) \\ & +\kappa_3\left(i\kappa_1^2 \mathcal{Q}_2 - i\,\mathcal{Q}_1 + \kappa_2 \mathcal{Q}_0\right)\,. \end{aligned} \tag{A.9}$$

Inserting (4.4) into the r.h.s. of (A.9), it becomes

$$\begin{aligned} v_2 v_3 \;=\;& -2i\kappa_1^5\, \mathfrak{R}_2\,(2\,|\,2) + 2i\kappa_1^3\, \mathfrak{R}_2\,(2\,|\,1\,) - 2\kappa_1^3\kappa_2\, \mathfrak{R}_2\,(2\,|\,0) + 2i\kappa_1^3\, \mathfrak{R}_2\,(1\,|\,2) \\ & -2i\kappa_1\, \mathfrak{R}_2\,(1\,|\,1\,) + 2\kappa_1\kappa_2\, \mathfrak{R}_2\,(1\,|\,0) - 2\kappa_1^3\kappa_2\, \mathfrak{R}_2\,(0\,|\,2) \\ & +2\kappa_1\kappa_2\, \mathfrak{R}_2\,(0\,|\,1\,) + i(2\kappa_1\kappa_2^2 + \kappa_1^2\kappa_3)\mathcal{Q}_2 - i\kappa_3 \mathcal{Q}_1 + \kappa_2\kappa_3 \mathcal{Q}_0\,. \end{aligned}$$

This is (4.14), as desired.

A.3 Obtaining Relation (4.17)

According to equations (4.1)–(4.3), we have

$$2v_1 v_5 \;=\; 4i\kappa_1\,\mathfrak{R}_2\,(0\,|\,v_1v_4+v_2v_3\,)+2\kappa_1\kappa_5\mathcal{Q}_0\,. \tag{A.10}$$

By (4.13) and (4.14)

$$\begin{aligned}
v_1v_4+v_2v_3 \;=\;& -2i\kappa_1^5\,\mathfrak{R}_2\,(2\,|\,2\,)+2i\kappa_1^3\,\mathfrak{R}_2\,(2\,|\,1\,)-2\kappa_1^3\kappa_2\,\mathfrak{R}_2\,(2\,|\,0\,)\\
&+2i\kappa_1^3\,\mathfrak{R}_2\,(1\,|\,2\,)-2i\kappa_1\,\mathfrak{R}_2\,(1\,|\,1\,)+2\kappa_1\kappa_2\,\mathfrak{R}_2\,(1\,|\,0\,)\\
&-8\kappa_1^3\kappa_2\,\mathfrak{R}_2\,(0\,|\,2\,)+4\kappa_1\kappa_2\,\mathfrak{R}_2\,(0\,|\,1\,)+3i(\kappa_1^2\kappa_3+\kappa_1\kappa_2^2)\mathcal{Q}_2\\
&-i\kappa_3\mathcal{Q}_1+(\kappa_2\kappa_3+\kappa_1\kappa_4)\mathcal{Q}_0+i\kappa_1\mathcal{A}_1\,.
\end{aligned}$$

Inserting this into (A.10), gives

$$\begin{aligned}
2v_1v_5 \;=\;& 8\kappa_1^6\,\mathfrak{R}_3\,(0\,|\,2\,|\,2\,)-8i\kappa_1^4\,\mathfrak{R}_3\,(0\,|\,2\,|\,1\,)-8i\kappa_1^4\kappa_2\,\mathfrak{R}_3\,(0\,|\,2\,|\,0\,)\\
&-8\kappa_1^4\,\mathfrak{R}_3\,(0\,|\,1\,|\,2\,)+8\kappa_1^2\,\mathfrak{R}_3\,(0\,|\,1\,|\,1\,)+8i\kappa_1^2\kappa_2\,\mathfrak{R}_3\,(0\,|\,1\,|\,0\,)\\
&-32i\kappa_1^4\kappa_2\,\mathfrak{R}_3\,(0\,|\,0\,|\,2\,)+16i\kappa_1^2\kappa_2\,\mathfrak{R}_3\,(0\,|\,0\,|\,1\,)\\
&-12(\kappa_1^3\kappa_3+\kappa_1^2\kappa_2^2)\,\mathfrak{R}_2\,(0\,|\,2\,)+4\kappa_1\kappa_3\,\mathfrak{R}_2\,(0\,|\,1\,)\\
&+4i\kappa_1(\kappa_2\kappa_3+\kappa_1\kappa_4)\,\mathfrak{R}_2\,(0\,|\,0\,)-4\kappa_1^2\,\mathfrak{R}_2\,(0\,|\,\mathcal{A}_1\,)+2\kappa_1\kappa_5\mathcal{Q}_0\,.
\end{aligned}$$

Let us now isolate all terms depending only on κ_1 and/or κ_2 in one single term, which we call $\mathcal{A}_2$:

$$\begin{aligned}
\mathcal{A}_2 \;:=\;& 8\kappa_1^6\,\mathfrak{R}_3\,(0\,|\,2\,|\,2\,)-8i\kappa_1^4\,\mathfrak{R}_3\,(0\,|\,2\,|\,1\,)-8i\kappa_1^4\kappa_2\,\mathfrak{R}_3\,(0\,|\,2\,|\,0\,)\\
&-8\kappa_1^4\,\mathfrak{R}_3\,(0\,|\,1\,|\,2\,)+8\kappa_1^2\,\mathfrak{R}_3\,(0\,|\,1\,|\,1\,)+8i\kappa_1^2\kappa_2\,\mathfrak{R}_3\,(0\,|\,1\,|\,0\,)\\
&+16i\kappa_1^2\kappa_2\,\mathfrak{R}_3\,(0\,|\,0\,|\,1\,)-32i\kappa_1^4\kappa_2\,\mathfrak{R}_3\,(0\,|\,0\,|\,2\,)\\
&-12\kappa_1^2\kappa_2^2\,\mathfrak{R}_2\,(0\,|\,2\,)-4\kappa_1^2\,\mathfrak{R}_2\,(0\,|\,\mathcal{A}_1\,)
\end{aligned} \tag{A.11}$$

and write

$$\begin{aligned}
2v_1v_5 \;=\;& \mathcal{A}_2-12\kappa_1^3\kappa_3\,\mathfrak{R}_2\,(0\,|\,2\,)+4\kappa_1\kappa_3\,\mathfrak{R}_2\,(0\,|\,1\,)\\
&+4i\kappa_1(\kappa_2\kappa_3+\kappa_1\kappa_4)\,\mathfrak{R}_2\,(0\,|\,0\,)+2\kappa_1\kappa_5\mathcal{Q}_0\,.
\end{aligned}$$

This is (4.17), as desired. Note that $\mathcal{A}_2$ is clearly quasi-periodic, being a sum of quasi-periodic functions.

A.4 Obtaining Relation (4.19)

Using relations (4.2)–(4.3), and (3.8) (which implies $\mathcal{Q}_0^{-1}\mathcal{Q}_i\,\mathfrak{R}_2\,(0\,|\,a\,)=\,\mathfrak{R}_2\,(i\,|\,a\,)$) we have

$$\begin{aligned}
2v_2v_4 \;=\;& 2\mathcal{Q}_0^{-1}\left(i\kappa_1^2\mathcal{Q}_2-i\mathcal{Q}_1+\kappa_2\mathcal{Q}_0\right)\left[2i\,\mathfrak{R}_2\,(0\,|\,v_1v_3\,)+i\,\mathfrak{R}_2\left(0\,\middle|\,v_2^2\,\right)+\kappa_4\mathcal{Q}_0\right]\\
=\;& -4\kappa_1^2\,\mathfrak{R}_2\,(2\,|\,v_1v_3\,)+4\,\mathfrak{R}_2\,(1\,|\,v_1v_3\,)+4i\kappa_2\,\mathfrak{R}_2\,(0\,|\,v_1v_3\,)\\
&-2\kappa_1^2\,\mathfrak{R}_2\left(2\,\middle|\,v_2^2\,\right)+2\,\mathfrak{R}_2\left(1\,\middle|\,v_2^2\,\right)+2i\kappa_2\,\mathfrak{R}_2\left(0\,\middle|\,v_2^2\,\right)\\
&+2\kappa_4\left(i\kappa_1^2\mathcal{Q}_2-i\mathcal{Q}_1+\kappa_2\mathcal{Q}_0\right)
\end{aligned}$$

Inserting above the expression for $v_1 v_3$ given in (A.3), we get

$$
\begin{aligned}
2v_2v_4 \;\; = \;\; & 8\kappa_1^6\,\mathfrak{R}_3\,(2\,|\,0\,|\,2\,) - 8\kappa_1^4\,\mathfrak{R}_3\,(2\,|\,0\,|\,1\,) - 8i\kappa_1^4\kappa_2\,\mathfrak{R}_3\,(2\,|\,0\,|\,0\,) \\
& -4\kappa_1^3\kappa_3\,\mathfrak{R}_2\,(2\,|\,0\,) - 8\kappa_1^4\,\mathfrak{R}_3\,(1\,|\,0\,|\,2\,) + 8\kappa_1^2\,\mathfrak{R}_3\,(1\,|\,0\,|\,1\,) \\
& +8i\kappa_1^2\kappa_2\,\mathfrak{R}_3\,(1\,|\,0\,|\,0\,) + 4\kappa_1\kappa_3\,\mathfrak{R}_2\,(1\,|\,0\,) - 8i\kappa_1^4\kappa_2\,\mathfrak{R}_3\,(0\,|\,0\,|\,2\,) \\
& +8i\kappa_1^2\kappa_2\,\mathfrak{R}_3\,(0\,|\,0\,|\,1\,) - 8\kappa_1^2\kappa_2^2\,\mathfrak{R}_3\,(0\,|\,0\,|\,0\,) + 4i\kappa_1\kappa_2\kappa_3\,\mathfrak{R}_2\,(0\,|\,0\,) \\
& -2\kappa_1^2\,\mathfrak{R}_2\left(2\,\middle|\,v_2^2\right) + 2\,\mathfrak{R}_2\left(1\,\middle|\,v_2^2\right) + 2i\kappa_2\,\mathfrak{R}_2\left(0\,\middle|\,v_2^2\right) \\
& +2\kappa_4\left(i\kappa_1^2\mathcal{Q}_2 - i\mathcal{Q}_1 + \kappa_2\mathcal{Q}_0\right) \,.
\end{aligned}
$$

We now isolate all terms depending only on κ_1 and/or κ_2 in one single term, which we call $\mathcal{A}_3$:

$$
\begin{aligned}
\mathcal{A}_3 \;\; := \;\; & 8\kappa_1^6\,\mathfrak{R}_3\,(2\,|\,0\,|\,2\,) - 8\kappa_1^4\,\mathfrak{R}_3\,(2\,|\,0\,|\,1\,) - 8i\kappa_1^4\kappa_2\,\mathfrak{R}_3\,(2\,|\,0\,|\,0\,) \\
& -8\kappa_1^4\,\mathfrak{R}_3\,(1\,|\,0\,|\,2\,) + 8\kappa_1^2\,\mathfrak{R}_3\,(1\,|\,0\,|\,1\,) + 8i\kappa_1^2\kappa_2\,\mathfrak{R}_3\,(1\,|\,0\,|\,0\,) \\
& -8i\kappa_1^4\kappa_2\,\mathfrak{R}_3\,(0\,|\,0\,|\,2\,) + 8i\kappa_1^2\kappa_2\,\mathfrak{R}_3\,(0\,|\,0\,|\,1\,) - 8\kappa_1^2\kappa_2^2\,\mathfrak{R}_3\,(0\,|\,0\,|\,0\,) \\
& -2\kappa_1^2\,\mathfrak{R}_2\left(2\,\middle|\,v_2^2\right) + 2\,\mathfrak{R}_2\left(1\,\middle|\,v_2^2\right) + 2i\kappa_2\,\mathfrak{R}_2\left(0\,\middle|\,v_2^2\right) \qquad \text{(A.12)}
\end{aligned}
$$

(note here that, by (A.5), v_2^2 depends only on κ_1 and κ_2). We can now write

$$
\begin{aligned}
2v_2v_4 \;\; = \;\; & \mathcal{A}_3 - 4\kappa_1^3\kappa_3\,\mathfrak{R}_2\,(2\,|\,0\,) + 4\kappa_1\kappa_3\,\mathfrak{R}_2\,(1\,|\,0\,) \\
& +2i(\kappa_1^2\kappa_4 + 2\kappa_1\kappa_2\kappa_3)\mathcal{Q}_2 - 2i\kappa_4\mathcal{Q}_1 + 2\kappa_2\kappa_4\mathcal{Q}_0 \,.
\end{aligned}
$$

This is the desired relation (4.19). Again, note that $\mathcal{A}_3$ is quasi-periodic, for it is a sum of quasi-periodic functions.

A.5 Obtaining Relation (4.34)

Let us explicitly write $\kappa_1 l_{n-1}$ in term (i) of (4.33) using equation (4.27). We have,

$$
\kappa_1 l_{n-1}(t) \;=\; i\kappa_1 \sum_{p=1}^{n-2} \mathfrak{R}_2\,(0\,|\,v_p v_{n-1-p}\,)_t \;=:\; A(t) + B(t)\,,
$$

where

$$
A(t) \;:=\; 2i\,\kappa_1\,\mathfrak{R}_2\,(0\,|\,v_1 v_{n-2}\,)\,, \quad B(t) \;:=\; i\kappa_1 \sum_{p=2}^{n-3} \mathfrak{R}_2\,(0\,|\,v_p v_{n-1-p}\,)\,. \qquad \text{(A.13)}
$$

The above expressions for $A(t)$ and $B(t)$ will now be worked individually. Let us start with $A(t)$. By the inductive hypothesis, we are allowed to use (4.1) and (4.3). We write

$$
\begin{aligned}
A(t) \;\; & = \;\; 2\,i\,\kappa_1^2\,\mathfrak{R}_2\left(0\,\middle|\, i\sum_{p=1}^{n-3} \mathfrak{R}_2\,(0\,|\,v_p v_{n-2-p}\,) + \kappa_{n-2}\mathcal{Q}_0\right) \\
& = \;\; -2\,\kappa_1^2 \sum_{p=1}^{n-3} \mathfrak{R}_3\,(0\,|\,0\,|\,v_p v_{n-2-p}\,) + 2\,i\,\kappa_1^2\kappa_{n-2}\,\mathfrak{R}_2\,(0\,|\,0\,)\,.
\end{aligned}
$$

Note that $A(t)$ is implicitly dependent on κ_{n-3}, namely through v_{n-3}. To make this dependence explicit we have to split the sum containing v_{n-3} and write v_{n-3} with the use of (4.3):

$$\begin{aligned}
A(t) &= -2\,\kappa_1^2 \sum_{p=2}^{n-4} \mathfrak{R}_3\left(0 \mid 0 \mid v_p v_{n-2-p}\right) - 4\,\kappa_1^2\, \mathfrak{R}_3\left(0 \mid 0 \mid v_1 v_{n-3}\right) \\
&\quad +2\,i\,\kappa_1^2 \kappa_{n-2}\, \mathfrak{R}_2\left(0 \mid 0\right) \\
&= -2\,\kappa_1^2 \sum_{p=2}^{n-4} \mathfrak{R}_3\left(0 \mid 0 \mid v_p v_{n-2-p}\right) - 4i\,\kappa_1^3 \sum_{p=1}^{n-4} \mathfrak{R}_4\left(0 \mid 0 \mid 0 \mid v_p v_{n-3-p}\right) \\
&\quad -4\,\kappa_1^3 \kappa_{n-3}\, \mathfrak{R}_3\left(0 \mid 0 \mid 0\right) + 2\,i\,\kappa_1^2 \kappa_{n-2}\, \mathfrak{R}_2\left(0 \mid 0\right) . \qquad \text{(A.14)}
\end{aligned}$$

We will now work on $B(t)$, equation (A.13). Using (4.1) and (4.3), we have

$$B(t) = 2i\kappa_1\, \mathfrak{R}_2\left(0 \mid v_2 v_{n-3}\right) + i\kappa_1 \sum_{p=3}^{n-4} \mathfrak{R}_2\left(0 \mid v_p v_{n-1-p}\right) . \qquad \text{(A.15)}$$

Next, we have to compute separately $v_2 v_{n-3}$. Using once more (4.2), (4.3) and (3.8) (which implies $\mathcal{Q}_0^{-1} \mathcal{Q}_i\, \mathfrak{R}_2\left(0 \mid a\right) = \mathfrak{R}_2\left(i \mid a\right)$), we get

$$\begin{aligned}
v_2 v_{n-3} &= \mathcal{Q}_0^{-1}\left(i\kappa_1^2 \mathcal{Q}_2 - i\mathcal{Q}_1 + \kappa_2 \mathcal{Q}_0\right) \left\{ i \sum_{p=1}^{n-4} \mathfrak{R}_2\left(0 \mid v_p v_{n-3-p}\right)_t + \kappa_{n-3} \mathcal{Q}_0(t) \right\} \\
&= -\kappa_1^2 \sum_{p=1}^{n-4} \mathfrak{R}_2\left(2 \mid v_p v_{n-3-p}\right) + \sum_{p=1}^{n-4} \mathfrak{R}_2\left(1 \mid v_p v_{n-3-p}\right) \\
&\quad +i\kappa_2 \sum_{p=1}^{n-4} \mathfrak{R}_2\left(0 \mid v_p v_{n-3-p}\right) + \kappa_{n-3}\left(i\kappa_1^2 \mathcal{Q}_2 - i\mathcal{Q}_1 + \kappa_2 \mathcal{Q}_0\right) .
\end{aligned}$$

This expression for $v_2 v_{n-3}$ has to be introduced into the first term of (A.15). The result is

$$\begin{aligned}
B(t) &= 2i\kappa_1 \sum_{p=1}^{n-4} \Big\{ -\kappa_1^2\, \mathfrak{R}_3\left(0 \mid 2 \mid v_p v_{n-3-p}\right) + \mathfrak{R}_3\left(0 \mid 1 \mid v_p v_{n-3-p}\right) \\
&\quad +i\kappa_2\, \mathfrak{R}_3\left(0 \mid 0 \mid v_p v_{n-3-p}\right) \Big\} + i\kappa_1 \sum_{p=3}^{n-4} \mathfrak{R}_2\left(0 \mid v_p v_{n-1-p}\right) . \\
&\quad +2i\kappa_1 \kappa_{n-3}\left[i\kappa_1^2\, \mathfrak{R}_2\left(0 \mid 2\right) - i\, \mathfrak{R}_2\left(0 \mid 1\right) + \kappa_2\, \mathfrak{R}_2\left(0 \mid 0\right)\right] . \qquad \text{(A.16)}
\end{aligned}$$

Since both A and B are quasi-periodic, we can now compute $\kappa_1 M(l_{n-1}) = M(A) + M(B)$. Using (A.14), (A.16), (4.6) and the already proven fact that $M\left(0 \mid 0\right) = 0$, we get

$$\kappa_1 M(l_{n-1}) = 2\kappa_1 \kappa_{n-3} M\left(0 \mid 1\right) + \mathcal{R}_n^{(1)} , \qquad \text{(A.17)}$$

where

$$\begin{aligned}\mathcal{R}_n^{(1)} \;:=\; & 2i\kappa_1 \sum_{p=1}^{n-4} \Bigg\{ -2\kappa_1^2 M(\,\mathfrak{R}_4\,(0\,|\,0\,|\,0\,|\,v_p v_{n-3-p}\,)) + M(\,\mathfrak{R}_3\,(0\,|\,1\,|\,v_p v_{n-3-p}\,)) \\ & -\kappa_1^2 M(\,\mathfrak{R}_3\,(0\,|\,2\,|\,v_p v_{n-3-p}\,)) + i\kappa_2 M(\,\mathfrak{R}_3\,(0\,|\,0\,|\,v_p v_{n-3-p}\,)) \Bigg\} \\ & -2\,\kappa_1^2 \sum_{p=2}^{n-4} M(\,\mathfrak{R}_3\,(0\,|\,0\,|\,v_p v_{n-2-p}\,)) + i\kappa_1 \sum_{p=3}^{n-4} M(\,\mathfrak{R}_2\,(0\,|\,v_p v_{n-1-p}\,))\,.\end{aligned} \tag{A.18}$$

This is the desired relation (4.34). By inspection, one verifies that $\mathcal{R}_n^{(1)}$ depends on the constants $\kappa_1, \ldots, \kappa_{n-4}$ only. Notice that the constant κ_{n-2} disappeared completely when we took the mean value of $A(t) + B(t)$, due to the crucial fact that $M(\mathcal{Q}_2) = 0$. This is very important, otherwise we would have in (A.17) an equation for <u>two</u> unknowns κ_{n-3} and κ_{n-2}.

A.6 Obtaining Relation (4.35)

The main point is to make the κ_{n-3} dependence of $l_{n-2}(t)$ explicit. Using (4.28) and (4.3) for $v_{n-3}(t)$, we can write

$$\begin{aligned} q^{-1} v_2 l_{n-2} \;&=\; i q^{-1} v_2 \sum_{p=1}^{n-3} \mathfrak{R}_2\,(0\,|\,v_p v_{n-2-p}\,) \;=\; i \sum_{p=1}^{n-3} \mathfrak{R}_2\,(q v_2\,|\,v_p v_{n-2-p}\,) \\ &=\; 2i\,\mathfrak{R}_2\,(q v_2\,|\,v_1 v_{n-3}\,) + i \sum_{p=2}^{n-4} \mathfrak{R}_2\,(q v_2\,|\,v_p v_{n-2-p}\,) \\ &=\; 2i\kappa_1\,\mathfrak{R}_2 \left(q v_2 \;\middle|\; i \sum_{p=1}^{n-4} \mathfrak{R}_2\,(0\,|\,v_p v_{n-3-p}\,) + \kappa_{n-3}\mathcal{Q}_0 \right) \\ &\quad\; + i \sum_{p=2}^{n-4} \mathfrak{R}_2\,(q v_2\,|\,v_p v_{n-2-p}\,) \\ &=\; -2\kappa_1 \sum_{p=1}^{n-4} \mathfrak{R}_3\,(q v_2\,|\,0\,|\,v_p v_{n-3-p}\,) + i \sum_{p=2}^{n-4} \mathfrak{R}_2\,(q v_2\,|\,v_p v_{n-2-p}\,) \\ &\quad\; + 2i\kappa_1\kappa_{n-3}\,\mathfrak{R}_2\,(q v_2\,|\,0\,)\;. \end{aligned}$$

According to (4.2),

$$\mathfrak{R}_2\,(q v_2\,|\,0\,) \;=\; i\kappa_1^2\,\mathfrak{R}_2\,(2\,|\,0\,) - i\,\mathfrak{R}_2\,(1\,|\,0\,) + \kappa_2\,\mathfrak{R}_2\,(0\,|\,0\,)$$

and, hence, $M(\,\mathfrak{R}_2\,(q v_2\,|\,0\,)) = -iM\,(1\,|\,0\,) = iM\,(0\,|\,1\,)$. Therefore,

$$M(q^{-1} v_2 l_{n-2}) \;=\; -2\kappa_1\kappa_{n-3} M\,(0\,|\,1\,) + \mathcal{R}_n^{(2)}\,,$$

where

$$\mathcal{R}_n^{(2)} \ := \ -2\kappa_1 \sum_{p=1}^{n-4} M(\mathfrak{R}_3\,(qv_2\ |\,0\ |\,v_p v_{n-3-p}\,)) + i \sum_{p=2}^{n-4} M(\mathfrak{R}_2\,(qv_2\ |\,v_p v_{n-2-p}\,))\,. \tag{A.19}$$

This is the desired equation (4.35). By inspection, one verifies that $\mathcal{R}_n^{(2)}$ depends on the constants $\kappa_1, \ldots, \kappa_{n-4}$ only.

B Proof of Convergence of the ϵ Expansion for Periodic f

Here we will present the proof of Theorem 2.3, i.e., the proof of convergence of the ϵ expansion of (2.5) for periodic f. It follows the ideas of [2], but technical adaptations are necessary. For the sake of simplification we shall consider here only the case where $F_0 = M(f) = 0$. The general case $F_0 \neq 0$ can be treated following the lines described in detail in [2].

In terms of the Fourier coefficients Q_m and $Q_m^{(2)}$, appearing in (2.11), of the Fourier coefficients $V_m^{(n)}$ of (2.12) and of the constants κ_n, relations (4.1)-(4.3), become

$$V_m^{(1)} \ = \ \kappa_1 Q_m, \quad V_m^{(2)} \ = \ \sum_{\substack{n_1 \in \mathbb{Z} \\ n_1 \neq 0}} \frac{Q_{m-n_1}\left(\kappa_1^2 Q_{n_1}^{(2)} - \overline{Q_{-n_1}^{(2)}}\right)}{n_1 \omega} \ + \ \kappa_2 Q_m, \tag{B.1}$$

$$V_m^{(n)} \ = \sum_{\substack{n_1,\, n_2 \in \mathbb{Z} \\ n_1 + n_2 \neq 0}} \frac{Q_{m-(n_1+n_2)}}{(n_1+n_2)\omega} \left(\sum_{p=1}^{n-1} V_{n_1}^{(p)} V_{n_2}^{(n-p)} \right) \ + \ \kappa_n Q_m, \quad \text{for } n \geq 3. \tag{B.2}$$

Of course, due to the choices of the constants κ_n described before, no secular terms appear.

By (2.13) and by an inductive argument, we will prove the following statement: for all $p \in \mathbb{N}$ and all $m \in \mathbb{Z}$ there are constants $K_p > 0$ such that

$$\left| V_m^{(p)} \right| \ \leq \ K_p \frac{e^{-\chi |m|}}{\ll m \gg^2}. \tag{B.3}$$

To show this, let us first recall the following result, proven in [2]:

Lemma B.1 *For $\chi > 0$ and $m \in \mathbb{Z}$ define*

$$\mathcal{B}(m) \equiv \mathcal{B}(m,\, \chi) \ := \ \sum_{n \in \mathbb{Z}} \frac{e^{-\chi(|m-n|+|n|)}}{\ll m-n \gg^2 \ \ll n \gg^2}.$$

Then one has

$$\mathcal{B}(m) \;\le\; B_0 \frac{e^{-\chi|m|}}{\ll m \gg^2},$$

for some constant $B_0 \equiv B_0(\chi) > 0$ *and for all* $m \in \mathbb{Z}$. $\square$

From (B.1) and (2.13), we have

$$\begin{aligned}
\left|V_m^{(1)}\right| \;&\le\; \mathcal{Q}\frac{e^{-\chi|m|}}{\ll m\gg^2},\\
\left|V_m^{(2)}\right| \;&\le\; \frac{2\mathcal{Q}^2}{\omega}\sum_{n_1\in\mathbb{Z}_*}\frac{e^{-\chi(|m-n_1|+|n_1|)}}{\ll m-n_1\gg^2\ll n_1\gg^2}\frac{1}{|n_1|} \;+\; |\kappa_2|\mathcal{Q}\frac{e^{-\chi|m|}}{\ll m\gg^2},
\end{aligned}$$

where we used $|\kappa_1| = 1$. Now

$$\sum_{n_1\in\mathbb{Z}_*}\frac{e^{-\chi(|m-n_1|+|n_1|)}}{\ll m-n_1\gg^2\ll n_1\gg^2}\frac{1}{|n_1|} \;\le\; \sum_{n_1\in\mathbb{Z}}\frac{e^{-\chi(|m-n_1|+|n_1|)}}{\ll m-n_1\gg^2\ll n_1\gg^2} \;\le\; B_0\frac{e^{-\chi|m|}}{\ll m\gg^2},$$

where the last inequality comes from Lemma B.1. Hence, we can write

$$\left|V_m^{(1)}\right| \;\le\; K_1\frac{e^{-\chi|m|}}{\ll m\gg^2} \qquad \text{and} \qquad \left|V_m^{(2)}\right| \;\le\; K_2\frac{e^{-\chi|m|}}{\ll m\gg^2},$$

for all $m \in \mathbb{Z}$, by choosing

$$K_1 \;:=\; \mathcal{Q} \qquad \text{and} \qquad K_2 \;:=\; \frac{2\mathcal{Q}^2 B_0}{\omega} + |\kappa_2|\mathcal{Q}.$$

To proceed, let us assume (B.3) for all $p = 1, \dots, n-1$. By (B.2) and (2.13), we have

$$\begin{aligned}
\left|V_m^{(n)}\right| \;\le\; & \frac{\mathcal{Q}}{\omega}\left(\sum_{\substack{n_1,\,n_2\in\mathbb{Z}\\ n_1+n_2\neq 0}}\frac{e^{-\chi(|m-n_1-n_2|+|n_1|+|n_2|)}}{\ll m-n_1-n_2\gg^2\ll n_1\gg^2\ll n_2\gg^2}\frac{1}{|n_1+n_2|}\right)\\
&\times\left(\sum_{p=1}^{n-1}K_pK_{n-p}\right) \;+\; |\kappa_n|\mathcal{Q}\frac{e^{-\chi|m|}}{\ll m\gg^2}, \qquad \text{for } n\ge 3.
\end{aligned}$$

Again, applying twice Lemma B.1,

$$\begin{aligned}
&\sum_{\substack{n_1,\,n_2\in\mathbb{Z}\\ n_1+n_2\neq 0}}\frac{e^{-\chi(|m-n_1-n_2|+|n_1|+|n_2|)}}{\ll m-n_1-n_2\gg^2\ll n_1\gg^2\ll n_2\gg^2}\frac{1}{|n_1+n_2|} \;\le\\
&\qquad\qquad \sum_{n_1,\,n_2\in\mathbb{Z}}\frac{e^{-\chi(|m-n_1-n_2|+|n_1|+|n_2|)}}{\ll m-n_1-n_2\gg^2\ll n_1\gg^2\ll n_2\gg^2} \;\le\; (B_0)^2\frac{e^{-\chi|m|}}{\ll m\gg^2}.
\end{aligned}$$

Therefore,

$$\left|V_m^{(n)}\right| \;\leq\; K_n \frac{e^{-\chi|m|}}{\ll m \gg^2}$$

for all $m \in \mathbb{Z}$, by choosing

$$K_n \;:=\; \frac{(B_0)^2 \mathcal{Q}}{\omega} \left(\sum_{p=1}^{n-1} K_p K_{n-p} \right) + \kappa_n^0 \mathcal{Q}, \tag{B.4}$$

where κ_n^0 is some suitably chosen upper bound for $|\kappa_n|$. We now turn our attention to $|\kappa_n|$, for which we have to find estimates using again the inductive hypothesis (B.3) for all $p = 1, \ldots, n-1$. The constants κ_1, κ_2 and κ_3 are are fixed by (4.36) and κ_n, $n \geq 4$, are given by (4.38), from which we get

$$|\kappa_n| \leq \frac{1}{4|M\,(0\mid 1)\,|} \left[\underbrace{\sum_{p=4}^{n-1} |M\,(v_p v_{n+3-p})|}_{T_1} + 2\underbrace{\left|\mathcal{R}_{n+3}^{(1)}\right|}_{T_2} + 2\underbrace{\left|\mathcal{R}_{n+3}^{(2)}\right|}_{T_3} + 2\underbrace{\left|M\,\left(q^{-1} v_3 l_n\right)\right|}_{T_4} \right], \tag{B.5}$$

for $n \geq 4$. We have to bound each of the terms T_i indicated above. Let us start with T_1.

Bound for T_1. By (2.12), $M\,(v_p v_{n+3-p}) = \sum_{m\in\mathbb{Z}} V_m^{(p)} V_{-m}^{(n+3-p)}$. Hence, by the inductive hypothesis (B.3), assumed for $p = 1, \ldots, n-1$, we have

$$T_1 \;=\; \sum_{p=4}^{n-1} |M\,(v_p v_{n+3-p})| \;\leq\; \eta_1 \sum_{p=4}^{n-1} K_p K_{n+3-p}, \tag{B.6}$$

where $\eta_1 := \displaystyle\sum_{m\in\mathbb{Z}} \frac{e^{-2\chi|m|}}{\ll m \gg^2}$.

Bound for T_2. Expression (A.18) involves sums of the mean value of functions like $\mathfrak{R}_k\,(a_1 \mid \cdots \mid a_k\,)$. Let us prove a general statement about such functions.

Proposition B.2 *For $k \geq 2$, let $a_1, \ldots, a_k$ be periodic functions with the same frequency ω, and such that $a_l(t) = \sum_{m\in\mathbb{Z}} A^{(l)}(m) e^{im\omega t}$, where the Fourier coefficients $A^{(l)}(m)$ satisfy*

$$\left|A^{(l)}(m)\right| \;\leq\; \alpha_l \frac{e^{-\chi|m|}}{\ll m \gg^2}, \tag{B.7}$$

for all $m \in \mathbb{Z}$ and all $l = 1, \ldots, k$, where $\alpha_l > 0$ and $\chi > 0$. Then, there is a positive constant β_k such that the Fourier coefficients $\mathfrak{R}_k\,(a_1 \mid \cdots \mid a_k\,)\,(m)$, $m \in \mathbb{Z}$, of $\mathfrak{R}_k\,(a_1 \mid \cdots \mid a_k\,)_t$ are bounded by

$$|\,\mathfrak{R}_k\,(a_1 \mid \cdots \mid a_k\,)\,(m)| \;\leq\; \beta_k\, \alpha_1 \cdots \alpha_k\, \frac{e^{-\chi|m|}}{\ll m \gg^2}\,. \tag{B.8}$$

□

Proof. Let us first consider the case $k = 2$. The Fourier coefficients of $\mathfrak{R}_2\left(a_1 \mid a_2\right)_t$ are given by

$$\mathfrak{R}_2\left(a_1 \mid a_2\right)(m) \; = \; \sum_{n\in\mathbb{Z}} A^{(1)}(m-n)\tilde{A}^{(2)}(n), \tag{B.9}$$

where

$$\tilde{A}^{(2)}(m) \; := \; \begin{cases} \dfrac{A^{(2)}(m)}{im\omega}, & \text{for } m \neq 0, \\[2ex] -\dfrac{1}{i\omega} \displaystyle\sum_{k\in\mathbb{Z}_*} \dfrac{A^{(2)}(k)}{k}, & \text{for } m = 0 \end{cases} .$$

From (B.7), it follows that

$$\left| A_I^{(2)}(m) \right| \; \leq \; \alpha_2 \frac{D_0}{\omega} \frac{e^{-\chi|m|}}{\ll m \gg^2},$$

where $D_0 := \sum_{m\in\mathbb{Z}} \frac{e^{-\chi|m|}}{\ll m\gg^2}$. Therefore, from (B.9), by (B.7) and by Lemma B.1, one has

$$\left| \mathfrak{R}_2\left(a_1 \mid a_2\right)(m) \right| \; \leq \; \alpha_1\alpha_2 \frac{B_0 D_0}{\omega} \; \frac{e^{-\chi|m|}}{\ll m\gg^2}.$$

This proves the statement for $k = 2$. The general case follows from (3.9), by induction. ■

As a corollary, one sees from (2.13) that the functions $\mathcal{Q}_0$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have Fourier coefficients bounded as $|\mathcal{Q}_i(m)| \leq \gamma_i \frac{e^{-\chi|m|}}{\ll m\gg^2}$ for some positive γ_i. Moreover, by the inductive hypothesis (B.3) and by Lemma B.1, the Fourier coefficients $(v_p v_q)(m)$ of product functions like $v_p(t)v_q(t)$, with $p,\ q = 1, \ldots , n-1$, are also bounded as

$$|(v_p v_q)(m)| \; \leq \; B_0\, K_p\, K_q\, \frac{e^{-\chi|m|}}{\ll m\gg^2} \tag{B.10}$$

for all $m \in \mathbb{Z}$. The consequence of all this is that for indices $i_j \in \{0,\ 1,\ 2\}$ and $p,\ q = 1, \ldots , n-1$ one has

$$\left| \mathfrak{R}_k\left(i_1 | i_2 | \cdots | i_{k-1} | v_p v_q\right)(m) \right| \;\; \leq \;\; \gamma_{i_1,\, i_2,\, \cdots,\, i_{k-1}}\; K_p\, K_q\; \frac{e^{-\chi|m|}}{\ll m\gg^2}, \qquad \forall m \in \mathbb{Z},$$

from some positive constants $\gamma_{i_1,\, i_2,\, \cdots,\, i_{k-1}}$, depending on the indices i_j.

Returning our attention to expression (A.18), we conclude that

$$\left| \mathcal{R}_{n+3}^{(1)} \right| \;\; \leq \;\; \eta_2 \sum_{p=1}^{n-1} K_p K_{n-p} + \eta_3 \sum_{p=2}^{n-1} K_p K_{n+1-p} + \eta_4 \sum_{p=3}^{n-1} K_p K_{n+2-p}\,, \tag{B.11}$$

for certain positive constants $\eta_2,\ \eta_3,\ \eta_4$.

Bound for T_3. Since qv_2 is a linear combination of the functions $\mathcal{Q}_0$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$, we conclude from (A.19) and from the previous arguments that

$$\left|\mathcal{R}^{(2)}_{n+3}\right| \;\leq\; \eta_5 \sum_{p=1}^{n-1} K_p K_{n-p} + \eta_6 \sum_{p=2}^{n-1} K_p K_{n+1-p}\,, \tag{B.12}$$

for certain positive constants $\eta_5,\ \eta_6$.

Bound for T_4. By (4.28), one has

$$M(q^{-1}v_3 l_n) \;=\; i\sum_{p=1}^{n-1} M(\,\Re_2\,(qv_3\mid v_p v_{n-p}\,)).$$

From (4.21) and (B.10), we see that both qv_3 and $v_p v_{n-p}$ satisfy the conditions of Proposition B.2. Hence, by (B.8), $M(\,\Re_2\,(qv_3\mid v_p v_{n-p}\,)) \leq \eta_7 K_p K_{n-p}$ for some positive constant η_7 and

$$M(q^{-1}v_3 l_n) \;\leq\; \eta_7 \sum_{p=1}^{n-1} K_p K_{n-p}. \tag{B.13}$$

We are finished with the bounds for the terms T_i of (B.5). If we collect (B.6), (B.11), (B.12) and (B.13) and return to (B.4), we conclude that there are positive constants $\Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4$ such that we can recursively define

$$\begin{aligned} K_n \;:=\; & \Gamma_1 \sum_{p=1}^{n-1} K_p K_{n-p} + \Gamma_2 \sum_{p=2}^{n-1} K_p K_{n+1-p} \\ & \qquad + \Gamma_3 \sum_{p=3}^{n-1} K_p K_{n+2-p} + \Gamma_4 \sum_{p=4}^{n-1} K_p K_{n+3-p}\,, \end{aligned}$$

for $n>4$, after fixing the convenient values for $K_1,\ K_2,\ K_3$ and K_4. Note that we can choose $K_1 = K_2 = K_3 = K_4$ taking $K_i = \max\{K_1,\ K_2,\ K_3,\ K_4\}$ for all $i = 1, \ldots, 4$. Defining $\Gamma := \max\{\Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4\}$, we can redefine the K_n so as to have the more convenient choice

$$\begin{aligned} K_n \;:=\; \Gamma \Bigg[& \sum_{p=1}^{n-1} K_p K_{n-p} + \sum_{p=2}^{n-1} K_p K_{n+1-p} \\ & \qquad + \sum_{p=3}^{n-1} K_p K_{n+2-p} + \sum_{p=4}^{n-1} K_p K_{n+3-p} \Bigg]. \end{aligned} \tag{B.14}$$

Expression (B.14) has an analogous one in [2], with the distinction that only the two first sums above occurred in the corresponding expression for K_n. From now on, we follow closely the steps of [2]. The first one is to show that K_n is a *non-decreasing* sequence. We have

$$\begin{aligned} K_{n+1}-K_n \;:=\; \Gamma\Bigg[\left(\sum_{a=1}^{4}K_a\right)K_n+\left(\sum_{a=1}^{3}K_a\right)(K_n-K_{n-a}) \\ +\sum_{p=4}^{n-1}K_p(K_{n+4-p}-K_{n-p})\Bigg]\,. \end{aligned}$$

Therefore, assuming inductively $K_1=K_2=K_3=K_4\leq\ldots\leq K_n$ implies $K_n\leq K_{n+1}$, thus proving that the sequence is non-decreasing. Next, we write (B.14) as

$$K_n=\Gamma\left[\sum_{a=1}^{3}\left(\sum_{b=a}^{3}K_b\right)K_{n-a}+\sum_{p=4}^{n-1}K_p\left(K_{n-p}+K_{n+1-p}+K_{n+2-p}+K_{n+3-p}\right)\right].$$

Since the sequence is non-decreasing, we have $K_{n-p}+K_{n+1-p}+K_{n+2-p}+K_{n+3-p}\leq 4K_{n+3-p}$ and $K_{n-a}\leq K_{n-1}$ for $a=1,2,3$. Hence, we may say that

$$\begin{aligned} K_n \;&\leq\; \Gamma\sum_{a=1}^{3}\left(\sum_{b=a}^{3}K_b\right)K_{n-1}+4\Gamma\sum_{p=4}^{n-1}K_pK_{n+3-p} \\ &=\; \tilde{\Gamma}\,K_{n-1}K_4+4\Gamma\sum_{p=4}^{n-1}K_pK_{n+3-p}\,, \end{aligned} \tag{B.15}$$

where $\tilde{\Gamma}:=\dfrac{\Gamma}{K_4}\displaystyle\sum_{a=1}^{3}\left(\sum_{b=a}^{3}K_b\right)$ is a positive constant. Adding up the positive quantity $\tilde{\Gamma}\displaystyle\sum_{p=4}^{n-2}K_pK_{n+3-p}$ to (B.15) and setting $\Lambda:=\max\{\tilde{\Gamma},\,4\Gamma\}$, we get

$$K_n\;\leq\;\Lambda\sum_{p=4}^{n-1}K_pK_{n+3-p}\,. \tag{B.16}$$

Let us now define another auxiliary sequence J_k such that $J_l=K_l$ for $l=1,\,2,\,3,\,4$ and

$$J_n\;:=\;\Lambda\sum_{p=4}^{n-1}J_pJ_{n+3-p}\,.$$

for $n>4$. It is a simple exercise to show from (B.16) that $K_n\leq J_n$ for all n. Now, let us consider the translated sequence $L_n=J_{n+2}$, $n\geq 1$. We have

$$L_n\;=\;\Lambda\sum_{p=4}^{n+1}J_pJ_{n+5-p}\;=\;\Lambda\sum_{p=4}^{n+1}L_{p-2}L_{n+3-p}\;=\;\Lambda\sum_{p=2}^{n-1}L_pL_{n+1-p}\,. \tag{B.17}$$

The sequence $\mathbf{c}_n$ defined by $\mathbf{c}_n = \sum_{p=2}^{n-1} \mathbf{c}_p \mathbf{c}_{n-p+1}$ for $n \geq 3$, with $\mathbf{c}_1 = \mathbf{c}_2 = 1$, defines the so-called "Catalan numbers", which can be expressed in a closed form as

$$\mathbf{c}_n \;=\; \frac{(2n-4)!}{(n-1)!(n-2)!}, \qquad n \geq 2\,.$$

By Stirling's formula, the $\mathbf{c}_n$'s have the following asymptotic behaviour: $\mathbf{c}_n \approx \frac{1}{16\sqrt{\pi}} \frac{4^n}{n^{3/2}}$, for n large. The existence of a connection between the Catalan numbers and the sequence L_n is evident from (B.17). Two distinctions are the factor Λ appearing in (B.17) and the fact that $L_1 = L_2 = K_3 = K_4$ are not necessarily equal to 1. One can, however, easily show that $L_n = (K_3)^{n-1}\, \Lambda^{n-2}\, \mathbf{c}_n$, $n \geq 2$. Hence, the following asymptotic behaviour can be established:

$$L_n \;\approx\; \frac{1}{16\sqrt{\pi} K_3 \Lambda^2}\; \frac{(4K_3\Lambda)^n}{n^{3/2}}, \qquad n \text{ large.}$$

Since $K_n \leq J_n = L_{n-2}$, we conclude that for n large $K_n \leq M_0 (M_1)^n$, for some positive constants $M_0,\ M_1$. This completes the proof of Theorem 2.3. ■

C Comments on the Fourier Coefficients

Relations (1.8) and (1.9) can be obtained in our case by repeating the analysis of [2]. One of the conclusions is that the Fourier coefficients of (1.9) are analytic functions of ϵ, for $|\epsilon|$ small enough. There is, however, a point to be noticed here. If we follow the steps of [2] and compute the Fourier expansion of $R(t)^{-2}$ to obtain the Fourier expansion for $S(t)$, this last function will contain terms like $\int_0^t e^{2i\Omega t} dt$, which behave like Ω^{-1}. In case (I), treated in [2], this is not problematic, since $\Omega = O(\epsilon)$ and since such terms are always multiplied by a factor ϵ (see (2.2)). In our case, however, $\Omega = O(\epsilon^3)$ and we have to look such terms more carefully.

Looking at the ϵ-expansion for $U_{12}(t)$, we have

$$U_{12}(t) = -ie^{-i\gamma(\epsilon)} \left[\epsilon W_1(t) + \epsilon^2 W_2(t) + O(\epsilon^3)\right],$$

where, after a lengthy computation, we get

$$\epsilon W_1(t) \;=\; \epsilon \sum_{n,\,m} \frac{Q_n^{(2)}}{i(n\omega + 2\Omega)} \left[\overline{Q_{n-m}} e^{i(m\omega+\Omega)t} - \overline{Q_{-m}} e^{i(m\omega-\Omega)t}.\right]$$

The problematic term is that proportional to ϵ/Ω, which appears for $n = 0$. However, this term is proportional to $Q_0^{(2)} = M(\mathcal{Q}_0) = 0$, by hypothesis and, hence, the Fourier coefficients of ϵW_1 are analytic on ϵ. Let us look now at $\epsilon^2 W_2$. After another lengthy computation, one gets

$$\begin{aligned}
\epsilon^2 W_2(t) \;=\; & \frac{2\kappa_1\epsilon^2}{i\omega} \sum_{p,\,n,\,m\neq 0} \left[\frac{Q^{(2)}_{n-m}\overline{Q_{n-p}}Q^{(2)}_m}{m(n\omega+2\Omega)} e^{i(p\omega+\Omega)t} - \frac{Q^{(2)}_{n-m}\overline{Q_{-p}}Q^{(2)}_m}{m(n\omega+2\Omega)} e^{i(p\omega-\Omega)t} \right] \\
& + \frac{\kappa_1\epsilon^2}{i\omega} \sum_{p,\,n,\,m\neq 0} \left[\frac{Q^{(2)}_{n}\overline{Q_{m-p}}Q^{(2)}_m}{m(n\omega+2\Omega)} e^{i(p\omega-\Omega)t} - \frac{Q^{(2)}_{n}\overline{Q_{m+n-p}}Q^{(2)}_m}{m(n\omega+2\Omega)} e^{i(p\omega+\Omega)t} \right].
\end{aligned}$$

Again, the dangerous terms are those with $n = 0$, since the are proportional to ϵ^2/Ω. However, one sees by inspection that such terms vanish in the expression above. In fact, in the last two sums they appear proportional to $Q^{(2)}_0$, which is zero. In the first two sums, they appear proportional to $\sum_{m\neq 0} \frac{Q^{(2)}_{-m}Q^{(2)}_m}{m}$, which is also zero, as one sees by interchanging $m \to -m$.

Our conclusion is that all dangerous terms like $1/\Omega$ appear multiplied at least by factors ϵ^3 and are, therefore, analytic.

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