ON LOCAL PERTURBATIONS OF SHRÖDINGER OPERATOR IN AXIS

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Abstract

We adduce the necessary and sufficient condition for arising of eigenvalues of Shrödinger operator in axis under small local perturbations. In the case of eigenvalues arising we construct their asymptotics.

1. Introduction

The questions addresses the existence of bound states and the asymptotics of associated eigenvalues (if they exist) for Shrödinger operator with small potential in axis are have been studied in [1]–[4]. The technique employed in these works based on the self-adjointness of the perturbed equation. In present paper it is considered a small perturbation which is arbitrary localized second-order operator and the necessary and sufficient conditions for arising of eigenvalues of perturbed operator are adduced. In the case of eigenvalues arising we construct their asymptotics. The main idea of the technique suggested giving a simple explanation of "non-regular" (optional) arising of eigenvalues under, obviously, regular perturbation is as follows. Instead of spectral parameter λ we introduce more natural frequency parameter k related to spectral one by the equality $\lambda = -k^2$, where k lies in a complex half-plane $\mathrm{Re}\, k > 0$. The solutions of both non-perturbed and perturbed equations are extended w.r.t. complex parameter on all complex plane. Under such extension the solution of non-perturbed problem has a pole at zero that moves under perturbation, while the residue at this pole (for both non-perturbed and perturbed problems) is a solution of corresponding homogeneous equation. For non-perturbed this residue is a constant which is considered as exponent with index $-kx$, where $k = 0$. Depending on side to

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which this pole moves, we obtain exponential increasing or decreasing residue for perturbed problem. As a result, if pole moves into the half-plane $\operatorname{Re} k > 0$ then the eigenvalue arises, while pole moving to the half-lane $\operatorname{Re} k \leq 0$ do not produce pole. The direction of moving is determined by the operator of perturbation.

The structure of the paper is as follows. In the second section we state the main result, in the third is adduced its proof. In the fourth section we demonstrate some examples illustrating the main statement of the paper.

2. Formulation of the main result

Hereinafter $W_{2,loc}^j(\mathbb{R})$ is a set of functions defined on $\mathbb{R}$ whose restriction to any bounded domain $D \subset \mathbb{R}$ belongs to $W_2^j(D)$, $\|\bullet\|_G$ and $\|\bullet\|_{j,G}$ are norms in $L_2(G)$ and $W_2^j(G)$, respectively. Next, let Q be an arbitrary fixed interval in $\mathbb{R}$, $L_2(\mathbb{R}; Q)$ be the subset of functions in $L_2(\mathbb{R})$ with supports in $\overline{Q}$, $\mathcal{L}_\varepsilon$ be linear operators mapping $W_{2,loc}^j(\mathbb{R})$ into $L_2(\mathbb{R}; Q)$ such that $\|\mathcal{L}_\varepsilon[u]\|_Q \leq C(\mathcal{L})\, \|u\|_{2,Q}$, where constant $C(\mathcal{L})$ does not depends on ε, $0 < \varepsilon << 1$,

$$\langle g \rangle = \int\limits_{-\infty}^{\infty} g\, dx, \qquad H_0 = -\tfrac{d^2}{dx^2}, \qquad H_\varepsilon = -\left(\tfrac{d^2}{dx^2} + \varepsilon \mathcal{L}_\varepsilon\right).$$

We define linear operators $A(k) : L_2(\mathbb{R}; Q) \to W_{2,loc}^2(\mathbb{R})$ and $T_\varepsilon^{(0)}(k) : L_2(\mathbb{R}; Q) \to L_2(\mathbb{R}; Q)$ in the following way:

$$A(k)g = -\frac{1}{2k} \int\limits_{-\infty}^{\infty} e^{-k|x-t|} g(t)\, dt, \qquad T_\varepsilon^{(0)}(k)g = \mathcal{L}_\varepsilon[A(k)g] + \frac{\langle g \rangle}{2k} \mathcal{L}_\varepsilon[1].$$

Denote by $\mathcal{B}(X, Y)$ the Banach space of linear bounded operators mapping Banach space X into Banach space Y, $\mathcal{B}(X) \stackrel{def}{=} \mathcal{B}(X, Y)$. We indicate by $\mathcal{B}^h(X, X)$ (by $\mathcal{B}^h(X)$) the set of holomorphic operator-valued functions whose values belongs to $\mathcal{B}(X, Y)$ (to $\mathcal{B}(X)$). We use the notation I for identity mapping and the notation S^t for a circle in $\mathbb{C}$ of radius t with center at zero. Since by definition of $T_\varepsilon^{(0)}(k)$ we have that $T_\varepsilon^{(0)}(k) \in \mathcal{B}^h(L_2(\mathbb{R}; Q))$,

$$T_\varepsilon^{(0)}(k)g = \frac{1}{2} \mathcal{L}_\varepsilon \left[\int\limits_{-\infty}^{\infty} g(t) |x - t|\, dt \right] + k T_\varepsilon^{(1)}(k) g, \qquad T_\varepsilon^{(1)}(k) \in \mathcal{B}^h(L_2(\mathbb{R}; Q)),$$

then we arrive at the following statement.

Lemma 2.1. *Let $S_\varepsilon(k)=(I+\varepsilon T_\varepsilon^{(0)}(k))^{-1}$. Then for all $R>0$ there exist $\varepsilon_0(R)>0$, such that for $\varepsilon<\varepsilon_0(R)$ and $k\in S^R$ the operator-valued function $S_\varepsilon(k)\in\mathcal{B}^h(L_2(\mathbb{R};Q))$, $S_\varepsilon(k)\underset{\varepsilon\to0}{\to} I$ uniformly on k, and the equation*

$$k-\frac{\varepsilon}{2}\langle S_\varepsilon(k)\mathcal{L}_\varepsilon[1]\rangle=0 \tag{2.1}$$

has a unique solution $k_\varepsilon\in S^R$, and also,

$$k_\varepsilon=\varepsilon\frac{1}{2}\left(m_\varepsilon^{(1)}+\varepsilon m_\varepsilon^{(2)}+O(\varepsilon^2)\right), \tag{2.2}$$

where

$$m_\varepsilon^{(1)}=\langle\mathcal{L}_\varepsilon[1]\rangle, \qquad m_\varepsilon^{(2)}=-\int_{-\infty}^{\infty}\mathcal{L}_\varepsilon\left[\int_{-\infty}^{\infty}|x-y|\mathcal{L}_\varepsilon[1](y)\,dy\right](x)\,dx. \tag{2.3}$$

Let us call the operator $\mathcal{L}_\varepsilon$ the real one, if $\mathrm{Im} <\overline{g}\mathcal{L}_\varepsilon[g]>=0$ for all $g\in W^2_{2,loc}(\mathbb{R})$. We denote $\Pi_s(t)=\{k:\ |\mathrm{Im}\,k|<sC(\mathcal{L}),\ \mathrm{Re}\,k>t\}$, and we indicate by $\Sigma(H_\varepsilon)$ the set of eigenvalues of operator H_ε. The aim of this paper is to prove the following statement.

Theorem 2.1. *If $\mathrm{Re}\,k_\varepsilon\le0$, then there exist $t(\varepsilon)\underset{\varepsilon\to0}{\to}\infty$, such that $\Sigma(H_\varepsilon)\subset\Pi_\varepsilon(t(\varepsilon))$. If, in addition, the operator $\mathcal{L}_\varepsilon$ is real, then $\Sigma(H_\varepsilon)\subset(t(\varepsilon),\infty)$.*

If $\mathrm{Re}\,k_\varepsilon>0$, then there exist $t(\varepsilon)\underset{\varepsilon\to0}{\to}\infty$, such that $\Sigma(H_\varepsilon)\backslash\Pi_\varepsilon(t(\varepsilon))=\{\lambda_\varepsilon\}$,

$$\lambda_\varepsilon=-k_\varepsilon^2, \tag{2.4}$$

and the associated single eigenfunction ϕ_ε has the form

$$\phi_\varepsilon=A(k_\varepsilon)S_\varepsilon(k_\varepsilon)\mathcal{L}_\varepsilon[1]. \tag{2.5}$$

If, in addition, the operator $\mathcal{L}_\varepsilon$ is real, then $\Sigma(H_\varepsilon)\backslash(t(\varepsilon),\infty)=\{\lambda_\varepsilon\}$.

Remark 2.1. The statements of Theorem 2.1 does not excludes the situation when $\mathrm{Re}\,k_\varepsilon\le0$ for some values of ε and $\mathrm{Re}\,k_\varepsilon>0$ for other those of k_ε (see example 4.3).

Directly from Lemma 2.1 (namely, from equation (2.1)) and Theorem 2.1 it follows

Corollary 2.1. *If $\mathcal{L}_\varepsilon[1]\equiv0$, then there exists $t(\varepsilon)\underset{\varepsilon\to0}{\to}\infty$, such that $\Sigma(H_\varepsilon)\subset\Pi_\varepsilon(t(\varepsilon))$. If, in addition, the operator $\mathcal{L}_\varepsilon$ is real, then $\Sigma(H_\varepsilon)\subset(t(\varepsilon),\infty)$.*

3. Proof of Theorem 2.1

Let us denote by $\mathcal{B}^m(X,Y)$ (by $\mathcal{B}^m(X)$) the set of meromorphic operator-valued functions with values in $\mathcal{B}(X,Y)$ ($\mathcal{B}(X)$). The set of linear operators mapping Banach space X into $W^2_{2,loc}(\mathbb{R})$ such that their restriction to any bounded set D belongs to $\mathcal{B}(X, W^2_2(D))$ is indicated by $\mathcal{B}(X, W^2_{2,loc}(\mathbb{R}))$. Similarly, we use the notation $\mathcal{B}^h(X, W^2_{2,loc}(\mathbb{R}))$ ($\mathcal{B}^m(X, W^2_{2,loc}(\mathbb{R}))$) for the set of operator-valued functions with values in $\mathcal{B}(X, W^2_{2,loc}(\mathbb{R}))$ such that for all bounded D they belongs to $\mathcal{B}^h(X, W^2_2(D))$ (to $\mathcal{B}^m(X, W^2_2(D))$). Next, let $P_\varepsilon(k)$ be the operator defined by the equality

$$P_\varepsilon(k)f = \varepsilon \frac{\langle S_\varepsilon(k)f\rangle\, S_\varepsilon(k)\mathcal{L}_\varepsilon[1]}{2k - \varepsilon\,\langle S_\varepsilon(k)\mathcal{L}_\varepsilon[1]\rangle} + S_\varepsilon(k)f,$$

$\mathcal{R}_\varepsilon(k) \stackrel{def}{=} A(k)P_\varepsilon(k)$, $\mathbb{C}_+ \stackrel{def}{=} \{z:\ \mathrm{Re}\, z > 0\}$.

Theorem 3.1. *For all $R > 0$ there exists $\varepsilon_0(k) > 0$ such that*

1). *$\mathcal{R}_\varepsilon(k) \in \mathcal{B}^m(L_2(\mathbb{R};Q), W^2_{2,loc}(\mathbb{R}))$ as $\varepsilon < \varepsilon_0$ and $k \in S^R$, and also, in S^R there is the only pole k_ε being a solution of the equation (2.1) and it is a first order pole; if, in addition, $k \in \mathbb{C}_+$, then $\mathcal{R}_\varepsilon(k) \in \mathcal{B}^m(L_2(\mathbb{R},Q); W^2_2(\mathbb{R}))$;*

2). *for all $f \in L_2(\mathbb{R};Q)$ the function $u_\varepsilon = \mathcal{R}_\varepsilon(k)f$ is a solution of the equation*

$$-H_\varepsilon u_\varepsilon = k^2 u_\varepsilon + f \qquad \mathbb{R}; \tag{3.1}$$

3). *the residue of the function u_ε at the pole k_ε is defined by the equality (2.5) up to a multiplicative factor, moreover, this factor is nonzero if $\langle f\rangle \neq 0$.*

Proof. By definition, $A(k) \in \mathcal{B}^m(L_2(\mathbb{R};Q), W^2_{2,loc}(\mathbb{R}))$, and also, $A(k)$ has a unique pole of first order at zero and $A(k) \in \mathcal{B}^h(L_2(\mathbb{R};Q), W^2_2(\mathbb{R}))$ for $k \in \mathbb{C}_+$. Then bearing in mind the definition of $\mathcal{R}_\varepsilon(k)$ and Lemma 2.1, we get consecutively $\mathcal{R}_\varepsilon(k)$ having no pole at zero and validity of statement 1) of Theorem being proved.

Let us proceed to the proof of the statement 2). We seek the solution of the equation (3.1) in the form

$$u_\varepsilon = A(k)g_\varepsilon, \tag{3.2}$$

where g_ε is some function belonging to $L_2(\mathbb{R};Q)$. Substituting (3.2) into (3.1), we deduce that (3.2) is a solution of (3.1) in the case

$$(I + \varepsilon T_\varepsilon(k))g_\varepsilon = f, \tag{3.3}$$

where
$$T_\varepsilon(k) = \mathcal{L}_\varepsilon A(k). \tag{3.4}$$
It follows from (3.4) and the definition of $\mathcal{L}_\varepsilon$ and $A(k)$ that the result of the action of the operator $T_\varepsilon(k)$ is as follows:
$$T_\varepsilon(k)g = -\frac{\langle g\rangle}{2k}\mathcal{L}_\varepsilon[1] + T_\varepsilon^{(0)}(k)g. \tag{3.5}$$

Let $R > 0$ be an arbitrary number and ε satisfies all assumptions of Lemma 2.1. Applying the operator $S_\varepsilon(k)$ to both hands of the equation (3.3) and taking into account (3.5), we obtain that
$$\left(g_\varepsilon - \varepsilon\frac{\langle g_\varepsilon\rangle}{2k}S_\varepsilon(k)\mathcal{L}_\varepsilon[1]\right) = S_\varepsilon(k)f. \tag{3.6}$$
Having integrated (3.6), we deduce
$$\langle g_\varepsilon\rangle\left(1 - \frac{\varepsilon}{2k}\left\langle S_\varepsilon(k)\mathcal{L}_\varepsilon[1]\right\rangle\right) = \left\langle S_\varepsilon(k)f\right\rangle. \tag{3.7}$$
The equality (3.7) allows us to determine $\langle g_\varepsilon\rangle$; substituting its value into (3.6), we easily get the formula
$$g_\varepsilon = P_\varepsilon(k)f. \tag{3.8}$$
The assertions (3.2) and (3.8) yield the validity of the statement 2). In its turn, the correctness of statement 3) is the implication from 1) and 2) and the definition of $\mathcal{R}_\varepsilon(k)$. The proof is complete.

We will use the notation $R_\varepsilon(\lambda)$ for the resolvent of the operator H_ε. It is well known fact that the set of eigenvalues coincide with the set of poles of the resolvent, while the coefficient of the pole (of highest order) is a projector into the space that is a span of eigenfunctions associated with this eigenvalue.

Lemma 3.1. *The number of poles of the resolvent $R_\varepsilon(\lambda)$, their orders and the dimensions of the residues at them are completely determined by the functions belonging to $L_2(\mathbb{R}; Q)$.*

Proof. Let F be an arbitrary function with compact support D. There is no loss of generality in assuming that $\{0\} \in Q$. We use symbols $R_+(\lambda)$ and $R_-(\lambda)$ for the resolvents of the Dirichlet problem for H_0 in the positive $(\mathbb{R}^+)$ and negative$(\mathbb{R}^-)$ real semi-axises respectively, by F_+ and F_- we denote the restrictions of F to these axises. We use symbol $\chi \in C^\infty(\mathbb{R})$ for the cut-off function vanishing in a neighbourhood of zero and equalling to one outside Q. Let $\mathbb{R}_+$ be the nonnegative real semi-axis, we also set $\mathbb{C}^+ = \mathbb{C}_+ \cup \mathbb{R}_+$.

Since the function $\lambda = -k^2$ establishes one-to-one correspondence from $\mathbb{C}^+$ onto $\mathbb{C}$, then for $\lambda \in \mathbb{C}$ (or, equivalently, for $k \in \mathbb{C}^+$)

$$R_\pm(\lambda)F_\pm(x) = R_\pm(-k^2)F_\pm(x) = \pm\frac{1}{2k}\int\limits_0^{\pm\infty}\left(e^{-k|x-t|} - e^{-k|x+t|}\right)F(t)\,dt.$$

On the other hand,

$$U_\pm(x;k) = \pm\frac{1}{2k}\int\limits_0^{\pm\infty}\left(e^{-k|x-t|} - e^{-k|x+t|}\right)F(t)\,dt$$

are holomorphic functions in $\mathbb{C}$ with values in $W^2_{2,loc}(\mathbb{R}^\pm)$ (i.e., their restrictions to all bounded domains G are holomorphic functions with values belonging to $W^2_{2,loc}(G)$). For this reason the function $\chi(R_+(-k^2)F_+ + R_-(-k^2)F_-)$ can be extended in $\mathbb{C}$ as holomorphic function with values in $W^2_{2,loc}(\mathbb{R})$. The solution of the equation

$$(H_\varepsilon - \lambda)U = F \quad \mathbb{R} \tag{3.9}$$

is sought in the form

$$U = u + \chi(R_+(-k^2)F_+ + R_-(-k^2)F_-), \tag{3.10}$$

where $-k^2 = \lambda$, $k \in \mathbb{C}^+$. Substituting (3.10) into (3.9), we obtain the equation $(H_\varepsilon - \lambda)u = f$ for u, where the function $f(x;\lambda) = f(x;-k^2)$ can be extended w.r.t. k in $\mathbb{C}$, that is a holomorphic function with values in $L_2(\mathbb{R};Q)$. Since the second term in (3.10) can be extended holomorphically in $\mathbb{C}$, then it implies the validity of the lemma being proved.

Theorem 3.2. *Let $R > 0$ be an arbitrary number, ε_0 and k_ε to satisfy Theorem 3.1, $\lambda = -k^2$. Then*

$$\mathcal{R}_\varepsilon(k)f = -R_\varepsilon(\lambda)f \tag{3.11}$$

for all $f \in L_2(\mathbb{R};Q)$ and for all $k \in \mathbf{C}_+ \cap S^R$ (or, equivalently, for all $\lambda \in S^{R^2}$).

If $\operatorname{Re} k_\varepsilon \le 0$, then $\Sigma(H_\varepsilon) \cap S^{R^2} = \emptyset$.

If $\operatorname{Re} k_\varepsilon > 0$, then $\Sigma(H_\varepsilon) \cap S^{R^2} = \{\lambda_\varepsilon\}$, where λ_ε and the associated single eigenfunction are determined by the equalities (2.4) and (2.5).

Proof. Since the function $\lambda = -k^2$ establishes one-to-one correspondence from $\mathbb{C}^+ \cap S(R)$ onto S^{R^2}, then the validity of the equality (3.11) follows from the statement 2) of Theorem 3.1 and the definition of the resolvent.

The correctness of the rest statement of the theorem begin proved follows from Theorem 3.1 and Lemma 3.1. The proof is complete.

Lemma 3.2. $\Sigma(H_\varepsilon) \subset \Pi_\varepsilon(-\varepsilon C(\mathcal{L}))$. *If the operator $\mathcal{L}_\varepsilon$ is real, then $\Sigma(H_\varepsilon) \subset [-\varepsilon C(\mathcal{L}), \infty)$.*

Proof. Let

$$\lambda_\varepsilon \in \Sigma(H_\varepsilon) \backslash \mathbb{R}_+. \tag{3.12}$$

Since $H_\varepsilon u = H_0 u$ outside $\overline{Q}$, then there exists normalized in $L_2(\mathbb{R})$ function $\phi_\varepsilon \in W_2^2(\mathbb{R})$, such that

$$H_\varepsilon \phi_\varepsilon = \lambda_\varepsilon \phi_\varepsilon. \tag{3.13}$$

Multiplying both hands of (3.3) by $\overline{\phi_\varepsilon}$ and integrating by part, we obtain the equality

$$\|\phi_\varepsilon'\|^2 - \varepsilon \left\langle \overline{\phi_\varepsilon} \mathcal{L}_\varepsilon \phi_\varepsilon \right\rangle = \lambda_\varepsilon. \tag{3.14}$$

Calculating the real and imaginary part of (3.14), employing the estimate $\|\mathcal{L}_\varepsilon \phi_\varepsilon\|_Q \le C(\mathcal{L}) \|\phi_\varepsilon\|_{2,Q}$ and bearing in mind (3.12), we conclude the statement of the lemma being proved is true.

It is easily seen that Theorem 2.1 is a direct implication of Theorem 3.2 and Lemma 3.2.

4. Examples

Example 4.1. Let $\mathcal{L}_\varepsilon[g] = Vg$, where $V \in C_0^\infty(Q)$. Then in view of (2.2), (2.3) and Theorem 2.1 we obtain, that an inequality $\operatorname{Re} \langle V \rangle < 0$ yields $\operatorname{Re} k_\varepsilon < 0$ and, therefore, the operator H_ε has no eigenvalue converging to zero, while opposite inequality $\operatorname{Re} \langle V \rangle > 0$ implies that such eigenvalue exists and satisfies the asymptotics

$$\lambda_\varepsilon = -\varepsilon^2 \frac{\langle V \rangle}{4} + O(\varepsilon^3). \tag{4.1}$$

In the case when $\langle V \rangle = 0$, taking into account that (in this case)

$$-\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} |x-y| V(x) V(y)\, dy dx = 2 \int\limits_{-\infty}^{\infty} \left(\int\limits_{-\infty}^{x} V(y) dy \right)^2 dx, \tag{4.2}$$

by the assertion (2.3), we get that

$$m_\varepsilon^{(1)} =< V >= 0, \qquad m_\varepsilon^{(2)} = 2 \int\limits_{-\infty}^{\infty} \left(\int\limits_{-\infty}^{x} V(y) dy \right)^2 dx. \tag{4.3}$$

If, in addition, $\operatorname{Im} V = 0$, then due to (1.2), (4.3) and Theorem 2.1 the eigenvalue exists and has the asymptotics

$$\lambda_\varepsilon = -\varepsilon^4 \left(\int\limits_{-\infty}^{\infty} \left(\int\limits_{-\infty}^{x} V(y)dy \right)^2 dx \right)^2 + (\varepsilon^5). \tag{4.4}$$

For real V the asymptotics (4.1), (4.4) have been derived in [1]. So, the asymptotics (4.1) is a generalization for the case of complex-valued potentials. Observe that for real V the inequality $\langle V \rangle \geq 0$ is an necessary and sufficient condition of the existence of eigenvalue of H_ε (what was proved in [1]). However, if V is a complex-valued function, then the assumption $\langle V \rangle = 0$ is not sufficient for the existence of the eigenvalue. Indeed, it is easy to see that if $V = u' + i2u'$, where $u \in C_0^\infty(Q)$ is a real function then

$$m_\varepsilon^{(1)} =< V >= 0, \qquad \operatorname{Re} m_\varepsilon^{(2)} = -6 \int\limits_{-\infty}^{\infty} u^2(x)dx < 0,$$

and by the assertion (2.2) and Theorem 2.1 the eigenvalue of H_ε does not exist.

Example 4.2. Let $\mathcal{L}_\varepsilon[g] = V_\varepsilon g$, where $V_\varepsilon = V + \varepsilon V_1$, and V, V_1 are real functions with supports in Q. Due to (2.2), (2.3), (4.2) and Theorem 2.1 the condition $\langle V_\varepsilon \rangle \geq 0$ is sufficient for existence of eigenvalue which has the asymptotics (4.1) as $\langle V \rangle > 0$ and the asymptotics

$$\lambda_\varepsilon = -\varepsilon^4 \left(\frac{1}{2} \langle V_1 \rangle + \int\limits_{-\infty}^{\infty} \left(\int\limits_{-\infty}^{x} V(y)dy \right)^2 dx \right)^2 + (\varepsilon^5),$$

as $\langle V \rangle = 0$. However, in distinction to classic case (real $V_\varepsilon = V$) the condition $\langle V_\varepsilon \rangle < 0$ is not sufficient for absence of eigenvalue. Indeed if $Q = (-\pi/2, \pi/2)$ and $V_\varepsilon = \sin x - \varepsilon \cos x$ then in view of (2.2), (2.3) $\langle V_\varepsilon \rangle = -2 < 0$, but $k_\varepsilon = \varepsilon^2 \frac{\pi - 2}{2} + O(\varepsilon^3)$. Hence, by Theorem 2.1 the eigenvalue exists.

Example 4.3. Let $\mathcal{L}_\varepsilon[g] = \exp\{i\varepsilon^{-1}\} V g$, where $V \in C_0^\infty(Q)$ is a real function and $\langle V \rangle > 0$. Then by (2.2), (2.3) and Theorem 2.1 we obtain that for all sufficient large n the eigenvalues are absent as $(3\pi/2 + 2\pi n - \delta)^{-1} < \varepsilon < (\pi/2 + 2\pi n + \delta)^{-1}$ while an eigenvalue exists as $(\pi/2 + 2\pi n - \delta)^{-1} < \varepsilon < (-\pi/2 + 2\pi n + \delta)^{-1}$ for each fixed $\delta > 0$ and satisfies the asymptotics

$$\lambda_\varepsilon = -\left(\varepsilon \cos \frac{1}{\varepsilon} \right)^2 \frac{1}{4} \langle V \rangle^2 + O(\varepsilon^3).$$

Example 4.4. Let

$$\mathcal{L}_\varepsilon = a_2\frac{d^2}{dx^2} + a_1\frac{d}{dx} + V_\varepsilon,$$

where a_j, $V_\varepsilon \in C_0^\infty(Q)$. Since $\mathcal{L}_\varepsilon[1] = V_\varepsilon$, then in the case $V_\varepsilon \equiv 0$ an eigenvalue is absent due to Corollary 2.1 and the equality $\langle\mathcal{L}_\varepsilon[1]\rangle = \langle V_\varepsilon\rangle$ implies that

1) if $V_\varepsilon = V$ and $\operatorname{Re} V \neq 0$, then k_ε has asymptotics derived in Example 4.1;

2) if $V_\varepsilon = \exp\{\varepsilon^{-1}\}V$ and $\langle V\rangle > 0$, then k_ε has asymptotics derived in Example 4.3.

Example 4.5. Let $\mathcal{L}_\varepsilon[g] = \varkappa(Q)\,\langle\rho g\rangle$ where $\rho \in C_0^\infty(Q)$, and $\varkappa(Q)$ is a characteristic function for Q, (i.e., this function equals to one for $x \in Q$ and vanishes for other x). Then by (2.2), (2.3), Theorem 2.1 ad Corrolary 2.1 an eigenvalue is absent if $\langle\rho\rangle = 0$ or Re $\langle\rho\rangle < 0$, and, if Re $\langle\rho\rangle > 0$, then an eigenvalue exists and has asymptotics

$$\lambda_\varepsilon = -\varepsilon^2\frac{1}{4}\left(|Q|\,\langle\rho\rangle\right)^2 + O(\varepsilon^3). \tag{4.5}$$

Example 4.6. Let

$$\mathcal{L}_\varepsilon[g] = \varkappa(Q)\int\limits_{-\infty}^{x}\rho(t)g(t)\,dt,$$

where $\rho \in C_0^\infty(Q)$. Then the assertions (2.2), (2.3) imply

$$k_\varepsilon = \varepsilon\frac{1}{2}\left(|Q|\,\langle\rho\rangle - \langle x\rho\rangle\right) + O(\varepsilon^2), \tag{4.6}$$

and, therefore, the eigenvalue exists if $(|Q|\,\langle\rho\rangle - \langle x\rho\rangle) > 0$, and it is absent if $(|Q|\,\langle\rho\rangle - \langle x\rho\rangle) < 0$. In particular, if ρ is an even function and $\langle\rho\rangle > 0$, then due to (2.4), (4.6) the eigenvalue has asymptotics (4.5).

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