ON THE SINGULAR SPECTRUM OF SCHRÖDINGER OPERATORS WITH DECAYING POTENTIAL

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Abstract. The relation between Hausdorff dimension of the singular spectrum of a Schrödinger operator and the decay of its potential has been extensively studied in many papers. In this work, we address similar questions from a different point of view. Our approach relies on the study of the so-called Krein systems. For Schrödinger operators, we show that some bounds on the singular spectrum, obtained recently by Remling, are optimal in $L^p(\mathbb{R}_+)$ scale.

Introduction

We consider a Schrödinger operator $L_q y = -y'' + qy$ on the positive half-line $\mathbb{R}_+$ with boundary condition $y'(0) + hy(0) = 0$. Assume that $q \in L^\infty(\mathbb{R}_+)$ is a real-valued function and $h \in \mathbb{R} \cup \{\infty\}$. Denote the spectral measure of the operator L_q by ρ.

Recently, Remling [11] proved the following theorem.

Theorem 0.1 ([11, 12]). *If $|q(x)| \le C(1+x)^{-\beta}$ with $1/2 < \beta \le 1$, then the support of the (possible) singular part of ρ has Hausdorff dimension less or equal to $2(1-\beta)$.*

Actually, the stronger result was obtained, that is, the set of all positive spectral parameters such that the transfer matrix is not bounded at infinity has Hausdorff dimension less or equal to $2(1-\beta)$. However, the presence of a non-trivial singular continuous part of ρ for some potentials was only guessed. More attention to the subject was attracted when Simon [15] asked the following question.

Question 0.1. *Do there exist potentials q on $\mathbb{R}_+$ so that $|q(x)| \le C(1+x)^{-(1/2+\varepsilon)}$, $(\varepsilon > 0)$ and the spectral measure of L_q has a non-trivial singular continuous part?*

The first example of a potential from $L^2(\mathbb{R}_+)$ with ρ having a singular continuous component for some h was given by Denisov [4]. Deift and Killip [3] proved that for potentials from $L^2(\mathbb{R}_+)$, the essential support of the absolutely continuous part of the spectral measure is the whole positive half-line. Later, Kiselev

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[9] constructed Schrödinger operators with potentials decaying arbitrarily slower than $C(1+x)^{-1}$ and having an embedded singular continuous component.

In [4], the construction was carried out in the opposite direction. It was started with a specific spectral measure. Then the analysis of the corresponding inverse spectral problem was used to establish the required properties of the potential.

In the present paper, we show that there exist potentials q so that ρ has a singular component with Hausdorff dimension exactly $2(1-\beta)$ and $q \in L^p(\mathbb{R}_+)$ for any $p > p_0(\beta)$ (see Section 3.2 for formulations). Our methods are essentially different from those of [11, 9]. In writing this paper, we were motivated by certain well-known results from the theory of orthogonal polynomials [7, 16]. Therefore, we start with the continuous analogs of orthogonal polynomials, the solutions of the so-called Krein systems. For these systems, we study the above-mentioned questions. It turns out that the problem can be reduced to the certain problem of minimization. We analyze this minimization problem using simple methods of complex analysis and approximation theory.

The plan of the paper is as follows. In section 1, we introduce some notation and discuss well-known results that are necessary in the sequel. In section 2, we study the Krein systems case. Results obtained for Krein systems are applied to Dirac and Schrödinger operators in section 3. Certain theorems for orthogonal polynomials are formulated in section 4.

We conclude the introduction with some notation. Given a measure σ on $\mathbb{R}$, let σ_s, σ_{ac} signify its singular and absolutely continuous components, respectively. The Lebesgue measure is denoted by m. The Hausdorff dimension of a measurable set $E \subset \mathbb{R}$ is denoted by $\dim_{\mathrm{H}} E$. As usual, $W^{2,m}(\mathbb{R}_+)$ and $H^p(\Omega)$ stand for standard Sobolev spaces on $\mathbb{R}_+$ and Hardy spaces on Ω, Ω being a domain in $\mathbb{C}$ with a smooth boundary. For $f \in L^2(\mathbb{R})$, $\hat{f}$ is its Fourier transform. C is a constant changing from one relation to another.

1. Preliminaries

1.1. In this subsection, we briefly discuss some simple properties of polynomials, orthogonal on the unit circle $\mathbb{T} = \{z : |z| = 1\}$. A detailed presentation of the subject can be found in [7, 16].

Let σ be a finite positive measure on $\mathbb{T}$. Let $\{\varphi_n\}$ be polynomials, orthonormal with respect to σ, that is, $\int_{\mathbb{T}} \varphi_n \overline{\varphi_m}\, d\sigma = \delta_{nm}$, δ_{nm} being the Kronecker symbol. We also consider monic orthogonal polynomials $\{\psi_n\}$, that is,

$$\int_{\mathbb{T}} \psi_n(t)\overline{\psi_m}(t)\, d\sigma(t) = k_n \delta_{nm},$$

where $\psi_n(z) = z^n + \dots$, and $k_n = ||\psi_n||^2_{2,\sigma}$. These polynomials can be explicitly computed. Consider momenta of σ, given by $c_k = \int_{\mathbb{T}} e^{ikt}\, d\sigma(t)$. Define the matrix

$$M_n = \begin{bmatrix} c_0 & c_1 & \dots & c_n \\ \overline{c_1} & c_0 & \dots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{c_n} & \overline{c_{n-1}} & \dots & c_0 \end{bmatrix}$$

and let $\Delta_n = \det M_n$. It is not difficult to see that

$$\psi_n(z) = \sum_{k=0}^{n} (-1)^k \frac{\Delta_{n+1,n+1-k}}{\Delta_{n-1}} z^{n-k}, \tag{1.1}$$

where $\Delta_{n+1,n+1-k}$ denotes the determinant of M_n with dropped $(n+1)$-th row and $(n+1-k)$-th column. On the other hand, the sequence $\{\psi_n\}$ generates a sequence $\{a_n\}$ of reflection (or Geronimus) coefficients by means of the relations

$$\begin{cases} \psi_{n+1}(z) &= z\psi_n(z) - \overline{a_n}\psi_n^*(z), \\ \psi_{n+1}^*(z) &= \psi_n^*(z) - a_n z\psi_n(z), \end{cases} \tag{1.2}$$

where $\psi_0(z) = \psi_0^*(z) = 1$ and $\psi_n^*(z) = z^n\overline{\psi_n(1/\overline{z})}$. Vice-versa, given a sequence $\{a_n\}$, $|a_n| < 1\, (n = 0, 1, \dots)$, we can define orthogonal polynomials $\{\psi_n\}$ uniquely; see [7] and [8, ch. 5] for details.

We say that σ is a Szegő measure, if $\log \sigma'_{ac} \in L^1(\mathbb{T})$. The following theorem is classical.

Theorem 1.1 ([7, 16])**.** *The following assertions are equivalent:*

i) σ is a Szegő measure.
ii) The series $\sum_{k=0}^{\infty} |\varphi_n(z)|^2$ converges at least for one (and hence, for all) $z \in \mathbb{D}$.
iii) There exists a subsequence $\{\varphi^*_{n_k}\}$ bounded at least for one (and hence, for all) $z \in \mathbb{D}$.
iv) The sequence $\{a_k\}$ lies in l^2.

In the above case, the limit $\pi(z) = \lim_{n\to\infty} \varphi_n^*(z)$, $z \in \mathbb{D}$, exists, and $\pi^{-1} \in H^2(\mathbb{D})$.

Furthermore, if σ is a Szegő measure, we have that (see [7, ch. 2])

$$\sum_{k=n}^{\infty} |a_k|^2 \le C \inf_{p\in\mathcal{P}_n} ||\pi_0 - p||^2_{2,\sigma}. \tag{1.3}$$

In this formula, $\mathcal{P}_n$ is the space of polynomials of degree less or equal to n, and $\pi_0 = \pi\chi_{\mathbb{T}\setminus E}$, where $E = \text{supp }\ \sigma_s$.

1.2. In this subsection, we introduce the so-called Krein systems and briefly discuss their properties. For more information on the topic, see [10, 2, 13].

By a Krein system (a K-system), we mean the system of differential equations

$$\left\{\begin{array}{rcl} P'(r,\lambda) & = & i\lambda P(r,\lambda)-\overline{A(r)}P_*(r,\lambda), \\ P_*'(r,\lambda) & = & -A(r)P(r,\lambda), \end{array}\right. \tag{1.4}$$

with boundary conditions $P(0,\lambda)=P_*(0,\lambda)=1$. We suppose that $A\in L^1_{loc}(\mathbb{R}_+)$, $\lambda\in\mathbb{C},\ r\in\mathbb{R}_+$.

It turns out that the K-system defines the unique positive measure σ, $\int_{\mathbb{R}}(1+\lambda^2)^{-1}d\sigma(\lambda)d\lambda<\infty$, with the following properties. Introduce the Fourier transform $\mathcal{F}: L^2(\mathbb{R}_+)\to L^2(\sigma)$ by the formula

$$(\mathcal{F}f)(\lambda)=\int_0^\infty f(r)P(r,\lambda)\,dr.$$

The inverse Fourier transform $\mathcal{F}^{-1}: L^2(\sigma)\to L^2(\mathbb{R}_+)$ is given by the relation

$$(\mathcal{F}^{-1}g)(r)=\int_{\mathbb{R}} g(\lambda)\overline{P(r,\lambda)}\,d\sigma(\lambda),$$

and we have $\mathcal{F}^{-1}\mathcal{F}f=f$ for any $f\in L^2(\mathbb{R}_+)$; see [10]. The above integrals should be understood in the L^2-sense.

Theorem 1.2 ([10]). *For any $f\in L^2(\mathbb{R}_+)$, the Parseval equality holds*

$$\int_{\mathbb{R}}|\mathcal{F}f(\lambda)|^2\,d\sigma(\lambda)=\int_0^\infty|f(s)|^2\,ds.$$

The parallels between K-systems and orthogonal polynomials may be extended much further. We discuss the construction in a very particular case to be used in the sequel. The general construction can be found in [10, 13]. We let $\sigma_0=m/(2\pi)$ and assume that $\operatorname{supp}(\sigma-\sigma_0)$ is compact. Introduce

$$H(t)=\int_{\mathbb{R}} e^{i\lambda t}\,d(\sigma-\sigma_0)(\lambda). \tag{1.5}$$

The function $H\in C^\infty(\mathbb{R})$ gives rise to an integral equation for the "resolvent" kernel Γ_r

$$\Gamma_r(t,\tau)+\int_0^r H(t-s)\Gamma_r(s,\tau)\,ds=H(t-\tau). \tag{1.6}$$

Remarkably, it occurs that

$$\begin{array}{rcl} P(r,\lambda) & = & e^{i\lambda r}\left(1-\displaystyle\int_0^r \Gamma_r(s,0)e^{-i\lambda s}\,ds\right), \\ P_*(r,\lambda) & = & 1-\displaystyle\int_0^r \Gamma_r(0,s)e^{i\lambda s}\,ds, \end{array} \tag{1.7}$$

and $A(r)=\Gamma_r(0,r)$. It is instructive to compare formulas (1.4) and (1.7) to (1.2) and (1.1), respectively.

Because $H(t) \in C^\infty(\mathbb{R}_+)$, the Fredholm formula for the resolvents $\Gamma_r(t,\tau)$ yields $A \in C^\infty(\mathbb{R}_+)$. We also have a weakened version of Theorem 1.1.

Theorem 1.3 ([10, 14, 5])**.** *The following assertions are equivalent:*

i) σ is a Szegő-type measure on $\mathbb{R}$, that is $(1+\lambda^2)^{-1}\log\sigma'_{ac} \in L^1(\mathbb{R})$.

ii) The integral $\int_0^\infty |P(r,\lambda)|^2\,dr$ converges at least for one (and hence, for all) $\lambda \in \mathbb{C}_+$.

iii) The function $P_*(.,\lambda)$ is bounded at least for one (and hence, for all) $\lambda \in \mathbb{C}_+$.

In the above cases, there exists a limit $\Pi(\lambda) = \lim_{r_n\to\infty} P_*(r_n,\lambda)$, $\lambda \in \mathbb{C}_+$, and $\left[(\lambda+i)\Pi\right]^{-1} \in H^2(\mathbb{C}_+)$.

Under the assumptions of the theorem, $2\pi\sigma'_{ac}(\lambda) = |\Pi(\lambda+i0)|^{-2}$; see [14].

1.3. We will need one simple approximation argument. Complete information on the subject can be found in [1]. We take $f \in L^\infty(\mathbb{R})$ and let

$$(f)_r(x) = r\int_{\mathbb{R}} K(r(x-s))f(s)\,ds, \tag{1.8}$$

where $K(x) = \dfrac{12}{\pi}\left\{\dfrac{\sin x/2}{x}\right\}^4$ and $r \geq 0$. For $f \in L^2(\sigma)$, we define a continuity modulus by the formula

$$\omega^2_{2,\sigma}(\delta,f) = \sup_{|h|\leq\delta}\int_{\mathbb{R}} |f(x+h)-f(x)|^2\,d\sigma(x).$$

The following lemma is a modification of [7, Lemma 3.1], and is quoted here for the reader's convenience only.

Lemma 1.1. *If $f \in L^2(\sigma)\cap L^\infty(\mathbb{R})$, then, for $r > 0$,*

$$||f-(f)_r||^2_{2,\sigma} \leq C\omega^2_{2,\sigma}\left(1/r,f\right).$$

Proof. It is clear that

$$(f)_r(x) - f(x) = \frac{12}{\pi r^3}\int_0^\infty \left\{\frac{\sin rs/2}{s}\right\}^4 A(s,x)\,ds,$$

where $A(s,x) = f(x+s)+f(x-s)-2f(x)$. Applying the Hölder inequality, we have

$$|(f)_r(x)-f(x)|^2 \leq \frac{12}{\pi r^3}\int_0^\infty \left\{\frac{\sin rs/2}{s}\right\}^4 |A(s,x)|^2\,ds.$$

Integration in x with respect to σ yields

$$||(f)_r - f||^2_{2,\sigma} \leq \frac{12}{\pi r^3}\int_0^\infty \left\{\frac{\sin rs/2}{s}\right\}^4 ||A(s,.)||^2_{2,\sigma}\,ds.$$

Since $\omega_{2,\sigma}(m\delta, f) \le (m+1)\omega_{2,\sigma}(\delta, f)$, we have

$$||A(s,.)||_{2,\delta}^2 \le \omega_{2,\sigma}^2(|s|, f) \le 2(r|s|+1)^2\omega_{2,\sigma}^2(1/r, f).$$

Observing that

$$\frac{12}{\pi r}\int_0^\infty \left\{\frac{\sin rs/2}{s}\right\}^4 s^2\, ds \le C,$$

with C that does not depend on r, we finish the proof of the lemma. □

We denote the Paley-Wiener space of entire functions of exponential type r by $\boldsymbol{\mathcal{F}}_r$, that is

$$\boldsymbol{\mathcal{F}}_r = \left\{F : F(\lambda) = \int_0^r e^{i\lambda s} f(s)\, ds,\ f \in L^2[0,r]\right\}.$$

Observe that if $f \in H^2(\mathbb{C}_+)$, then $(f)_r \in \boldsymbol{\mathcal{F}}_{2r}$. Indeed, $\hat f$ is supported on $\mathbb{R}_+$, $\widehat{K(rx)}$ is supported on $[-2r, 2r]$ (see [1], Sect. 71), and, consequently, $\widehat{(f)_r} = \widehat{K(rx)}\hat f$ lives on $[0, 2r]$.

2. Krein systems

2.1. Suppose that the measure σ of a K-system satisfies the Szegő-type condition (see Theorem 1.3). Our goal is to understand how the properties of the singular and absolutely continuous parts of σ and their mutual location influence the properties of the coefficient A. We distinguish two different cases. In the first case, σ_{ac} and σ_s are well-agreed. In the second, a "good" σ_{ac} and a singular component σ_s are chosen more or less independently. By "good" we mean that the density σ'_{ac} is piecewise smooth, bounded above and bounded below from zero.

In the first case, we have the following theorem.

Theorem 2.1. *There exists a K-system with the properties:*

i) $\dim_{\mathrm{H}}$ supp $\sigma_s = 2(1-\beta)$, $1/2 < \beta < 1$.

ii) The corresponding coefficient A satisfies the inequality

$$\int_x^\infty |A(s)|^2\, ds \le \frac{C}{(1+x)^{2\beta-1}}.$$

Notice that the bound for the above integral matches perfectly the estimate from [11] for Schrödinger operators.

As expected, the bound in the second case is worse.

Theorem 2.2. *There exists a K-system with the properties:*

i) $\dim_{\mathrm{H}}$ supp $\sigma_s = 2(1-\beta)$ *with* $1/2 < \beta < 1$, *and*

$$\sigma'_{ac}(x) = 1/(2\pi)\begin{cases} 1/2, & |x| \le 1, \\ 1, & |x| > 1. \end{cases}$$

ii) The coefficient A lies in $L^p(\mathbb{R}_+)$ with any $p > 4/(2+\gamma_0)$, where $\gamma_0 = -\log(1-\beta)/(\log 2 - \log(1-\beta))$.

Remark 2.1. *If* $\dim_{\mathrm{H}} \operatorname{supp}\ \sigma_s \to 0$, *we get* $A \in L^p(\mathbb{R}_+)$ *with any* $p > 4/3$.

2.2. The proofs of both theorems rely on several lemmas which are proved in this subsection.

Lemma 2.1. *If $A \in W^{1,2}(\mathbb{R}_+)$ is a real-valued coefficient of a K-system, then*

$$\lim_{y\to+\infty} y^2 \int_t^\infty P^2(\tau, iy)\, d\tau = \int_t^\infty A^2(\tau)\, d\tau \tag{2.1}$$

for $t > 0$.

Proof. Take the K-system (1.4) with $\lambda = iy$ and introduce $Q(t) = e^{yt}P(t, iy)$. Clearly, Q is a solution to

$$\begin{cases} Q' &= -Ae^{yt}P_*, \\ P_*' &= -Ae^{-yt}Q \end{cases}$$

with boundary conditions $Q(0) = P_*(0) = 1$. We have

$$Q \;=\; 1 - \int_0^t A(\tau)e^{y\tau}P_*(\tau)\, d\tau, \tag{2.2}$$

$$P_* \;=\; 1 - \int_0^t A(\tau)e^{-y\tau}\left(1 - \int_0^\tau A(\xi)e^{y\xi}P_*(\xi)\, d\xi\right) d\tau. \tag{2.3}$$

Plug relation (2.3) in (2.2) and express P through Q. This gives

$$\begin{aligned} P \;&=\; e^{-yt} - e^{-yt}\int_0^t A(\tau)e^{y\tau}\Big[1 - \int_0^\tau A(\xi)e^{-y\xi}\, d\xi \\ &+ \int_0^\tau A(\xi)e^{-y\xi}\int_0^\xi A(\eta)e^{y\eta}P_*(\eta)\, d\eta d\xi\Big]\, d\tau = I_1 - I_2 + I_3 - I_4. \end{aligned} \tag{2.4}$$

Now, we estimate integrals $\int_t^\infty |I_i|^2\, d\tau$. It is clear that

$$\int_t^\infty |I_1|^2\, d\tau \le C\frac{e^{-2yt}}{y}.$$

Furthermore,

$$I_2 = \frac{1}{y}\left(A(t) - A(0)e^{-yt} - e^{-yt}\int_0^t A'(\tau)e^{y\tau}\, d\tau\right) = I_2^1 + I_2^2 + I_2^3,$$

and, obviously, $||I_2^2||_2 \le C/(y\sqrt{y})$. By the Young inequality for convolutions, $||I_2^3||_2 \le C/y^2$ if y is big enough. For I_3 from (2.4), we obtain

$$|I_3| \le \frac{e^{-yt}}{\sqrt{y}}||A||_2 \int_0^t |A(\tau)|e^{y\tau}\, d\tau.$$

Consequently, $||I_3||_2 \le C/(y\sqrt{y})$. Equality (2.3) implies that $|P_*(x,iy)| \le C$ uniformly in x and y; see [5]. Hence, we get $||I_4||_2 \le C/y^2$.

To compute the left-hand side of (2.1), we represent $|P(\tau,iy)|^2$ with the help of (2.4). The above estimates yield the claim of the lemma. $\square$

In the lemma below, we use notations introduced in Section 1.3. We also let $k_{iy}(\lambda) = -(2\pi i)^{-1}(\lambda+iy)^{-1}$ and $\Pi_0 = k_{iy}\Pi$.

Lemma 2.2. *Let the measure σ of a Krein system satisfy a Szegő-type estimate. Then*

$$\inf_{F\in\boldsymbol{\mathcal{F}}_r} ||\Pi_0 - F||^2_{2,\sigma} = \frac{1}{4\pi^2|\Pi(iy)|^2}\int_r^\infty |P(x,iy)|^2\,dx. \tag{2.5}$$

Proof. We begin with computation of $\mathcal{F}^{-1}\Pi_0$. Notice that

$$\left(\mathcal{F}^{-1}\Pi_0\right)(x) = \int_{\mathbb{R}} \Pi_0(\lambda)\overline{P(x,\lambda)}\,d\sigma(\lambda) = \frac{1}{2\pi}\overline{\left(\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{P(x,\lambda)}{\Pi(\lambda)}\frac{1}{\lambda-iy}\,d\lambda\right)}.$$

Taking into account the assumptions of the lemma and relations (1.7), we see that the last integral can be calculated by the Cauchy formula. Hence

$$(\mathcal{F}^{-1}\Pi_0)(x) = \frac{1}{2\pi}\overline{\left(\frac{P(x,iy)}{\Pi(iy)}\right)}.$$

Using the simple identity [10]

$$|P_*(r,\lambda)|^2 - |P(r,\lambda)|^2 = 2\,\mathrm{Im}\,\lambda\int_0^r |P(s,\lambda)|^2 ds,$$

we can see that

$$\int_{\mathbb{R}} |\Pi_0(\lambda)|^2 d\sigma(\lambda) = \int_0^\infty |\mathcal{F}^{-1}\Pi_0|^2 dx.$$

Therefore, Π_0 belongs to the range of $\mathcal{F}$.

Pick a function $f_r \in L^2[0,r]$ and extend it to $[r,+\infty)$ by zero. It follows that

$$\mathcal{F}^{-1}(\Pi_0 - \mathcal{F}f_r)(x) = \frac{1}{2\pi\overline{\Pi(iy)}}\left(\overline{P(x,iy)} - 2\pi\overline{\Pi(iy)}f_r(x)\right).$$

Using the Parseval equality (see Theorem 1.2) and observing that $\mathcal{F}f_r \in \boldsymbol{\mathcal{F}}_r$, we conclude the proof. $\square$

The next lemma is an elementary remark on the behavior of outer functions of some special form. Let

$$g(z) = \exp\left\{\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{h_0(x)\,dx}{x-z}\right\},$$

where $h_0 \in L^1(\mathbb{R})$ is a real-valued function, $f = g(.+h)/g$.

Lemma 2.3. *Suppose that $h_0 \in L^1(\mathbb{R}), g \in H^\infty(\mathbb{C}_+)$, and $g^{-1} \in H^2(\mathbb{C}_+)$. Then*

$$\begin{aligned} &i) \quad f(iy) = 1 + o(1/y), \ y \to +\infty, \\ &ii) \quad \int_{\mathbb{R}} |k_{iy}|^2 |f-1|^2 \, dx \le \int_{\mathbb{R}} |k_{iy}|^2 ||f|^2 - 1| \, dx + o\left(1/y^2\right). \end{aligned}$$

Proof. Obviously,

$$f(z) = \exp\left\{\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1}{x-z}(h_0(x+h) - h_0(x))\, dx\right\},$$

and $k_{iy} f \in H^2(\mathbb{C}_+)$. By dominated convergence theorem,

$$\lim_{y\to+\infty} y \int_{\mathbb{R}} \frac{1}{x-iy}(h_0(x+h) - h_0(x))\, dx = i\int_{\mathbb{R}} (h_0(x+h) - h_0(x))\, dx = 0,$$

and the first claim of the lemma is proved. As for the second one, we see that

$$||k_{iy}(f-1)||^2 = ||k_{iy} f||^2 - \frac{2}{4\pi y}\operatorname{Re} f(iy) + \frac{1}{4\pi y} = ||k_{iy} f||^2 - \frac{1}{4\pi y} + o\left(1/y^2\right),$$

since $(k_{iy} f, k_{iy}) = f(iy)/(4\pi y)$ and $(k_{iy}, k_{iy}) = 1/(4\pi y)$ under the assumptions of the lemma. ☐

Remark 2.2. *If h_0 is an even function, g is real for $z \in i\mathbb{R}_+$.*

2.3. In this section, for given $1/2 < \beta < 1$, we construct a nonnegative function w with certain special properties. This function gives rise to the measure σ. The measure, in turn, generates the K-system from Theorem 2.1.

Let $E^0 = [-1, 1]$. In the first step, we set $E^1 = E^0 \backslash J_0$, where J_0 is the open middle interval of E_0, and $|J_0| = \beta|E_0|$. In the $(n+1)$-th step, we represent E^n as $E^n = \cup_{k=1}^{2^n} I_{nk}$, $|I_{nk}| = 2((1-\beta)/2)^n$. Similarly, we define $E^{n+1} = \cup_{k=1}^{2^n}(I_{nk}\backslash J_{nk})$, where J_{nk} are open middle intervals of I_{nk}, and $|J_{nk}| = \beta|I_{nk}|$, etc.

Consider $E_\beta = \cap_{k=0}^\infty E^k$. The Hausdorff dimension of E_β is readily seen to be

$$\dim_{\mathrm{H}} E_\beta = \frac{\log 2}{\log 2 - \log(1-\beta)}. \tag{2.6}$$

For $M > 0$, we define the function $w = w_\beta$ as follows:

$$w(x) = 2\pi \begin{cases} \min\{1, M|x+1|^\gamma\}, & x \le -1, \\ 0, & x \in E_\beta, \\ M\min\{|x-a|^\gamma, |x-b|^\gamma\}, & x \in J_{nk} = (a, b), \\ \min\{1, M|x-1|^\gamma\}, & x \ge 1, \end{cases}$$

where

$$0 < \gamma \le \gamma_0 = \frac{-\log(1-\beta)}{\log 2 - \log(1-\beta)}. \tag{2.7}$$

In particular, w is even. It lies in $\mathrm{Lip}_\gamma(\mathbb{R})$, and $w = 0$ on E_β.

Furthermore, for the chosen γ, we have $w^{-1} \in L^1[-1,1]$, and, consequently, $\dfrac{\log w(x)}{1+x^2} \in L^1(\mathbb{R})$. Indeed, denoting by J_n an arbitrary interval J_{nk} (they are of the same length), we see that

$$\begin{aligned} \int_{-1}^{1} w^{-1}\,dx &\le \frac{C}{M}\sum_{n=0}^{\infty} 2^n \int_0^{|J_n|/2} x^{-\gamma}\,dx \\ &\le \frac{C}{M}\sum_{n=0}^{\infty} 2^n \left(\frac{1-\beta}{2}\right)^{n(1-\gamma)} < \infty, \end{aligned} \tag{2.8}$$

since $2((1-\beta)/2)^{1-\gamma} < 1$ under condition (2.7). Moreover, we can fix M big enough to ensure that

$$\int_{-2}^{2} w^{-1}\,dx < 4/(2\pi).$$

Proof of Theorem 2.1. Given $1/2 < \beta < 1$, we construct E_β and the function w as described above. Consider $d\sigma = w^{-1}dx + d\sigma_s$, where σ_s is a finite singular measure on E_β so that

$$\int_{-2}^{2} d\sigma(x) = \int_{-2}^{2}\left(w^{-1}\,dx + d\sigma_s(x)\right) = 4/(2\pi).$$

In particular, σ_s can be singular continuous. The measure σ generates the Krein system with coefficient A; see Section 1.2. Since $\log[w/(2\pi)] \in L^1(\mathbb{R})$, there exists an outer function $\Pi \in H^\infty(\mathbb{C}_+)$ such that $w = 2\pi|\Pi|^2$ on $\mathbb{R}$. Consequently, $\Pi_0 = k_{iy}\Pi \in H^2(\mathbb{C}_+)$.

By Lemma 2.2, we have

$$\begin{aligned} \int_n^{\infty} |P(x,iy)|^2\,dx &= C \inf_{F\in\boldsymbol{\mathcal{F}}_n} ||\Pi_0 - F||^2_{2,\sigma} \\ &\le ||\Pi_0 - (\Pi_0)_{n/2}||^2_{2,\sigma} \le C\omega^2_{2,\sigma}\left(1/n,\Pi_0\right). \end{aligned} \tag{2.9}$$

The remark given after the lemma explains why $(\Pi_0)_{n/2}$ lies in $\boldsymbol{\mathcal{F}}_n$. On the other hand, by the definition of $\omega_{2,\sigma}$,

$$\omega^2_{2,\sigma}\left(1/n,\Pi_0\right) = \sup_{|h|\le 1/n}\left\{\frac{1}{2\pi}\left|\left|\frac{\Pi_0(.+h)-\Pi_0}{\Pi}\right|\right|_2^2 + ||\Pi_0(.+h)-\Pi_0||^2_{\sigma_s}\right\}.$$

These norms are not difficult to estimate. Recalling that $\operatorname{supp}\sigma_s = E_\beta \subset [-1,1]$ and $w = 2\pi|\Pi|^2 = 0$ on E_β, we get

$$||\Pi_0(.+h)-\Pi_0||^2_{\sigma_s} = \int_{\mathbb{R}} |\Pi_0(x+h)|^2\,d\sigma_s(x) \le \frac{C}{n^\gamma}\int_{-1}^{1}\frac{d\sigma_s(x)}{x^2+y^2}.$$

Introduce the function $f(x) = \Pi(x+h)/\Pi(x)$. Note that $k_{iy}f \in H^2(\mathbb{C}_+)$. We have

$$\left\|\frac{\Pi_0(.+h)-\Pi_0}{\Pi}\right\|_2^2 \leq C\Big\{\int_{\mathbb{R}} |k_{iy}|^2|f-1|^2\,dx + \int_{\mathbb{R}} |k_{iy}(x+h)-k_{iy}(x)|^2 dx\Big\}.$$

For the second integral, we readily get

$$\int_{\mathbb{R}} |k_{iy}(x+h)-k_{iy}(x)|^2 dx \leq \frac{1}{n^2}\,O(1/y^3). \tag{2.10}$$

The following representation for Π holds [10]

$$\Pi(z) = \exp\left\{\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\ln(2\pi\sigma'_{ac})}{z-t}\,dt\right\},$$

where $z \in \mathbb{C}_+$. So, with the help of Lemma 2.3, we obtain

$$\int_{\mathbb{R}} |k_{iy}|^2|f-1|^2\,dx \leq \int_{\mathbb{R}} |k_{iy}|^2|w(x+h)-w(x)|\frac{dx}{w(x)} + o\left(1/y^2\right).$$

Since $w \in \mathrm{Lip}_\gamma(\mathbb{R})$ and $w(x+h) = w(x)$ outside the compact $[-2,2]$, the last inequality yields

$$\int_{\mathbb{R}} |k_{iy}|^2|f-1|^2\,dx \leq \frac{C}{n^\gamma}\int_{-2}^{2} \frac{1}{x^2+y^2}\frac{dx}{w(x)} + o\left(1/y^2\right).$$

From [4, Section 1], we infer $A \in W^{1,2}(\mathbb{R}_+)$. Thus, combining (2.9) with Lemma 2.1, we conclude that

$$\int_n^\infty A(s)^2\,ds \leq \frac{C}{n^\gamma}\lim_{y\to+\infty} y^2\Big\{\int_{-2}^{2}\frac{d(\sigma_{ac}+\sigma_s)}{x^2+y^2} + o(y^{-2})\Big\} \leq \frac{C}{n^\gamma}.$$

The theorem is proved. □

Notice that $\gamma_0 + \dim_{\mathrm{H}}\, E_\beta = 1$.

2.4. The proof of Theorem 2.2 is close in spirit to that of Theorem 2.1. Once again, we begin with choosing parameters that define the measure of a K-system. Suppose we have the set E_β, $1/2 < \beta < 1$ (see Section 2.3). Let $w_n = w_{\beta,n}$ be

$$w_n(x) = \begin{cases} \min\{1, r_n|x+1|^\gamma\}, & x \leq -1, \\ 0, & x \in E_\beta, \\ \min\{1, r_n|x-a|^\gamma, r_n|x-b|^\gamma\}, & x \in J_{nk} = (a,b), \\ \min\{1, r_n|x-1|^\gamma\}, & x \geq 1, \end{cases}$$

where $\{r_n\}$ is a sequence such that $r_n > 0$, $r_n \nearrow +\infty$, $\gamma > 0$. The precise choice of $\{r_n\}$ and γ will be made during the proof of the theorem.

Repeating, in essence, computations from (2.8), we deduce that

$$\int_{-2}^{2} w_n(x)^{-1}\,dx \le C,$$

provided $0<\gamma\le\gamma_0$ (see (2.7)). Taking $j_n = \dfrac{\log(C_0 r_n)}{\gamma(\log 2-\log(1-\beta))}$, we deduce

$$\begin{aligned}\int_{-2}^{2}\log\frac{1}{w_n(x)}\,dx \quad &\le C\sum_{k=1}^{j_n}2^k\int_0^{r_n^{-1/\gamma}}\log\frac{1}{r_n x^\gamma}\,dx\\ &+C\sum_{k=j_n+1}^{\infty}2^k\int_0^{(1-\beta)^k2^{-k}}\log\frac{1}{r_n x^\gamma}\,dx \le C\frac{\log r_n}{r_n^{\gamma_0/\gamma}}.\end{aligned}$$

Proof of Theorem 2.2. Take measure σ, $d\sigma = d\sigma_{ac}+d\sigma_s$, where σ_{ac} is given in the formulation of the theorem and σ_s is a finite singular measure supported on E_β so that

$$\int_{-1}^{1} d\sigma = 2/(2\pi).$$

Again, σ_s can be singular continuous. For any n, consider the outer functions $v_n\in H^\infty(\mathbb{C}_+)$ with the property $w_n=|v_n|^2$ and normalized as in Remark 2.2. We also take an outer function $\Pi\in H^\infty(\mathbb{C}_+)$ so that ${\sigma'_{ac}}^{-1} = 2\pi|\Pi|^2$. Moreover, let $v_{n0}=k_{iy}v_n\Pi$ and $\Pi_0 = k_{iy}\chi_{\mathbb{R}\backslash E_\beta}\Pi$.

In a trivial way,

$$\inf_{F\in\boldsymbol{\mathcal{F}}_N}||\Pi_0-F||_{2,\sigma}^2 \le C\left\{||\Pi_0-v_{n0}||^2_{2,\sigma}+\inf_{F\in\boldsymbol{\mathcal{F}}_N}||v_{n0}-F||^2_{2,\sigma}\right\}. \tag{2.11}$$

We obtain a bound for the first summand in the last sum

$$\begin{aligned}2\pi||\Pi_0-v_{n0}||^2_{2,\sigma} \quad &= ||k_{iy}(v_n-1)||_2^2 = \left(||k_{iy}v_n||^2-\frac{1}{4\pi y}\right)\\ &+\frac{2}{4\pi y}(1-\mathrm{Re}\ v_n(iy)) \le \frac{2}{4\pi y}(1-v_n(iy))\end{aligned}$$

because $||v_n||_\infty = 1$ and $v_n(iy)\in\mathbb{R}$ (see Remark 2.2). Furthermore, using the fact that $h_n=-\log w_n$ is supported on $[-2,2]$ and that $1-e^{-x}\le x$ for $x\in\mathbb{R}$, we proceed as follows

$$\begin{aligned}1-v_n(iy) \quad &= 1-\exp\left(-\frac{1}{2\pi}\int_{\mathbb{R}}\frac{y}{x^2+y^2}h_n(x)\,dx\right)\\ &\le \frac{1}{2\pi}\int_{-2}^{2}\frac{yh_n(x)}{x^2+y^2}\,dx.\end{aligned}$$

Hence, we come to

$$||\Pi_0 - v_{n0}||_2^2 \le C \int_{-2}^{2} \frac{h_n(x)}{x^2+y^2}\, dx. \tag{2.12}$$

We turn to the second summand at the right-hand side of (2.11). By Lemma 1.1, we get

$$\begin{aligned}\inf_{F\in\boldsymbol{\mathcal{F}}_N} ||v_{n0} - F||_{2,\sigma}^2 \;&\le\; C\omega_{2,\sigma}^2\left(1/N, v_{n0}\right) \le C \sup_{|h|\le 1/N} \Big\{ \int_{\mathbb{R}} |v_{n0}(x+h) - v_{n0}(x)|^2\, dx \\ &+ \int_{\mathbb{R}} |v_{n0}(x+h)|^2\, d\sigma_s(x)\Big\}. \end{aligned}\tag{2.13}$$

For the first integral, we have

$$\begin{aligned}\int_{\mathbb{R}} & |v_{n0}(x+h) - v_{n0}(x)|^2\, dx \le C\Big\{\int_{\mathbb{R}} |k_{iy}(x+h) - k_{iy}(x)|^2\, |\Pi v_n|^2 dx \\ + & \int_{\mathbb{R}} |k_{iy}|^2 |v_n(x+h) - v_n(x)|^2\, dx + \int_{\mathbb{R}} |k_{iy}|^2 |\Pi(x+h) - \Pi(x)|^2\, dx\Big\}.\end{aligned}$$

The first summand in the sum is bounded as in (2.10).

We set $f_n = v_n(.+h)/v_n$ and obtain a bound for the second summand with the help of Lemma 2.3

$$\begin{aligned}\int_{\mathbb{R}} |k_{iy}|^2 |f_n - 1|^2 |v_n|^2\, dx \;&\le\; \int_{\mathbb{R}} |k_{iy}|^2 \big||f_n|^2 - 1\big|\, dx + o\left(1/y^2\right) \\ &\le\; \int_{\mathbb{R}} |k_{iy}|^2 \left||v_n(x+h)|^2 - |v_n(x)|^2\right| \frac{dx}{|v_n(x)|^2} + o\left(1/y^2\right) \\ &\le\; \frac{Cr_n}{N^\gamma} \int_{-2}^{2} \frac{1}{x^2+y^2} \frac{dx}{|v_n(x)|^2} + o\left(1/y^2\right), \end{aligned}\tag{2.14}$$

that is because $w_n = |v_n|^2$, and the function $|v_n(.+h)|^2 - |v_n|^2$ is supported on $[-2, 2]$.

Let $f = \Pi(x+h)/\Pi(x)$. Applying Lemma 2.3 once again, we get

$$\int_{\mathbb{R}} |\Pi(x+h) - \Pi(x)|^2 |k_{iy}|^2 dx \le C\left(\int_{\mathbb{R}} |k_{iy}|^2 \big||f|^2 - 1\big| dx + o(1/y^2)\right).$$

Clearly,

$$\int_{\mathbb{R}} |k_{iy}|^2 \big||f|^2 - 1\big| dx = \int_{\mathbb{R}} |k_{iy}|^2 |{\sigma'_{ac}}^{-1}(x+h) - {\sigma'_{ac}}^{-1}(x)| dx \le Ch \int_{-2}^{2} \frac{dx}{x^2+y^2}.$$

For the second integral in (2.13), we have

$$\int_{\mathbb{R}} |v_{n0}(x+h)|^2 \, d\sigma_s(x) \le \frac{Cr_n}{N^\gamma} \int_{-1}^{1} \frac{d\sigma_s(x)}{x^2+y^2}. \tag{2.15}$$

Take $N = n$. Hence, (2.11), (2.12), (2.14), (2.15), and Lemma 2.1 yield

$$\begin{aligned}
\int_n^\infty A(s)^2 \, ds &\le C \lim_{y\to+\infty} y^2 \Big\{ \int_{-2}^{2} \frac{\log w_n(x)^{-1} \, dx}{x^2+y^2} \\
&+ \frac{r_n}{n^\gamma} \int_{-2}^{2} \frac{1}{x^2+y^2} \frac{dx}{w_n(x)} + o\left(1/y^2\right) + \frac{1}{ny^2} + \frac{r_n}{n^\gamma} \int_{-1}^{1} \frac{d\sigma_s(x)}{x^2+y^2} \Big\} \\
&\le C \left\{ \frac{\log r_n}{r_n^{\gamma_0/\gamma}} + \frac{r_n}{n^\gamma} \right\}.
\end{aligned}$$

To optimize the result, pick $\gamma = \gamma_0 - \varepsilon$, $r_n = n^\alpha$ with $\alpha = \gamma_0/2$, and arbitrarily small $\varepsilon > 0$. The bound becomes

$$\int_n^\infty A(s)^2 \, ds \le \frac{C}{n^{\gamma_0/2-\varepsilon}}.$$

From [4], we have $A \in L^\infty(\mathbb{R}_+)$. Thus, $A \in L^p(\mathbb{R}_+)$ with any $p > p_0 = 4/(2+\gamma_0)$. $\square$

3. Schrödinger and Dirac operators

3.1. In this section, we apply results obtained for Krein systems to Dirac and one-dimensional Schrödinger operators.

Assume that we have a K-system (1.4) with coefficient A. We define the functions $\phi(x,\lambda) = \mathrm{Re}\, e^{-i\lambda x} P(2x,\lambda)$ and $\psi(x,\lambda) = \mathrm{Im}\, e^{-i\lambda x} P(2x,\lambda)$. An easy computation shows that the functions ϕ, ψ are solutions of the following Dirac system

$$\left\{ \begin{array}{lcl} \phi' &=& -\lambda\psi - a_1\phi + a_2\psi, \\ \psi' &=& \lambda\phi + a_2\phi + a_1\psi, \end{array} \right. \tag{3.1}$$

with boundary conditions $\phi(0) = 1, \psi(0) = 0$. Above, $a_1(x) = 2\,\mathrm{Re}\, A(2x)$ and $a_2(x) = 2\,\mathrm{Im}\, A(2x)$. This allows us to say (see [10]) that $\rho_{Dir} = 2\sigma$, where ρ_{Dir} is the spectral measure of the Dirac system (3.1).

Using this simple relation between the measures of Krein and Dirac systems, we immediately obtain the following corollaries of Theorem 2.1 and Theorem 2.2.

Theorem 3.1. *There exists a Dirac system (3.1) with the properties:*

i) The coefficient $a_2(x) = 0$ *for all* $x > 0$.
ii) $\dim_{\mathrm{H}}$ supp $\sigma_{Dir,s} = 2(1-\beta)$, $1/2 < \beta < 1$.

iii) The coefficient a_1 *satisfies the inequality*

$$\int_x^\infty |a_1(s)|^2\, ds \le \frac{C}{(1+x)^{2\beta-1}}.$$

Theorem 3.2. *There exists a Dirac system (3.1) with the properties:*

i) The coefficient $a_2(x) = 0$ *for all* $x > 0$.

ii) $\dim_{\mathrm{H}} \operatorname{supp} \sigma_{Dir,s} = 2(1-\beta)$ *with* $1/2 < \beta < 1$, *and*

$$\sigma'_{Dir,ac}(x) = 1/(2\pi) \begin{cases} 1/2, & |x| \le 1, \\ 1, & |x| > 1. \end{cases}$$

iii) The coefficient a_1 *is in* $L^p(\mathbb{R}_+)$ *for any* $p > 4/(2+\gamma_0)$, γ_0 *is given by* (2.7).

3.2. When the coefficient a_1 is absolutely continuous and $a_2 = 0$, we deduce from (3.1) that

$$\begin{aligned} \psi'' - q\psi + \lambda^2\psi &= 0, \\ \phi'' - q_1\phi + \lambda^2\phi &= 0, \end{aligned} \tag{3.2}$$

where $q = a_1^2 + a_1'$, $q_1 = a_1^2 - a_1'$. The corresponding boundary conditions are

$$\begin{aligned} &\psi(0) = 0, \quad \psi'(0) = \lambda, \\ &\phi(0) = 1, \quad \phi'(0) + a_1(0)\phi(0) = 0. \end{aligned}$$

Therefore, the spectral measure ρ_{Sch} of the Schrödinger operator

$$L_q y = -y'' + qy \tag{3.3}$$

with boundary condition $y(0) = 0$ is related to σ by the formula

$$\rho_{Sch}(\lambda) = 4\int_0^{\lambda^{1/2}} \xi^2 d\sigma(\xi), \tag{3.4}$$

where $\lambda > 0$.

Let a real-valued coefficient A be as in Theorems 2.1 or 2.2 and $a_1(x) = 2A(2x)$. The arguments from [4, Section 2] show that q lies not only in $L^2(\mathbb{R}_+)$, but also in $W^{m,2}(\mathbb{R}_+)$ for any integer m. Indeed, if the first derivatives of A at $x = 0$ vanish, then the first trace formulas for the K-system and trace identities for the Schrödinger operator coincide. In the general case, the trace formulas for K-systems involve derivatives of A at $x = 0$. Still, the approximation argument used in [4] works and we have that $q(x) \in W^{m,2}(\mathbb{R}_+)$ for any m. A simple interpolation argument yields the following statement:

Theorem 3.3. *There exists a potential* q *with the properties:*

i) The spectral measure of L_q *(see* (3.3)*) has a singular continuous component* $\rho_{Sch,s}$ *such that* $\dim_{\mathrm{H}} \operatorname{supp} \rho_{Sch,s} = 2(1-\beta)$ *with* $1/2 < \beta < 1$.

ii) The following estimate holds

$$\int_x^\infty |q(s)|^2 ds \le \frac{C}{(1+x)^\gamma},$$

where $0 \le \gamma < 2\beta - 1$.

Sketch of the proof. Take A as in Theorem 2.1. By [4], the coefficient $A \in W^{1,2}(\mathbb{R}_+)$ and is therefore bounded. Since $q = a_1' + a_1^2$, it suffices to show that

$$\int_x^\infty |a_1'(s)|^2 ds \le \frac{C}{(1+x)^\gamma}. \tag{3.5}$$

Since $q \in W^{m,2}(\mathbb{R}_+)$, we can easily prove that $a_1 \in W^{m,2}(\mathbb{R}_+)$ for any integer m as well. It suffices to use an inequality from [6]

$$\|f'\|_2 \le C_m \|f\|_2^{1-1/m} \|f^{(m)}\|_2^{1/m}$$

to obtain (3.5). The required properties of ρ_{Sch} now follow from Theorem 2.1 and (3.4). □

Similarly, the following corollary of Theorem 3.3 can be proved.

Theorem 3.4. *Fix any* $p_0 > 4/3$. *There exists a potential* q *such that*

i) The spectral measure ρ_{Sch} *of the Schrödinger operator* (3.3) *has a nontrivial singular continuous component for some boundary condition at* $x = 0$.
ii) The positive density $\rho'_{Sch,ac}$ *is piecewise constant.*
iii) $q \in L^p(\mathbb{R}_+)$ *for any* $p > p_0$.

4. Orthogonal polynomials

The following theorems are counterparts of Theorems 2.1 and 2.2 for orthogonal polynomials on $\mathbb{T}$. Their proofs follow word-by-word the proofs of the results for Krein systems. The only difference is that we need to use inequality (1.3) instead of Lemmas 2.1 and 2.2. That is why the arguments below are omitted.

Theorem 4.1. *There exists a measure* σ *with the properties:*

i) $\dim_{\mathrm{H}} \operatorname{supp} \sigma_s = 2(1-\beta)$, $1/2 < \beta < 1$.
ii) The sequence $\{a_n\}$ *of reflection coefficients is such that*

$$\sum_{k=n}^\infty |a_k|^2 \le \frac{C}{n^{2\beta-1}}.$$

In fact, some ingredients of the proof of the above theorem can be found in [7].

Theorem 4.2. *There exists a singular continuous measure* σ_s *so that*

i) $\dim_{\mathrm{H}} \operatorname{supp} \sigma_s = 2(1-\beta)$ *with* $1/2 < \beta < 1$.
ii) The sequence $\{a_n\}$ *associated to* $d\sigma = dm/2 + d\sigma_s$ *lies in* l^p *for any* $p > 4/(2+\gamma_0)$, *where* γ_0 *is given by* (2.7).

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