Multicomponent dynamical systems: SRB measures and phase transitions

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Abstract. We discuss a notion of phase transitions in multicomponent systems and clarify relations between deterministic chaotic and stochastic models of this type of systems. Connections between various definitions of SRB measures are considered as well.

1 Introduction

The aim of the present paper is twofold: to study the notion of phase transitions in multicomponent systems and to clarify the relations between deterministic chaotic and stochastic models of this type of systems. We also discuss the differences in the approaches to multicomponent systems in statistical physics and in dynamical systems theory. In the former case it is basically a system of interacting particles, while in the later case each component of a multicomponent systems can have a nontrivial (local) dynamics, which leads to more rich evolution and thus to more rich statistical properties. Our definition of a multicomponent system (see Section 4) assumes that it consists of components (local susbsystems) which have their intrinsic dynamics and besides interact with each other. In particular, we argue that in distinction to situations considered in statistical physics, phenomenona similar to phase transitions can appear even in systems with a finite number of components (degrees of freedom). Indeed, in the absence of interactions dynamics of particle systems is trivial. Besides, some interesting phenomena, even in the presence of interactions, may appear only in the limit when the number of particles goes to infinity. Moreover, often additional assumptions like indecomposability (see e.g. [8]), are used to emphasize that the situation under study is impossible in a finite dimensional setting. On the other hand, as we shall show in Section 2, already the simplest one-dimensional dynamical systems satisfying the indecomposability assumption (and even the assumption of topological transitivity) may be non ergodic, which shows that restrictions of this type are not quite reasonable in the context of general dynamical systems.

A recent progress in the analysis of chaotic spatially extended dynamical systems allows to advance in answering to a long standing question how to define exactly phase transitions rigorously and what are the conditions for phase transitions in this type of systems and more generally in multicomponent systems. The problem with the definition of the phase transition phenomenon is that this notion is used in different ways in statistical physics (see review in [5]), moreover various existing approaches to this notion for spatially extended systems [2, 1, 3, 8] lead to different statements about the existence of phase transitions. As it was already mentioned in [3, 2, 8] it is essential to make distinction between qualitative changes in the topological behaviour of a system, called bifurcations in the dynamical systems theory, and changes to measure-theoretical properties which we shall identify with phase transitions. In the present paper by the phase transition we shall mean a change of a number of SRB measures (see the definition and discussion further) which one can naturally identify with phases in statistical physics. In fact, this is the same general idea, which was used earlier in a number of papers

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[3, 2, 8], however differences in the definitions of the SRB measures (which we shall discuss in detail) lead again to different statements about the phase transitions.

If local components of our multicomponent system are identical and the interaction is translationally invariant one can discuss finite dimensional approximations $(T^{(d)}, X^d)$ (where X^d is a direct product of d identical copies of X) with certain boundary conditions (e.g. with periodic boundary conditions). Assuming that we are able to study ergodic properties of those finite dimensional approximations for any $d < \infty$, one of the major problems is to analyze how their limiting (as $d \to \infty$) behavior corresponds to the dynamics of the entire (infinite) system. In particular, it might be possible that for each finite d there is only one SRB measure μ_d, but their limit points do not coincide with the SRB measures of the multicomponent system.

2 SRB measures

Let (X, ρ) be a compact metric space with a certain reference measure m on it, e.g. a finite dimensional unit cube or torus with the Lebesgue measure. Consider a nonsingular with respect to the measure m map T from the space X into itself (i.e. $m(T^{-1}A) = 0$ whenever $m(A) = 0$). The pair (T, X) defines a deterministic dynamical system. To study statistical properties of this dynamical system we need to consider its action in the space of measures. Let $\mathcal{M}(X)$ be the space of probabilistic measures on X equipped with the topology of weak convergence of measures. Then the induced map $T^* : \mathcal{M}(X) \to \mathcal{M}(X)$ is defined as follows:

$$T^*\mu(A) := \mu(T^{-1}A) \tag{2.1}$$

for any measure $\mu \in \mathcal{M}(X)$ and any Borel set $A \subseteq X$.

We shall call a measure μ_T a *natural* measure for the map T if there exists an open subset $U \subseteq X$ (called the basin of attraction for the measure μ_T) such that for any measure $\mu \in \mathcal{M}(X)$ absolutely continuous with respect to the reference measure m and having its support in U we have:

$$\frac{1}{n} \sum_{k=0}^{n-1} T^{*k}\mu \stackrel{n\to\infty}{\longrightarrow} \mu_T \tag{2.2}$$

In other words, the measure μ_T is a stable fixed point of the dynamics of absolutely continuous initial measures. A similar definition has been used e.g. in [6, 1].

Observe that from the point of view of the action of the map T in the space of measures a natural measure (not necessary unique) is nothing more than a stable (with respect to absolutely continuous initial conditions in the space of measures) fixed point of the induced map T^*, i.e. an attractor. This object is well known in ergodic theory of dynamical systems and corresponds to one of the definitions of SRB measures (see e.g. [1, 4, 6, 8]), which we shall discuss in a moment. Observe also that one of the advantages of this definition is that it works without any changes for true random Markov chains as well. Indeed, let T^* be the transfer operator of a Markov chain with the phase space X. This operator generates the dynamics of measures on $\mathcal{M}(X)$, i.e. this is a conjugate operator to the Markov operator (transition matrix) of the Markov chain under consideration. Then the relation (2.2) defines the notion of the natural (SRB) measure in this true random setting as well.

In the literature one can find three main different approaches to a definition of SRB measures in the deterministic setting. We already mentioned one of them (natural measure).

The second definition is very close to the previous one, with the only difference that one considers pathwise convergence of sample measures, rather than a convergence of orbits of the induced map. Namely, by SRB measure in this case one means the common limit as $n \to \infty$ for m-almost all points $x \in U$ of $\frac{1}{n}\sum_{k=1}^{n-1} \delta_{T^n x}$, where δ_x stands for the δ-measure at the point x.

Observe that both these definitions are based on Gibbs idea of construction of stationary measures and therefore they both are closely related to the statistical physics formalism and to the well known Bogolyubov-Krylov approach in dynamical systems theory.

The third definition of SRB measure is based on a completely different observation, namely that for some 'good' dynamical systems with strong stochastic properties (for example, uniformly hyperbolic systems) an SRB measure, which corresponds to any of the above definitions, has a marginal distribution absolutely continuous with respect to the Lebesgue measure on the so called unstable foliation of the dynamical system. This is indeed a very important statistical feature of the dynamics, but it is well defined only in the case of hyperbolic dynamical systems and therefore it is not clear what is the reason to use it as a definition of the SRB measure for more general dynamical systems. Denote by μ_T, μ_T^p and μ_T^c the versions of SRB measures corresponding to the three definitions given above respectively. The following simple statement describes connections between these definitions. It shows also that the first version, which we call the natural invariant measure is, indeed, more natural than the others.

Theorem 2.1 *(a) If μ_T^p exists then μ_T is also well defined and $\mu_T = \mu_T^p$.*

(b) If μ_T is ergodic and invariant then μ_T^p is well defined and $\mu_T^p = \mu_T$, however without ergodicity this might not hold.

(c) If μ_T^c is stable with respect to the dynamics of absolutely continuous initial measures then $\mu_T = \mu_T^c$.

(d) The existence of the μ_T or μ_T^p does not imply the existence of μ_T^c.

Proof. We start from the assertion (a). By definition for a m-a.a. point $y \in U \subseteq X$ we have the week convergence of the sequence of measures $\frac{1}{n}\sum_{k=1}^{n-1} \delta_{T^k y} \to \mu_T^p$, i.e. for any continuous function $\phi : X \to \mathbb{R}^1$ and m-a.a. $y \in U$ we have

$$\int \phi(x)\, d\left(\frac{1}{n}\sum_{k=1}^{n-1} \delta_{T^k y}\right) = \frac{1}{n}\sum_{k=1}^{n-1} \phi(T^k y) \stackrel{n\to\infty}{\longrightarrow} \int \phi\, d\mu_T^p.$$

Choose a measure $\mu \in \mathcal{M}(X)$ with a support in U being absolutely continuous with respect to m and consider its Cesaro averages: $\mu_n := \frac{1}{n}\sum_{k=1}^{n-1} T^{*k}\mu$. Then using the above convergence and absolute continuity of the measure μ we get

$$\begin{aligned} \int \phi\, d\mu_n &= \int \phi\, d\left(\frac{1}{n}\sum_{k=1}^{n-1} T^{*k}\mu\right) = \int \frac{1}{n}\sum_{k=1}^{n-1} \phi(T^k x)\, d\mu \\ &\stackrel{n\to\infty}{\longrightarrow} \int (\int \phi\, d\mu_T^p)\, d\mu = \int \phi\, d\mu_T^p, \end{aligned}$$

which proves the first assertion.

It is of interest that even if the measure μ_T^p exists and is unique it might be non ergodic. To show this consider the following example, proposed by G. Del Magno:

$$Tx := \begin{cases} (1 - \sin(\pi x - \pi/2))/2 & \text{if } 0 < x < 1, \\ x & \text{if } x \in \{0, 1\} \end{cases} . \tag{2.3}$$

One can easily show that the locally maximal attractor in this example consists of two fixed points at 0 and 1 and that for any initial point $x \in (0, 1)$

$$\frac{1}{n}\sum_{k=1}^{n-1} \delta_{T^n x} \stackrel{n\to\infty}{\longrightarrow} \frac{1}{2}(\delta_0 + \delta_1) = \mu_T^p.$$

On the other hand, this measure is nonergodic, since both points 0 and 1 are fixed points.

The first part of the assertion (b) follows immediately from Birkhoff ergodic theorem. Observe that in order to apply this theorem we need the measure μ_T to be invariant. If the map T is continuous this is certainly correct (it is enough to apply the induced map to the both sides of the limit construction in the definition of μ_T), but for a general nonsingular map this fact is not an immediate consequence of the definition of the natural measure. To demonstrate this consider the following one dimensional map from the unit interval into itself: $Tx = x/2$ for all $x \in (0,1]$ with the only discontinuity at the origin: $T0 = 1$. In this example under the action of the induced map T^* any probabilistic measure converges to the δ-measure at 0, but the map T has no invariant measure at all.

To finish the proof of this assertion we need to show that the natural measure might not coincide with μ_T^p, which at first sight looks rather doubtful. Observe, that the example of the map (2.3) shows that even being unique the measure μ_T might be nonergodic. Indeed, again by the same argument as above the images of any absolutely continuous probabilistic measure converge to $\frac{1}{2}(\delta_0 + \delta_1)$, which is nonergodic. Still in this case the measures μ_T and μ_T^p coincide.

To demonstrate that these two measures might not coincide, we consider the following one-dimensional map introduced recently in [9]:

$$Tx := \begin{cases} x + 4x^3 & \text{if } 0 \le x < 1/2 \\ x - 4(1-x)^3 & \text{if } 1/2 \le x \le 1 \end{cases} . \tag{2.4}$$

This map has two neutrally unstable fixed points 0 and 1. It has been shown [9] that for any sufficiently small $\delta > 0$ for Lebesgue a.a. points $x \in [0,1]$:

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{[0,\delta]}(T^k x) = 1, \qquad \liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{[0,\delta]}(T^k x) = 0,$$

where 1_A stands for the characteristic function of the set A. Therefore a measure μ_T^p does not exist in this case. On the other hand, in [11] it has been shown that for any probabilistic absolutely continuous measure μ the sequence of measures $T^{*n}\mu \to (\delta_0 + \delta_1)/2 =: \mu_T$ weakly as $n \to \infty$, which proves that the natural measure μ_T is well defined. Observe, however, that there are invariant sets $\{0\}$ and $\{1\}$ having μ_T-measure 1/2 each, which contradicts to ergodicity as in the example (2.3).

The statement (c) is a trivial corollary to the definition of the natural measure. The last assertion (d) follows from the observation that in the case of a globally contracting map the measures μ_T and μ_T^p coincide with a δ-measure being the only invariant measure of the system, while the measure μ_T^c does not exist. □

Observe that the measure μ_T^c may also exist but be not finite. It happens already for the most closest to uniformly hyperbolic so called almost Anosov diffeomorphisms, i.e. for diffeomorphisms which are uniformly hyperbolic away from a finite set of points [7]. Similar results for the case of one-dimansional neutral maps are also well known (see e.g. [9] and references therein).

In the literature dedicated to phase transitions one can find the assumption of indecomposability (see e.g. [8]) introduced to stress the necessity to consider only infinite dimensional systems. Roughly speaking the indecomposability means the following: if we have two finite pieces of trajectories $\{T^k x\}_{k=1}^n$ and $\{T^k y\}_{k=1}^m$ on the locally maximal attractor of the map T, then $\forall \varepsilon > 0$ there exists a point $z = z(\varepsilon) \in X$ and an integer N such that $\rho(T^k x, T^k z) < \varepsilon$ for any $k = 1, \ldots, n$ and $\rho(T^k y, T^{N+k} z) < \varepsilon$ for any $k = 1, \ldots, m$. It is not hard to understand that if the map T is continuous this is equivalent to the assumption of *topological transitivity* of the map, i.e. that for any two open subsets A, B having nonempty intersections with the locally maximal attractor there exists a number N such that $T^N A \cap B \ne \emptyset$, which in turn is equivalent to the presence of a trajectory densely covering the locally maximal attractor.

Let us show now that in distinction to models of classical statistical physics even elementary one-dimensional dynamical system might be nonergodic but topologically transitive. We formulate this statement as a lemma, but in fact it is already proven above.

Lemma 2.1 *The maps (2.3) and (2.4) are topologically transitive but their unique SRB measures are non ergodic.*

3 Deterministic models of Markov chains

The aim of this section is to study connections between Markov chains and piecewise linear maps.

Let T be a nonsingular map from the unit interval $[0,1]$ into itself and let $\bar{\Delta} := \{\Delta_i\}_{i=1}^W$ be a partition of $[0,1]$ into disjoint intervals. The number W of elements of the partition Δ might be infinite. This partition is called a *special* partition for the map T if the restriction of the map T to the inner part of any interval $\bar{\Delta}_i$ is a diffeomorphism onto its image. The map T is called *Markov* if there exists a special partition Δ (which is also called Markov one) for which the following property holds: $T\Delta_i = \cup_{j\in I(i)}\Delta_j$ for each i. Given a Markov partition $\bar{\Delta}$ consider a subset of a set of probabilistic measures

$$\mathcal{M}_u([0,1],\bar{\Delta}) := \{\mu \in \mathcal{M}([0,1]) : \quad \frac{\mu(I)}{\mu(I')} = \frac{|I|}{|I'|}, \quad \forall i \text{ and } \forall \text{ intervals } I, I' \subset \Delta_i\},$$

where $|I|$ stands for the Lebesgue measure of the interval I. In other words, $\mathcal{M}_u([0,1],\bar{\Delta})$ corresponds to the set of piecewise uniform distributions on intervals $\bar{\Delta}$, i.e. the restriction of any measure from this set to an interval Δ_i for each i is proportional to the Lebesgue measure. Observe that if the map T is Markov and is mixing, then the natural measure is unique and its density with respect to the reference measure is a piecewise constant function on the elements of the partition.

Theorem 3.1 *For any given transition matrix P of a Markov chain with N states (where N can be infinite) there exists a piecewise linear one-dimensional map T with a Markov partition $\bar{\Delta}'$ such that the restriction of the induced map T^* to $\mathcal{M}_u([0,1],\bar{\Delta}')$ is equivalent to the left action of the matrix P in the space of distributions.*

Proof. Let $\bar{\Delta}$ be any partition of the unit intervals into N subintervals. (For example, if $N < \infty$ one can choose a partition into N equal intervals, while if $N = \infty$ one can consider a countable partition into intervals $\Delta_i := (2^{-i}, 2^{-i+1}]$ for $i = 1, 2, \ldots$.) For a given integer i let $I_i := \{i_j\}_j$ be the collection of indices such that $P_{ii_j} > 0$. Consider now a subpartition of the interval Δ_i into intervals $\{\Delta_{ij}\}_j$ of lengths $|\Delta_{ij}| = P_{ii_j}|\Delta_i|$. This refined partition, consisting of intervals Δ_{ij}, will be a special partition Δ' for a piecewise linear map $T : [0,1] \to [0,1]$ defined as follows: on each interval Δ_{ij} the map T is a linear map from this interval onto the interval Δ_{i_j}. On the other hand, since for any pair of indices i, j the image under the action of the map T of the interval Δ_{ij} is an interval from the original partition $\bar{\Delta}$, and hence a union of intervals from the partition $\bar{\Delta}'$, we indeed have shown that this partition is Markov.

It remains to discuss the equivalence of the restricted induced map T^* with the left action of the transition matrix P. Observe that the density of a measure $\mu \in \mathcal{M}_u([0,1],\bar{\Delta}')$ is well defined and is a piecewise constant function on intervals Δ'_i. Associate to the measure μ a vector $\bar{p}(\mu)$ with components $(\bar{p}(\mu))_i := \mu(\Delta'_i)$. Since the map T constructed above is Markov, each of the intervals Δ'_i is mapped under the action of T onto a union of intervals from the partition $\bar{\Delta}'$. Moreover, the map μ is linear on each element of the partition and thus $(\bar{p}(T^*\mu))_i = (\bar{p}(\mu)P)_i$ for each index i. □

Corollary 3.1 *Let a number of states N of the Markov chain be finite and let for each i a number of positive elements in the i-th row of the matrix P do not exceed K. Then the number of elements in the special partition of the one-dimensional map constructed above is at most NK.*

One might ask if it is possible to construct a continuous version of the map representing say a finite Markov chain. Indeed, at the first sight, it looks like using a rearrangement of the elements of the partition Δ and changing their lengths this should be possible. However the following example of a 3-state Markov chain shows that in general there is no continuous deterministic model of this type and illustrates also the procedure described in the proof of above result.

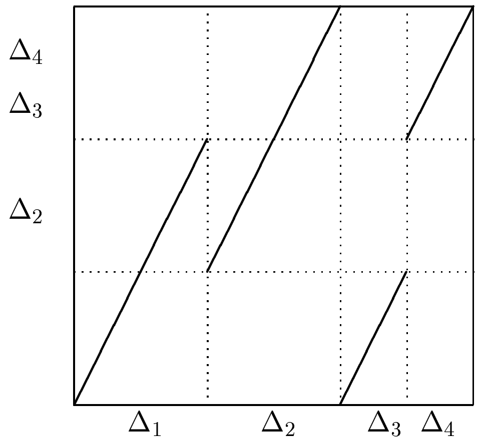

Figure 1: Deterministic model of the Markov chain (3.1).

Consider a Markov chain with three states and the following transition matrix:

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}. \tag{3.1}$$

Then the construction described in the proof of Theorem 3.1 leads to the following map (see Fig.1 below):

$$Tx := \begin{cases} 2x & \text{if } 0 \le x < 1/3 \\ 2x - 1/3 & \text{if } 1/3 \le x < 2/3 \\ 2x - 4/3 & \text{if } 2/3 \le x < 5/6 \\ 2x - 1 & \text{if } 5/6 \le x \le 1. \end{cases}$$

In this case we can make a smaller number of the elements of the partition, i.e. 4 instead of 6, but a simple geometric argument shows that one cannot rearrange these intervals to make the map be continuous, even if we consider it on a circle instead of the interval. Indeed, the structure of the transition matrix (3.1) P is such that the interval(s) corresponding to at least one of the tree states of the Markov chain should be mapped to the intervals corresponding to the two other states, inevitably having a gap between them.

Another and a more interesting example is the deterministic model of a random walk on nonnegative integers, defined for any positive i by transition probabilities $p_{i,i-1}, p_{i,i}, p_{i,i+1}$ to go to the left, to remain in the current position, and to go to the right respectively, and for $i = 0$ by transition probabilities $p_{0,0}$ and $p_{0,1}$. The corresponding map is shown on Fig.2(a). Observe that $p_{i,j} = |\Delta_j|/|T\Delta_i|$ for any pair of indices i, j such that $|i - j| \le 1$.

It is not hard to realize also a (space) homogeneous random walk on nonnegative integers which is define by transition probabilities $p_{i,i-1} = p_L, p_{i,i} = 1 - p_L - p_R, p_{i,i+1} = p_R$ for all $i > 0$ and $p_{0,0} = 1 - p_R, p_{0,1} = p_R$. The corresponding map is shown on Fig.2(b) and the transition probabilities are equal to $p_L = |\Delta_{i,i-1}|/|\Delta_i|$ and $p_R = |\Delta_{i,i+1}|/|\Delta_i|$, where $\Delta_i = \Delta_{i,i-1} \cup \Delta_{i,i} \cup \Delta_{i,i+1}$. Observe that in this case the restriction of the map to Δ_i is only piecewise linear (in distinction to the previous case).

While being sufficiently general Markov chains with a countable number of states do not describe the dynamics of the so called probabilistic cellular automata, which we are going to discuss now. Let G be a finite or countable graph, and let v be a function with a finite number of values (not larger than $K < \infty$) defined on the vertices of this graph, which we shall denote by the same letter G, i.e. $v : G \to \{1, 2, \ldots, K\}$. We assume also that the graph G is locally finite, i.e. for a vertex $g \in G$ we define its neighborhood $O(g)$ as the union of vertices to which it is connected. We assume that the graph G is locally finite, i.e. for each vertex $g \in G$ its neighborhood $O(g)$ consists of at most $L < \infty$ vertices of the graph G. A probabilistic cellular automaton on the graph G is defined as follows. For

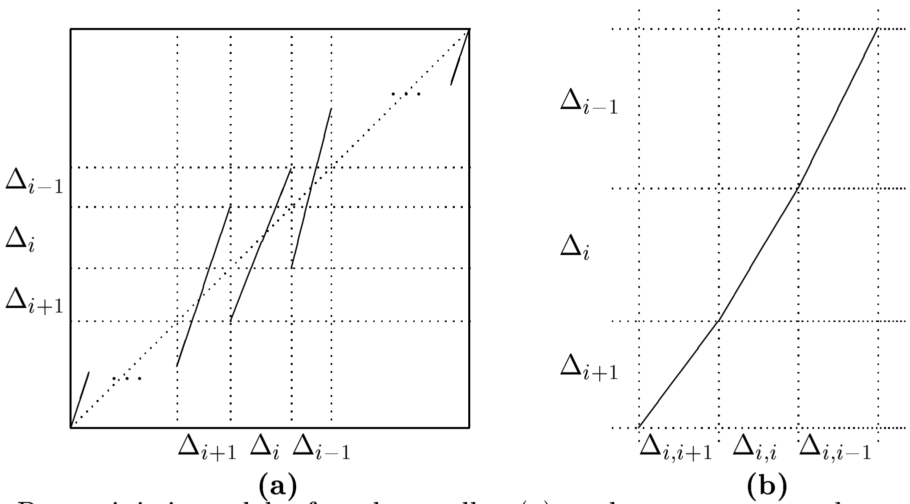

Figure 2: Deterministic models of random walks. (a) nonhomogeneous random walk, (b) homogeneous random walk with $\Delta_i = \Delta_{i,i-1} \cup \Delta_{i,i} \cup \Delta_{i,i+1}$.

each $g \in G$, each value of $v(g)$, and each configuration of values of the function $v(O(g))$ we define transition probabilities $p(\cdot, i)$ for $i \in \{1, 2, \ldots, K\}$ and thus the dynamics of values $v(g)$. (Observe that the number of these configurations is finite).

Let P_M be the transfer operator corresponding to a Markov chain M acting on the state space X_M, i.e. $P_M : \mathcal{M}(X_M) \to \mathcal{M}(X_M)$. We shall say that this Markov chain is *equivalent* to a dynamical system (T, X) if there is a subspace $\mathcal{M}_M(X)$ of the space of probabilistic measures on X and a homeomorphism $\pi : \mathcal{M}(X_M) \to \mathcal{M}_M(X)$ such that for any two measures $\mu \in \mathcal{M}(X_M)$ and $\nu \in \mathcal{M}_M(X)$ we have $\pi(P_M \mu) = T^*(\pi\mu)$ and $\pi P_M(\nu) = T^*(\pi\nu)$. Observe that the equivalence of the restricted induced map to the left action of the transition matrix in Theorem 3.1 is a special case of this general definition.

Theorem 3.2 *For any probabilistic cellular automaton on a locally finite graph G there exists a deterministic dynamical system described by a countable number of maps, whose dynamics is equivalent to the dynamics of the probabilistic cellular automaton.*

Proof. The idea of the construction of the equivalent dynamical system for a probabilistic cellular automaton is the following: in each vertex of the graph we define a number of maps (corresponding to all possible configurations of states of neighbors and itself). A choice of the map is given by the configuration of these states.

Observe, that for a given vertex $g \in G$ a total number of various configurations of values of the function v in its neighborhood $O(g)$ cannot exceed $K^L < \infty$. On the other hand, for any given configuration of $v(O(g))$ the construction in the proof of the previous theorem can be applied to define a piecewise linear map $T_{v(O(g))}$ from a unit interval into itself with the Markov partition consisting of at most K^K intervals Δ_i, which has the dynamics "equivalent" to the transition probabilities of our cellular automaton. Consider now a multicomponent system whose phase space $X := [0, 1]^G$ is a direct product of unit intervals, at each vertex $g \in G$ we have a finite number of one-dimensional maps $T_{v(g)}$ and for any point $\bar{x} = (x_g) \in X$ we define its image as follows. Let $x_{g'} \in \Delta_{i(g')}$ for any $g' \in O(g)$ then the map $T_{v(O(g))}(x_g)$, corresponding to the configuration $\{i(g')\}$, defines the value of the g-th coordinate of $\bar{x}$. $\square$

Remark 3.2 Using the same argument as above we can construct a deterministic model with $K = \infty$ if we assume additionally that for any $i \in \{1, 2, \ldots, K\}$ only a finite number of transition probabilities (in the definition of the cellular automaton) are positive, however the assumption of the local finiteness of the graph G we cannot drop.

A particular case of this construction has been used in [8]. Note that our description of the deterministic model corresponding to an arbitrary probabilistic cellular automaton is somewhat simpler and more explicit than the one in proposed in the cited paper.

Let us consider briefly an inverse problem – an approximation of a general dynamical system by a finite (countable) state Markov chain. In distinction to the case of piecewise linear Markov maps (considered above) in the general case one cannot construct an "equivalent" finite or countable state Markov chain, however an approximation is still possible. One of the simplest approaches here was proposed in [14]. Let T be a nonsingular map from a compact metric space (X, ρ) into itself and let (m) be a reference measure on X. Consider a partition $\{\Delta_i\}_{i=1}^W$ of the phase space X and associate to it a W-state Markov chain with transition probabilities

$$p_{ij} := \frac{m(T^{-1}\Delta_j \cap \Delta_i)}{m(\Delta_i)}.$$

It has been conjectured that a number of statistical features of the map T can be obtained in the limit as the diameter of the partition $\{\Delta_i\}_{i=1}^W$ vanishes of the corresponding features of the above Markov chains (see a review of recent results in this field and generalizations of this procedure in [1]).

4 Multicomponent dynamical systems and phase transitions

Let $\mathcal{N}$ be a finite or countable collection of indices. Then by $(X^{\mathcal{N}}, \rho^{\mathcal{N}}, m^{\mathcal{N}})$ we denote a direct product of compact metric spaces (X_i, ρ_i) with given reference measures m_i on them, i.e. $X^{\mathcal{N}} := \otimes_{i \in \mathcal{N}} X_i$, $\rho^{\mathcal{N}}(\bar{x}, \bar{y}) := \max \rho_i(x_i, y_i)$, and $m^{\mathcal{N}} := \otimes_{i \in \mathcal{N}} m_i$, and consider a nonsingular map $T^{(\mathcal{N})} : X^{\mathcal{N}} \to X^{\mathcal{N}}$ from this space into itself. Consider also a collection of "local" maps T_i acting on the i-th copy (X_i, ρ_i, m_i) of our "local" phase space. We shall call the pair $(T^{(\mathcal{N})}, X^{\mathcal{N}})$ a *multicomponent dynamical system* if its action can be decomposed as a superposition of an "interaction" and the direct product of "local" maps: $\mathcal{T}^{(\mathcal{N})} = \mathcal{I}^{(\mathcal{N})} \circ T^{\mathcal{N}}$, where $T^{\mathcal{N}} := (\otimes_{i \in \mathcal{N}} T_i)$ is a direct product of maps (from our collection of $\{T_i\}_{i \in \mathcal{N}}$). In the most interesting spatially homogeneous case (when all spaces (X_i, ρ_i, m_i) coincide) we assume also that the map : $X^{\mathcal{N}} \to X^{\mathcal{N}}$ which describes the "interaction" between local components (systems (T_i, X_i)) of the multicomponent dynamical system should be identical on the "diagonal" set

$$X^{\mathcal{N}}_{\text{diag}} := \{\bar{x} \in X^{\mathcal{N}} : \quad x_i = x_j \ \forall i, j\}.$$

The reason of this assumption is that when all the coordinates of a point $\bar{x} \in X^{\mathcal{N}}$ are the same the interaction cannot change them, i.e. $\mathcal{I}^{(\mathcal{N})}\bar{x} = \bar{x}$ for any $\bar{x} \in X^{\mathcal{N}}_{\text{diag}}$. For a multicomponent dynamical system with a finite number of components this property can be considered as a substitute for a translation invariance.

The first part of this definition, in fact, is not really restrictive: any map $\bar{T}$ from $X^{\mathcal{N}}$ into itself can be represented as $\bar{T} \equiv \bar{T} \circ (\otimes_{i \in \mathcal{N}} \mathrm{Id})$, where Id is an identical map. On the other hand, this representation certainly contradicts to the second part, which assumes that the "interaction" ($\bar{T}$ in this representation) cannot change identical elements. Note that this is one of major differences between multicomponent models considered in statistical physics, where the main object of interest is the evolution of systems of particles (and therefore the "local dynamics" is not defined at all, since the dynamics of individual particles without interactions with others is trivial), and in the dynamical systems theory.

A typical example of an admissible "interaction" is a space homogeneous finite range (depending on a finite number $2K$ of neighbors) coupling:

$$(\mathcal{I}_\varepsilon \bar{x})_i := (1 - \varepsilon)\bar{x}_i + \varepsilon \sum_{j=-K}^{K} a_j \bar{x}_{i+j}, \tag{4.1}$$

where the parameter $\varepsilon > 0$ describes the "interaction" strength and $a_i \geq 0$, $\sum_{i=-K}^{K} a_i = 1$ are constants defining the interaction. Observe that formula (4.1) describes a convex hall of values of coordinates of $\bar{x}$ in the K-neighborhood of the i-th coordinate.

One might argue that our definition of the multicomponent system does not cover the case when the interaction is defined in terms of original vectors $\bar{x}$ rather than in terms of their images under dynamics, i.e. $\mathcal{T}\bar{x}$. In particular, there is an ubiquity of important examples [12] of multicomponent systems obtained under space discretizations of partial differential equations, where the system has the following form:

$$(\bar{T}_\varepsilon \bar{x})_i = (1-\varepsilon)T_i\bar{x}_i + \varepsilon \sum_{j=-K}^{K} a_j \bar{x}_{i+j},$$

i.e. the interaction acts on $\bar{x}$ rather than on $\mathcal{T}\bar{x}$. Formally we cannot decompose this system into the superposition of the local dynamics described by the maps T_i and the interaction $\mathcal{I}_\varepsilon$. However we can construct an equivalent system, which satisfies our definition, by "doubling" of the local systems. In the new system for each local component (T_i, X_i) we consider its 'delayed' copy acting on the phase space $Y_i \equiv X_i$ so that the local dynamics of the i-th pair of coordinates is defined as $\tilde{T} : (\bar{x}_i, \bar{y}_i) \to (T_i\bar{x}_i, \bar{x}_i)$. The interaction map $\Phi(\bar{x}, \bar{y}) := (\bar{x}', \bar{y}')$ is defined as

$$\bar{x}_i' := (1-\varepsilon)\bar{x}_i + \varepsilon \sum_{j=-K}^{K} a_j \bar{y}_{i+j},$$

$$\bar{y}_i' := (1-\varepsilon)\bar{y}_i + \varepsilon \sum_{j=-K}^{K} a_j \bar{y}_{i+j}.$$

Thus the projection of the dynamics of the multicomponent system $(\Phi \circ \tilde{T}, \ \otimes (X_i \otimes Y_i))$ to its x-components coincides with the system $(\bar{T}_\varepsilon, \ \otimes X_i)$.

An important example of formula (4.1) is the so called diffusive coupling:

$$(\mathcal{I}_\varepsilon \bar{x})_i = (1-\varepsilon)\bar{x}_i + \frac{\varepsilon}{3}(\bar{x}_{i-1} + \bar{x}_i + \bar{x}_{i+1}),$$

i.e. the space homogeneous finite range coupling with $K = 1$ and $a_i \equiv 1/3$. This case corresponds to the discretization of Laplacian, indeed, we have:

$$-\varepsilon\bar{x}_i + \frac{\varepsilon}{3}(\bar{x}_{i-1} + \bar{x}_i + \bar{x}_{i+1}) = \frac{\varepsilon}{3}((\bar{x}_{i+1} - \bar{x}_i) - (\bar{x}_i - \bar{x}_{i-1})) = \frac{\varepsilon}{3}(\nabla\bar{x})_i.$$

Our definition of a natural measure is well defined for a multicomponent dynamical system if a number of components $|\mathcal{N}| < \infty$. In this case the triple $(X^{\mathcal{N}}, \rho^{\mathcal{N}}, m^{\mathcal{N}})$ is again a finite dimensional metric space with a certain reference measure and thus we can use all previous definitions. In the case of infinite dimension ($|\mathcal{N}| = \infty$) the problem is that any probabilistic measure absolutely continuous with respect to $m^{\mathcal{N}}$ should coincide with it, which does not give much freedom in the choice of initial measures. Therefore to be able to work with infinite dimensional multicomponent dynamical systems we need to modify the way how we choose initial measures.

Let $\mathcal{L} \subseteq \mathcal{N}$ be a subset of the set of indices $\mathcal{N}$. Denote by $\pi_{\mathcal{L}}^* : \mathcal{M}(X^{\mathcal{N}}) \to \mathcal{M}(X^{\mathcal{L}})$ – the projection operator in the space of probabilistic measures, defined as $\pi_{\mathcal{L}}^*\mu := \int \mu \ d(\otimes_{i\in(\mathcal{N}\setminus\mathcal{L})} m_i)$ for any measure $\mu \in \mathcal{M}(X^{\mathcal{N}})$. Since there is a natural enclosure of spaces $\mathcal{M}(X^{\mathcal{L}})$ into the space $\mathcal{M}(X^{\mathcal{N}})$ we can choose a family of metrics $\mathrm{dist} = \mathrm{dist}_{\mathcal{L}}$ acting on all considered spaces of measures such that for any measure $\mu \in \mathcal{M}(X^{\mathcal{N}})$ we have

$$\mathrm{dist}(\pi_{\mathcal{L}}^*\mu, \mu) \stackrel{|\mathcal{L}|\to|\mathcal{N}|}{\longrightarrow} 0. \tag{4.2}$$

We shall say that a measure $\mu \in \mathcal{M}(X^{\mathcal{N}})$ is *smooth* if its marginals $\mu_i := \pi_{\{i\}}\mu$ are absolutely continuous with respect to the reference measures m_i for any $i \in \mathcal{N}$ and define an infinite dimensional generalization of the natural measure μ_T as a common limit of Cesaro means $\frac{1}{n}\sum_{k=0}^{n-1}(\mathcal{T}^{(\mathcal{N})*})^n\mu$ of all smooth measures $\mu \in \mathcal{M}(X^{\mathcal{N}})$ having a support in a direct product $\otimes_{i\in\mathcal{N}}U_i$ of some open sets $U_i \subseteq X_i, \ i \in \mathcal{N}$. Here $\otimes_{i\in\mathcal{N}}U_i$ plays the role of the basin of attraction of the measure μ_T.

To define the notion of phase transition we consider a family of multicomponent systems $(T[\gamma], X)$ depending on a certain parameter γ. We shall say that this family has *the phase transition* at the point $\gamma = \gamma_0$ if the number of natural measures changes when the parameter γ crosses the value $\gamma = \gamma_0$. Observe that in the case of multicomponent systems this may happen in two different ways.

Assume that for $\gamma < \gamma_0$ each finite dimensional approximation $(\mathcal{T}^{(\mathcal{L})}[\gamma], X^{\mathcal{L}})$ have a finite number $N[\gamma]$ of natural measures $\mu_{\mathcal{L}}[\gamma]$ which does not depend on $\mathcal{L}$, and that any natural measure of the complete system $\mu_{\mathcal{N}}[\gamma]$ is a (weak) limit of measures $\mu_{\mathcal{L}}[\gamma]$. The main way how the phase transition may happen is that for all $\mathcal{L}$, such that $|\mathcal{L}|$ is large enough, each system $(\mathcal{T}^{(\mathcal{L})}[\gamma], X^{\mathcal{L}})$ goes through the phase transition as the value of γ crosses γ_0, i.e. the number $\mu_{\mathcal{L}}[\gamma]$ changes, and the same happen to the complete system. However, in the infinite dimensional case there is also another possibility: finite dimensional approximations do not demonstrate any phase transition, but their limit points either fail to correspond to the natural measures of $T^{(\mathcal{N})}$ at the parameter value γ_0, or a new natural measure of the complete system appears which does not belong to the set of limit points of natural measures of finite dimensional approximations.

With a slight abuse of notation we denote by $\mathcal{T}^{(\mathcal{L})}$ a $|\mathcal{L}|$-dimensional approximation of our infinite-dimensional map $\mathcal{T}^{(\mathcal{N})}$ for a given finite subset $\mathcal{L} \subset \mathcal{N}$ of the set of indices. To define this approximation explicitly we need to take into account boundary conditions, namely we have to choose the states on the remaining infinite-dimensional part of the phase space $X^{\mathcal{N} \setminus \mathcal{L}}$. Note that this can be done in various ways. Two of them are the most common ones: fixed boundary conditions, when the corresponding coordinates of the vector $(\bar{x})_j$ for $j \in (\mathcal{N} \setminus \mathcal{L})$ are preserved at given values, and periodic boundary conditions. The following simple result shows that the choice of boundary conditions can change even very rough characteristics of the dynamics.

Lemma 4.1 *Let $X_i = [0,1]$ and $T_i x = x + \frac{1}{6}x(x-1)^2$ for all $i \in \mathcal{N} = \mathbb{Z}^1$ and let $\mathcal{I}_\varepsilon$ be the diffusive interaction. For a finite subset of integers $\mathcal{L}$ denote by $\mathcal{I}_\varepsilon^{(\mathcal{L},y)}$ the finite dimensional approximation of the interaction with boundary conditions fixed at the value y for all coordinates not belonging to the set $\mathcal{L}$. Then for any $0 < \varepsilon < 1/2$ the multicomponent system $\mathcal{I}_\varepsilon^{(\mathcal{L},y)} T^{\mathcal{L}}, [0,1]^{\mathcal{L}})$ has the only one attractor (and thus the only one ergodic SRB measure) if $y = 0$ and has $2^{|\mathcal{L}|}$ attractors and ergodic SRB measures if the boundary condition $y = 1$ and $\varepsilon > 0$ is small enough.*

Proof. The map $T_i x = x + \frac{1}{6}x(x-1)^2$ has two fixed points: the stable fixed point at 0 and the neutrally unstable fixed point at 1. A straightforward calculation shows that in the case of zero boundary conditions $y = 0$ for any $\varepsilon > 0$ the fixed point 0 remains the only global attractor of the system, while in the case of $y = 1$ for any $\varepsilon > 0$ the fixed point at 1 becomes stable and if $\varepsilon < 1/2$ then the fixed point at 0 remains stable as well. □

The main question we shall be interested in this section is the possibility that for each finite subset of indices $\mathcal{L}$ (and a certain choice of boundary conditions) the corresponding finite dimensional approximation $\mathcal{T}^{(\mathcal{L})}$ has only one natural measure, while the entire system has several of them.

In the statistical physics literature there are numerous examples of multiparticle systems when this phenomenon takes place. One important class of such examples is the so called voter models in cellular automata theory. In this case each local component has only two states and no "local" dynamics: the behavior of the system is described in terms of the random "interaction", namely the future state of the local coordinate is determined by the present state of a certain number of its neighbors (including the local component itself) with a small random error. Therefore according to our definition this is not a multicomponent dynamical system. The presence of phase transition for these system has been shown by [13] under certain assumptions on the interaction. Later a deterministic version of this model has been considered in [8]. Observe that any voter model is a probabilistic cellular automaton. Therefore by making use of the argument from Section 3 one can immediately construct the corresponding deterministic one-dimensional Markov map describing this process. In fact, the construction in [8] basically follows this idea.

On the other hand, to the best of our knowledge, examples of multicomponent dynamical systems with phase transitions are not known and the only promising candidate for that is the so called case of "mean field" interaction, when each local subsystem interacts with all others (see, e.g. [10]). In what follows we shall give sufficient conditions under which phase transitions cannot occur.

Theorem 4.1 *Assume that for any finite subset $\mathcal{L} \subset \mathcal{N}$ of the set of indices there exists the only one natural measure $\mu_{\mathcal{L}}$ of the induced map $\mathcal{T}^{(\mathcal{L})^*} := \pi_{\mathcal{L}} \mathcal{T}^{(\mathcal{N})^*} \pi_{\mathcal{L}}$, and there exists a constant $C < \infty$ and two functions $\phi, \psi : \mathbb{R}^1 \to \mathbb{R}^1$ such that for any two smooth measures $\mu, \nu \in \mathcal{M}(X^{\mathcal{N}})$ and any two finite subsets $\mathcal{L} \subset \mathcal{L}' \subset \mathcal{N}$ we have*

$$\operatorname{dist}(\mathcal{T}^{(\mathcal{L})^*}\mu, \mathcal{T}^{(\mathcal{L})^*}\nu) \leq C \operatorname{dist}(\mu, \nu), \tag{4.3}$$

$$\operatorname{dist}(\mathcal{T}^{(\mathcal{L})^*}\mu, \pi^*_{\mathcal{L}}\mathcal{T}^{(\mathcal{L}')^*}\mu) \leq \psi(|\mathcal{L}|) \stackrel{|\mathcal{L}| \to \infty}{\longrightarrow} 0 \quad \textit{uniformly on } \mu, \tag{4.4}$$

$$\operatorname{dist}(\mathcal{T}^{(\mathcal{L})^{*^n}}\mu, \mu_{\mathcal{L}}) \leq \phi(n) \stackrel{n \to \infty}{\longrightarrow} 0 \quad \textit{uniformly on } \mu, \mathcal{L}. \tag{4.5}$$

Assume also that for any measure $\mu \in \mathcal{M}(X^{\mathcal{N}})$ we have

$$\operatorname{dist}(\mathcal{T}^{(\mathcal{L})^*}\mu, \mathcal{T}^{(\mathcal{N})^*}\mu) \stackrel{|\mathcal{L}| \to |\mathcal{N}|}{\longrightarrow} 0, \tag{4.6}$$

then the multicomponent dynamical system $\mathcal{T}^{(\mathcal{N})}$ also has the only one natural measure.

Proof. We will show that there is a weak limit of the sequence of natural measures $\mu_* := \mu_{\mathcal{L}}$ as $\mathcal{L} \to \mathcal{N}$ and this limit is the only one natural measure of the map $\mathcal{T}^{(\mathcal{N})}$, i.e. $\mu_* := \mu_{\mathcal{T}^{(\mathcal{N})}}$.

Consider a sequence of growing enclosed finite subsets $\mathcal{L} \subset \mathcal{N}$. For any two finite subsets $\mathcal{L} \subset \mathcal{L}' \subset \mathcal{N}$ from this sequence and any positive integer n by the triangle inequality we have

$\pi^*_{\mathcal{L}}\mathcal{T}^{(\mathcal{L}')^*} \circ \mathcal{T}^{(\mathcal{L})^{*^{n-1}}}\mu_{\mathcal{L}'})$

$+ \operatorname{dist}(\pi^*_{\mathcal{L}}\mathcal{T}^{(\mathcal{L}')^*} \circ \mathcal{T}^{(\mathcal{L})^{*^{n-1}}}\mu_{\mathcal{L}'},$

$\pi^*_{\mathcal{L}}\mathcal{T}^{(\mathcal{L}')^*} \circ \mathcal{T}^{(\mathcal{L}')^{*^{n-1}}}\mu_{\mathcal{L}'})$

$\leq \phi(n) + \psi(|\mathcal{L}|) + C \operatorname{dist}(\mathcal{T}^{(\mathcal{L})^{*^{n-1}}}\mu_{\mathcal{L}'}, \pi^*_{\mathcal{L}}(\mathcal{T}^{(\mathcal{L}')^{*^{n-1}}}\mu_{\mathcal{L}'})$

$\leq \ldots \leq \phi(n) + \frac{C^n - 1}{C - 1}\psi(|\mathcal{L}|). Therefore the sequence of unique natural measures \{\mu_{\mathcal{L}}\}$ is fundamental and thus there is a subsequence $\{\mathcal{L}_i\}_i$ such that $\mu_{\mathcal{L}_i} \stackrel{i \to \infty}{\longrightarrow} \mu_*$. By making use of the relation (4.6), we see that the limit measure μ_* is an invariant measure of the map $\mathcal{T}^{(\mathcal{N})}$. Thus it remains to prove that this measure is unique and natural.

Taking a limit $\mathcal{L}' = \mathcal{L}_i \to \mathcal{N}$ in the previous inequality we get

$$\operatorname{dist}(\mu_{\mathcal{L}}, \pi^*_{\mathcal{L}}\mu_*) \leq \phi(n) + \frac{C^n - 1}{C - 1}\psi(|\mathcal{L}|).$$

For a given smooth measure $\mu \in \mathcal{M}(X^{\mathcal{N}})$ using the same argument as above we get

$$\operatorname{dist}(\mathcal{T}^{(\mathcal{L})^{*^n}}\mu, \pi^*_{\mathcal{L}}\mathcal{T}^{(\mathcal{N})^{*^n}}\mu) \leq \text{ Const}\, \phi(n) + \text{ Const}^{\,n}\psi(|\mathcal{L}|),$$

which can be made to be arbitrary small by a proper choice of $n, |\mathcal{L}| \to \infty$. Now using the assumption (4.5) we get that the measure μ_* is indeed the natural measure for $T_{\mathcal{N}}$. □

Corollary 4.2 *Let a family of multicomponent systems $\mathcal{T}[\gamma]$ satisfy the assumptions of Theorem 4.1 uniformly in $\gamma \in [\gamma_1, \gamma_2]$. Then there are no phase transitions in the interval $[\gamma_1, \gamma_2]$.*

Observe that in the proof of this theorem we never used the (spatial) decomposition of the multicomponent system into the local components and interaction. Assume now that local components are identical.

Theorem 4.2 *Assume that $X_i \equiv X$, $T_i \equiv T$ for all $i \in \mathcal{N}$, and the map T is nonsingular w.r.t. the reference measure $m_i \equiv m$. Assume also that the interaction $\mathcal{I}$ is local, i.e. the value of $(\mathcal{I}\bar{x})_i$ depends only on a finite number of 'neighboring' components x_i of the vector $\bar{x}$. Then the statement of Theorem 4.1 remains valid if instead of the assumptions (4.3, 4.4) we shall use simpler assumptions:*

$$\operatorname{dist}(\mathcal{I}^*\mu, \mathcal{I}^*\nu) \leq \text{ Const } \operatorname{dist}(\mu, \nu) \tag{4.7}$$

$$\operatorname{dist}(\mathcal{I}^{(\mathcal{L})^*}\mu,\ \pi^*_{\mathcal{L}}\mathcal{I}^{(\mathcal{L}')^*}\mu \leq \psi(|\mathcal{L}|) \stackrel{|\mathcal{L}| \to \infty}{\longrightarrow} 0, \tag{4.8}$$

for any $\mathcal{L}' \supset \mathcal{L}$ and any $\mu, \nu \in \mathcal{M}(X^{\mathcal{N}})$, preserve the assumption (4.5) and drop (4.6).

Proof. Observe that $\mathcal{T}^{(\mathcal{L})^*} = \mathcal{I}^{(\mathcal{L})^*} T^{\mathcal{L}^*}$. Therefore, using (4.7), we get

$$\begin{aligned}
\operatorname{dist}(\mathcal{T}^{(\mathcal{L})^*}\mu, \mathcal{T}^{(\mathcal{L})^*}\nu) &\leq \operatorname{dist}(\mathcal{I}^{(\mathcal{L})^*}(T^{\mathcal{L}^*}\mu),\ \mathcal{I}^{(\mathcal{L})^*}(T^{\mathcal{L}^*}\nu)) \\
&\leq \text{Const dist}(T^{\mathcal{L}^*}\mu,\ T^{\mathcal{L}^*}\nu) \leq \text{Const}' \cdot \operatorname{dist}(\mu, \nu).
\end{aligned}$$

A similar argument shows that the inequality (4.4) is satisfied if (4.8) holds:

$$\begin{aligned}
\operatorname{dist}(\mathcal{T}^{(\mathcal{L})^*}\mu, \pi_{\mathcal{L}}^*\mathcal{T}^{(\mathcal{L}')^*}\mu) &\leq \operatorname{dist}(\mathcal{I}^{(\mathcal{L})^*}(T^{\mathcal{L}^*}\mu),\ \pi_{\mathcal{L}}^*\mathcal{I}^{(\mathcal{L}')^*}(T^{\mathcal{L}'^*}\mu)) \\
&\leq \psi(|\mathcal{L}|) \overset{|\mathcal{L}|\to\infty}{\longrightarrow} 0.
\end{aligned}$$

It remains to show that the inequality (4.6) is satisfied automatically under our assumptions. It again follows from the decomposition into local maps and interaction by making use of (4.2) and (4.8) that

$$\begin{aligned}
\operatorname{dist}(\mathcal{T}^{(\mathcal{L})^*}\mu,\ \mathcal{T}^{(\mathcal{N})^*}\mu) &= \operatorname{dist}(\mathcal{I}^{(\mathcal{L})^*}(T^{\mathcal{L}^*}\mu),\ \pi_{\mathcal{L}}^*\mathcal{I}^{(\mathcal{N})^*}(T^{\mathcal{N}^*}\mu)) \\
&\leq \text{Const dist}(T^{\mathcal{L}^*}\mu, T^{\mathcal{N}^*}\mu) \overset{|\mathcal{L}|\to|\mathcal{N}|}{\longrightarrow} 0.
\end{aligned}$$

□

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