NATIONAL ACADEMY OF SCIENCES OF BELARUS
STATE COMMITTEE ON SCIENCE AND TECHNOLOGIES OF REPUBLIC OF BELARUS
ASSOTIATION «NOMATECH»
INSTITUTE OF RESOURCE SAVING PROBLEMS
V.A.BELYI METAL-POLYMER RESEARCH INSTITUTE OF NASB

MATERIALS. TECHNOLOGIES. TOOLS
1999, #4, pp. 5-13

MECHANICS OF STRESSEDLY-DEFORMED STATE

Exact analytical solution for 1D elastic homogeneous finite lumped line vibration

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This paper was published first in the journal of National Academy of Sciences of Belarus "Materials. Technologies. Tools", **4** (1999), 4, pp.5-13 (in Russian). This is the second of a large cycle devoted to the new method to obtain the exact analytical solutions for vibrant elastic systems, both lumped and distributed. Though the material stated in it is quite simple, as well as the investigated problems, you can make sure from the presented solutions that in the exact analytical form and in all their completeness these solutions were not obtained up to now. It is important that we give the checks here that these solutions are exact and complete, and the connection between the lumped and distributed systems as well. Of a great importance is that in vibration systems we can pass from the lumped to distributed systems but in no case vice versa, as is conventional and in which the investigators are absolutely sure. Also important is that these results are basic for the further investigations which we carry out for much more complicated models. There we often refer to this introducing paper, so we decided to load it also to the Archive, to make its English version available for the English-language readers. You can read some of the further papers, visiting for free our electronic journal "SELF Transaction" at the URL

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We will be thankful if you express us your opinion and the more grateful for your wish to go on with us in the investigation of this huge and prospective theme.

Abstract

We will analyse in this paper the principal demerits of conventional approaches to the problem of vibrant 1D homogeneous finite lumped line and present the exact analytical solutions for forced and free vibrations in finite lines with free ends and with the free end and fixed start. We will analyse these solutions and their distinctions from the conventional concept on the vibration pattern in such lines. We will give the check of presented solutions proving them to be complete and exact

Keywords: Mathematical physics, Wave physics, Dynamics, Finite elastic lumped lines, ODE systems, Microwave vibrations in elastic lines

Classification by MSC 2000: 30E25; 70E55; 70J35; 70J60; 70K40; 70F40

Classification by PASC 2001: 02.60.Lj; 05.10.-a; 05.45.-a; 45.30.+s; 46.15.-x; 46.25.Cc; 46.40.-f; 46.40.Fr

1. Introduction

In [1] we investigated the exact analytical solutions for infinite 1D elastic lumped lines and showed that on distinct of conventional solutions, the exact analytical solutions have generally three vibration regimes: periodical, aperiodical and critical. With it the vibration pattern essentially changes dependently on the line type (semi-finite or infinite) and on the vibration conditions, i.e., are they forced or free. Generally, finite lines have the same distinctions, but also some others are inherent in them. In this paper we will study these features.

2. The drawbacks of conventional methods

Two basic approaches are currently applied to find exact solutions for vibrations in 1D finite elastic lumped lines. They are the allowed modes method (see, e.g., [2]) and Krylov method (see, e.g., [3], [4] or [5]). To see their drawbacks, analyse each briefly.

In the allowed modes method the standard system of equations

$$\frac{d^2 y_r}{\partial t^2} = \frac{T}{ma}\left(y_{r+1} + y_{r-1} - 2y_r\right) \tag{1}$$

is taken as the basis. We will study it after Pain [2] relatively to the transversal vibrations (see Fig. 1); the essence of this technique for those longitudinal is the same. The solution of this system of equations is sought in the following form:

$$y_r = A_r e^{j\omega t} \tag{2}$$

with the following boundary conditions for a fixed-ends finite line:

$$A_0 = A_{n+1} = 0\,; \qquad r = 0,1,2,\ldots(n+1)\ . \tag{3}$$

Substituting y_r to the initial system (1), we yield a system of algebraic equations of a following type:

$$A_{r+1} - \left(2 - \frac{ma\omega^2}{T}\right)A_r + A_{r-1} = 0\ , \tag{4}$$

which can be naturally presented as the system of ratios constant for given line parameters and frequency ω:

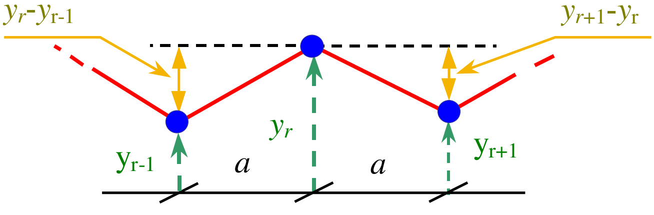

Fig. 1. Conventional model for transversal vibrations calculation in an infinite lumped line

6

$$\frac{A_{r-1} + A_{r+1}}{A_r} = \frac{2\omega_0^2 - \omega^2}{\omega_0^2}\ , \tag{5}$$

where

$$\omega_0 = \sqrt{\frac{T}{ma}} \ .$$

Then one assumes that A_r can be presented as a product of some constant C into the sine of some angle Θ_p constant for the given value ω_p , i.e., for some allowed circular frequency of possible vibrations

$$A_r = C \sin \Theta_p \ , \tag{6}$$

where $C = const$ and $\Theta_p = f\left(\omega_p\right)$.

In its turn, one determines the parameter Θ_p from the boundary conditions (3), whence

$$\Theta_p = \frac{p\pi}{n+1}; \qquad p = 1,2,3,\ldots,n \ . \tag{7}$$

Finally, proceeding from the obtained value Θ_p, one finds the resolved circular frequencies ω_p, substituting (6) and (7) into (5):

$$\omega_p^2 = 2\omega_0^2\left(1-\cos\frac{p\pi}{n+1}\right); \qquad \omega_{\lim} = 2\omega_0 \ . \tag{8}$$

One can immediately see from this brief analysis that the method of resolved modes can give the exact solutions only in case of free vibrations and only for a fixed-ends-line. If the forced vibrations were present in the line, the modelling system of differential equations cannot be reduced to the system of algebraic equations (4), because at least in one equation of this system there will be present the external force parameters breaking the uniformity of (4). And if at least one of the line ends is unfixed, then, as we will show below, we cannot indicate a priori the vibration amplitude value at the line ends, because in lumped lines, on distinct from the known solutions for those distributed, the vibration maximum will not take place at the free ends. Thus, as we see, the limitations inherent in the allowed modes method are quite essential, so we cannot think the problem of elastic lumped lines to be completely solved by this approach.

The second approach to this problem solution is Krylov method grounded on the matrix theory tool which bases on the energy conservation for the connected-bodies-system free vibrations that leads one to the system of differential equations similar to (1):

$$\sum_{k=1}^{N}\left(a_{jk}\ddot{q}_k + c_{jk}q_k\right) = 0 \ , \tag{9}$$

where a_{jk} , c_{jk} are the constant parameters characterising the investigated system of bodies; q_k is the k th body location in the generalised co-ordinates; N is the bodies quantity in the investigated system.

Further, assuming that

$$q_k = A_k \sin\left(\omega t + \varepsilon\right) , \tag{10}$$

where A_k , ε are some invariable values characterising the vibrations of a bodies structure, the system of equations (9) transforms to the determinant of a following type:

$$\left\| c_{jk} - a_{jk}\omega^2 \right\| = 0 \ . \tag{11}$$

In the view of seeking the exact solutions, this method has the following complication. The characteristic equation (11) is an algebraic equation of $2N$ power with respect to ω; hence, it has $2N$ roots, i.e. N eigenvalues ω_k $\left(k = 1,2\ldots,N\right)$. With it, as is known, one can obtain the exact solution of an algebraic equation only up to $N = 4$ (in our case, the bi-N algebraic equation). Thus, the matrix approach to solve this problem for free vibrations is much limited by a small number of connected bodies. In case of forced vibrations, Krylov method is based on solving the problem for free vibrations, so the solution is sought by variation of the constant. With it one naturally obtains the line spectrum of forced vibrations.

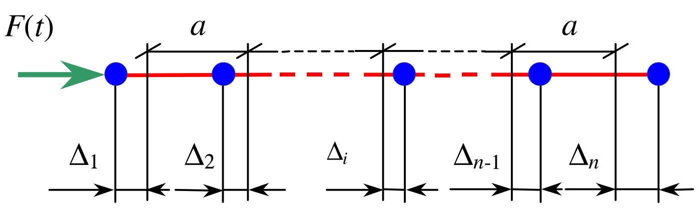

Fig. 2. Mechanical model of a finite lumped line having the start and end unfixed

At the same time, as we will show below, in finite lines the forced vibrations spectrum has a continued pattern with the infinite resonances at the frequencies corresponding to the bodies structure natural vibrations. Furthermore, in a finite structure of connected bodies the forced vibrations have three regimes that cannot be determined on the basis of natural frequencies.

3. Analysis of exact complete solutions and checking the results obtained for finite lines

Above we indicated the substantial limitations inherent in existing approaches. We can easy overcome them in frames of method presented as the basic in [1], when obtaining the complete exact solutions for infinite elastic lumped lines. To show it, consider first the solutions obtained with this method for a finite unfixed-ends line, i.e., namely for the line whose exact solution one cannot obtain in general case, when the number of elements is large, neither by the resolved modes method nor by Krylov method.

7

Forced vibrations of a line having the start and end unfixed (Fig. 2). The following system of differential equations

$$\begin{cases} m\dfrac{d^2\Delta_1}{dt^2} = F(t) + s(\Delta_2 - \Delta_1) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_i}{dt^2} = s(\Delta_{i+1} + \Delta_{i-1} - 2\Delta_i) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_n}{dt^2} = s(\Delta_{n-1} - \Delta_n) \end{cases} \tag{12}$$

corresponds to this model. The same as for infinite lines, (12) has in general case three basically different solutions depending on the ratio of β to 1, where

$$\beta = \sqrt{\frac{\omega^2 m}{4s}}\ .$$

For finite lines the *periodic* solution corresponding to $\beta < 1$ has the form

$$\Delta_i = -\frac{F_0 \cos\left[2(n-i)+1\right]\tau}{\omega\sqrt{ms}\sin 2n\tau} e^{j\omega t}\ , \tag{13}$$

where $\tau = \arcsin\beta$.

This solution describes standing waves, which is determined by the constant phase delay of vibrations for all line elements. But unlike the conventional solutions (6), the exact solution structure has basically other form. With

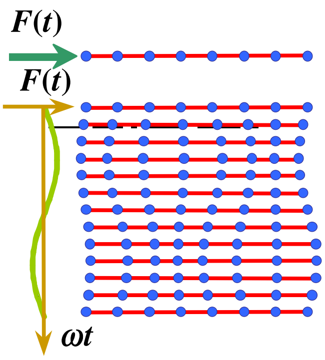

Fig. 3 The typical vibrations in the periodical regime in an ideal finite homogeneous elastic lumped line with the unfixed ends under the external longitudinal harmonic force acting on the first line element. Parameters of the line: m = 0,01kg; s = 100N/m; a = 0,01m; n = 8; F_0 = 0,6N; f = 15 Hz

$$2n\tau = p\pi \tag{14}$$

an infinite resonance will take place in the line. But in other cases, the vibration amplitude is finite and determined by the value $\sin 2n\tau$. With

$$2n\tau = \frac{\pi}{2} + p\pi \tag{15}$$

it will be minimal but not zero, as we used to think for resonant lines. The typical vibration pattern is shown in Fig. 3. But the main that follows from the obtained solutions is that line vibrations are not limited by the critical frequency.

In the *aperiodical* regime corresponding to $\beta > 1$ the exact solution takes the form

$$\Delta_i = (-1)^i \frac{F_0\left[\gamma_+^{2(n-i)+1} - \gamma_-^{2(n-i)+1}\right]}{\omega\sqrt{ms}\left[\gamma_+^{2n} - \gamma_-^{2n}\right]} e^{j\omega t} \ , \tag{16}$$

where

$\gamma_- = \beta - \sqrt{\beta^2 - 1}$ and $\gamma_+ = \beta + \sqrt{\beta^2 - 1}$.

The same as in case of infinite lines in [1], we can see the anti-phase vibrations, but damping not so fast as in infinite lines. In infinite lines the damping distributes along the entire line. In [1], when analysing free vibrations, we mentioned the resemblance between vibrations for finite and infinite lines and that in the aperiodical case the damping begins from the point of energy input into a vibrant system. In Fig. 4 one can see just this phenomenon, when the damping passes from the first to the last element, distributing uniformly along the entire line, which confirms the analogy indicated in [1].

For the *critical* regime corresponding to $\beta = 1$, the changes are also interesting:

$$\Delta_i = (-1)^i \frac{F_0\left[2(n-i)+1\right]}{4sn} e^{j\omega t} \ . \tag{17}$$

In a finite line these vibrations have a damping pattern, not remaining their amplitude, as in infinite lines. But unlike the aperiodical regime, the damping has not a power-type but linear pattern. We can see these features in Fig. 4.

8

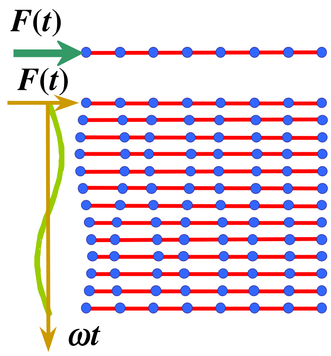

Fig. 4 The typical vibrations in the critical regime in an ideal finite homogeneous elastic lumped line with the unfixed ends under the external longitudinal harmonic force acting on the first line element. Parameters of the line: m = 0,01kg; s = 100N/m; a = 0,01m; n = 8; F_0 = 0,6N; $f = f_{crit}$ = 31,8 Hz

The presented solutions emphasise again a vast diversity of vibration forms and patterns dependently on the line type and parameters.

Check the correspondence of presented solutions (13), (16) and (17) to (12). It will be sufficient to substitute them sequentially into the first, arbitrary i th and the last equations of this system.

For the *periodic* regime on the left part of the first equation of (12) we yield

$$m\frac{d^2\Delta_1}{dt^2}=2\beta\frac{\cos(2n-1)\tau}{\sin 2n\tau}F_0\,e^{j\omega t}\ , \tag{18}$$

and on the right part

$$F(t)+s(\Delta_2-\Delta_1)=F_0\,e^{j\omega t}\left\{1-\frac{\left[\cos(2n-3)\tau-\cos(2n-1)\tau\right]}{2\beta\sin 2n\tau}\right\}=$$

$$=F_0\,e^{j\omega t}\left\{1-\frac{\sin(2n-2)\tau}{\sin 2n\tau}\right\}=2\beta\frac{\cos(2n-1)\tau}{\sin 2n\tau}F_0\,e^{j\omega t}\ . \tag{19}$$

For the left part of the i th equation

$$m\frac{d^2\Delta_i}{dt^2}=2\beta\frac{\cos\left[2(n-i)+1\right]\tau}{\sin 2n\tau}F_0\,e^{j\omega t}\ , \tag{20}$$

and for the right part

$$s(\Delta_{i+1}+\Delta_{i-1}-2\Delta_i)=\frac{F_0\,e^{j\omega t}}{2\beta\sin 2n\tau}\left\{\left[2(n-i)-1\right]\tau+\cos\left[2(n-i)+3\right]-\right.$$

$$\left.-2\cos\left[2(n-i)+1\right]\tau\right\}==-\frac{F_0\,e^{j\omega t}}{\beta\sin 2n\tau}\left\{\cos\left[2(n-i)+1\right]\tau\cdot\cos 2\tau-\right.$$

$$\left.-\cos\left[2(n-i)+1\right]\tau\right\}=2\beta\frac{\cos\left[2(n-i)+1\right]\tau}{\sin 2n\tau}F_0\,e^{j\omega t}\ . \tag{21}$$

For the left part of the last equation of the system

$$m\frac{d^2\Delta_n}{dt^2}=2\beta\frac{\cos\tau}{\sin 2n\tau}F_0\,e^{j\omega t}\ , \tag{22}$$

and for the right part

$$s\left(\Delta_{n-1}-\Delta_n\right)=-\frac{F_0\,e^{j\omega t}}{2\beta\sin 2n\tau}\left\{\cos 3\tau-\cos\tau\right\}=$$

$$=\frac{2F_0\,e^{j\omega t}}{\beta\sin 2n\tau}\cos\tau\cdot\sin^2\tau=2\beta\frac{\cos\tau}{\sin 2n\tau}F_0\,e^{j\omega t}\ . \tag{23}$$

Thus, the periodic solution satisfies (12) completely.

For the *aperiodical* regime it is sufficient to check, whether the solutions satisfy any equation of the system, because, as we showed in [1], this regime, the same as that critical, is one of the forms of the common solution, and if the periodical solution satisfied the system, then the aperiodical and critical ones satisfy too. So make a control check – for example, for the first, the most complicated equation of the system.

For its left part we yield

$$m\frac{d^2\Delta_1}{dt^2}=2\beta\frac{\left[\gamma_+^{(2n-1)}-\gamma_-^{(2n-1)}\right]}{\left[\gamma_+^{2n}-\gamma_-^{2n}\right]}F_0\,e^{j\omega t}\ ; \tag{24}$$

and for the right one

$$F\left(t\right)+s\left(\Delta_2-\Delta_1\right)=F_0\,e^{j\omega t}\left\{1-\frac{\left[\gamma_+^{(2n-3)}-\gamma_-^{(2n-3)}+\gamma_+^{(2n-1)}-\gamma_-^{(2n-1)}\right]}{2\beta\left[\gamma_+^{2n}-\gamma_-^{2n}\right]}\right\}=$$

$$=F_0\,e^{j\omega t}\left[1-\frac{\gamma_+^{2n-3}\left(\gamma_+^2+1\right)-\gamma_-^{2n-3}\left(\gamma_-^2+1\right)}{2\beta\left(\gamma_+^{2n}-\gamma_-^{2n}\right)}\right]=$$

$$=\frac{F_0\,e^{j\omega t}}{\left[\gamma_+^{2n}-\gamma_-^{2n}\right]}\left\{\gamma_+^{2(n-1)}\left(\gamma_+^2+1\right)-\gamma_-^{2(n-1)}\left(\gamma_-^2+1\right)\right\}=2\beta\frac{\left[\gamma_+^{(2n-1)}-\gamma_-^{(2n-1)}\right]}{\left[\gamma_+^{2n}-\gamma_-^{2n}\right]}F_0\,e^{j\omega t}\ . \tag{25}$$

Similarly, on the left part of the first equation for *critical* regime

$$m\frac{d^2\Delta_1}{dt^2}=\frac{\left(2n-1\right)}{n}F_0\,e^{j\omega t} \tag{26}$$

and on the right part

$$F\left(t\right)+s\left(\Delta_2-\Delta_1\right)=F\left(t\right)e^{j\omega t}\left\{1+\frac{\left(2n-3\right)+\left(2n-1\right)}{4n}\right\}=\frac{\left(2n-1\right)}{n}F\left(t\right)e^{j\omega t}\ . \tag{27}$$

As we can see, if the solutions were complete and exact, they satisfy as the general equation of the system as the starting and finishing equations as the equation containing the external force. With it, we can see from the equations reduction pattern that the way to reduce is different in each case, because in the solutions themselves their own specific multipliers corresponding to each equation of the modelling system are present; in this way they provide the identity in each case. This is one more interesting property of the obtained solutions, whence it follows that one and the same solution satisfies few types of differential equations at the same time. This was the main complication and main reason of failures when many generations of scientists sought the exact solutions.

9

Note one more property of the obtained solutions. If in a finite line in Fig. 2 we retain only one element, i.e. $n=1$, then $i=1$ and the solution of (13) will transform to the form

$$\Delta_1=-\frac{2\beta\cos\tau}{\omega^2 m\sin 2\tau}F_0\,e^{j\omega t}=-\frac{F_0}{\omega^2 m}e^{j\omega t}\ . \tag{28}$$

It means, when reducing the number of line elements to 1, the solution naturally transforms to the expression describing the motion of a solitary element without constraints. Further we will multiply see the solutions property to transform from more complicated models to those simple, which is an important advantage of the exact solutions.

Free vibrations of a line with unfixed start and end. As we know, the homogeneous system of differential equations without external force $F(t)$

$$\begin{cases} m\dfrac{d^2\Delta_1}{dt^2} = s(\Delta_2 - \Delta_1) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_i}{dt^2} = s(\Delta_{i+1} + \Delta_{i-1} - 2\Delta_i) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_n}{dt^2} = s(\Delta_{n-1} - \Delta_n) \end{cases} \quad (29)$$

corresponds to the elastic line free vibrations.

On distinct from the forced vibrations, in such system the vibrations can arise not in the entire range of frequencies but only in its definite part, and their spectrum will be discrete. For the investigated unfixed-ends model, the conditions of vibrations are determined by the expression

$$\sin 2n\tau_p = 0 \, , \quad (30)$$

where $\tau_p = \dfrac{p\pi}{2n} = \arcsin\beta = \arcsin\sqrt{\dfrac{\omega^2 m}{4s}}$.

Proceeding from (30), the natural (allowed) frequencies are determined by the expression

$$\omega_p = \sqrt{\frac{4s}{m}}\sin\frac{p\pi}{2n}; \quad p = 1,2,3,...,(n-1) \, . \quad (31)$$

In accord with the allowed frequencies (31), the solution of (29) has the form

$$\Delta_i = \frac{A\cos\left[2(n-i)+1\right]\tau_p}{\cos(2n-1)\tau_p} e^{j\omega t} \, . \quad (32)$$

As we see, the structure of the amplitude expression (32) basically differs from such in (6). In (32) the amplitude is presented by the ratio of trigonometric functions with a complex system of angular shifts, while (6) being the basis of the conventional solution had the form of a trivial trigonometric function. Such structure is typical for all lines where the reflection processes take place, even in a semi-infinite line presented in [1], because it is just the result of taking into account the process of multiple reflections from the boundaries. Due to this, the vibration amplitude becomes dependent on the line length and on the number of an allowed mode, not only on the element number as in (6); the denominator of (32) determines it.

To determine the stated dependence pattern, conveniently transform the terms of (32) characterising the vibration amplitude, noting (30). We yield

$$\frac{A\cos\left[2(n-i)+1\right]\tau_p}{\cos(2n-1)\tau_p} = \frac{A\cos\left[(2i-1)p\pi/2n\right]}{\cos\left[p\pi/2n\right]} \, . \quad (33)$$

The expression (33) shows that with the increasing number of the mode, the denominator monotonously decreases but does not vanish, since

$$p_{\max} = (n-1) < n \, . \quad (34)$$

In consequence of it, the amplitude grows monotonously but does not reach the infinite value at the maximally possible mode. With the growing number of elements, this amplitude maximum increases too and turns into infinity only when transiting to a distributed line.

Thus, we see that with free vibrations, only finite vibrations can arise in lumped lines, though the values of natural frequencies coincide with the resonance conditions (14) for forced vibrations. Practically it means the known fact that if some solitary pulse has acted upon a finite system, then finite vibrations arise in it, and if this pulse repeats with resonance frequency, it will lead to the line destruction.

Besides, as it follows from the carried out analysis, if in the regime of free vibrations the amplitude monotonously changed with the mode number, then with the forced vibrations in the band of periodical regime, the amplitude can multiply reach the maximum and minimum dependently on the length and parameters of a line. This difference has the following reason. The forced vibration spectrum has a continuous pattern, while in free vibrations the line spectrum is inherent; this causes the monotonous discrete sampling standing out of the continuous periodical regularity.

In the light of revealed essential distinctions between the vibration types, we should note that the solution (32) is, nevertheless, similar to (13) in its block structure.

10

Thus we see that the exact analytical solutions (32) essentially differ from the conventional solutions.

As before, we can easy check that the expressions (32) really are the solutions, substituting them immediately into the first, ith and last equations of the modelling system (29). For the left part of the first equation

$$m\frac{d^2\Delta_1}{dt^2} = -\omega^2 mA\, e^{j\omega t} \tag{35}$$

and for that right

$$s\left(\Delta_2 - \Delta_1\right) = sA\, e^{j\omega t}\left[\frac{\cos(2n-3)\tau_p - \cos(2n-1)\tau_p}{\cos(2n-1)\tau_p}\right] =$$

$$= 2\beta sA\, e^{j\omega t}\left[\frac{\sin(2n-2)\tau_p}{\sin(2n-1)\tau_p}\right] = 2\beta sA\, e^{j\omega t}\left[\frac{\sin 2n\tau_p \cos 2\tau_p - \cos 2n\tau_p \sin 2\tau_p}{\cos 2n\tau_p \cos\tau_p + \sin 2n\tau_p \sin\tau_p}\right] =$$

Because at the condition (30) $\cos 2n\tau_p = \pm 1\,;\, \sin 2n\tau_p = 0$, consequently,

$$= -4\beta sA \sin\tau_p\, e^{j\omega t} = -\omega^2 mA\, e^{j\omega t}\ . \tag{36}$$

For the left part of the ith equation

$$m\frac{d^2\Delta_i}{dt^2} = -\omega^2 mA\frac{\cos\left[2(n-i)+1\right]\tau_p}{\cos(2n-1)\tau_p}\, e^{j\omega t}\ , \tag{37}$$

and for that right

$$s\left(\Delta_{i+1} + \Delta_{i-1} - 2\Delta_i\right) =$$

$$= \frac{sA\, e^{j\omega t}}{\cos(2n-1)\tau_p}\left\{\cos\left[2(n-i)-1\right]\tau_p + \cos\left[2(n-i)+3\right]\tau_p - 2\cos\left[2(n-i)+1\right]\tau_p\right. =$$

$$= \frac{2sA\, e^{j\omega t}\cos\left[2(n-i)+1\right]\tau_p}{\cos\left[2n-1\right]\tau_p}\left[\cos 2\tau_p - 1\right] = -\,\omega^2 mA\frac{\cos\left[2(n-i)+1\right]\tau_p}{\cos(2n-1)\tau_p}\, e^{j\omega t}\ . \tag{38}$$

For the left part of the last equation

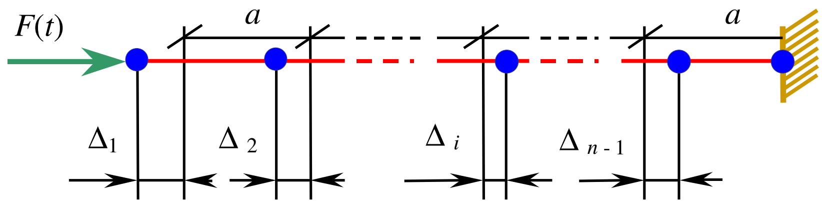

Fig. 5. Mechanical model of a finite lumped line having the unfixed start and rigidly fixed end

$$\frac{d^2\Delta_n}{dt^2} = -\frac{\omega^2 mA\cos\tau_p}{\cos(2n-1)\tau_p}e^{j\omega t} \tag{39}$$

and for the right one

$$s(\Delta_{n-1}-\Delta_n) = \frac{sA\,e^{j\omega t}}{\cos(2n-1)\tau_p}(\cos 3\tau_p - \cos\tau_p) =$$

$$= -\frac{sA\,e^{j\omega t}}{\cos(2n-1)\tau_p}\sin 2\tau_p \sin\tau_p = -\frac{\omega^2 mA\cos\tau_p}{\cos(2n-1)\tau_p}e^{j\omega t}\ . \tag{40}$$

Proceeding from the described above and considering the theorem of uniqueness of a solution of differential equation, we can surely say that the presented solutions are exact, which substantially changes the concept of the vibration processes pattern in finite lines. The same as in infinite lines considered in [1], the solutions transform in correspondence with the line features, but remaining the solution block structure. We can make it sure, considering a finite line having an unfixed start and fixed end.

The forced and free vibrations of an elastic line having unfixed start and fixed end under an external harmonic force $F(t)$ action is presented in Fig. 5. The following system of differential equations

$$\begin{cases} m\dfrac{d^2\Delta_1}{dt^2} = F(t) + s(\Delta_2 - \Delta_1) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_i}{dt^2} = s(\Delta_{i+1} + \Delta_{i-1} - 2\Delta_i) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_{n-1}}{dt^2} = s(\Delta_{n-2} - 2\Delta_{n-1}) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_n}{dt^2} = 0 \end{cases} \tag{41}$$

corresponds to this model. Three solutions correspond to this system:
for the periodical regime at $\beta < 1$

$$\Delta_i = -\frac{F_0 \sin 2(n-i)\tau}{\omega\sqrt{ms}\cos(2n-1)\tau}e^{j\omega t}\ . \tag{42}$$

The vibration diagram for the periodical regime is presented in Fig. 6. For the aperiodical regime at $\beta > 1$

11

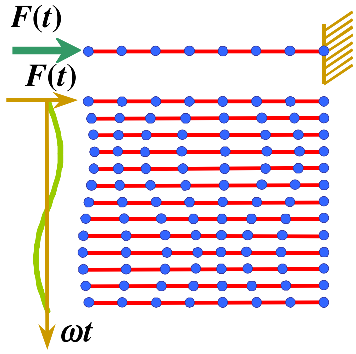

Fig. 6. The typical vibrations in the periodical regime in an ideal finite homogeneous elastic lumped line with the rigidly fixed end under the external longitudinal harmonic force acting on the first line element. Parameters of the line: m = 0,01 kg; s = 100 N/m; a = 0,01 m; n = 8; F_0 = 0,3 N; f = 15 Hz

$$\Delta_i = (-1)^i \frac{F_0\left[\gamma_+^{2(n-1)} - \gamma_-^{2(n-1)}\right]}{\omega\sqrt{ms}\left[\gamma_+^{2n-1} - \gamma_-^{2n-1}\right]} e^{j\omega t} \ . \tag{43}$$

Finally, for that critical at $\beta = 1$ the solution has the following form:

$$\Delta_i = (-1)^i \frac{F_0(n-i)}{s(2n-1)} e^{j\omega t} \ . \tag{44}$$

The system of equations describing the free vibrations in an investigated line differs from (41) by the force absence in the first equation; it has the form

$$\begin{cases} m\dfrac{d^2\Delta_1}{dt^2} = s(\Delta_2 - \Delta_1) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_i}{dt^2} = s(\Delta_{i+1} + \Delta_{i-1} - 2\Delta_i) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_{n-1}}{dt^2} = s(\Delta_{n-2} - 2\Delta_{n-1}) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_n}{dt^2} = 0 \ . \end{cases} \tag{45}$$

The given vibrations arise in an elastic system at the condition

$\cos(2n-1)\tau_p = 0$, (46)

where $\tau_p = \dfrac{(2p-1)\pi}{2(2n-1)} = \arcsin\beta = \arcsin\sqrt{\dfrac{\omega^2 m}{4s}}$.

Proceeding from (46), the natural frequencies are determined by the expression

$$\omega_p = \sqrt{\frac{4s}{m}} \sin\left[\frac{(2p-1)\pi}{2(2n-1)}\right]; \quad p = 1, 2, 3, \dots (n-1) \ , \tag{47}$$

and the equation of vibration has the form

$$\Delta_i = \frac{A\sin 2(n-i)\tau_p}{\sin 2(n-1)\tau_p} e^{j\omega t} . \tag{48}$$

In comparison with the line having an unfixed end, in the presented solutions the substantial changes really have taken place, though the block structure in the form of ratio of trigonometric functions has retained.

For the *forced* vibrations, the changes concern to the resonance frequency, because the conditions of its rise gained the form

$$(2n-1)\tau = \frac{\pi}{2} + p\pi . \tag{49}$$

The same for the conditions determining the amplitude minimum:

$$(2n-1)\tau = p\pi . \tag{50}$$

It follows from the conditions (49) and (50) that due to fixing one of the line ends, in the amplitude dependence with respect to τ the maximums and minimums shift took place, and not by a quarter of a period, as one could expect, but some more, because on the left parts of these expressions we see the value $(2n-1)\tau$, not $2n\tau$, as it was in (14) and (15). This difference disappears as the number of line elements growing and as the gradual transition to the distributed line. When reverse transition, this difference is non-restored.

For *free* vibrations, the amplitude dependence on the mode number is determined by the conditions which we can set up after the following transformation of the amplitude part of solution (48), noting (46):

$$\frac{A\sin\left[2(n-i)\right]\tau_p}{\sin 2(n-1)\tau_p} = \\ = \frac{A\sin\left[\frac{(2p-1)\pi}{2} - \frac{(2i-1)(2p-1)\pi}{2(2n-1)}\right]}{\sin\left[\frac{(2p-1)\pi}{2} - \frac{(2p-1)\pi}{2(2n-1)}\right]} = \frac{A\cos\left[\frac{(2i-1)(2p-1)\pi}{2(2n-1)}\right]}{\cos\left[\frac{(2p-1)\pi}{2(2n-1)}\right]} . \tag{51}$$

The obtained expression (51) shows that as the number of mode p increases the denominator of the expression decreases similarly to (33). It means, the amplitude of free vibrations, the same as in a free-end line, increases monotonously with the increasing mode number and reaches the maximum at $p_{\max}$, but for a lumped line it never reaches an infinite value.

To complete this analysis, make certain that the presented expressions are the solutions. It is simple to do so by their direct substitution to any related equation of the modelling systems, by analogy with the above models. For example, substitute (48) to the last but one equation of (45). We yield:

12

for the left part

$$m\frac{d^2\Delta_{n-1}}{dt^2} = -\frac{\omega^2 mA\sin 2\tau_p}{\sin 2(n-1)\tau_p} e^{j\omega t} , \tag{52}$$

and for the right one

$$s\left(\Delta_{n-2} - 2\Delta_{n-1}\right) = \frac{sA\, e^{j\omega t}}{\sin 2(n-1)\tau_p}\left(\sin 4\tau_p - 2\sin 2\tau_p\right) =$$

$$= \frac{2sA\sin 2\tau_p \, e^{j\omega t}}{\sin 2(n-1)\tau_p}\left(\cos 2\tau_p - 1\right) = -\frac{\omega^2 mAdin2\tau_p}{\sin 2(n-1)\tau_p} e^{j\omega t} . \qquad (53)$$

As wee see, (48) quite simply satisfies the incomplete differential equation of (45) describing a finite fixed-end line, the same as (32) satisfied the differential equation of the system (29) for a finite unfixed-end line.

4. Completeness of the obtained solutions

In the present paper, the same as in [1], we gave the solutions for two types of finite lines that cover the entire range of frequencies and describe both forced and free vibrations. All given solutions are exact and satisfying the related systems of differential equations. This corroborates the completeness and exactness of the presented solutions, as well as the method ability to investigate vibration processes not only for infinite but also for finite homogeneous elastic lines.

5. Validity to use a distributed line as an analogue

Analysing the exact analytical solutions for two presented finite lines, we practically did not consider the pattern of vibration process, since this is much alike the conventional concept. But there are some distinctions.

For example, one usually thinks that if an elastic line end was free, the end element bunching displacement takes place, and if the end was rigidly fixed, one will see a node, but as for displacement bunching, there will be some difficulties connected with the obtained exact solutions. In the view of the first (i=1) and last (i=n) line elements vibration, from the expression (33) determining the free vibrations amplitude for a line with unfixed ends, we yield the following expressions for the amplitude: for i=1

$$A_{\max} = \frac{A\cos(p\pi/2n)}{\cos(p\pi/2n)} = A \ , \qquad (54)$$

which is quite natural, proceeding from the statement of the given problem, and for i=n

$$A_{\max} = \frac{A\cos\left[(2n-1)\pi/2n\right]}{\cos(p\pi/2n)} . \qquad (55)$$

It means that for the last line element the amplitude is some less than the value corresponding to the bunching displacement. And the difference vanishes as one transits to a distributed line.

Generally, the shown difference between the lumped and distributed lines is typical. Before, in that number in [1], we noted it, when considered the phase delay τ being basic in the view of the line vibration pattern exact description. This all evidences that if we strive to describe the processes exactly, then generally it would be incorrect to use a distributed line as the basic model analogous for that lumped. The more that the shown difference is not limited to the phase and amplitude deviations but reflect in the vibration pattern itself.

5. Conclusions

The carried out investigation has revealed the essential drawbacks in the known solutions for 1D elastic lines, because of which one cannot think these solutions covering for finite lines.

The exact analytical solutions obtained by the presented original method show that in general case in a finite ideal elastic line, the same as in that infinite, three vibration regimes take place, and they essentially differ in their properties. The *periodical* regime typically produces a standing non-damping wave. The *aperiodical* regime makes the vibration process damping along the entire line in which the neighbouring elements vibrate in anti-phase. In distinction from infinite lines, the *critical* regime is characterised by vibration process damping along the entire line, with the anti-phase vibration of the neighbouring elements. But unlike the aperiodical regime, the element-to-element damping occurs not as the power-type but as the linear regularity. All revealed vibration regimes are different forms of the general solution, which completely holds true the validity of theorem of uniqueness of solution of differential equation. In finite lines the revealed regimes are inherent only in forced vibrations. Free vibrations can exist only in the band of periodical regime.

13

Analysing the structure of the obtained exact solutions, we have revealed that they contain specific summands and multipliers enabling each solution to satisfy at the same time a few types of differential equations being a part of the modelling system. For finite lines, the solutions have general structure – a ratio of trigonometric functions as sines and cosines whose arguments depend on the line length and parameters, on the frequency of vibration process and on the element location in the line. The completeness of obtained solutions is determined by the following way. On one hand, their assemblage satisfies both homogeneous and heterogeneous systems of the modelling equations; and on the other hand, the solutions cover the entire range of frequencies from zero to infinity.

For free vibrations in a finite line, there is typical the discrete frequency band of permissible vibrations in which the vibration amplitude increases monotonously as the mode number increases, but in lumped lines never reaches the infinite value. As opposed, with forced vibrations in a finite line, as the frequency increases the amplitude multiply reaches the infinite value but never vanishes, and the vibration spectrum is continuous. With it the values of permissible frequencies of free vibrations coincide with the values of resonance frequencies for forced vibrations, and the block structure of solutions is the same.

For finite lumped unfixed-end lines, the last element vibration amplitude is not maximal, as it is conventionally thought. Its value is shifted by the angle τ. When transiting to distributed lines, this difference disappears. In this connection finite lumped lines cannot be correctly modelled by distributed lines, because with the limiting process a number of features disappears and is non-restored with the reverse transition.

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This paper was submitted on 21.06.99.

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