Localization transition of $(d+1)$-friendly walkers

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Abstract

Friendly walkers is a stochastic model obtained from one-dimensional simple random walks $\{\widetilde{S}^k_n\}_{n\geq 0}$, $k = 1, 2, \dots, d+1$ by introducing "non-crossing condition": $\widetilde{S}^1_j \geq \widetilde{S}^2_j \geq \dots \geq \widetilde{S}^{d+1}_j, j = 1, 2, \dots, n$ and "reward for collisions" characterized by parameters $\beta_1, \dots, \beta_d \geq 0$. Here, the reward for collisions is described as follows. If there are exactly m collisions at time j, i.e., $m = \sharp\{1 \leq k \leq d : \widetilde{S}^k_j = \widetilde{S}^{k+1}_j\} \geq 1$, then the probabilistic weight for the walkers increases by multiplicative factor $\exp(\beta_m) \geq 1$. We study the localization transition of this model in terms of the positivity of the free energy. In particular, we prove the existence of the critical surface in the d-dimensional space for the parameters $(\beta_1, \dots, \beta_d)$.

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Contents

1 Introduction

1.1 The model

Friendly walkers is a stochastic model studied mainly by physicists in connection with the Domany-Kinzel model, directed percolation, wetting and various other models; See [2, 5, 6] and references therein. Roughly speaking, the $(d+1)$-friendly walkers (of the length n) is obtained from one-dimensional simple random walks $\{\widetilde{S}_n^k\}_{n\geq 0}$, $k = 1, 2, \ldots, d+1$ by introducing the following additional rules:

- Non-crossing condition; the walkers are conditioned to preserve the order $\widetilde{S}_j^1 \geq \widetilde{S}_j^2 \geq \ldots \geq \widetilde{S}_j^{d+1}$, $j = 1, 2, \ldots, n$.

- Reward for collisions; if there are exactly m collisions at time j, i.e., $m = \sharp\{1 \leq k \leq d : \widetilde{S}_j^k = \widetilde{S}_j^{k+1}\} \geq 1$, then the probabilistic weight for the walkers increases by multiplicative factor $\exp(\beta_m) \geq 1$.

By the non-crossing condition, the walkers repel each other to avoid violating the order. On the other hand, the reward for collisions introduces an attractive interaction among the walkers, which encourages the walkers to stay close.

Our formulation of this model is based on a $(d+1)$-dimensional random walk $(\widetilde{S}_n, \widetilde{P}_d^x)$ such that the coordinates $\{\widetilde{S}_n^k\}_{n\geq 0}$, $k = 1, 2, \dots, d+1$, are independent simple random walk on $\mathbb{Z}$. To be consistent with the non-crossing condition and to ensure the possibility of collisions, we always take the starting point x from the set;

$$\mathbb{Z}_{\geq}^{d+1} \stackrel{\text{def.}}{=} \{x = (x^j)_{j=1}^{d+1} \in \mathbb{Z}^{d+1} \,;\, \frac{x^j - x^{j+1}}{2} \in \mathbb{N},\ j = 1, \dots, d\}, \tag{1.1}$$

where $\mathbb{N} = \{0, 1, 2, \dots\}$. We will refer to the number d as the "dimension" of the model. For $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in [0, \infty)^d$, the reward for collisions at time n is defined by;

$$\chi_n(\beta) = \begin{cases} \beta_m, & \text{if } \sharp\{1 \leq k \leq d : \widetilde{S}_n^k = \widetilde{S}_n^{k+1}\} = m \geq 1 \\ 0 & \text{if } \sharp\{1 \leq k \leq d : \widetilde{S}_n^k = \widetilde{S}_n^{k+1}\} = 0. \end{cases} \tag{1.2}$$

We then define

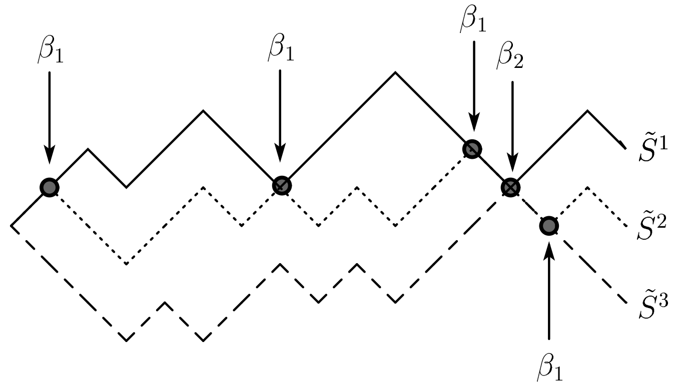


Figure 1: *An example of 3-friendly walkers.*

$$L_n(\beta) = \sum_{j=0}^{n-1} \chi_j(\beta), \quad n \geq 1, \tag{1.3}$$

$$z_{n,d}(\beta) = \exp(L_n(\beta)) 1\{\widetilde{S}_j \in \mathbb{Z}_{\geq}^{d+1}, j = 1, 2, \dots, n\}, \quad n \geq 1, \tag{1.4}$$

$$z_{0,d}(\beta) = 1, \tag{1.5}$$

where $1\{\cdots\}$ denotes the indicator function.

In this paper, we are concerned with the existence and the the positivity of the *free energy*:

$$\psi_d(\beta) = \lim_{n\to\infty} \frac{1}{n} \ln Z_{n,d}^x(\beta), \tag{1.6}$$

where $Z_{n,d}^x(\beta)$ is the partition function[1] :

$$Z_{n,d}^x(\beta) = \widetilde{P}_d^x[z_{n,d}(\beta)]. \tag{1.7}$$

For $d = 1$, it is known [3, 4] that

$$\psi_1(\beta_1) = \begin{cases} 0, & \text{if } \beta_1 \le \ln\frac{4}{3}, \\ \ln\{\frac{e^{\beta_1}}{4}(1+\sqrt{\frac{e^{\beta_1}}{e^{\beta_1}-1}})\} > 0, & \text{if } \beta_1 > \ln\frac{4}{3}. \end{cases} \tag{1.8}$$

So, we are mainly concerned with the higher dimensional case: $d \ge 2$.

The system is said to be *localized* if $\psi_d(\beta) > 0$ and *delocalized* if $\psi_d(\beta) = 0$. Plausibility of this terminology might be explained as follows. Consider a probability measure μ_n^x, $x \in \mathbb{Z}_{\ge}^{d+1}$, defined by

$$\mu_n^x(d\omega) = \frac{1}{Z_{n,d}^x(\beta)} \widetilde{P}_d^x[z_{n,d}(\beta) : d\omega]. \tag{1.9}$$

We look at the paths under this probability measure. Then, as is usual the case with models in statical mechanics, we see competition of energy ($=-L_n$, in this case) and entropy.

- The strategy for walkers to minimize the energy (i.e., maximize the reward) is to travel together, so that they can collect as much reward as possible. In this case, the "width" $\widetilde{S}_n^{d+1} - \widetilde{S}_n^1$ of the group of walkers should remain small as $n \nearrow \infty$ (localization). On the other hand, this strategy lowers the entropy considerably, and therefore, can be optimal only when β_j's are large so that the gain in energy exceeds the loss in entropy.

- The entropy is maximized when the walkers travel separately as they would do if $\beta_j \equiv 0$, with only a small number of collisions which can be ignored

[1] In the sequel, we often omit the subscript d when we deal with a generic dimension d.

in a macroscopic scale. In this case, the “width” $\widetilde{S}_n^{d+1} - \widetilde{S}_n^1$ should diverge as $n \nearrow \infty$ (delocalization). On the other hand, this strategy does not let the walkers pick up much reward, and therefore, can be optimal only when β_j's are small so that the gain in entropy makes up the loss in energy.

Then problem now is to determine which strategy becomes “typical” depending on the choice of β_j's. The answer to this question is believed to be given by the positivity of the free energy mentioned above. In fact, Isozaki and Yoshida [4] prove for $d = 1$ the following;

- For $\beta_1 > \ln(4/3)$, the profile of the width $(\widetilde{S}_k^2 - \widetilde{S}_k^1)_{k=1}^n$ remains bounded and converges to an exponentially mixing Markov chain.
- For $\beta_1 \leq \ln(4/3)$, the width $(\widetilde{S}_k^2 - \widetilde{S}_k^1)_{k=1}^n$ diverges like $\sqrt{n}$, and, if properly scaled (i.e., divided by $\sqrt{n}$), converges to Brownian meander if $\beta_1 < \ln(4/3)$ and to reflecting Brownian motion if $\beta_1 = \ln(4/3)$.

For $d \geq 2$, our knowledge of the localization transition is in much more primitive stage, i.e., at the level of the free energy rather than the pathwise description. However, the results in this paper provide fairly precise information about when localization should occur.

1.2 Main Results

The main result of this paper is the following theorem. We introduce an event

$$\Delta_n = \{\widetilde{S}_n^1 = \widetilde{S}_n^2 = \ldots = \widetilde{S}_n^{d+1}\}$$

and its partition function

$$Z_{n,d}^{x,0}(\beta) = \widetilde{P}_d^x[z_{n,d}(\beta) : \Delta_n]. \tag{1.10}$$

Theorem 1.1 *For $d \geq 2$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_d) \in [0, \infty)^d$, the following limits exsit and are independent of $x \in \mathbb{Z}_{\geq}^{d+1}$;*

$$\psi_d(\beta) = \lim_{n\to\infty} \frac{1}{n} \ln Z_{n,d}^x(\beta), \quad \psi_d^0(\beta) = \lim_{n\to\infty} \frac{1}{n} \ln Z_{n,d}^{x,0}(\beta). \tag{1.11}$$

Moreover, these limits are related as;

$$\psi_d(\beta) = \max_{1 \le k \le d} \psi_k^0(\beta). \tag{1.12}$$

Although ψ_k and ψ_k^0, $1 \le k \le d$ are functions of $(\beta_1, \beta_2, \ldots, \beta_k)$, we often regard them as functions of $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$.

Let us now turn to the positivity of the free energy. For higher dimensions, we have a set of thermodynamic parameters $(\beta_1, \ldots, \beta_d)$, so that we should have a critical surface in d-dimensional space of the paramerters. To describe the critical surface, we introduce;

$$\begin{aligned} W_{n,d}(\beta_1, \ldots, \beta_{d-1}) &= \widetilde{P}_d^0[z_{n,d}(\beta_1, \ldots, \beta_{d-1}, 0) : \Delta_1^c \cap \ldots \cap \Delta_{n-1}^c \cap \Delta_n], & (1.13) \\ \widehat{W}_{1,d}(\beta_1, \ldots, \beta_{d-1}) &= \sum_{n \ge 1} W_{n,d}(\beta_1, \ldots, \beta_{d-1}) \in (0, \infty]. & (1.14) \end{aligned}$$

Theorem 1.2 *Define a concave, decreasing function* $\beta_d^{\mathrm{crit}} : [0,\infty)^{d-1} \longrightarrow [-\infty, \infty)$ *by*

$$\beta_d^{\mathrm{crit}}(\beta_1, \ldots, \beta_{d-1}) = \begin{cases} \ln \frac{4}{3}, & \textit{if } d = 1, \\ -\ln\left(\widehat{W}_{1,d}(\beta_1, \ldots, \beta_{d-1})\right) \in [-\infty, \infty) & \textit{if } d \ge 2. \end{cases} \tag{1.15}$$

Then, it enjoys the following properties;

(a)

$$\beta_d^{\mathrm{crit}}(\beta_1^*, \ldots, \beta_{d-1}^*) \ge \beta_d^*, \quad \textit{where } \beta_j^* = \ln \frac{2^{j+1}}{j+2} > 0 \quad j = 1, 2, \ldots. \tag{1.16}$$

(b) $\psi_d^0(\beta_1, \ldots, \beta_d) > 0$ *if and only if* $\beta_d > \beta_d^{\mathrm{crit}}(\beta_1, \ldots, \beta_{d-1})$.

(c) $\psi_d(\beta_1, \ldots, \beta_d) > 0$ *if and only if* $\beta_k > \beta_k^{\mathrm{crit}}(\beta_1, \ldots, \beta_{k-1})$ *for some* $k = 1, \ldots, d$.

Remark 1.1 Part (a) of the above theorem can be made more precise;

$$\beta_d^{\mathrm{crit}}(\beta_1^*, \ldots, \beta_{d-1}^*) \begin{cases} = \beta_d^* & \text{if } d = 1, 2, \\ > \beta_d^* & \text{if } d \ge 3. \end{cases}$$

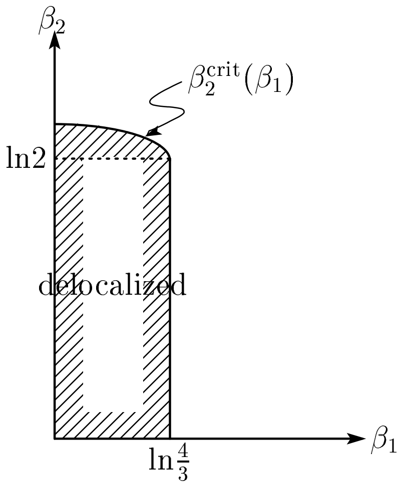


Figure 2: *The delocalized phase for 3-friendly walkers.*

The proof is based on the following observations. If $\beta = (\beta_j^*)_{j=1}^d$, then the measure (1.9) defines a reversible Markov chain which is recurrent for $d = 1, 2$ and is transient for $d \geq 3$.

Consider now a special case $\beta_k = k\beta_1$, $k = 1, \ldots, d$. Then we get the "friendly walkers" in the sense of [5] (with $p = \exp(-\beta_1)$ and $\tau = 1$ in notations there), for which we have the following.

Corollary 1.1

$$\psi_d\left(\beta_1, 2\beta_1, \ldots, d\beta_1\right) > 0 \ \textit{ if and only if } \ \beta_1 > \ln(4/3). \tag{1.17}$$

Proof. The "if" part follows immediately from Theorem 1.2 (c). Suppose that $\beta_1 \leq \beta_1^*$. Since $\beta_k^* > k\beta_1^* \geq k\beta_1$, we see that

$$\begin{aligned}
\beta_d^{\rm crit}(\beta_1, 2\beta_1, \ldots, (d-1)\beta_1) \ &\geq \ \beta_d^{\rm crit}(\beta_1^*, \ldots, \beta_{d-1}^*) \\
&\geq \ \beta_d^* \\
&> \ d\beta_1,
\end{aligned}$$

and hence that $\psi_d\left(\beta_1, 2\beta_1, \ldots, d\beta_1\right) = 0$. $\square$

2 Preliminaries

2.1 A reformulation of the model

Our original formulation of the friendly walkers was based on a $(d+1)$-dimensional random walk conditioned to stay below diagonal. We now explain that the model can be reformulated in terms of a d-dimensional nearest neighbor random walk conditiond to stay in the first quadrant $\mathbb{N}^d$. Define a map $\Theta_d : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d$ by

$$\Theta_d(y^1, y^2, \dots, y^{d+1}) = (\frac{y^1 - y^2}{2}, \frac{y^2 - y^3}{2}, \dots, \frac{y^d - y^{d+1}}{2}).$$

Then the non-crossing condition reads:

$$S_j = (S_j^1, S_j^2, \dots, S_j^d) = \Theta_d \widetilde{S}_j \in \mathbb{N}^d, \quad j = 1, 2, \dots, n. \tag{2.1}$$

It is often convenient to look at this d-dimensional nearest neighbor random walk $S_\cdot$, rather than $\widetilde{S}_\cdot$ itself. We denote[2] by P_d^x the law of the d-dimensional random walk $S_\cdot$ starting from x. The transition function of $S_\cdot$ can be described as follows. Let $\mathcal{R}_d$ be the set of points $x = (x_1, x_2, \dots, x_d) \in \{-1, 0, 1\}^d$ satisfying

$$\sum_{j=k}^{\ell} x_j \in \{-1, 0, 1\}, \quad 1 \le k < \ell \le d. \tag{2.2}$$

The transition function $p(x, y)$ of the random walk $\{S_j\}_{j \ge 0}$ is given by

$$p(x, y) = \begin{cases} 2^{-d}, & \text{if } y = x \\ 2^{-d-1}, & \text{if } y - x \in \mathcal{R}_d \setminus \{0\}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Recall that the random variable $z_{n,d}$ is a function of differences $(\widetilde{S}_\cdot^{k+1} - \widetilde{S}_\cdot^k)_{k=1}^{d+1}$. We can therefore regard $z_{n,d}$ as a function of $S_\cdot$, so that we have

$$Z_{n,d}^x = P_d^{\Theta_d(x)}[z_{n,d}], \quad x \in \mathbb{Z}_{\ge}^{d+1}.$$

[2]We often omit the subscript d when we deal with a generic dimension d.

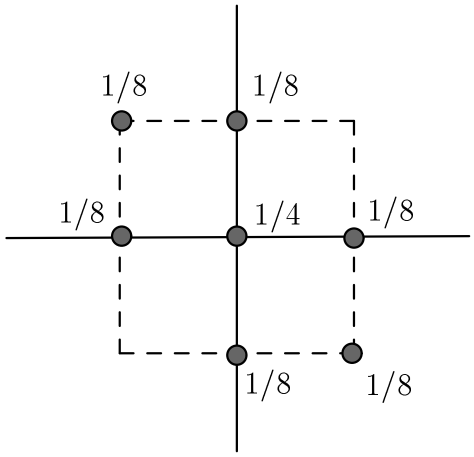


Figure 3: *The transition function $p(0,\cdot)$ for $d=2$.*

2.2 The free energy of small excursions

Lemma 2.1 **(a)** *The following upper and lower limits are independent of $x \in \mathbb{Z}^{d+1}_{\geq}$;*

$$\psi_d(\beta) \stackrel{\text{def.}}{=} \limsup_{n\to\infty} \frac{1}{n} \ln Z^x_{n,d}(\beta), \quad \liminf_{n\to\infty} \frac{1}{n} \ln Z^x_{n,d}(\beta)$$

(b) *The following limit exists and is independent of $x \in \mathbb{Z}^{d+1}_{\geq}$;*

$$\psi^0_d(\beta) = \lim_{n\to\infty} \frac{1}{n} \ln Z^{x,0}_{n,d}(\beta). \tag{2.4}$$

Proof. **(a)** We set $Z^{x,y}_n = P^x[z_n : S_n = y]$ in this proof. Take $x, y \in \mathbb{N}^d$, then

$$R^x_m \stackrel{\text{def.}}{=} \{z \in \mathbb{N}^d \ ; \ Z^{x,z}_m > 0\} \ni y \ \text{ for some } m \geq 1.$$

If $n > m$, we have

$$\begin{aligned} Z^x_n &= \sum_{z \in R^x_m} Z^{x,z}_m Z^z_{n-m} \\ &\geq \left(\min_{z \in R^x_m} Z^{x,z}_m \right) Z^y_{n-m} \end{aligned}$$

and therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \ln Z^x_n \geq \limsup_{n\to\infty} \frac{1}{n} \ln Z^y_n \quad \liminf_{n\to\infty} \frac{1}{n} \ln Z^x_n \geq \liminf_{n\to\infty} \frac{1}{n} \ln Z^y_n.$$

(b) The same calculation as above proves that the following upper and lower limits are independent of $x \in \mathbb{Z}_{\geq}^{d+1}$;

$$\limsup_{n\to\infty} \frac{1}{n} \ln Z_{n,d}^{x,0}(\beta), \quad \liminf_{n\to\infty} \frac{1}{n} \ln Z_{n,d}^{x,0}(\beta).$$

Therefore, it is enough to consider the case of $x = 0$, in which the existence of the limit is guaranteed by the superadditivity;

$$Z_{n+n'}^{00}(\beta) \geq Z_n^{00}(\beta) Z_{n'}^{00}(\beta), \quad n, n' \geq 1. \tag{2.5}$$

□

We set

$$\widehat{W}_s(\beta) = \sum_{n\geq 1} s^n W_n(\beta), \quad \widehat{Z}_s^{00}(\beta) = \sum_{n\geq 1} s^n Z_{n,d}^{00}(\beta). \tag{2.6}$$

Then we show the following lemma.

Lemma 2.2

$$\widehat{Z}_s^{00}(\beta) = \sum_{m\geq 1} [e^{\beta_d} \widehat{W}_s(\beta)]^m. \tag{2.7}$$

In particular, $\psi_d^0(\beta)$ is characterized by the relation;

$$\exp\left(-\psi_d^0(\beta)\right) = \sup\{s : e^{\beta_d} \widehat{W}_s(\beta) < 1\}. \tag{2.8}$$

Proof. We set $\eta_0^0 \equiv 0$ and $\eta_0^j = \inf\{k > \eta_0^{j-1} : S_k = 0\}$, $j \geq 1$. We then have that

$$W_n(\beta) = P^0[z_{n,d}(\beta_1, \beta_2, \ldots, \beta_{d-1}, 0) : \eta_0^1 = n], \tag{2.9}$$

and hence by the Markov property that

$$\begin{aligned}
Z_n^{00}(\beta) &= P^0[z_n(\beta) : S_n = 0] \\
&= \sum_{m=1}^{n} \sum_{0<i_1<\cdots<i_{m-1}<n} P^0[z_n(\beta) : \eta_0^1 = i_1, \eta_0^2 = i_2, \ldots, \eta_0^m = i_m] \\
&= \sum_{m=1}^{n} e^{\beta_d m} \sum_{0<i_1<\cdots<i_{m-1}<n} \prod_{k=1}^{m} P^0[z_{i_k - i_{k-1}}(\beta_1, \ldots, \beta_{d-1}, 0) : \eta_0 = i_k - i_{k-1}] \\
&= \sum_{m=1}^{n} e^{\beta_d m} \sum_{\substack{j_1, j_2, \ldots, j_m \geq 1 \\ j_1 + j_2 + \cdots + j_m = n}} W_{j_1}(\beta) W_{j_2}(\beta) \ldots W_{j_m}(\beta),
\end{aligned}$$

where we put $i_0 = 0$ and $i_m = n$. The desired equality (2.7) is now immediately obtained by computing the generating function of the right-hand-side. □

We introduce an event

$$\Theta_{n,M} = \{S_n = 0, \eta_0^i - \eta_0^{i-1} \leq M,\ 0 \leq i \leq \tau_n\} \tag{2.10}$$

and its partition function:

$$Z_n^{00}(\beta, M) = \widetilde{P}_d^0[z_{n,d}(\beta) : \Theta_{n,M}], \tag{2.11}$$

where η_0^i, $i \geq 0$ are stopping times defined in the proof of Lemma 2.2 and

$$\tau_n(\widetilde{S}) = \max\{m : \eta_0^m \leq n\}.$$

The event $\Theta_{n,M}$ can be understood as the "phase of small excursions". The following lemma relates $\psi_d^0(\beta)$ with $Z_n^{00}(\beta, M)$.

Lemma 2.3

$$\psi_d^0(\beta) = \lim_{M\to\infty} \lim_{n\to\infty} \frac{1}{n} \ln Z_{n,d}^{00}(\beta, M). \tag{2.12}$$

Proof. The first limit (in n) on the right-hand-side exists by the superadditivity while the second one (in M) by monotonicity. To identify the limit, take any $c < \psi_d^0(\beta)$. Then, by Lemma 2.2, we can take a positive integer $M(c)$ such that

$$\sum_{n=1}^{M(c)} e^{-cn} W_{n,d}(\beta) e^{\beta_d} \geq 1. \tag{2.13}$$

By the same procedure to show Lemma 2.2 we have

$$\lim_{n\to\infty} \frac{1}{n} \ln Z_{n,d}^{00}(\beta, M(c)) \geq c. \tag{2.14}$$

This proves (2.12). □

3 Proof of Theorems

3.1 Proof of Theorem 1.1

We begin by presenting three lemmas, which the proof of Theorem 1.1 is based on. The proofs of these lemmas are postponed to Section 4.

For $\ell \geq 0$, we introduce an event

$$\Xi_n^\ell = \{|S_j| > \ell, 1 \leq j \leq n\}$$

and its partition function:

$$V_{n,d}^x(\beta) = P_d^x[z_{n,d}(\beta) : \Xi_n^\ell]. \tag{3.1}$$

It is then easy to see that the following upper limit is independent of the choice of $x \in \{x \in \mathbb{N}^d \ ; \ |x| \geq \ell\}$;

$$\varphi_d^\ell(\beta) = \limsup_{n\to\infty} \frac{1}{n} \ln V_{n,d}^x(\beta). \tag{3.2}$$

(c.f.the proof of Lemma 2.1) . $\varphi_d^\ell(\beta)$ can be understood as free energy for "a large excursion from $0 \in \mathbb{N}^d$ ". We then have the following lemma, which decomposes the free energy $\psi_d(\beta)$ into the contribution from small excursions (i.e., $\psi_d^0(\beta)$) and that from a large excursion (i.e., $\varphi_d^\ell(\beta)$).

Lemma 3.1

$$\psi_d(\beta) = \psi_d^0(\beta) \vee \varphi_d^\ell(\beta), \quad \ell \geq 0. \tag{3.3}$$

The next lemma says that the free energy for a large enough excursion is absorbed in those for lower dimensional subsystems.

Lemma 3.2 *Suppose that $d \geq 2$. Then*

$$\limsup_{\ell\to\infty} \varphi_d^\ell(\beta) \leq \max_{1\leq k\leq d-1} \psi_k(\beta). \tag{3.4}$$

Lastly, we present the following

Lemma 3.3 *If $1 \leq k \leq d$, then*

$$\liminf_{n\to\infty} \frac{1}{n} \ln \widetilde{P}_d^0[z_{n,d}(\beta) : \Delta_{n,k}] \geq \psi_k^0(\beta), \tag{3.5}$$

where

$$\Delta_{n,k} = \{\widetilde{S}_n^1 = \ldots = \widetilde{S}_n^{k+1}\}.$$

Proof of Theorem 1.1: By Lemma 3.1 and Lemma 3.2,

$$\psi_d(\beta) \leq \psi_d^0(\beta) \vee \max_{1\leq k\leq d-1} \psi_k^0(\beta) = \max_{1\leq k\leq d} \psi_k^0(\beta). \tag{3.6}$$

Combining Lemma 3.3 and the above, we have Theorem 1.1.

3.2 Proof of Theorem 1.2

(b): By (2.8), the positivity of $\psi_d^0(\beta)$ is equivalent to that

$$\widehat{W}_s(\beta) > \exp\left(-\beta_d\right) \text{ for some } s < 1. \tag{3.7}$$

On the other hand, we have by monotone convergence theorem that

$$\lim_{s \nearrow 1} \widehat{W}_s(\beta) = \widehat{W}_1(\beta) = \exp\left(-\beta_d^{\mathrm{crit}}(\beta_1, \ldots, \beta_{d-1})\right). \tag{3.8}$$

If $\beta_d > \beta_d^{\mathrm{crit}}(\beta_1, \ldots, \beta_{d-1})$, then we see from (3.8) that (3.7) holds. Conversely, if (3.7) holds true, then $\widehat{W}_1(\beta) > \exp(-\beta_d)$, and hence $\beta_d > \beta_d^{\mathrm{crit}}(\beta_1, \ldots, \beta_{d-1})$. Since

$$p(x, \mathbb{N}^d)e^{\beta_m^*} = 1 \ \text{ if } \ \sharp\{k \in \{1, 2, \ldots, d\} : x^k = 0\} = m \geq 1,$$

it is not difficult to see that

$$Z_n^x(\beta_1^*, \ldots, \beta_d^*) = Z_1^x(\beta_1^*, \ldots, \beta_d^*) \quad n \geq 1$$

and hence that $\psi_d(\beta_1^*, \ldots, \beta_d^*) = 0$. We therefore have $\beta_d^* \leq \beta_d^{\mathrm{crit}}(\beta_1^*, \ldots, \beta_{d-1}^*)$ by part (b).

(c): This follows from part (b) and (1.12). □

4 Proof of Lemmas

4.1 Proof of Lemma 3.1

We put $D_n = \{0 \leq j \leq n : |S_j| \leq \ell\}$, $d_n = \max D_n$ and

$$k_{m,n} = 1\{|S_{m+1}| > \ell, |S_{m+2}| > \ell, \ldots, |S_n| > \ell\}, \quad 0 \leq m < n.$$

Since for $0 \leq m < n$,

$$z_n(\beta)1\{d_n = m\} = z_m(\beta)1\{|S_m| \leq \ell\}z_{n-m}(\beta)k_{m,n},$$

we have

$$\begin{aligned} Z_n^0(\beta) &= P^0[\sum_{m=0}^{n-1} z_n(\beta)1\{d_n = m\} + z_n(\beta)1\{d_n = n\}] \\ &= \sum_{m=0}^{n-1} P^0[z_m(\beta), |S_m| = \ell, P_{S_m}[z_{n-m} : \Xi_{n-m}^\ell]] + P^0[z_n(\beta), |S_n| = \ell]. \end{aligned}$$

We now set

$$\begin{aligned}
Z_n^\ell(\beta) &= P^0[z_n(\beta) : |S_n| = \ell], \\
\overline{V}_n^\ell(\beta) &= \max_{x \in \mathbb{N}^d, |x|=\ell} V_{n,d}^x(\beta), \\
\underline{V}_n^\ell(\beta) &= \min_{x \in \mathbb{N}^d, |x|=\ell} V_{n,d}^x(\beta).
\end{aligned}$$

We then have

$$Z_n^0(\beta) \le \sum_{m=0}^{n-1} Z_m^\ell(\beta) \overline{V}_{n-m}^\ell(\beta) + Z_n^\ell(\beta), \tag{4.1}$$

$$Z_n^0(\beta) \ge \sum_{m=0}^{n-1} Z_m^\ell(\beta) \underline{V}_{n-m}^\ell(\beta) + Z_n^\ell(\beta), \tag{4.2}$$

$$\varphi_d^\ell(\beta) = \limsup_{n\to\infty} \frac{1}{n} \ln \overline{V}_n^\ell(\beta) = \limsup_{n\to\infty} \frac{1}{n} \ln \underline{V}_n^\ell(\beta), \tag{4.3}$$

$$\psi_d^0(\beta) = \limsup_{n\to\infty} \frac{1}{n} \ln Z_n^\ell(\beta) \tag{4.4}$$

(c.f.the proof of Lemma 2.1) . The relation (3.3) is now clear from (4.1), (4.2),(4.3) and (4.4). □

4.2 Proof of Lemma 3.2

It is convenient to introduce the following notations;

$$Z_{n,d}^{\mathrm{uni}}(\beta) = \sup_{x \in \mathbb{Z}_{\ge}^{d+1}} Z_{n,d}^x(\beta), \tag{4.5}$$

$$V_{n,d}^\ell(\beta) = \sup_{x \in \mathbb{N}^d, |x| \ge \ell} V_{n,d}^x(\beta). \tag{4.6}$$

It would be clear that the Lemma 3.2 is a consequence of the following two inequalities;

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \frac{1}{n} \ln V_{n,d}^\ell(\beta) \le \limsup_{n\to\infty} \frac{1}{n} \ln Z_{n,d-1}^{\mathrm{uni}}(\beta), \tag{4.7}$$

$$\limsup_{n\to\infty} \frac{1}{n} \ln Z_{n,d}^{\mathrm{uni}}(\beta) \le \max_{1\le k\le d} \psi_k(\beta) \tag{4.8}$$

Proof of (4.7): We let $\psi_{d-1}^{\mathrm{uni}}$ denote the right-hand-side of (4.7) and take $0 \le s < \exp(-\psi_{d-1}^{\mathrm{uni}})$. We will show that if ℓ is sufficiently large, then

$$\widehat{V}_{s,d}^\ell(\beta) = \sum_{n\ge1} s^n V_{n,d}^\ell(\beta) < \infty, \tag{4.9}$$

which implies (4.7). We now define a sequence of stopping times $\sigma^i, i \geq 0$ by $\sigma^0 = 0$ and $\sigma^i = \inf\{n > \sigma^{i-1} \ ; \ S_n^d = 0\}$, $i \geq 1$. By the Markov property, we have for any x with $|x| \geq \ell$ that

$$
\begin{aligned}
&V_{n,d}^{x,\ell}(\beta) = P^x[z_{n,d}(\beta) : \Xi_n^\ell] \\
&= P^x[z_{n,d}(\beta) : \Xi_n^\ell, \sigma^1 > n] \\
&+ \sum_{m=1}^{n} \sum_{0<i_1<\cdots<i_{m-1}\leq n} P^x[z_{n,d}(\beta) : \Xi_n^\ell, \sigma^1 = i_1, \cdots, \sigma^{m-1} = i_{m-1}, \sigma^m > n] \\
&= P^x[z_{n,d}(\beta) : \Xi_n^\ell, \sigma > n] \\
&+ \sum_{m=1}^{n} \sum_{\substack{j_1,j_2,\dots,j_m\geq 1 \\ j_1+j_2+\cdots+j_m=n}} P^x[z_{j_1}(\beta) : \Xi_{j_1}^\ell, \sigma = j_1, P^{S_{j_1}}[z_{j_2}(\beta) : \Xi_{j_2}^\ell, \sigma = j_2, \\
&\cdots P^{S_{j_{m-2}}}[z_{j_{m-1}}(\beta) : \Xi_{j_{m-1}}^\ell, \sigma = j_{m-1}, P^{S_{j_{m-1}}}[z_{j_m}(\beta) : \Xi_{j_m}^\ell, \sigma > j_m]]...]].
\end{aligned}
$$

We put

$$
\begin{aligned}
U_n^\ell(\beta) &= \max_{x\in\mathbb{N}^d:|x|\geq\ell} P^x[z_{n,d}(\beta) : \Xi_n^\ell, \sigma > n], \quad \widehat{U}_s^\ell(\beta) = \sum_{n\geq 1} s^n U_n^\ell(\beta), \\
W_n^\ell(\beta) &= \max_{x\in\mathbb{N}^d:|x|\geq\ell} P^x[z_{n,d}(\beta) : \Xi_n^\ell, \sigma = n], \quad \widehat{W}_s^\ell(\beta) = \sum_{n\geq 1} s^n W_n^\ell(\beta).
\end{aligned}
$$

We then have that

$$
V_n^\ell(\beta) \leq U_n^\ell(\beta) + \sum_{m=1}^{n} \sum_{\substack{j_1,j_2,\dots,j_m\geq 1 \\ j_1+j_2+\cdots+j_m=n}} W_{j_1}^\ell(\beta)\dots W_{j_{m-1}}^\ell(\beta) U_{j_m}^\ell(\beta),
$$

and therefore that

$$
\widehat{V}_s^\ell(\beta) \leq \widehat{U}_s^\ell(\beta) + \widehat{U}_s^\ell(\beta) \sum_{m\geq 1} \widehat{W}_s^\ell(\beta)^m. \tag{4.10}
$$

Hence (4.9) is obtained from the following:

$$
\widehat{U}_s^\ell(\beta) < \infty \ \text{ and } \ \lim_{\ell\to\infty} \widehat{W}_s^\ell(\beta) < 1. \tag{4.11}
$$

Consider now a projection map;

$$
x \in \mathbb{N}^d \mapsto \underline{x} = (x_j)_{j=1}^{d-1} \in \mathbb{N}^{d-1}.
$$

Note that $P_d^x[z_{d-1,n}(\beta)] = P_{d-1}^x[z_{d-1,n}(\beta)]$ and that $z_{n,d}(\beta) \le z_{n,d-1}(\beta)$ on the event $\{\sigma \ge n\}$. Therefore,

$$\begin{aligned} P_d^x[z_{n,d}(\beta) : \Xi_n^\ell, \sigma > n] &\le P_d^x[z_{d-1,n} : \Xi_n^\ell, \sigma > n] \\ &\le Z_{n,d-1}^{\mathrm{uni}}(\beta). \end{aligned}$$

This implies that

$$\widehat{U}_s^\ell(\beta) \le \sum_{n\ge 1} s^n Z_{n,d-1}^{\mathrm{uni}}(\beta). \tag{4.12}$$

As for $\widehat{W}_s^\ell(\beta)$, we have

$$\widehat{W}_s^\ell(\beta) \le \sum_{n\ge \frac{\ell}{d}} s^n Z_{n,d-1}^{\mathrm{uni}}(\beta). \tag{4.13}$$

by a similar calculation as above, together with an observation:

$$P_d^x[z_{n,d}(\beta) : \Xi_n^\ell, \sigma = n] = 0 \quad \text{if } n < \frac{\ell}{d}.$$

We now obtain (4.11) from (4.12),(4.13) and the choice of s. □

Proof of (4.8): We prove (4.8) by the induction on the dimension d. For $d = 1$, (4.8) can be obtained from [4, Lemma 4.2]. Suppose that we have (4.8) up to $d-1$. Then, it follows from (4.7) and the induction hypothesis that

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \frac{1}{n} \ln V_{n,d}^\ell(\beta) \le \max_{1\le k\le d-1} \psi_k(\beta). \tag{4.14}$$

On the other fand, it is not difficult to repeat the proof of Lemma 3.1 to see that

$$\widehat{Z}_{s,d}^{\mathrm{uni}}(\beta) \le \widehat{V}_{s,d}^\ell(\beta) + \widehat{V}_{s,d}^\ell(\beta)\widehat{Z}_{s,d}^\ell(\beta) + \widehat{V}_{s,d}^\ell(\beta)^2 \widehat{Z}_{s,d}^\ell(\beta). \tag{4.15}$$

We now obtain (4.8) from (4.4), (4.14) and (4.15). □

4.3 Proof of Lemma 3.3

Lemma 4.1 *For any $\varepsilon > 0$ and $M > 0$, there exists an $L \ge 1$ such that*

$$\widetilde{P}_d^0[z_{n,d}(\beta), \Delta_n \cap \Lambda_d(L,\varepsilon)] \ge \frac{1}{2} Z_n^{00}(\beta, M),\ n > L, \tag{4.16}$$

where

$$\Lambda_d(L,\varepsilon) = \cap_{j\ge L}\{\widetilde{S}_j^{d+1} > -\varepsilon j\}.$$

Proof. First we introduce spaces of $(d+1)$-dimensional finite paths $\mathcal{W}(n)$, $\mathcal{W}_+(n)$ $n \geq 1$ defined by

$$\begin{aligned}\mathcal{W}(n) = \{\widetilde{w} = \{\widetilde{w}_j\}_{j=1}^n : |\widetilde{w}_j^i - \widetilde{w}_{j-1}^i| = 1,\ 1 \leq j \leq n, 1 \leq i \leq d+1,\\ \widetilde{w}_0^1 = \widetilde{w}_0^i,\ \widetilde{w}_n^1 = \widetilde{w}_n^i,\ 2 \leq i \leq d+1\},\\ \mathcal{W}_+(n) = \{\widetilde{w} \in \mathcal{W}(n) : \widetilde{w}_0 = 0, \Theta_d \widetilde{w}_j \neq 0, 1 \leq j \leq n-1\}.\end{aligned}$$

For $\widetilde{w}' \in \mathcal{W}(n_1)$ and $\widetilde{w}'' \in \mathcal{W}(n_2)$, $n_1, n_2 \geq 1$, $\widetilde{w}' \cdot \widetilde{w}''$ represent the path in $\mathcal{W}(n_1+n_2)$ defined by

$$(\widetilde{w}' \cdot \widetilde{w}'')_j = \begin{cases} \widetilde{w}_j', & \text{if } 0 \leq j \leq n_1, \\ \widetilde{w}_{n_1}' + \widetilde{w}_{j-n_1}'' - \widetilde{w}_0'', & \text{if } n_1 \leq j \leq n_1+n_2. \end{cases} \tag{4.17}$$

Suppose that $\eta_0^m[S.] = n$. Then we define $\widetilde{w}(i) \in \mathcal{W}_+(\eta_0^i - \eta_0^{i-1})$, $1 \leq i \leq m$ as

$$\widetilde{w}_j(i) = \widetilde{S}_{\eta_0^{i-1}+j} - \widetilde{S}_{\eta_0^{i-1}}, \quad 0 \leq j \leq \eta_0^i - \eta_0^{i-1}. \tag{4.18}$$

It is clear that $\widetilde{S}_j = (\widetilde{w}(1) \cdot \widetilde{w}(2) \cdots \widetilde{w}(m))_j$, for $0 \leq j \leq n$. We introduce the map r from $\mathcal{W}_+(n)$ to $\mathcal{W}_+(n)$ defined by

$$(r\widetilde{w})_j = \widetilde{w}_{n-j} - \widetilde{w}_n, \quad 0 \leq j \leq n,\ \widetilde{w} \in \mathcal{W}_+(n). \tag{4.19}$$

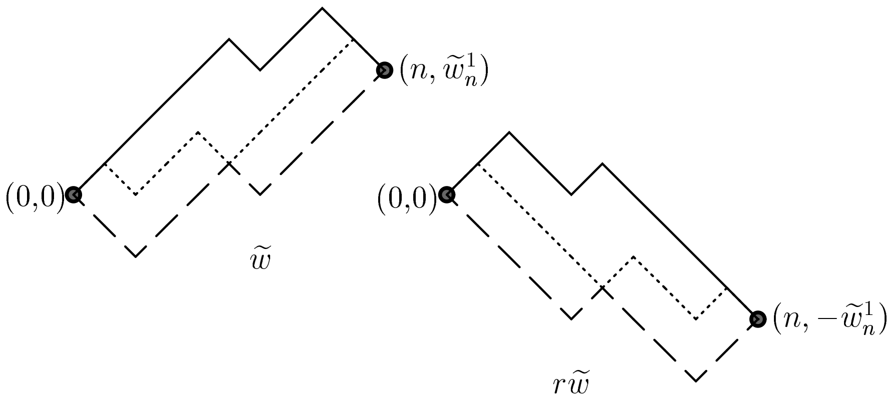


Figure 4: *Examples of $\widetilde{w}$ and $r\widetilde{w}$.*

For $\xi \in \{-1,1\}^m$ we define $\widetilde{S}^\xi$ inductively by

$$\widetilde{S}^\xi_j = \begin{cases} \widetilde{S}^\xi_{\eta_0^{i-1}} + \widetilde{w}(i)_{j-\eta_0^{i-1}}, & \text{if } \eta_0^{i-1} \le j \le \eta_0^i,\ \xi(i) = 1,\ 1 \le i \le m, \\ \widetilde{S}^\xi_{\eta_0^{i-1}} + (r\widetilde{w}(i))_{j-\eta_0^{i-1}}, & \text{if } \eta_0^{i-1} \le j \le \eta_0^i,\ \xi(i) = -1,\ 1 \le i \le m, \\ \widetilde{S}^\xi_n + \widetilde{S}_j - \widetilde{S}_n, & \text{if } j > n. \end{cases}$$

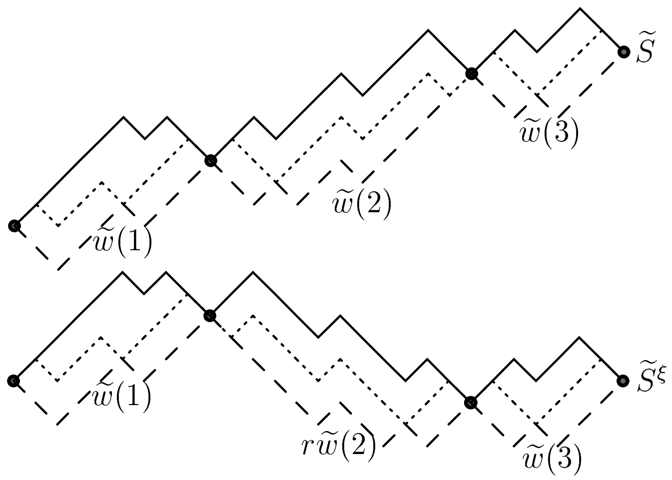


Figure 5: *Examples of* $\widetilde{S}$ *and* $\widetilde{S}^\xi$*, in the case that* $n = 3$*,* $\xi(1) = 1$*,* $\xi(2) = -1$ *and* $\xi(3) = 1$*.*

Note that

$$z_n(\beta)[\widetilde{S}_\cdot] = z_n(\beta)[\widetilde{S}^\xi_\cdot], \quad \xi \in \{-1,1\}^m. \tag{4.20}$$

We put $T_i(\widetilde{S}) = \widetilde{S}^1_{\eta_0^i} - \widetilde{S}^1_{\eta_0^{i-1}}$. Then we see that $T_i(\widetilde{S}) = \widetilde{S}^\ell_{\eta_0^i} - \widetilde{S}^\ell_{\eta_0^{i-1}} = \widetilde{w}^\ell(i)_{\eta_0^i - \eta_0^{i-1}}$, $1 \le \ell \le d+1$, and

$$T_i(\widetilde{S}^\xi) = \xi(i) T_i(\widetilde{S}). \tag{4.21}$$

Hence,

$$\widetilde{P}^0_d[z_n(\beta) : \Delta_n \cap \Lambda_d(L,\varepsilon)] \ge \sum_{m=1}^n \widetilde{P}^0_d[z_n(\beta) : \Lambda_d(L,\varepsilon) \cap \Theta_{n,M,m}], \tag{4.22}$$

where $\Theta_{n,M,m} = \Theta_{n,M} \cap \{\tau_n = m\}$. Note that

$$\widetilde{S} \in \Theta_{n,M,m} \Longleftrightarrow \widetilde{S}^{\xi} \in \Theta_{n,M,m}$$

and therefore that the right-hand-side of (4.22) equals

$$\sum_{m=1}^{n} \frac{1}{2^m} \sum_{\xi \in \{-1,1\}^m} \widetilde{P}_d^0[z_n(\beta) : \{\widetilde{S}^{\xi} \in \Lambda_d(L,\varepsilon)\} \cap \Theta_{n,M,m}].$$

If $\eta_0^i - \eta_0^{i-1} \leq M$, $0 \leq i \leq \tau_n = m$, then

$$(\widetilde{S}^{\xi})_j^{d+1} \geq \sum_{i=1}^{\tau_j} \xi(i) T_i(\widetilde{S}) - M, \quad L \leq j \leq n.$$

Thus

$$\widetilde{P}_d^0[z_n(\beta) : \Delta_n \cap \Lambda_d(L,\varepsilon)] \geq \sum_{m=1}^{n} \widetilde{P}_d^0[z_n(\beta) \rho_{L,m}(\widetilde{S}) : \Theta_{n,M,m}]. \tag{4.23}$$

where

$$\rho_{L,m}(\widetilde{S}) = \frac{1}{2^m} \sum_{\xi \in \{-1,1\}^m} 1\{\sum_{i=1}^{\tau_j} \xi(i) T_i(\widetilde{S}) > -\varepsilon j + M, \ L \leq j\}$$

Since $m \geq \frac{n}{M}$, $\tau_L \geq \frac{L}{M}$ and $\sum_{i=1}^{j} \xi(i) T_i(\widetilde{S})$ is a mean-zero martingale with bounded increments, we can use Azuma's inequality [1, page85] to have a constant L, independent of $\widetilde{S}$ and m such that

$$\rho_{L,m}(\widetilde{S}) \geq \frac{1}{2} \tag{4.24}$$

We now obtain (4.16) by plugging (4.24) into (4.23). $\Box$

Proof of Lemma 3.3: We may assume that $k \leq d-1$. For $\varepsilon \in (0, \frac{1}{2d})$ we define $q_1, q_2, \dots, q_{d+1} \in (0,1)$ by

$$q_i = \frac{1}{2}, \ 1 \leq i \leq k+1, \quad q_i = \frac{1}{2} - (i-k-1)\varepsilon, \ k+2 \leq i \leq d+1,$$

We then consider independent random walks $\{\widetilde{S}_n^i\}_{n \geq 0}$, $i = 1, 2, \dots, d+1$, whose transition functions are give by

$$q^i(x,y) = \begin{cases} q_i, & \text{if } y = x+1, \\ 1 - q_i, & \text{if } y = x-1, \\ 0, & \text{otherwise,} \end{cases} \tag{4.25}$$

$i = 1, 2, \ldots, d+1$. We let $\widetilde{P}_{d,\varepsilon}$ denote the law of the $d+1$-dimensional random walk $\widetilde{S}_\cdot = (\widetilde{S}^i_\cdot)_{i=1}^{d+1}$. We see from a simple comparison of the transition probability that

$$\widetilde{P}^0_d[A] \geq (1 + 2d\varepsilon)^{-nd} \widetilde{P}^0_{d,\varepsilon}[A]$$

for any $A \in \sigma[\widetilde{S}_1, \ldots, \widetilde{S}_n]$. Hence, to prove Lemma 3.3 it is enough to show that

$$\liminf_{n\to\infty} \frac{1}{n} \ln \widetilde{P}^0_{d,\varepsilon}[z_{n,d}(\beta) : \Delta_{n,k}] \geq \psi^0_k(\beta), \quad \text{for any } \varepsilon \in (0, 1/2d). \tag{4.26}$$

To do so, take any $M \geq 1$ and then $L = L(M) \geq 1$ by Lemma 4.1 so that

$$\widetilde{P}^0_k[z_{n,k}(\beta) : \Delta_{n,k} \cap \Lambda_k(L, \varepsilon)] \geq \frac{1}{2} Z^{00}_{n,k}(\beta, M), \ n > L.$$

We put

$$\Gamma_k(L, \varepsilon) = \cap_{j \geq L} \{\widetilde{S}^{d+1}_j < \widetilde{S}^d_j < \cdots < \widetilde{S}^{k+2}_j < -\varepsilon j\}.$$

By the independence of $\widetilde{S}^i$, $i = 1, \ldots, d+1$ and the definition of $z_{n,d}(\beta)$, we have

$$\begin{aligned}
\widetilde{P}_{d,\varepsilon}[z_{n,d}(\beta)] &\geq \widetilde{P}_{d,\varepsilon}[z_{n,d}(\beta), \Gamma_k(L, \varepsilon) \cap \Delta_{n,k} \cap \Lambda_k(L, \varepsilon)] \\
&= \widetilde{P}^0_{d,\varepsilon}[\Gamma_k(L, \varepsilon)] \widetilde{P}^0_k[z_{n,k}(\beta) : \Delta_{n,k} \cap \Lambda_k(L, \varepsilon)] \\
&\geq C(L) \widetilde{P}^0_k[z_{n,k}(\beta) : \Delta_{n,k} \cap \Lambda_k(L, \varepsilon)] \\
&\geq \tfrac{1}{2} C(L) Z^{00}_{n,k}(\beta, M).
\end{aligned} \tag{4.27}$$

where $C(L) = \widetilde{P}^0_\varepsilon[\Gamma_k(L, \varepsilon)] > 0$. We therefore have that

$$\liminf_{n\to\infty} \frac{1}{n} \ln \widetilde{P}^0_{d,\varepsilon}[z_{n,d}(\beta) : \Delta_{n,k}] \geq \lim_{n\to\infty} \frac{1}{n} \ln Z^{00}_{n,k}(\beta, M) \tag{4.28}$$

By Lemma 2.3, the desired inequality (4.26) is now obtained by letting $M \to \infty$ in (4.28). $\square$

5 Concluding remarks

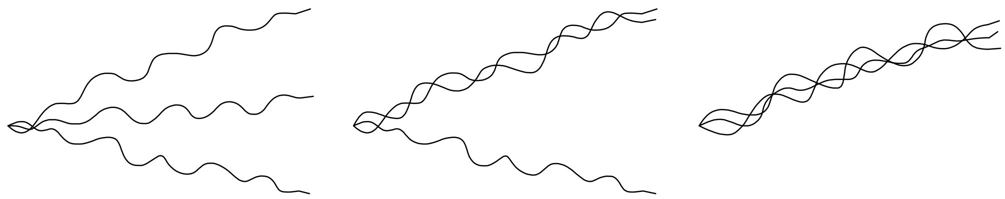
In this section we discuss the asymptotic behavior of the friendly walkers as $t \to \infty$. By virtue of Theorem 1.1 we can consider the following phases $\mathcal{D}_k, k = 1, \ldots, d+1$:

$$\mathcal{D}_1 = \{\beta \in [0, \infty)^d : \psi_d(\beta) = 0\},$$

$$\mathcal{D}_k = \{\beta \in [0,\infty)^d : \psi^0_{k-1}(\beta) > \psi^0_j(\beta),\ 1 \le j \le d, j \ne k-1\},$$

$k = 2, 3, \dots, d+1$. The phase $\mathcal{D}_1$ is the delocalized phase, the phase $\mathcal{D}_k$ is that of k-walkers collision, for $k = 2, 3, \dots, d+1$, and the complement of $\cup_{k=1}^{d+1}\mathcal{D}_k$ is the region of critical values. We call $\mathcal{D}_{d+1}$ the completely localized phase, and $\mathcal{D}_k$, $k = 2, 3, \dots, d$ intermediate phases. For example, the 3-friendly walkers could have the following three phases:

- (The delocalized phase $\mathcal{D}_1$) The walkers collide finitely many often and the each width between two walkers diverges.
- (The intermediate phase $\mathcal{D}_2$) Two of three walkers collide infinitely many often but the other walker does collide finitely many often.
- (The completely localized phase$\mathcal{D}_3$) Three walkers collide together infinitely many often.



(1) (2) (3)

Figure 6: *Examples of trajectories of 3-friendly walkers in* (1) the delocalized phase, (2) the intermediate phase, and (3) the completely localized phase.

Define functions $\widehat{\beta}^{\rm crit}_j : [0,\infty)^{j-1} \longrightarrow [-\infty,\infty)$, $j = 2, 3, \dots, d$ by

$$\widehat{\beta}^{\rm crit}_j(\beta) = \widehat{\beta}^{\rm crit}_j(\beta_1, \dots, \beta_{j-1}) = -\ln\left(\widehat{W}_{e^{-\psi_{j-1}(\beta)},j}(\beta_1, \dots, \beta_{j-1})\right),$$

where

$$\widehat{W}_{s,j}(\beta_1, \dots, \beta_{j-1}) = \sum_{n\ge 1} s^n W_{n,j}(\beta_1, \dots, \beta_{j-1}),\ s > 0.$$

It is obvious that

$$\widehat{\beta}^{\rm crit}_j(\beta) = \beta^{\rm crit}_j(\beta), \quad \text{if } \psi_{j-1}(\beta) = 0, \tag{5.1}$$

and

$$\beta \in \mathcal{D}_k \text{ if and only if } \beta_{k-1} > \widehat{\beta}_{k-1}^{\mathrm{crit}}(\beta),\ \beta_j < \widehat{\beta}_j^{\mathrm{crit}}(\beta), k \le j \le d, \tag{5.2}$$

for $2 \le k \le d-1$,

$$\beta \in \mathcal{D}_{d+1} \text{ if and only if } \beta_d > \widehat{\beta}_d^{\mathrm{crit}}(\beta). \tag{5.3}$$

Therefore, the phase diagram can be determined by the functions $\widehat{W}_{e^{-\psi_{k-1}(\beta)},k}(\beta)$, $k = 2, 3, \ldots, d$. For instane, the intermideate phase $\mathcal{D}_k$ would not exist if

$$\widehat{W}_{e^{-\psi_{k-1}(\beta)},k}(\beta) = \infty, \quad \text{for any } \beta \text{ with } \psi_{k-1}(\beta) > 0.$$

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