SOME FEATURES OF THE FORCED VIBRATIONS MODELLING FOR 1D HOMOGENEOUS ELASTIC LUMPED LINES

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This is the third leading paper of a large cycle devoted to our new method to obtain the exact analytical solutions for vibrant elastic lumped and distributed systems. Its initial version was published in "Materials. Technologies. Tools", the journal of National Academy of Sciences of Belarus, **5** (2000), 3, pp.14-19 (in Russian). In the present version we retained generally the original part of the published paper, so the English-language readers can consider it as the elaborated version of the published paper. We have essentially widen the conventional methods analysis; in our view, it presents more brightly the advantages which our method adds to the calculation scope and which we embody in our investigations, both published and being about publishing in the nearest future.

You can find this and other papers of this cycle in our electronic journal SELF Transactions (for free) at http://angelfire.lycos.com/la3/selftrans , as well as many other interesting results, both there and on our web site at http://angelfire.lycos.com/la3/SELFlab . We will be very grateful if you visit it and express us your opinion of our results or your interest to join us in our research.

Abstract

In this paper we survey the conventional methods to calculate the systems modelling vibrant 1D elastic lumped lines, in comparison with the new non-matrix method to obtain the exact analytical solutions for such systems. We consider the features arising when an external force acts on an interior element of such system. We analyse the conditions of the limiting process to the distributed lines and derive the conditions, when a lumped line can be modelled by the methods conventional for distributed lines.

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1. Introduction

We know the multiple types of vibrations, though they are caused by a small number of parameters: such as the system elements masses, the constraints stiffness, the type of dissipative forces and some others. The dependence of vibrations on the model structure conditions it and makes necessary to use a broad gamut of the basic models describing these processes. Even the most general classification divides the models into 1D, 2D and 3D, lumped and distributed, homogeneous and heterogeneous, dissipative and not, having one or few degrees of freedom, forced and free vibrations etc. For each of the mentioned types of basic models a special approach has been developed, and for most of them even several different approaches whose results often do not match. The more, most of the methods have quite narrow specialisation; it essentially hinders the general studying of the vibration process regularities. As we will show below, the methods used for distributed systems cannot give the exact analytical solutions for the entire complex of lumped models; the methods used for infinite models are inapplicable to those finite, and so on.

Many methods which one conventionally thinks analytical can be actually classified as numerical by their essence. They cannot give a complete pattern of all interconnections of a dynamical processes. To move over this limitation, we have developed the original non-matrix method to obtain exact analytical solutions. In [1], [2] and [3] we presented some results which we have obtained for infinite and finite elastic lumped lines. Stating them, we proceeded, on one hand, from the mind that it will be convenient to present the new method, beginning with the simple models easy comparable with the known results in some particular cases. On the other hand, "in the engineering viewpoint, the particular case is important when the oscillators are put in series so that the nth oscillator is connected only with the previous $(n-1)$th and next $(n+1)$th ones. As an example we can take a shaft with the disks fitted on it. The disks behave as the vibrant masses, and the between-the-disks shaft sections serve as the elastic constraints" [4, p.277]. The integrating feature of the models considered in the mentioned sources is that the external force acted always on the start of line. But practically we often meet a case when the external force acts on the interior elements of a line. This difference essentially effects both on the pattern of solution and on the pattern of forced vibrations produced in the line.

In this paper we will analyse the features of vibrations caused by the mentioned reason. In this way we will develop the solutions obtained by the new method, extending them to the more complicated models of elastic systems.

2. Comparative analysis of conventional basic methods

With all diversity of the used methods, we can emphasise a few being basic for them all.

To construct the mathematical model per se, one uses mainly two basic methods: the Lagrange method based on the energy balance, and the method of force balance on the elastic system elements. The first of them gives the scope to state the problem directly in the generalised co-ordinates for the systems having many degrees of freedom; this is advantageous in the fundamental research of complex vibration processes. The second method is more visual and simple, when constructing the particular models of elastic systems, so it is used more often in the applied calculation. Both methods give identical results, so at this stage of mathematical modelling the difficulties are rare, they arise when seeking the solutions. Analysing them, we will confine ourselves to ideal 1D elastic lines related to the subject of this paper.

The conventional methods to find analytical solutions are usually based on one of three approaches to solving the problem of vibrant elastic systems:

- the direct use of the matrix methods (see, e.g., [4]-[10]);
- the integral equations method (see, e.g., [11]);
- using the modified methods basing on the matrix properties; the oscillatory, Voronoy, Toeplitz matrixes etc. are related to them (see, e.g., [12], [13]);

- the indirect methods based on the revealed regularities in the particular modelling systems of differential equations (see, e.g., [4], [14], [15]).

In their turn, the problems of forced vibrations are usually based on the investigation of a homogeneous system of differential equations corresponding to free vibrations in an elastic model, due to the fact that the resonance frequencies coincide for the forced and free vibrations. So it will be efficient to begin our analysis with this type of methods.

In the matrix methods, the following structure of solutions is inherent. Using the Lagrange equation of the second kind

$$\frac{d}{dT}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = -\frac{\partial \Pi}{\partial \dot{q}_j}, \tag{1}$$

(where $T = \frac{1}{2}\sum_{j,k} a_{jk}\dot{q}_j\dot{q}_k$ is the system kinetic energy; $\Pi = \frac{1}{2}\sum_{j,k} c_{jk}q_jq_k$ is the system potential energy; q_j and q_k are the generalised co-ordinates of system) one yields the system of ordinary linear second-order differential equations

$$\sum_{k=1}^{s} T = a_{jk}\dot{q}_k + c_{jk}q_k = 0\,. \tag{2}$$

One seeks its solution in the form

$$q_k = A_k \sin\left(\lambda t + \varepsilon\right), \tag{3}$$

where A_k, λ and ε are the constants that are to be determined. Substituting (3) into (2), after an obvious reduction, one obtains an algebraic system of a following type:

$$\sum_{k=1}^{s}\left(a_{jk}\dot{q}_k + c_{jk}q_k\lambda^2\right)A_k = 0\,. \tag{4}$$

"This system to have a non-trivial solution, it is necessary and sufficient, its determinant to be zero:

$$\Delta = \left\| c_{jk} - a_{jk}\lambda^2 \right\| = 0 \quad " \tag{5}$$

[8, p.529]. "The characteristic equation (5) is an algebraic equation of the $2s$-order with respect to λ, hence, it has $2s$ solutions, i.e. $2s$ eigenvalues $\lambda_\alpha\left(\alpha = 1, 2 \ldots 2s\right)$. After one substitutes the solution λ_α to the system (4), this system will determine the relationship between the 'amplitudes' C_k:

$$\sum_{k=1}^{s}\left(a_{jk}\lambda_\alpha^2 + c_{jk}\right)C_k = 0; \quad \left(j = 1, \ldots s\right). \tag{6}$$

One can write the general solution of (2) as a real (or imaginary) part of the sum of partial solutions, i.e., as

$$q_k = \mathrm{Re}\sum_{\alpha=1}^{2s} C_k^\alpha e^{\lambda_\alpha t} \quad \left(k = 1, \ldots s\right), \tag{7}$$

where the amplitudes C_k^α (or rather their relationship) are determined by (6), and solutions λ_α – by (5)" [7, p.263].

As we see from this consideration, the matrix method, though it is thought analytical, is numerical in its essence, since in general case all main vibration parameters can be obtained only as the numbers. "The solution of these equations requires first to find the proper determinants. Their form and any sizeable number of lines make it practically impossible or extremely laborious" [6, p.98]. "It stands to reason that we could not find the determinant of $\left(4n+4\right)^2$ elements directly ($4n$ integration conditions plus 4 boundary conditions), and it would be useless, since its root values might be calculated only numerically, not with a literal presentation" [6, p.191]. Even to find the frequency and shape of the first main oscillation of a system, one uses different approximate methods, since the exact solution of a problem is impossible at a large number of degree of freedom" [9, p.162]. Above all, when calculating the finite elastic systems, this demerit strongly limits the matrix methods applicability and causes the indirect analytical methods to calculate the vibration processes main parameters. As we mentioned before, these methods are based on some regularities revealed for the specific mathematical models, so their applicability is narrower.

One of the mostly used indirect methods is described by K. Magnus. According to this method, "to calculate the natural vibrations, consider an infinite 1D elastic line. At equal masses $m_p = m$ and spring stiffnesses $c_p = c$ we obtain

$$m\ddot{x}_p = -c\left(x_p - x_{p-1}\right) - c\left(x_p - x_{p+1}\right) . \tag{8}$$

We will seek the known to be existing main vibration of the *p*th mass in the form

$$x_p = X_p \omega s\left(\omega t - \varphi\right) .$$

Substituting this solution into (8), we will yield

$$\left[\left(2c - \omega^2 m\right) X_p - c\left(X_{p-1} + X_{p+1}\right)\right] \cos\left(\omega t - \varphi\right) = 0 \quad " \tag{9}$$

[[4, pp.278-282]. After some modification, (9) reduces to

$$\chi_p = \left(2 - \eta^2\right)\chi_{p-1} - \chi_{p-2} \ , \tag{10}$$

where

$$\eta = \frac{\omega^2 m}{a}; \qquad \chi_p = X_p / X_1; \qquad \left(p = 1, 2, \ldots n\right) .$$

"From this we can sequentially calculate all χ_p, since the boundary conditions, i.e. the conditions at both ends of a chain, are known" [4, p.279]. Further the author limits himself, considering the case when both ends of an elastic line are fixed rigidly, i.e.

$$X_0 = X_{n+1} = 0 . \tag{11}$$

Then (10) lead us to the following result:

$$\chi_1 = 1; \quad \chi_2 = \eta^2 + 2; \quad \chi_3 = \eta^4 - 4\eta^2 + 3; \quad \chi_4 = -\eta^6 + 6\eta^4 - 10\eta^2 + 4 . \tag{12}$$

These functions "are often used to calculate the natural frequencies of vibrant chains (particularly, [16]). The natural frequencies can be calculated as zero values of (*n*+1)th frequency function (12). It means, for the unitless natural frequency η_q

$$\chi_{n+1}\left(\eta_q\right) = 0$$

is true. Thus, the natural frequencies distinguish by the feature that for them the boundary condition $\chi_{n+1} = 0$ specified at the line end is true automatically. The zero tables for the frequency functions up to *n*=11 are given in [16, vol. II, chapter XIII]" [4, p.279].

From this brief description we can see the demerits of the method. To find the natural frequencies, we have to solve the power equation all the same; as is known, the equations higher than the 4th power (in this case of 8th power) cannot be solved exactly in the analytical form. The author gives also the analytical method. "We can express the natural frequency also in the explicit form. In order to do so, try to find the solution of

$$X_p = C \sin p\alpha \ . \tag{13}$$

With it the first condition of (11) is satisfied automatically. To satisfy the second one, the equality

$$\alpha = \frac{\pi q}{n+1}$$

or

$$(n+1)\alpha = q\pi \quad (q = 1, 2, ...) \tag{14}$$

has to be true. On the other hand, when substituting (13) into (9), we obtain

$$C \sin p\alpha \left(2 - \eta^2 - 2\cos\alpha\right) = 0 \,. \tag{15}$$

As the values C=0 and $\sin p\alpha = 0$ are not of our interest, this condition can be true only when

$$\eta^2 = 2\left(1 - \cos\alpha\right) = 4\sin^2\left(\alpha/2\right) \tag{16}$$

or

$$\omega = \eta\omega_0 = 2\omega_0 \sin\left(\alpha/2\right) , \tag{17}$$

$$\omega_q = 2\omega_0 \frac{\pi q}{2\left(n+1\right)} \quad " \tag{18}$$

[4, p.280].

Despite the effective transformation, the cited method does not resolve finally the problems of seeking an analytical solution. Even if ω_q is determined exactly, the vibration amplitude has to be calculated on the basis of polynomials (12), with the complication growing as n grows. And η will be numerical all the same, so this method cannot be thought rigorously analytical. Furthermore, the technique to find ω_q analytically is limited not only by homogeneous finite lines but also by the boundary conditions. As it was shown above, for the fixed ends the method gives a solution, but if at least one end is loaded or free, the entire structure of this technique will be destroyed. Actually, as is shown in [1], for a line with unfixed ends "according to the allowed frequencies, the solution of this system has the form

$$\Delta_i = \frac{A\cos\left[2\left(n-i\right)+1\right]\tau_p}{\cos\left(2n-1\right)\tau_p} e^{j\omega t}; \quad i = 1, 2, ...n \quad " , \tag{19}$$

where Δ_i is the i-th element shift, A is the first element vibration amplitude, $\tau_p = \arcsin\sqrt{\omega^2 m/4s}$ is the parameter corresponding to the allowed frequencies ω_p which are calculated analytically in the process of solution, and s is the line stiffness coefficient. We can see from (19) that at i=n, at the end of a line, the vibration amplitude will not be maximal, as we used to think for a distributed line, but will be shifted by some phase τ_p depending on the allowed mode number. This effect will be especially perceptible at high frequencies, when τ_p is comparable with 1. And the denominator of (19) will also effect on the amplitude. It means, doing not knowing the exact analytical solutions presented in [1], one cannot state a priori the boundary conditions for a free-ends line. However without this the entire Magnus method does not work.

True, for today it is practicable to extend the boundary conditions which are true for distributed lines to those lumped. In particular, this procedure is used in [6, pp.149-150] and [11, pp.48-49] when studying a shaft with n fitted disks and a beam having n lumped masses. None the less, as is shown in [1], (19) fully satisfies the conventional modelling system of differential equations. Therefore, when calculating the exact analytical solutions for a lumped line, we have to consider this difference.

As regards to the known methods to describe the infinite models, this direction is even less clear. The solutions are mainly limited by finding the phase delay of a process along an infinite elastic line by the dispersion characteristics (see, e.g., [14, p.169], [15, pp.106-107]). At the same time Pain writes: "The utmost ultrasonic frequency achieved for today is approximately 10 times less than $\nu = c_0 / 2\pi\alpha$ (the critical frequency). At the range from $5\cdot 10^{12}$ to $1\cdot 10^{13}$ Hz one should expect many interesting experimental results" [14, p.169]. Actually, the results presented in [2] show that in the infinite elastic systems both forced and free vibrations can exist. They can take place not only at the subcritical regime (less than critical frequency), as in finite lines, but also at the overcritical band, when the entire vibration energy localises at a small region, but is not distributed along the entire infinite elastic line. In this case the energy accumulated in an infinite line can be finite. This conclusion is especially important, since many real processes, e.g., in crystals, can be modelled with the help of an ideal infinite model [15, p.105]. The more, such modelling is valid also for the finite elastic lines loaded by a wave impedance. So this is a great demerit that for today in this area one cannot take into account the line features, the external force application point, the description of the pattern of processes etc. To a definite extent this gap was filled in [2]. In the present paper we will study one more important factor – the external force application point influence. But the material interesting and important for the researchers and engineers is well more voluminous and requires a special attention to this research direction.

Passing to the analysis of the methods to calculate the forced vibrations in an elastic system, we should mark first of all that the matrix methods can be divided into two basic ones: the direct solution method (see e.g. [7, pp.296-297]) and the normal co-ordinates method (see e.g. [5], [8, pp.539-560]).

In accord to this first, one seeks the solution of a system of a following type:

$$\sum_{k=1}^{s}\left\{a_{jk}\ddot{\xi}_k + \left(b_{jk}-\gamma_{jk}\right)\dot{\xi}_k + c_{jk}\xi_k\right\} = Q_j^e\left(t\right) \tag{20}$$

in the form

$$\xi_k = a_k^e e^{i\Delta\beta_k^e} e^{j(\omega_e t+\beta_e)} \quad \left(j = 1,2,\ldots,s\right). \tag{21}$$

"This system of the linear heterogeneous equations has the following solution:

$$a_k^e e^{i\Delta\beta_k^e} = \frac{\Delta^k\left(\lambda_e\right)}{\Delta\left(\lambda_e\right)}, \tag{22}$$

where $\Delta\left(\lambda_e\right)$ is the characteristic determinant of (20) taken at the value λ_e, and $\Delta^k\left(\lambda_e\right)$ is the determinant obtained from the characteristic equation by way of substituting the *k*th column elements composed of $Q_1^e, Q_2^e, \ldots Q_s^e$ " [7, p.296].

In (22) the same problems are inherent as in the above method for the free vibrations calculation. For elastic systems having a large number of elements, one has first calculate numerically all λ_e, and then, again numerically, to calculate the determinants in (22). It means that in the essence this method is not analytical too, because it does not indicate in analytical form the pattern of the vibration process dependence on the acting force frequency and the system parameters.

The second matrix method bases on the modification of the initial generalised co-ordinates of a system to the normal co-ordinates:

$$\delta q_i = \sum_{\alpha=1}^{N}\frac{\partial q_i}{\partial\Theta_\alpha}\delta\Theta_\alpha \tag{23}$$

where Θ_α are the normal co-ordinates). In these co-ordinates, the generalised force has a form

$$P_\alpha = \sum_{j=1}^{N} Q_j \frac{\partial q_i}{\partial\Theta_\alpha} = \sum_{j=1}^{N}\Delta_j\left(\lambda^2\right)Q_j . \tag{24}$$

"Using the expression for the kinematic and potential energy in the normal co-ordinates, find

$$A_{\alpha\alpha}\ddot{\Theta}_\alpha + C_{\alpha\alpha}\Theta_\alpha = P_\alpha(t) \quad " \tag{25}$$

[8, p.539].

The equation (25) is similar to a modelling differential equation for a single body having ideal elastic constraints. "We can integrate (25) using the symbolic method of integration. If $P(t)$ was the integrable function with respect to time t_1, we can present the integral of (25) in the following form:

$$\Theta_\alpha = C_{1\alpha}\cos\lambda_\alpha t + C_{2\alpha}\sin\lambda_\alpha t + \frac{1}{A_{\alpha\alpha}\lambda_\alpha}\int_{t_0}^{t} P_\alpha(t_1)\sin\lambda_\alpha(t-t_1)\,dt_1, \tag{26}$$

where C_{1a} and C_{2a} are the integration constants, λ_α is the free vibration frequency and t_0 is the initial moment of time. The term containing $P_\alpha(t)$ is a particular solution of (25)" [8, p.539].

Using this technique, A.N. Krylov [6, pp.161-173] has considered the problem of forced vibrations of a shaft with n fitted gears. In the case "when at the points for which the variable x values being $a_1, a_2, a_3, \dots a_n$ there have been applied the forces $Q_1, Q_2, Q_3, \dots Q_n$ and the pairs whose moments are $M_1, M_2, M_3, \dots M_n$" [6, p.166], "the elastic line equation, when the end of the shaft x=0 is supported, will be

$$y = BT(x) + DV(x) + \Phi(x). \tag{27}$$

But when this end is embedded,

$$y = CU(x) + DV(x) + \Phi(x) \tag{28}$$

and when it is free,

$$y = AS(x) + BT(x) + \Phi(x) \tag{29}$$

(where $S(x) = \frac{1}{2}(\cosh x + \cos x)$; $T(x) = \frac{1}{2}(\sinh x + \sin x)$; $U(x) = \frac{1}{2}(\cosh x - \cos x)$; $V(x) = \frac{1}{2}(\sinh x - \sin x)$ are so-called unit matrixes, which Caushy used in many works), and $\Phi(x)$ is in all these cases one and the same function determined as

$$\Phi(x) = \frac{1}{\alpha^3}\int f(\xi)V(a,(x-\xi))d\xi. \tag{30}$$

The constant arbitrary values are determined by the boundary conditions for the shaft end corresponding to the value x=1" [6, p.166].

"The problem is solvable, if all forces and moments are *given*. But for a rotating shaft neither forces nor moments cannot be given, there are known only their expressions through the masses and inertia moments of the pulleys and sags and inclinations at the places of their fixation" [6, p.167]. In this case, considering the system of equations composed for the boundaries of all homogeneous sections of a shaft, Krylov comes to a conclusion: "We can see from the formulas that, beginning with y_0, we can compose sequentially all other functions that always have the form

$$Bx_1 + Dx_2,$$

where x_1 and x_2 are the functions composed by a definite way for the known functions with respect to the argument kx. Therefore it is obvious that the boundary conditions will have the form

$$\begin{aligned} BG(k) + DH(k) &= 0 \\ BL(k) + DN(k) &= 0, \end{aligned} \tag{31}$$

and the equation for calculation the critical frequencies will be

$$\Delta(k) = G(k)\cdot N(k) - H(k)\cdot L(k) = 0\ . \tag{32}$$

This expression will be so complicated that one can solve it only by sequential approximations, needing for them only a scope to calculate the particular values $\Delta(k)$ specified by the particular value k" [6, p.173].

The similar results are obtained when one uses the integral equations method. 'If an integral equation of a vibrant system with continuously and discretely distributed masses has been composed, then, using the Fredholm method, we can find the eigenvalues. The essence of this method is that the eigenvalues are found as the roots of Fredholm series

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n C_n}{n!}\lambda^n\ , \tag{33}$$

where the coefficients of this series having (n–1) discrete masses are calculated as

$$C_1 = \int_0^{l_n} K(x,s)ds + \sum_{i=1}^{n-1} K(s_i,s)$$

$$C_2 = \int_0^{l_n}\int_0^{l_n} \begin{vmatrix} K(x,x) & K(x,s) \\ K(s,x) & K(s,s) \end{vmatrix} dsdx + 2\sum_{i=1}^{n-1}\int_0^{l_n} \begin{vmatrix} K(x,s) & K(x,s_i) \\ K(s_i,s) & K(s_i,s) \end{vmatrix} dx + \tag{34}$$

$$+\sum_{j=1}^{n-1}\sum_{i=1}^{n-1} \begin{vmatrix} K(s_j,s_j) & K(s_j,s_i) \\ K(s_i,s_j) & K(s_i,s_i) \end{vmatrix}\ .$$

Actually, the eigenvalues calculation needs to construct preliminary the Green function and to find the complicated coefficients C_n, especially when calculating a high-order eigenvalue. Note also that there exists a whole class of the boundary problems for which it is impossible to construct the so-named generalised Green function" [11, p.38]. Using this method, Kuhta and Kravtchenko reduced a particular problem of forced vibrations to the numerical modelling. "To calculate the roots $\Delta(\rho) = 0$ (the frequencies of the considered problem), one estimates their lower boundary. Then, combining the chord technique with the selection technique which is suitably realised by computing in the case of as simple as multiple roots, one determines the necessary number of frequencies in the increasing order" [11, p.47].

The considered matrix method has all above demerits, since it works only at the known values λ_α which one has to find numerically. But not only the common demerits unite these methods. In their essence, they are variations of one and the same approach, since, on one hand, "since the characteristic equation has 2s roots λ_α, then $\Delta(\lambda_\alpha)$ can be presented as the product

$$\Delta(\lambda_\alpha) = a\prod_{\alpha=1}^{2s}(\lambda_e - \lambda_\alpha)\ , \tag{35}$$

where a is some constant" [7, p.296]. On the other hand, the introduction of the main co-ordinates amounts to the simultaneous reducing of two quadratic forms T and U to the canonical form [7, p.266]. We can conclude from it that both matrix methods can predict in analytical form only the fact that "if the attenuation is small, then each amplitude α_k^e will have s resonance peaks at s frequencies $\omega_c \approx \omega_\alpha \quad (\alpha = 1,2,...,s)$. These maximums turn into infinity if the energy dissipation is absent, i.e., $\mu_e \to 0$. In this case

$$\alpha_\alpha^e \sim \frac{1}{\prod_{\alpha=1}^{s}\left(\omega_\alpha^2 - \omega_e^2\right)} \text{ "} \tag{36}$$

[7, p.297].

The techniques using the Voronoy, Toeplitz and other particular matrixes have the essential drawbacks too. Despite all attempts to put these matrixes in order and to reduce them to the diagonal form, these techniques work well, only if the succession of numbers $p_0 > 0, p_1 \geq 0, p_2 \geq 0,...$ was specified. Then at the condition $P_n = \sum_0^n p_k (n = 0,1,2,...)$ [17, p.120] one can try to find the solutions. However the task is just to find these p_0, p_1, p_2,..., and especially p_0, which determine the line start reaction to an external action. But to find these parameters, even in case of simple models, as we showed in [1], one cannot use a priori some stable phase delay, because even in cases of semi-finite and finite lines the line element parameters being under the force action are different. Furthermore, these techniques, the same as the integral and classical matrix, see only one regime – under the critical vibrations.

As a counterpoise to this, the exact analytical solutions obtained by the original non-matrix method in [1] show not one but three vibration regimes: periodical $(\omega < \omega_0)$, critical $(\omega = \omega_0)$ and aperiodical $(\omega > \omega_0)$. In particular, for a line having one (right) fixed end, these solutions have the following form:

at $\omega < \omega_0$

$$\Delta_i = \frac{F_0 \sin 2(n-i)\tau}{\omega\sqrt{ms}\cos(2n-1)\tau} e^{j\omega t}, \tag{37}$$

at $\omega = \omega_0$

$$\Delta_i = (-1)^i \frac{F_0 (n-i)}{s(2n-1)} e^{j\omega t} \tag{38}$$

and at $\omega > \omega_0$

$$\Delta_i = (-1)^i \frac{F_0\left[\gamma_+^{2(n-i)} - \gamma_-^{2(n-i)}\right]}{\omega\sqrt{ms}\left[\gamma_+^{2n-1} - \gamma_-^{2n-1}\right]} e^{j\omega t}, \tag{39}$$

where $\gamma_+ = \beta + \sqrt{\beta^2 - 1}$ and $\gamma_- = \beta - \sqrt{\beta^2 - 1}$.

Comparing (37)-(39) with (36), we see them clearly determined as to the external force parameters as to the elastic line parameters as to the resonance frequencies. With it, when n growing, the difficulty of analysis does not increase. These solutions can be easy extended to a distributed line, which is unrealisable analytically by the matrix methods. Outwardly one can even doubt, whether (37)-(39) are reliable? As [1] shows, (37)-(39) completely satisfy the conventional modelling system of differential equations. And the aperiodical regime is not so much unexpected. The indirect methods, though they give the applicability limitation and incomplete results, in particular cases also corroborate the physical reality of this vibration regime. As an example, consider the original indirect method described by Magnus [4, pp.282-285]. It will be the more interesting for us that when presenting his method, Magnus studied a finite line with the fixed right end too, so we can compare the results.

Magnus has drawn his attention to the relation between α and η in (17); true, in this expression α took the discrete values (14). Besides, considering the forced vibrations, he lifted the fixation of the left end of an elastic line (filter) and made it to move periodically as

$$x_0 = x_e = X_e \cos\Omega t \ . \tag{40}$$

In doing so, he naturally violated one of the boundary conditions that had to lead him to the definite difficulties. Getting them round, he supposed that "with it in the motion equations (8) nothing will change for the individual masses, so we can seek the periodical solution having the same frequency as the excitation, and coming or in phase or in anti-phase with the excitation, supposing

$$x_p = X_p \cos\Omega t \ . \tag{41}$$

As a merit of Magnus method we can mark, he did not follow the conventional way, introducing the boundary conditions for an unfixed end, but suggested a complex way to get over the discrepancy of solutions. "We can now write the cited expression (13) for the amplitude X_p as

$$X_p = C\sin\left(p\alpha + \beta\right) . \tag{42}$$

The introduced constant β gives us the scope to 'adjust' the solution to the boundary condition in the line beginning. The value α makes it impossible, because when substituting (42) into the amplitude relationship (9), we see that the relation between α and η is determined by (16). The distinction from the previous item relation is, when studying the natural vibrations, we have first to find η as a relative natural frequency, and vice versa, for the forced vibrations the relative frequency of excitation η is known" [4, p.282]. On this grounds Magnus has replaced the discrete relation between α and η (14) by the continuous relation $\alpha(\eta)$.

Further, "at the boundary condition given here, taking into account (42), the following requirements are imposed on the amplitudes:

$$\begin{aligned} &X_0 = X_e = C\sin\beta \\ &X_{n+1} = 0 = C\sin\left[\left(n+1\right)\alpha + \beta\right] \quad " \end{aligned} \tag{43}$$

[4, p.283]. The result is

$$x_p = X_e \frac{\sin\left(n+1-p\right)\alpha}{\sin\left(n+1\right)\alpha}\cos\Omega t \ . \tag{44}$$

To compare (44) with (37), note that X_e corresponds to the amplitude of the first element (40), $0 \le p \le n+1$, while $1 \le i \le n$. Noting these features, on the basis of (37), we obtain

$$X_0 = \frac{F_0 \sin 2\left(n+1\right)\tau}{\omega\sqrt{ms}\cos\left(2n+3\right)} . \tag{45}$$

Noting moreover that according to (16) and (19)

$$\alpha = 2\tau \tag{46}$$

and substituting (45) into (37), we obtain

$$\Delta_p = X_0 \frac{\sin\left(n+1-p\right)\alpha}{\sin\left(n+1\right)\alpha} e^{j\omega t} \ , \tag{47}$$

which fully corresponds to (44).

The demerits of Magnus method are seen from the consideration. If lifting the fixation from the second end of a line, the condition (43) will be violated and the whole method will not work. Besides, the Magnus method has established the relationship between the vibration amplitudes of the *p*th and the first body, but did not establish the relation between this body vibration and the external force parameters. As we see from (45), this relationship is quite complex. Basically, this demerit reflects the impossibility to specify vibrations exactly at the free end of a line with the help of boundary conditions, since, as it follows from (45), this amplitude vitally depends on the external force parameters. None the less, despite these demerits, the Magnus method completely corroborates (37)-(39) validity in the particular case of a finite line having one end free. And not only in the band of periodical regime. Basing on (44), Magnus considered further also aperiodical (overcritical) regime. "First of all we see that for all frequencies $\Omega > 2\omega_0$, i.e. for all $\alpha^* = \alpha - i\pi$, the signs before the amplification coefficients alternate, so the chain masses always vibrate in anti-phase with the neighbouring masses" [4, p.284]. "It follows from the hyperbolic sine function behaviour that in the most general case for each mass, with the growing α^*, the more this mass is remote from the chain start the more the amplification coefficient value decreases. For the last mass of a chain $(p = n)$ the amplification coefficient is

$$V_n = (-1)^n \frac{sh\alpha^*}{sh(n+1)\alpha^*}. \tag{48}$$

At quite large n this function decreases so much with the growing frequency that practically we can say, the frequency higher than that boundary is cut off. The chain does not pass the frequencies $\Omega > 2\omega_0$, it works as a low-frequency filter" [4, p.285]. Blakemore [15] in his calculation also obtains the anti-phase vibrations in critical regime for an infinite 1D crystalline lattice. But he considers neither critical nor aperiodical regimes, thinking, due to the incompleteness of his solutions, that the phase delay at the overcritical domain will exceed π. And due to the strong absorption, "the waves having angular frequency exceeding $\omega_m = 2v_0 / a$ cannot exist in an imaginary 1D crystal" [15, p.110]. None the less, in many problems of the applied mechanics, solid physics etc., not only the energy transmission by an elastic line but also the process of energy accumulation and redistribution within the line is important. The local accumulation and redistribution of the vibration energy is inherent in the aperiodical regime. This is just the case when, for example, under an external force the reaction at the support is absent, even under the dynamical load in the excitation region being critical for the elastic constraints. This is a very important aspect, when studying the fatigue processes in elastic systems. We should note here, at the periodical regime the neighbouring elements vibrate in the anti-phase – it means that the constraints are loaded maximally. In this view the case is important when the external force acted on the line interior elements. With it both supports will not experience the load, while in the excitation region the critical vibrations can take place, crushing the internal constraints of an elastic line. And when the external force "action radius" dependence on the frequency and elastic system parameters is complicated, it is important to obtain the solutions of modelling system equations in the analytical form. These indicate the most exactly the measure of each factor effect and allow choosing qualitatively the elastic line parameters dependently on the external force type.

Unfortunately, neither in practical investigation (see e.g. [6], [16]) nor in the fundamental research on the basis of integral equations (see e.g. [11], [12]) this vibration regime is not considered and taken into account. "The resolving of the boundary problems of the vibration theory reduces, in the essence, to the calculation of eigenvalues caused by the natural frequencies or other parameters of the studied system and to the calculation of eigenfunctions (vibration forms). If the eigenvalues and eigenfunctions have been determined, we can think the boundary problem solved... At present a great amount of approximative methods to calculate the eigenvalues has been developed, however they all are quite laborious, give a scope to find only the first eigenvalues and, the main, do not generalise the studying of the systems having discrete or continuous mass distribution. So, as many authors correctly think, the eigenvalues calculus is up to now one of the most important and laborious problems that many researchers are still involved in, in that number we can list Kellatz, Guld, Wilkinson in their lately published fundamental monographs devoted to the problems of natural vibrations. We can find the endeavours to establish the mathematical unification and synthesis of 1D discrete and continuous problems as long ago as in the works by Euler and Lagrange. However up to now setting up the analogies between such boundary problems is under process, for example, in the works by Krain, Atkinson and others" [11, pp.3-4].

In view of the new analytical method, we can figuratively explain the high-order eigenvalues problem, using the model consisting of three sections of elastically linked masses. If the first k_1 bodies have the element mass m_1, the second k_2 bodies have the mass $m_2>m_1$ and the rest $\left(n-k_1-k_2\right)\neq k_1$ bodies − again m_1, then, when the natural frequencies are subcritical for all sections, the resonance frequencies eigenvalues transformed by the mutual influence of sections will correspond to each section. At the frequencies overcritical for a 'heavy' section, the line definitively 'divides' into three regions. The first and the third ones will remain the periodical vibration regime, and the second one − aperiodical. If the vibrations are free, we can expect the 'partial' resonances at any of three sections − two in the subcritical region and two in the outer sections − in the overcritical region. But if the vibrations are forced, then the solution will depend on the external force application point.

So, describing the line by the unified matrix of integral method and doing not taking into consideration the possible transitions in some sections (or individual masses) to the aperiodical regime, we will naturally run into the insurmountable problems in describing the processes in a line. And the fact that neither matrix method nor integral equation method do not reveal the overcritical vibration regime evidences only some incompleteness of these methods and necessity to develop and to consider the features vibration process. The new non-matrix method can be also useful. Some results obtained with its help were presented in [1]-[3], and the solutions analysed in this paper have been obtained by this method too.

3. Impact of the external force application point

Finite line with unfixed ends. To make convenient the comparison with solutions presented in [2], consider a finite free-ends line on whose kth element ($1\le k\le n$) an external harmonic force acts. The general form of such line is given in Fig. 1. A modelling system of equations has the form

$$\begin{cases} m\dfrac{d^2\Delta_1}{dt^2}=s(\Delta_2-\Delta_1) \\ m\dfrac{d^2\Delta_2}{dt^2}=s(\Delta_3+\Delta_1-2\Delta_2) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_k}{dt^2}=F(t)+s(\Delta_{k+1}+\Delta_{k-1}-2\Delta_k) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_n}{dt^2}=s(\Delta_{n-1}-\Delta_n)\ , \end{cases} \tag{49}$$

where $F\left(t\right)=F_0e^{j\omega t}$ is the external force acting on the kth element of the line, m is the mass of the line element, s is the constraints stiffness coefficient and Δ_i is the momentary shift of the ith element from the state of rest.

This system has the following solution:

for the periodical regime $\left(\beta<1\right)$

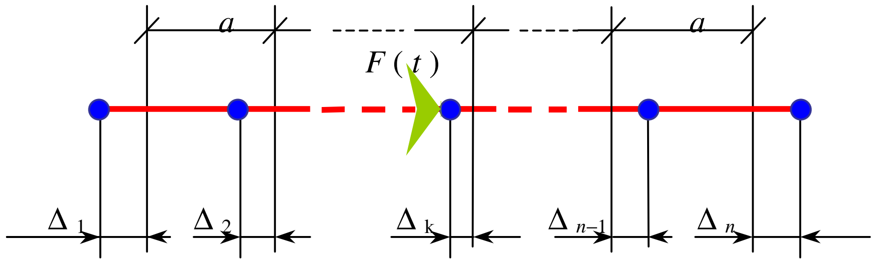

Fig.1. General form of finite elastic line under external force affecting upon its *k*th element

$$\Delta_i = -\frac{F_0\, e^{j\omega t}}{\omega\sqrt{ms}\cos\tau\ \sin 2n\tau}\times\begin{cases}\cos(2i-1)\tau\ \ \cos[2(n-k)+1]\tau;\ 1\le i\le k\\ \cos(2k-1)\tau\ \ \cos[2(n-i)+1]\tau;\ k\le i\le n\ ,\end{cases} \tag{50}$$

for the aperiodical regime $(\beta > 1)$

$$\Delta_i = (-1)^{(i-k+1)}\frac{F_0\, e^{j\omega t}}{\omega\sqrt{ms}(\gamma_+-\gamma_-)(\gamma_+^{2n}-\gamma_-^{2n})}\times\begin{cases}(\gamma_+^{(2i-1)}-\gamma_-^{(2i-1)})(\gamma_+^{[2(n-k)+1]}-\gamma_-^{[2(n-k)+1]});\ 1\le i\le k\\ (\gamma_+^{(2k-1)}-\gamma_-^{(2k-1)})(\gamma_+^{[2(n-i)+1]}-\gamma_-^{[2(n-i)+1]});\ k\le i\le n\end{cases} \tag{51}$$

and for the critical regime $(\beta = 1)$

$$\Delta_i = (-1)^{(i-k+1)}\frac{F_0\, e^{j\omega t}}{4sn}\times\begin{cases}(2i-1)[2(n-k)+1];\ 1\le i\le k\\ (2k-1)[2(n-i)+1];\ k\le i\le n\ ,\end{cases} \tag{52}$$

where

$$\gamma_- = \beta-\sqrt{\beta^2-1}\ ;\ \gamma_+ = \beta+\sqrt{\beta^2-1}\ ;\ \tau = \arcsin\beta\ ;\ \beta = \sqrt{\frac{\omega^2 m}{4s}}\ . \tag{53}$$

Comparing those with the corresponding solutions (13), (16) and (17) presented in [2], we see that in (50)-(52) the additional multipliers appear and the solutions themselves bifurcate into two ones describing the pattern of elements before and after the *k*th one, which reflects the formation of two standing waves in one line. With it the vibration diagram will essentially vary, depending on the number *k* and the frequency. Thus, in the periodical regime with

$$(2k-1)\tau = \frac{\pi}{2}+p\pi\ ,\qquad \text{or}\qquad [2(n-k)+1]\tau = \frac{\pi}{2}+p\pi \tag{54}$$

the vibration amplitude vanishes in some line sections, while in the other sections the vibrations remain. This is shown in Fig. 2 presenting the pattern of the process. In the left diagrams of Figures 2a and 2b, the vibrations in the left and right parts of the line are absent, while in the left diagram of Fig. 2c we see the standing waves of the complex structure. Depending on the vibration frequency, the pattern of standing waves can essentially vary, too. For example, with decreasing frequency there can form the standing waves having the oblique inflections divergent to both sides of the *k*th element.

In the aperiodical regime, the vibration structure also depends on the number *k*, though, in distinction from the periodical regime, vibrations themselves are non-annihilated in any line section. This is well seen in the right diagrams of Figures 2 a, b, c.

At the same time, the solutions presented here are more general than the corresponding solutions in [2]. Thus, for the periodical regime at $k=1$, the first expression of (50) takes the following form:

$$\Delta_i = -\frac{F_0\cos(2n-1)\tau}{\omega\sqrt{ms}\sin 2n\tau}\ e^{j\omega t}\ , \tag{55}$$

.

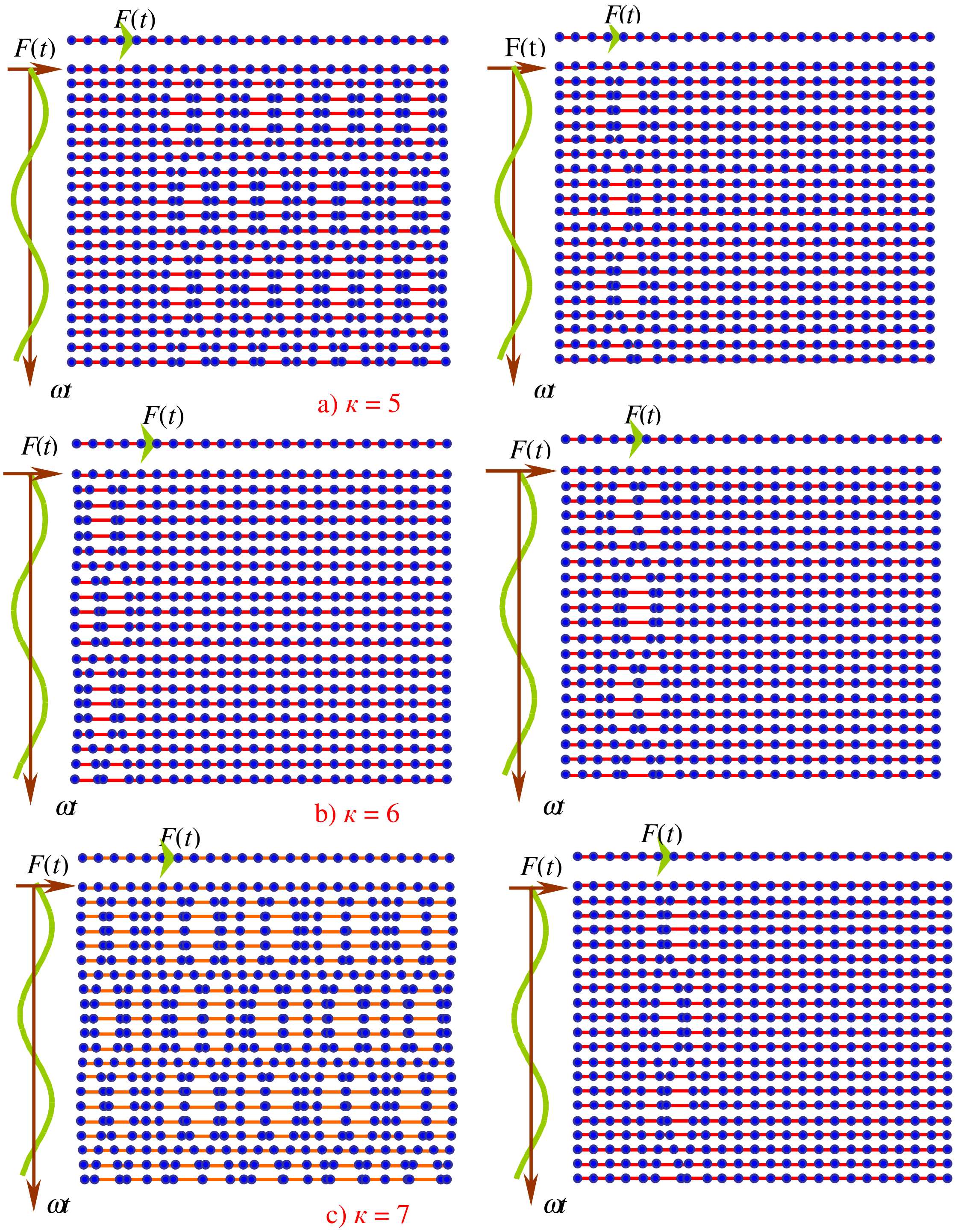

Fig. 2. Diagrams of forced vibrations in finite homogeneous elastic line with unfixed ends, under external force acting on the interior elements of a line. The periodical vibration regime is on the left, the aperiodical one is on the right. The parameters of line: m = 0,01 kg; s = 100 N/m; a = 0,01 m; n = 24; F_0 = 0.48 N, f = 30 Hz – in periodic regime; F_0 = 2,0 N, f = 35 Hz – in aperiodical one. The critical frequency f = 31,8 Hz

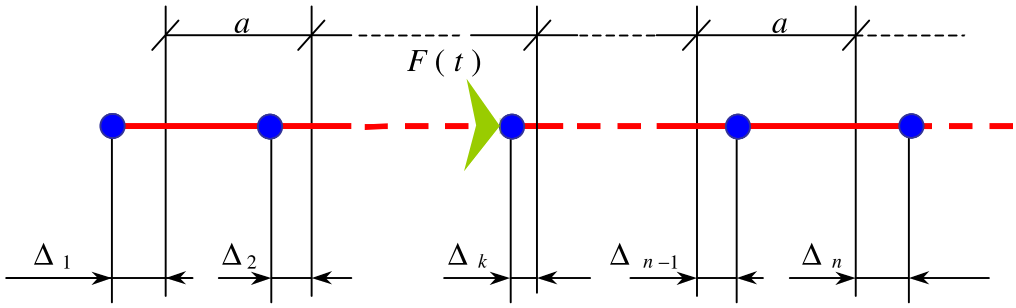

Fig.3. General form of semi-infinite elastic line under external force affecting upon its kth element

which corresponds to the second expression of the given system at i = 1, k = 1. And the second expression itself takes the form

$$\Delta_i = -\frac{F_0 \cos[2(n-i)+1]\tau}{\omega\sqrt{ms}\sin 2n\tau} \quad e^{j\omega t}\,, \tag{56}$$

corresponding to the periodic solution for a finite free-ends line in [1].

Note that when the solutions transform, the definite multipliers disappear; this makes the reverse transition impossible out of direct using the method being the base for these models calculation*Semi-infinite line with the free start*. The external force acting on the line interior elements essentially changes the vibration pattern in a semi-finite elastic line too. To prove it, consider the solution for a semi-finite elastic line with the free start; its analogue was considered in [2]. Its general form is presented in Fig. 3. The modelling system of equations has the following form:

$$\begin{cases} m\dfrac{d^2\Delta_1}{dt^2} = s(\Delta_2 - \Delta_1) \\ m\dfrac{d^2\Delta_2}{dt^2} = s(\Delta_3 + \Delta_1 - 2\Delta_2) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_i}{dt^2} = s(\Delta_{i+1} + \Delta_{i-1} - 2\Delta_i) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_k}{dt^2} = F(t) + s(\Delta_{k+1} + \Delta_{k-1} - 2\Delta_k) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m\dfrac{d^2\Delta_n}{dt^2} = s(\Delta_{n+1} + \Delta_{n-1} - 2\Delta_n) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases} \tag{57}$$

The system (57) has the following form of solution:

in the periodical regime $(\beta < 1)$

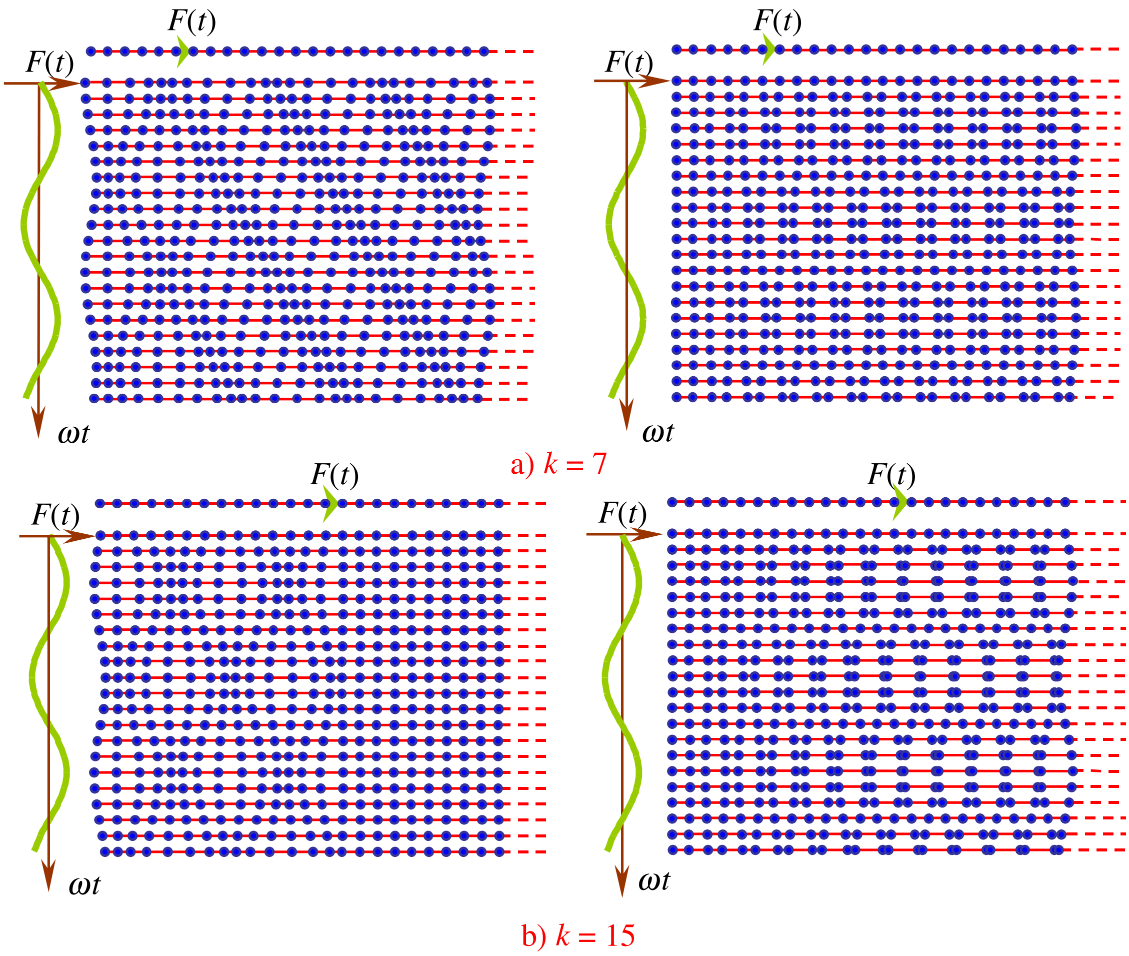

Fig.4. Diagrams of forced vibrations in a semi-finite elastic line under external force acting on the interior elements of a line. The periodical vibration regime is on the left, the critical one – on the right. The parameters of line: m = 0,01 kg; s = 100 N/m; a = 0,01 m; n = 24; F_0 = 0.24 N, f =15 Hz – in periodical regime; F_0 = 0,06 N, f = 31,8 Hz – in critical one

$$\Delta_i = -j\frac{F_0 e^{j\omega t}}{2s\beta\cos\tau}\times\begin{cases}\cos(2i-1)\tau \quad e^{-j(2k-1)\tau} \quad 1\le i\le k\\ \cos(2k-1)\tau \quad e^{-j(2i-1)\tau};\ k\le i\le\infty\end{cases}, \tag{58}$$

in the aperiodical regime $(\beta > 1)$

$$\Delta_i = (-1)^{(i+k-1)}\frac{F_0 e^{j\omega t}}{2s\beta(\gamma_+-\gamma_-)}\times\begin{cases}(\gamma_+^{(2i-1)}-\gamma_-^{(2i-1)})\gamma_-^{(2k-1)};\ 1\le i\le k\\ (\gamma_+^{(2k-1)}-\gamma_-^{(2k-1)})\gamma_-^{(2i-1)};\ k\le i\le\infty\end{cases} \tag{59}$$

and in the critical one $(\beta = 1)$

$$\Delta_i = (-1)^{(i+k-1)}\frac{F_0 e^{j\omega t}}{2s}\times\begin{cases}(2i-1);\ 1\le i\le k\\ (2k-1);\ k\le i\le\infty\end{cases}. \tag{60}$$

The same as in previous example, the force acting on the line interior element has bifurcated the solution into two intervals. On the interval $1\le i\le k$, in the periodical regime, there has arisen a standing wave having some phase delay $(2k-1)\tau$ depending on the number of the picked out element *k*, and on the interval $k \le i \le \infty$ the progressive wave has retained, and its amplitude at the line specified parameters also depends on the number *k*. Hence, at

$$(2k-1)\tau = \frac{\pi}{2} + p\pi \tag{61}$$

the progressive wave amplitude vanishes, and the vibrations are located in a line quasi-finite section bounded by the line end and the external force application point. At the same time, unlike the finite line, vibrations in a quasi-finite section are non-suppressible.

These distinctions are reflected in the diagrams presented in Fig. 4.

Going on comparing finite and semi-finite lines, note that at $k = 1$, i.e. in case when the force acts upon the first line element, the solutions (58)–(60) transform into the corresponding solutions of [2] losing the multipliers being specific namely for the given structure of the generalising model, which makes the reverse transition also impossible.

4. The feature of the line heterogeneity at the external force application point

On the presented examples we can run down a number of common features caused by the external force acting on the line interior elements. In both cases the point of application mattered as a heterogeneity on which the vibrations pattern transforms. Due to it, in a finite line there have formed two sections with different vibration patterns. In a semi-finite line there formed a quasi-bounded section with the standing wave, and in a infinite line in [2] there arose two progressive waves propagating in opposite directions, which enables us speaking about the feature of a line heterogeneity at the external force application point.

5. The progressive wave parameters in an infinite elastic line

Investigating the features of models for homogeneous elastic lines, we should note one more peculiarity connected with the parameters of progressive waves propagating along the infinite lumped lines. For these lines the wavelength λ has quite conventional pattern, because not always it is connected with the specific lumped masses; i.e., far from always we can select two bodies at a distance of wavelength vibrating in phase. However, the body-to-body phase delay equal to 2τ indicates also the definite wavelength fully characterising the line processes.

To determine λ mathematically, we have to suppose that at the wavelength distance from a picked out element there exists some fictitious line element vibrating in phase with this picked out element.

Noting that, in case of non-damping process, the distance between the elements vibrating in phase does not depend on the amplitude, the wavelength is determined by the expression

$$\lambda = \frac{2\pi}{2\tau} a = \frac{\pi a}{\tau} . \tag{62}$$

On the grounds of (62), the propagation velocity v in the line is determined by the expression

$$v = \frac{\lambda\omega}{2\pi} = \frac{a\omega}{2\tau} . \tag{63}$$

It is typical that in distinction from distributed lines, in the investigated models the propagation velocity does not remain constant and depends on the vibration frequency. Therefore, when a complicated-spectral-composition mechanical pulse is fed to the line input, this pulse does not hold its structure when

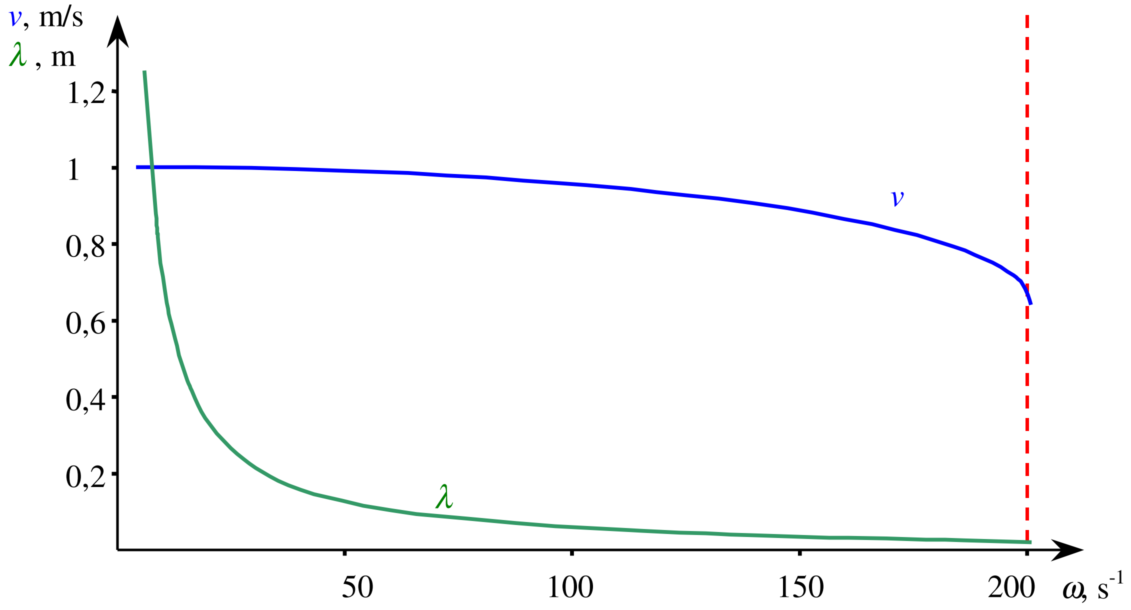

Fig.5. Plot of $v(\omega)$, $\lambda(\omega)$. Parameters of elastic line: m = 0,01 kg; s = 100 N/m; a = 0,01 m; ω_{crit}= 200 rad/s; λ_{crit}= 0,02 m

propagating along the line. And only at $\beta << 1$ the propagation velocity stabilises, coming in the limit to the known value

$$v = \sqrt{\frac{T}{\rho}} \; . \tag{64}$$

With it the wavelength will be

$$\lambda = \frac{2\pi}{\omega}\sqrt{\frac{T}{\rho}} \; , \tag{65}$$

which also corresponds to the known value.

Furthermore, the presented regularity of the phase velocity of wave propagation with respect to frequency much complicates, or rather makes impossible to obtain straight the exact analytical solutions for the acting complex-form pulses, particularly the rectangular pulses, out of their decomposing into a spectrum. This is especially important to note, because many authors try currently to solve the dynamical problems, giving as the statement just rectangular pulses. Hence, with such approach, the researchers make themselves in advance disable to obtain a high-quality solution.

When the frequency tends to the second boundary value of the band related to the periodical regime, i.e., when $\beta \to 1$,

$$\lambda = \lim_{\beta \to 1} \frac{\pi a}{\arcsin\beta} = 2a \; . \tag{66}$$

Correspondingly, the velocity

$$v = \lim_{\beta \to 1} \frac{a\omega}{2\arcsin\beta} = \frac{a\omega_{crit}}{\pi} = \frac{2a}{\pi}\sqrt{\frac{s}{m}} = v_{crit} \; , \tag{67}$$

because, proceeding from (53), at $\beta = 1$

$$\omega_{crit} = 2\sqrt{\frac{s}{m}} \; . \qquad (68)$$

It means, when transiting to the critical regime and further to that aperiodical, the wavelength tends to the double distance between the non-excited elements of a line; it corresponds to the anti-phase vibrations which we observed both in [2] and in the present paper. Similarly, the propagation velocity does not vanish with the limiting process, but stabilises at some ultimate value v_{crit}. To illustrate it, in Fig. 5 the typical regularities $\lambda(\omega)$ and $v(\omega)$ are presented.

It presents a difficulty to investigate more detailed the group and phase velocities in the frames of an ideal elastic line model, because the fact itself that the elements vibrate in anti-phase in the critical and aperiodical regimes does not evidence yet that the wave propagation velocity in these regimes is absent. We showed it obviously in our paper [18], when analysing the wave propagation processes in a resistant line. Passing to the limit at the vanishing line resistance, we showed that in an ideal line the phase velocity in the aperiodical regime grows linearly with the growing frequency from the minimal value (67). Generally, the group velocity also exists at the overcritical frequency, but in transition to an ideal line its value actually turns into infinity. Just this last caused the incorrect conclusion settled in the literature that in elastic lines the group velocity does not exist at the overcritical frequencies. Up to now the scientists used in modelling some fragmentary results accessible in the absence of the exact analytical solutions, and only for ideal lines (see, e.g., [15]). However the exact analytical solutions show that if a line had a least impedance, the group velocity of the wave propagation will exist, though its value will be very large. More detailed, see in the indicated paper.

6. The limiting process to a distributed line

Investigating the limiting process for the vibration parameters, in view of the practice of modelling, it is important to determine the conditions, when the line elements can be considered as those distributed.

To determine them, consider the parameter β. As λ and v were determined, (53) takes the form

$$\beta^2 = \frac{\omega m}{4s} = \frac{\pi^2 v^2 m}{\lambda^2 s} = \frac{\pi^2 v^2 \rho}{T}\,\frac{a^2}{\lambda^2} \; , \qquad (69)$$

where $\rho = m/a$ is the line elements density, and $T = sa$ is the line tension. Since the transition to a distributed line occurs at $\beta \to 0$, we can write the sought condition so:

$$\beta = \pi v \frac{a}{\lambda}\sqrt{\frac{\rho}{T}} << 1 \; ,$$

whence

$$\frac{a}{\lambda} << \frac{1}{\pi v}\sqrt{\frac{T}{\rho}} \qquad (70)$$

follows. Noting that, when transiting to a distributed line, the velocity varies slightly, we can simplify (70):

$$\frac{a}{\lambda} << \frac{1}{\pi} < \frac{1}{\pi v}\sqrt{\frac{T}{\rho}} \; ,$$

whence

$$\frac{a}{\lambda} << \frac{1}{\pi} . \tag{71}$$

It means that only when the wavelength well exceeds the distance between the line elements, this line parameters can be considered as those distributed.

The limiting process to a distributed line can be effected on the basis of obtained solutions at the condition $\beta \to 0$, if the main parameters characterising the line were presented in the form convenient for it, i.e.,

$$\rho = \frac{m}{a} \; ; \; s = \frac{T}{a} \; ; \; n = \frac{x_0}{a} \; , \tag{72}$$

where x_0 is the distance from the reading-point to the point of rest of the considered line element.

Besides, the necessary requirement is, the values ρ, T and x_0 to remain finite in the limiting process. For example, for the solution (50) we yield

$$\delta_i = x_i - x_{0i} = -\frac{F_0 e^{j\omega t}}{\omega\sqrt{T\rho} \; \sin\omega l\sqrt{\rho/T}} \times \begin{cases} \cos\omega x_{0i}\sqrt{\rho/T} \quad \cos\omega(l - x_{0k})\sqrt{\rho/T}; \; 1 \le x_{oi} \le x_{ok} \\ \cos\omega x_{0k}\sqrt{\rho/T} \quad \cos\omega(l - x_{0i})\sqrt{\rho/T}; \; x_{0k} \le x_{0i} \le l \; , \end{cases} \tag{73}$$

where x_{0i} is the investigated point location in the non-excited state; x_{0k} is the external force application point location, in the non-excited state too; x_i is the momentary location of investigated point in the excited state; l is the line length, and δ_i is the line element shift.

As follows from (73), when transiting to a distributed line, there retains the main distinction caused by the external force application point sectioning the line. Consequently, the vibration structure remains on the whole and, at definite conditions, the vibrations are produced only in a part of a line. With it, when passing to the limit, the expression (50) transforms irreversibly, which makes the reverse transition impossible.

When investigating the conditions of transition to a distributed line, it is interesting to consider, how the solutions for critical and aperiodical regimes transform. Conveniently investigate this not on the basis of particular solutions but at the condition of these regimes realisation. When $a \to 0$, noting (53) and (72), we yield

$$\lim_{a\to 0} \beta = \lim_{a\to 0} a\sqrt{\frac{\omega^2\rho}{4T}} = 0 \; , \tag{74}$$

whence it follows that of all three vibration regimes only that periodical can be realised with the limiting process. For two other regimes the conditions of existence will be simply absent.

Thus, analysing the transition to a distributed line, we see disappearing the number of properties inherent namely in lumped lines, which makes non-identical these models description. We can pass from the lumped line description to that distributed, but vice versa we cannot. There is allowable only some particular approximation, when the condition (71) was fulfilled.

7. Conclusions

In the course of the carried out investigation we have revealed that not only the line finiteness, the way of the ends fixation or vibration regime has a great effect on the vibration pattern of an external force application point. This point is able, particularly, to be a border of a quasi-finite section where the standing wave forms, to form the complicated-structure-vibrations and conditions at which the vibration process completely vanishes in one of the sections of line, while other sections retain the vibrations. This all gives the grounds to indicate the feature of heterogeneity which the external force application point causes in the line.

We also have revealed that in lumped lines the progressive wave propagation velocity decreases non-linearly and monotonously, with the growing vibration frequency, reaches the minimal value at ω_{crit} and retains it in the aperiodical regime.

Similarly, the wavelength in the line decreases nonlinearly to the double distance between the line elements, and it retains this value in the aperiodical regime.

At the limit passing to the distributed line, the aperiodical and critical vibration regimes become impossible. Besides, at this passing, the main distinctions of the solutions, caused by an external force action on the interior elements, retain. At the same time, when transiting to a distributed line, the solutions transform irreversibly, which makes the reverse transition impossible.

We have established also that a lumped line can be modelled by means of that distributed only when the ratio of between-the-elements distance to the wavelength is considerably less than π^{-1}.

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