Supersymmetry and Brownian motion on supermanifolds

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Abstract

An anticommuting analogue of Brownian motion, corresponding to fermionic quantum mechanics, is developed, and combined with classical Brownian motion to give a generalised Feynman-Kac-Itô formula for paths in geometric supermanifolds. This formula is applied to give a rigorous version of the proofs of the Atiyah-Singer index theorem based on supersymmetric quantum mechanics. It is also shown how superpaths, parametrised by a commuting and an anticommuting time variable, lead to a manifestly supersymmetric approach to the index of the Dirac operator. After a discussion of the BFV approach to the quantization of theories with symmetry, it is shown how the quantization of the topological particle leads to the supersymmetric model introduced by Witten in his study of Morse theory.

1 Introduction

This survey concerns a battery of generalised probabilistic techniques, originally motivated by path integration in fermionic and supersymmetric quantum physics, which may be brought to bear on some significant geometric operators.

A theory of Brownian motion in spaces with anticommuting coordinates is developed, together with the corresponding stochastic calculus. This is combined with conventional Brownian motion to provide a mathematically rigorous version of the direct and intuitive proofs of the Atiyah-Singer index theorem based on supersymmetric quantum mechanics [1, 2], and also of the quantum tunneling calculations which are necessary to build Witten's link between supersymmetry and Morse Theory [3].

The connection between anticommuting variables and geometry arises via Clifford algebras. It seems to have first been observed in the context of canonical anticommutation relations for fermi fields (which are essentially Clifford algebra relations) that such algebras have a simple representation in terms of differential operators on spaces of functions of anticommuting variables. It was thus in the physics literature that functions of anticommuting variables were first considered, beginning with the work of [4] and ideas of Schwinger [5], and extensively developed by Berezin [6] and by DeWitt [7].

Anticommuting variables are not used to model physical quantities directly; their use is motivated by the algebraic properties of the function spaces of these variables. In application to physics, results which are real or complex numbers emerge after what has become known as Berezin integration (defined in equation (9)) which essentially takes a trace. The approach to fermions using anticommuting variables is particularly useful in the context of supersymmetry symmetry because bose and fermi degrees of freedom, which are related by symmetry transformations, can then be handled in the same way. Similarly, when using BRST techniques to consider theories with symmetry, the use of anticommuting variables to handle ghost degrees of freedom provides a unifying approach.

At this stage a comment on the prefix 'super' is appropriate. Originally used in the physical term *supersymmetry*, to describe a symmetry which mingles bosonic and fermionic degrees of freedom, the word has migrated into mathematics to describe a generalisation or extension of a classical object to an object which is in some sense Z_2-graded, with an even part usually associated with the classical, commuting object and an odd part associated with some anticommuting analogue.

A generic superspace is a space with coordinates some of which are commuting and others anticommuting. There are two approaches to superspace, a concrete one which generalises the actual space, and an abstract one which generalises the algebra of functions on such a space. Here generally the concrete approach will be used because it allows simpler super analogues of the classical concepts considered; however it should be born in mind that the function algebras used in the concrete approach will normally correspond to the generalised algebras in the more abstract approach, so that the difference is more apparent than real.

The anticommuting analogue of probability theory developed here does not mirror all aspects of classical probability theory, but is largely restricted to features which relate directly to the heat kernels of differential operators.

In section 2 of this survey superspace is described, the concept of supersmooth function is defined together with the related notions of differentiation and integration. The construction of a particular class of supermanifold, obtained by building odd dimensions onto a manifold using the data of a vector bundle, is also given. These supermanifolds play a key role in the geometric applications of the anticommuting probability theory, because they carry natural classes of supersmooth functions which correspond to spaces of forms and spinors on the underlying manifold. Section 3 introduces fermionic and supersymmetric quantum mechanics, explaining quantization in terms of functions of commuting and anticommuting variables. Systems are described whose Hamiltonians correspond to a number of geometric Laplacians.

Section 4 introduces the anticommuting analogue of probability measure, and in particular fermionic Wiener measure. In section 5 fermionic Wiener measure is combined with classical Wiener measure to give a super Wiener measure for paths in superspace. The stochastic calculus for the corresponding super Brownian paths is developed. Next, in section 7, these techniques are used to define Brownian paths on supermanifolds and derive a Feynman-Kac-Itô formula for certain Hamiltonians.

The concept of supersymmetry implies more than the mere presence of anticommuting and commuting variables; an odd operator, known as the supercharge, which squares to the Hamiltonian or Laplacian of the theory is required. (In some cases the Hamiltonian is the sum of squares of more than one supercharge.) Supersymmetric quantum mechanical models are described in section 3, while geometric examples of supercharges are the Hodge-De Rham operator and the Dirac operator which are considered in later sections. Using the 'super' stochastic machinery, these operators can be analysed. In section 8 the methods are applied to make rigorous the supersymmetric proofs of the Atiyah-Singer index theorem given by Alvarez-Gaumé [1] and by Friedan and Windey [2], while in section 11 it is used to verify the key instanton calculation in Witten's linkage of Morse theory with supersymmetric quantum mechanics. The Dirac operator is studied in section 9, where a further super construction, the Brownian super path parametrised by both a commuting time t and an anticommuting time τ, is introduced. Anticommuting variables may also be used to handle the ghost degrees of freedom used in the quantization of theories with symmetry. These ideas are briefly described in section 10, and the used in the following section to show that quantization of the topological particle leads to the supersymmetric model used by Witten to analyse Morse theory.

Other work involving anticommuting analogues of probability measures includes that of Applebaum and Hudson [8], Barnet, Streater and Wilde [9], Haba and Kupsch [10]. An alternative approach to considering paths in superspace is to consider differential forms on loop space, as in the work of Jones and Léandre [11].

The work in this survey shows how an extended notion of path integral may be applied to a variety of quantum mechanical systems. When considering quantum field theory the standard approach is by functional integrals, but an interesting alternative approach, involving Brownian motion over loop groups, and over torus groups and higher dimensional objects, has been considered in papers by Brzezniak and Elworthy [12], by Brzezniak and Léandre [13], and by Léandre [14]. Here too fermionic analogues might be possible.

2 Superspace and supermanifolds

This section introduces the analysis and geometry of anticommuting variables. As explained in the introduction, we use a concrete approach to the concept of supermanifold. We begin with some algebraic notions, leading to the key concept of supercommutative superalgebra.

Definition 2.1 A *super vector space* is a vector space V together with a direct sum decomposition

$$V = V_0 \oplus V_1. \tag{1}$$

The subspaces V_0 and V_1 are referred to respectively as the *even* and *odd* parts of V.

We will normally consider homogeneous elements, that is elements x which are either even or odd, with parity denoted by ϵ_X so that $\epsilon_X = i$ if x is in $\mathbb{R}_{S,i}$, $i = 0, 1$. (Arithmetic of parity indices $i = 0, 1$ is always modulo 2.)

Definition 2.2 (a) A *superalgebra* is a super vector space $A = A_0 \oplus A_1$ which is also an algebra which satisfies $A_i A_j \subset A_{i+j}$.

(b) The superalgebra is *supercommutative* if, for all homogeneous X, Y in A, $XY = (-1)^{(\epsilon_X \epsilon_Y)} YX$.

The effect of this definition is that in a superalgebra the product of an even element with an even element and that of an odd element with an odd element are both even while the product of an odd element with an even element is odd; if the algebra is supercommutative then odd elements anticommute, and the square of an odd element is zero.

The basic supercommutative superalgebra used is the real Grassmann algebra with generators $\mathbf{1}, \beta_1, \beta_2, \ldots$ and relations

$$\mathbf{1}\beta_i = \beta_i \mathbf{1} = \beta_i, \qquad \beta_i \beta_j = -\beta_j \beta_i. \tag{2}$$

This algebra, which is denoted $\mathbb{R}_S$, is a superalgebra with $\mathbb{R}_S := \mathbb{R}_{S,0} \oplus \mathbb{R}_{S,1}$ where $\mathbb{R}_{S,0}$ consists of linear combinations of products of even numbers of the anticommuting generators, while $\mathbb{R}_{S,1}$ is built similarly from odd products.

It is useful to introduce multi-index notation: for a positive integer n, let M_n denote the set of all multi-indices of the form $\mu := \mu_1 \ldots \mu_k$ with $1 \leq \mu_1 < \ldots < \mu_k \leq n$ together with the empty multi-index $\emptyset$; also let $|\mu|$ denote the length of the multi-index μ, and for any suitable n-component object $\xi^1, \ldots, \xi^n$ define $\xi^\emptyset := \mathbf{1}$ (the appropriate unit for the objects ξ^i) and $\xi^\mu := \mathbf{1}\xi^{\mu^1} \ldots \xi^{\mu_{|\mu|}}$. The set M_∞ is defined in a similar way, but with no upper limit on the indices. A typical element A of $\mathbb{R}_S$ may then be expanded as

$$A = \sum_{\lambda \in M_\infty} A_\lambda \beta_\lambda \tag{3}$$

where each coefficient A_μ is a real number. For each $\lambda \in M_\infty$ there is a *generator projection* map

$$P_\lambda : \mathbb{R}_S \to \mathbb{R}, \quad A \mapsto A_\lambda, \tag{4}$$

a particular case of this being the augmentation map

$$\epsilon : \mathbb{R}_S \to \mathbb{R}, \quad A \mapsto A_\emptyset. \tag{5}$$

Clearly $\mathbb{R}_S$ is an infinite-dimensional vector space. However our use of $\mathbb{R}_S$ is purely algebraic, so that we do not need to equip it with a norm, and the following notion of convergence is sufficient:

Definition 2.3 A sequence $\text{x}_k, k = 1, \ldots$ of elements of $\mathbb{R}_S$ is said converge to the limit x in $\mathbb{R}_S$ if each coefficient $\text{x}_{k,\lambda}$ in the expansion $\text{x}_k = \sum_{\lambda \in M_\infty} \text{x}_{k,\lambda}\beta_\lambda$ converges to the coefficient x_λ in the expansion $\text{x} = \sum_{\lambda \in M_\infty} \text{x}_\lambda \beta_\lambda$.

The Grassmann algebra $\mathbb{R}_S$ is used to build (m,n)-dimensional superspace $\mathbb{R}_S^{m,n}$ in the following way:

Definition 2.4 (m,n)-dimensional *superspace* is the space

$$\mathbb{R}_S^{m,n} = \underbrace{\mathbb{R}_{S\,0} \times \ldots \times \mathbb{R}_{S\,0}}_{m \text{ copies}} \times \underbrace{\mathbb{R}_{S\,1} \times \ldots \times \mathbb{R}_{S\,1}}_{n \text{ copies}}. \tag{6}$$

A typical element of $\mathbb{R}_S^{m,n}$ is written $(x;\xi)$ or $(x^1, \ldots, x^m; \xi^1, \ldots, \xi^n)$, where the convention is used that lower case Latin letters represent even objects and lower case Greek letters represent odd objects, while small capitals are used for objects of mixed or unspecified parity.

Consider first functions with domain the space $\mathbb{R}_S^{0,n}$ of which a typical element is $\xi := (\xi^1, \ldots, \xi^n)$, that is, functions of purely anticommuting variables. (For simplicity it will be assumed that n is an even number.) We will consider functions on this space which are multinomials in the anticommuting variables. Using the multi-index notation introduced above, a multinomial function F may then be expressed in the standard form

$$\begin{aligned} \text{F} : \mathbb{R}_S^{0,n} &\longrightarrow \mathbb{R}_S \\ (\xi^1, \ldots, \xi^n) &\mapsto \sum_{\mu \in M_n} \text{F}_\mu \xi^\mu \end{aligned} \tag{7}$$

where the coefficients F_μ are real numbers. Such functions will be known (anticipating the terminology for functions of both odd and even variables) as *supersmooth*. More general supersmooth functions, with the coefficients F_μ taking values in $\mathbb{C}$, $\mathbb{R}_S$, or some other algebra are also possible.

Differentiation of supersmooth functions of anticommuting variables is defined by linearity together with the rule

$$\frac{\partial \xi^\mu}{\partial \xi^j} = \begin{cases} (-1)^{k-1}\xi^{\mu_1} \ldots \widehat{\xi^k} \ldots \xi^{\mu_{|\mu|}}, & \text{if } j = \mu_k \text{ for some } k,\ 1 \le k \le |\mu|, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

where the caret $\widehat{\ }$ indicates an omitted factor.

Integration of functions of purely anticommuting variables is defined algebraically by the Berezin rule [4, 6]:

$$\int_B d^n\xi\, F(\xi) = F_{1\ldots n}, \tag{9}$$

where $F(\xi) = \sum_{\mu\in M_n} F_\mu \xi^\mu$ as in (7), so that $F_{1\ldots n}$ is the coefficient of the highest order term.

This integral can be used to define a Fourier transform with useful properties. If F is a supersmooth function on $\mathbb{R}_S^{0,n}$, then the Fourier transform $\hat{F}$ of F is defined by

$$\hat{F}(\rho) = \int_B \mathrm{d}^n\xi\, F(\xi) \exp i\rho.\xi \tag{10}$$

where $\rho.\xi = \sum_{i=1\ldots n} \rho^i\xi^i$. A simple calculation establishes the Fourier inversion theorem

$$\hat{\hat{F}} = F. \tag{11}$$

Any linear operator K on the space of supersmooth functions of purely anticommuting variables has integral kernel $[K]$ taking $\mathbb{R}_S^{0,n}\times\mathbb{R}_S^{0,n}$ into $\mathbb{R}_S$ defined by

$$KF(\xi) = \int_B d^n\theta\, [K](\xi,\theta)F(\theta). \tag{12}$$

Using the Fourier inversion theorem we see that

$$\begin{aligned}\delta(\xi,\theta) &= \int_B \mathrm{d}^n\rho\, \exp i\left(-\rho.(\xi-\theta)\right)\\ &= \prod_{i=1}^n (\xi^i - \theta^i)\end{aligned} \tag{13}$$

is the kernel of the identity operator, that is

$$\int \mathrm{d}^n\theta\delta(\xi,\theta)F(\theta) = F(\xi). \tag{14}$$

More generally, if K is a differential operator acting on supersmooth functions on $\mathbb{R}_S^{0,n}$, then

$$[K](\xi,\theta) = K_\xi\delta(\xi,\theta) \tag{15}$$

where the subscript ξ indicates that derivatives are taken with respect to this variable.

The link between anticommuting variables and Clifford algebras, which underpins the various constructions in this paper, arises from the operators

$$\psi^i = \xi^i + \tfrac{\partial}{\partial \xi^i}, \qquad i = 1, \ldots, n \tag{16}$$

which satisfy the anticommutation relations

$$[\psi^i, \psi^j] = \psi^i\psi^j + \psi^j\psi^i = \delta^{ij}. \tag{17}$$

(In the super algebra context, the commutator of two operators A and B is defined by $[A, B] = AB - (-1)^{(\epsilon_A \epsilon_B)} BA$.)

If μ is a multi-index in M_n then

$$[\psi^\mu](\xi, \theta) = {}_{B}\!\!\int \mathrm{d}^n\rho (\xi + i\rho)^\mu \exp i\left(-\rho.(\xi - \theta)\right), \tag{18}$$

a fact which leads to the fermionic Feynman-Kac formulae exploited in this paper.

In the case of operators acting on a graded algebra a useful and natural quantity is the supertrace. If γ is the operator on the algebra in question which acts as 1 on even elements and -1 on odd elements, the supertrace is of an operator K is defined to be the trace of the operator γK. It may readily be shown that integration of the kernel of K at coincident points gives the supertrace, that is

$$\mathrm{Supertrace} K = {}_{B}\!\!\int d^n\xi\, [K](\xi, \xi) \tag{19}$$

while the standard trace may be obtained from the formula

$$\mathrm{Trace} K = {}_{B}\!\!\int d^n\xi\, [K](\xi, -\xi). \tag{20}$$

More general supertraces also occur, where γ is replaced by some other involution.

In order to extend the notion of supersmooth to functions on the more general superspace $\mathbb{R}_S^{m,n}$, we must take note of the fact that an even Grassmann variable is not simply a real or complex variable. However the necessary class of functions can be captured by defining supersmooth functions on $\mathbb{R}_S^{m,0}$ as extensions by Taylor expansion from smooth functions on $\mathbb{R}^m$.

Definition 2.5 The function $F : \mathbb{R}_S^{m,0} \to \mathbb{R}_S$ is said to be *supersmooth* if there exists a function $\tilde{F} : \mathbb{R}^m \to \mathbb{R}_S$ whose combination with each generator projection (c.f. (4)) is smooth, such that

$$\begin{aligned} F(x^1, \dots, x^m) &= \tilde{F}(\epsilon(x)) + \sum_{i=1}^{n} (x^i - \epsilon(x^i)\mathbf{1}) \frac{\partial \tilde{F}}{\partial x^i}(\epsilon(x)) \\ &\quad + \sum_{i,j=1} (x^i - \epsilon(x^i)\mathbf{1})(x^j - \epsilon(x^j)\mathbf{1}) \frac{\partial^2 \tilde{F}}{\partial x^i \partial x^j}(\epsilon(x)) \dots . \end{aligned} \tag{21}$$

(Although this Taylor series will be infinite, it gives well defined coefficients for each β_λ in the expansion (3), so that the value of F is a well-defined element of $\mathbb{R}_S$.)

A supersmooth function on the general superspace $\mathbb{R}_S^{m,n}$ can now be defined.

Definition 2.6 A function $F : \mathbb{R}_S^{m,n} \to \mathbb{R}_S$ is said to be supersmooth if there exist supersmooth functions $F_\mu, \mu \in M_n$ of $\mathbb{R}_S^{m,0}$ into $\mathbb{R}_S$ such that

$$F(x, \xi) = \sum_{\mu \in M_n} F_\mu(x) \xi^\mu \tag{22}$$

for all (x, ξ) in $\mathbb{R}_S^{m,n}$.

Integration of supersmooth functions is defined by a combination of Berezin integration and conventional Riemann integration.

Definition 2.7 If $F : \mathbb{R}_S^{m,n} \to \mathbb{R}_S$ is supersmooth and $V \subset \mathbb{R}^m$, then the integral of F over V is defined to be

$$\int_{S\,V} d^m x \, d^n \xi \, F(x, \xi) = \int_V d^m x \left(\int_B d^n \xi \, F(x, \xi) \right). \tag{23}$$

Using the Berezinian, which is the superdeterminant of the matrix of partial derivatives, a change of variable rule may be obtained which is valid for functions of compact support.

Up to this point we have used the prefix super merely to indicate the presence of an anticommuting extension of some classical commuting object. The concept of supersymmetry involves the further feature that the Hamiltonian (or, geometrically, the Laplacian) of the system is the square of an odd operator known as the supercharge, or the sum of squares of several supercharges. Many examples of this are given in the following section on supersymmetric quantum mechanics. The corresponding time evolution may also have a square root, which will be defined by introducing an odd parameter τ in conjunction with the usual time parameter t, and defining some rather special objects on the $(1,1)$-dimensional superspace parametrised by $(t;\tau)$. The starting point is the superderivative $D_{(t;\tau)} = \frac{\partial}{\partial\tau} + \tau\frac{\partial}{\partial t}$ acting on functions $F(t;\tau)$ on the superspace $\mathbb{R}_S^{1,1}$. Since $(\frac{\partial}{\partial\tau})^2 = 0$, this has the property that

$$(D_{(t;\tau)})^2 = \frac{\partial}{\partial t}, \tag{24}$$

so that $D_{(t;\tau)}$ is a square root of the generator of time translations. It is also possible to introduce a notion of integration between even and odd limits of a function F on $\mathbb{R}_S^{1,1}$ in the following way:

$$\int_0^\tau \int_0^t D[s;\sigma] F(s;\sigma) = \int_B d\sigma \int_0^{t+\sigma\tau} ds F(s;\sigma). \tag{25}$$

It may then be shown by direct calculation that this integral provides a square root of the fundamental theorem of calculus in the sense that that

$$\int_0^\tau \int_0^t D[s;\sigma] D_{(s;\sigma)} F(s;\sigma) = F(t;\tau) - F(0;0). \tag{26}$$

If we now introduce a superpath $X : \mathbb{R}_S^{1,1} \to \mathbb{R}_S^{p,q}$ and let

$$DX^i = D[s;\sigma] D_{(s;\sigma)} X^i(s,\sigma), i = 1, \dots p+q, \tag{27}$$

then by using the chain rule for derivatives together with (26) the integral along X of the gradient of a function G on $\mathbb{R}_S^{p,q}$ can be expressed as the difference between the values of G at the endpoints of the super path:

$$\int_0^\tau \int_0^t DX^i \partial_i G(X(s;\sigma)) = G(X(t;\tau)) - G(X(0;0)). \tag{28}$$

(The usual summation convention is applied for repeated indices.) In section 9 a stochastic version of this result is established and applied to the Dirac operator.

In many applications of anticommuting variables the simple superspace $\mathbb{R}_S^{m,n}$ must be replaced by a more general supermanifold. In this article it will be sufficient to consider supermanifolds constructed in a standard way from the data of a smooth vector bundle E over a smooth manifold M. (A theorem of Batchelor [15] shows that all smooth supermanifolds may be obtained in this way.) The idea of the construction is to patch together local pieces of the supermanifold using the change of coordinate functions of the manifold and the transition functions of the bundle. A careful definition would need an excursion into supermanifold theory [16]; here it will suffice to define change of coordinate functions, a full description of the patching construction may be found in [17].

Suppose that M has dimension m and E has dimension n; the supermanifold $\mathcal{S}(M,E)$ is then (m,n)-dimensional. If $\{U_\alpha | \alpha \in \Lambda\}$ is an open cover of M by sets which are both coordinate neighbourhoods of M and local trivialization neighbourhoods of E, then for each α in Λ there are local coordinates $(x_\alpha^1, \ldots, x_\alpha^m; \xi_\alpha^1, \ldots, \xi_\alpha^n)$ for the supermanifold; change of coordinates on any overlap between charts is defined in terms of the coordinate maps $\phi_\alpha : U_\alpha \to \mathbb{R}^m$ of M and the vector bundle transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n|\mathbb{R})$ by

$$\begin{aligned} x_\beta^i(x_\alpha;\xi_\alpha) &= \phi_\beta^i(x_\alpha) \qquad i = 1, \ldots, m \\ \xi_\beta^j(x_\alpha;\xi_\alpha) &= \sum_{k=1}^{n} g\,{}_{\alpha\beta}{}^j{}_k(x_\alpha)\xi_\alpha^k \qquad j = 1, \ldots, n. \end{aligned} \tag{29}$$

A particular example of this construction is the supermanifold $\mathcal{S}(M,TM)$ obtained from the tangent bundle of a manifold m. Explicitly, if M has dimension m, then $\mathcal{S}(M,TM)$ has dimension (m,m), and local coordinates $(x_\alpha^1, \ldots, x_\alpha^m; \xi_\alpha^1, \ldots, \xi_\alpha^m)$ which change according to the rule

$$\begin{aligned} x_\beta^i(x_\alpha;\xi_\alpha) &= \phi_\beta^i(x_\alpha) \qquad i = 1, \ldots, m \\ \xi_\beta^j(x_\alpha;\xi_\alpha) &= \sum_{k=1}^{m} \frac{\partial x_\beta^j}{\partial x_\alpha^k}(x_\alpha)\,\xi_\alpha^k \qquad j = 1, \ldots, m. \end{aligned} \tag{30}$$

Supersmooth functions on this supermanifold are then naturally identified with forms on M; in local coordinates this identification may be expressed as

$$\sum_{\mu \in M_m} f_\mu(x)\xi^\mu \leftrightarrow \sum_{\mu \in M_m} f_\mu(x) dx^\mu, \tag{31}$$

so that Berezin integration on the supermanifold corresponds to the standard integration of top forms on the manifold and the exterior derivative takes the form $d = \xi^i \frac{\partial}{\partial x^i}$.

For geometric applications of superspace path integration, the significant supermanifold is $\mathcal{S}(O(M), T(M) \otimes E)$, the underlying manifold being the orthonormal frame bundle $O(M)$ of a Riemannian manifold (M, g), and the vector bundle over $O(M)$ being the bundle induced by projection of $O(M)$ onto M of the product of the tangent bundle of M and a bundle E over M. There is a natural definition of Brownian motion on this supermanifold, based on the Brownian motion on manifolds defined by Elworthy [18] and by Ikeda and Watanabe [19], whose construction is described in section 5.

A further supermanifold with geometric applications is the supermanifold $\mathcal{S}(M, T(M) \otimes E)$, where E is an n-dimensional Hermitian vector bundle over m; if one takes local coordinates $(x^1, \ldots, x^m; \xi^1, \ldots, \xi^m, \eta^1, \ldots, \eta^n)$, then supersmooth functions which are linear in the η variables correspond to forms on M twisted by E. Given a metric g on M and a connection A for E, the Hodge-de Rham operator takes the form

$$d + \delta = \psi^i \left(\frac{\partial}{\partial x^i} - \Gamma_{ij}{}^k \xi^j \frac{\partial}{\partial \xi^k} - A_{i\,r}{}^s \eta^r \frac{\partial}{\partial \eta^s} \right) \tag{32}$$

where $\Gamma_{ij}{}^k$ are the Christoffel symbols for the metric g and the Clifford algebra operators ψ^i are now adapted to curved space, taking the form $\psi^i = \xi^i + g^{ij}(x) \frac{\partial}{\partial \xi^j}$. The corresponding Weitzenbock formula is then

$$-2(d+\delta)^2 = B - R^j_i(x) \xi^i \frac{\partial}{\partial \xi^j} - \frac{1}{2} R_{ki}{}^{jl}(x) \xi^i \xi^k \frac{\partial}{\partial \xi^j} \frac{\partial}{\partial \xi^k} + \frac{1}{4} [\psi^i, \psi^j] F_{ij\,r}{}^s(x) \eta^r \frac{\partial}{\partial \eta^s} \tag{33}$$

where $R_{ki}{}^{jl}$ are the components of the curvature of g, $F_{ij\,r}{}^s$ are the components of the curvature of A and B is the twisted Bochner Laplacian

$$B = g^{ij} \left(D_i D_j - \Gamma_{ij}{}^k D_k \right) \tag{34}$$

with $D_i = \frac{\partial}{\partial x^i} - \Gamma_{ij}{}^k \xi^j \frac{\partial}{\partial \xi^k} - A_{i\,r}{}^s \eta^r \frac{\partial}{\partial \eta^s}$. (A proof of this result, generalising the proof given in [20], may be found in [21].)

The Dirac operator for a spin manifold M of even dimension m may also be represented as a differential operator on a supermanifold. In this case the supermanifold is constructed from the vector bundle $E_{O(M)}$ associated to the bundle of orthonormal frames of M via the vector representation of

$SO(m)$. The supermanifold $\mathcal{S}(M, E_{O(M)})$ has local coordinates $(x^i; \eta^a)$, $a, i = 1, \ldots, m$, using the convention that coordinate indices are from the middle of the alphabet while orthonormal frame indices are from the beginning of the alphabet.

Defining operators $\psi^a = \eta^a + \frac{\partial}{\partial \eta^a}$ we see that $\psi^a\psi^b + \psi^b\psi^a = 2\delta^{ab}$, so that the supersmooth functions over a point of M define a 2^m-dimensional representation of the Clifford algebra on $\mathbb{R}^m$. The Dirac representation can obtained by considering supersmooth functions with values in $\mathbb{R}_S \otimes \mathbb{C}$, and restricting to the $2^{m/2}$-dimensional subspace of functions which satisfy the n conditions

$$\overline{\psi}^{2r-1}\overline{\psi}^{2r} f = if, \quad r = 1, \ldots, m/2, \tag{35}$$

with $\overline{\psi}^a = \eta^a - \frac{\partial}{\partial \eta^a}$.

Functions satisfying these conditions will be referred to as Dirac functions. It is also useful to define the projection operator P which projects arbitrary functions onto Dirac functions by setting

$$P = P_1 \ldots P_{n/2} \tag{36}$$

where, for each $r = 1, \ldots, n/2$, P_r is the operator which satisfies

$$\begin{aligned} &P_r\left(g(\eta^1, \ldots, \eta^{2r-2})(a + b\eta^{2r-1} + c\eta^{2r} + d\eta^{2r-1}\eta^{2r})h(\eta^{2r+1}, \ldots, \eta^n)\right) = \\ &\quad g(\eta^1, \ldots, \eta^{2r-2})\left(\tfrac{a-id}{2}(1 - i\eta^{2r-1}\eta^{2r}) + \tfrac{b+ic}{2}(\eta^{2r-1} - i\eta^{2r})\right) h(\eta^{2r+1}, \ldots, \eta^n). \end{aligned} \tag{37}$$

3 Fermionic and supersymmetric quantum mechanics

In this section we describe the quantum mechanical models whose heuristic path integral quantization is constructed rigorously in this survey. We begin with a purely fermionic model, and investigate various supersymmetric models, starting in flat space and then moving to curved space where we develop the models used section 8 for the supersymmetric proof of the index theorem and in section 11 for the study of Morse theory.

In the canonical quantization of n-dimensional particle mechanics, the classical observables p_i (momentum) and x^i (position) are replaced by quantum operators which satisfy the canonical commutation relations $[x^i, p_j] = i\delta^i_j$. The standard representation is the Schrödinger representation, with $x^i, i = 1, \ldots, m$ realised as the multiplication operator on $\mathcal{L}^2(\mathbb{R}^m)$, and $p_i = -i\frac{\partial}{\partial x^i}$. For fermionic operators ψ^i the canonical anticommutation relations can be represented as in an analogous manner on functions of anticommuting variables ξ^i by setting $\psi^i = \xi^i + \frac{\partial}{\partial \xi^i}$, so that $[\psi^i, \psi^j] = \delta^{ij}$.

When fermions and bosons are both present, wave functions are functions $F(x, \xi)$, with time evolution determined as usual by the Schrödinger equation $i\frac{\partial F}{\partial t} = HF$ (or, in Euclidean time, $\frac{\partial F}{\partial t} = -HF$), where units are used in which Plank's constant $\hbar$ is equal to 1. In flat space the standard Hamiltonians take the form $H = \frac{1}{2m}p_ip_i + V(x, \xi)$. The free Hamiltonian, on which the super Wiener measure developed in sections 4 and 5 is based, is $H_0 = \frac{1}{2m}p_ip_i$; the fermionic contribution to the free Hamiltonian is zero. (Generally we will use $m = 1$.)

The defining feature of a supersymmetric theory is that the Hamiltonian H has the form $H = \frac{1}{2}[Q, Q] = Q^2$ (or, more generally, $H = \frac{1}{2}\sum[Q_i, Q_i] = \sum Q_i^2$) where Q (or Q_i) is an odd 'supercharge'. The Lagrangian for such a theory is symmetric under a group of transformations which includes transformations of fermions into bosons and vice versa. The simplest example of a supersymmetric Hamiltonian is the free Hamiltonian $H_0 = Q_0^2$ with $Q_0 = \psi^i p_i$. In sections 8, 9 and 11 geometric examples of supersymmetric Hamiltonians are used.

4 Grassmann probability spaces and fermionic Wiener measure

In this section we begin by defining a notion of Grassmann probability space and random variable, based on the standard finite-dimensional Berezin integral (9) for functions of anticommuting variables. The Berezin integral is essentially algebraic; it is neither the limit of a sum nor an antiderivative, and lacks the positivity properties necessary for the use of standard measure theoretic techniques. The approach to defining an analogue of probability measure taken here, which was first developed in [22], is to use finite dimensional marginal distributions and Kolmogorov consistency conditions.

A particular example, Grassmann Wiener space, together with the corresponding fermionic Brownian motion, is then constructed and its relationship to the heat kernel of fermionic Hamiltonians is demonstrated.

Definition 4.1 An *n-Grassmann probability space of weight w*, where w is an element of $\mathbb{R}_S$, consists of

(a) a set A;

(b) for each finite subset B of A a supersmooth function $F_B : (\mathbb{R}_S^{0,n})^B \to \mathbb{R}_S$ such that if $B = \{b_1, \dots, b_N\}$ and $B' = \{b_1, \dots, b_{N-1}\}$ then

(i)
$$\int_B \mathrm{d}^n\theta_{b_1} \dots \mathrm{d}^n\theta_{b_N}\, F_B(\theta_{b_1}, \dots \theta_{b_N}) = w, \tag{38}$$

(ii)
$$\int_B \mathrm{d}^n\theta_{b_N}\, F_B(\theta_{b_1}, \dots, \theta_{b_N}) = F_{B'}(\theta_{b_1}, \dots, \theta_{b_{N-1}}). \tag{39}$$

Such a space will be denoted $\left(\mathbb{R}_S^{(0,n),A}, \{F_B\}\right)$.

Having built Kolmogorov consistency conditions into the definition, a notion of Grassmann random variable can be defined by a limiting process. The definition of such random variables is rather cumbersome because the generalised measure we are using does not have the usual positivity properties.

Definition 4.2 Suppose that $\mathcal{F}$ is a class of functions on $\mathbb{R}_S^{r,s}$. Then an (r,s)-dimensional Grassmann random variable of class $\mathcal{F}$ on a Grassmann probability space $\left(\mathbb{R}_S^{(0,n),A}, \{F_B\}\right)$ consists of

(a) a sequence $B_1 = \{b_{1,1}, \dots, b_{1,|B_1|}\}, B_2 = \{b_{2,1}, \dots, b_{2,|B_2|}\}, \dots$ of finite subsets of A, where $|B_k|$ denotes the number of elements in the set B_k;

(b) a sequence $G_1, G_2, \dots$ of supersmooth functions, with $G_k : (\mathbb{R}_S^{0,n})^{B_k} \to \mathbb{R}_S^{r,s}$, such that for each function H in $\mathcal{F}$ the sequence

$$\begin{aligned} I_k H = & \\ & \int_B \mathrm{d}^n\theta_{b_{k,1}} \dots \mathrm{d}^n\theta_{b_{k,|B_k|}}\, F_B(\theta_{b_{k,1}}, \dots \theta_{b_{k,|B_k|}}) \\ & \qquad \times H(G_k(\theta_{b_{k,1}}, \dots \theta_{b_{k,|B_k|}})) \end{aligned} \tag{40}$$

tends to a limit in $\mathbb{R}_S$ as $k \to \infty$. (The notion of convergence in $\mathbb{R}_S$ is given in Definition 2.3.)

The limit of this sequence is the *Grassmann expectation* of $F(G)$, and is denoted $\mathbb{E}_G[F(G)]$.

We now construct fermionic Wiener measure, which is the key construction underpinning the probabilistic methods described in this survey. The finite dimensional marginal distributions which determine this measure are built from the heat kernel of the free fermionic Hamiltonian. Since this Hamiltonian is zero, the measure is in fact built from Grassmann delta functions (13). In order to obtain a Feynman-Kac formula which can handle differential operators of quite a general class, the measure is over paths in phase space, that is, the Fourier transform variables are not integrated out before defining the measure.

Definition 4.3 (a) Let A be the interval $[0, \infty)$. Then n-dimensional fermionic Wiener space is defined to be the $2n$-Grassmann probability space $((\mathbb{R}_S^{0,2n})^A, \{F_B\})$ such that, given $B = \{t_1, \ldots, t_N\} \subset A$ with $0 \leq t_1 < t_2 < \ldots < t_N$,

$$\begin{aligned} &F_B(\theta_1, \rho_1, \ldots, \theta_N, \rho_N) \\ &\quad = \exp i\left(-\rho_1.\theta_1 - \rho_2.(\theta_2 - \theta_1) - \ldots - \rho_N.(\theta_N - \theta_{N-1})\right) \end{aligned} \tag{41}$$

where $\rho_k, \theta_k, k = 1 \ldots N$ are in $\mathbb{R}_S^{0,n}$. (Here we use ρ_k, θ_k rather than ρ_{t_k}, θ_{t_k} because the ordering of the elements of B is specified.)

(b) The corresponding stochastic process (θ_t, ρ_t) is called *fermionic Brownian motion.*

The following example illustrates this measure in action.

Example 4.4 For any supersmooth function F of n anticommuting variables with $F(\xi) = \sum_{\mu \in M_n} F_\mu \xi^\mu$ define the operator $F(\psi)$ to be $\sum_{\mu \in M_n} F_\mu \psi^\mu$ where ψ^i are as before the Clifford algebra or fermionic operators $\psi^i = \xi^i + \frac{\partial}{\partial \xi^i}$. Let $F_1, \ldots, F_N$ and G be supersmooth functions of n anticommuting variables and $t_1, \ldots, t_N$ be real numbers with $0 < t_1 < \ldots < t_N$. The action of the operator $K = F_1(\psi) \ldots F_N(\psi)$ on the function G is then given by

$$K\, G(\xi^1, \ldots, \xi^n) = \mathbb{E}_{\mathcal{G}}[F_1(\theta_{t_1} + i\rho_{t_1}) \ldots F_N(\theta_{t_N} + irho_{t_N})G(\theta_{t_N})]. \tag{42}$$

This result follows from repeated application of (18) .

We end this section with a lemma that is useful when estimating fermionic integrals.

Lemma 4.5 *Let* $\mathcal{F}$ *be a finitely defined random variable on* n*-dimensional fermionic Wiener space, with*

$$\mathcal{F}(\theta_{t_1}, \rho_{t_1}, \ldots, \theta_{t_N}, \rho_{t_N}) = \sum_{\mu_1 \in M_n} \sum_{\nu_1 \in M_n} \cdots \sum_{\mu_N \in M_n} \sum_{\nu_N \in M_n} F_{\mu_1\nu_1\ldots\mu_N\nu_N} \theta_{t_1}^{\mu_1} \rho_{t_1}^{\nu_1} \ldots \theta_{t_N}^{\mu_N} \rho_{t_N}^{\nu_N}, \tag{43}$$

then

$$|\mathbb{E}_{\mathcal{G}}[\mathcal{F}]| \leq \sum_{\mu_1 \in M_n} \sum_{\nu_1 \in M_n} \cdots \sum_{\mu_N \in M_n} \sum_{\nu_N \in M_n} |F_{\mu_1\nu_1\ldots\mu_N\nu_N}|. \tag{44}$$

Outline of proof The result follows from the Berezin integration rule, together with the fact that each term in the expansion of $\exp(-i\rho.\theta)$ occurs with coefficient of size exactly one.

5 Super Wiener measure and Itô integration along super Brownian paths

In this section we begin by combining some ideas from conventional measure theory with the anticommuting probability measures of the preceding section to construct a notion of super probability space, together with its associated random variables. Again we use finite dimensional marginal distributions as the basis of the construction.

Definition 5.1 An (m, n)-*super probability space of weight* w, where w is an element of $\mathbb{R}_S$, consists of

(a) a set A;

(b) for each finite subset B of A a supersmooth function $\mathcal{F}_B : (\mathbb{R}_S^{m,n})^B \to \mathbb{R}_S$ such that if $B = \{b_1, \ldots, b_N\}$ and $B' = \{b_1, \ldots, b_{N-1}\}$ then

(i)

$$\int_{\mathbb{R}^m}^{S} \mathrm{d}^m_{b_1} x \mathrm{d}^n \theta_{b_1} \dots \mathrm{d}^m x_{b_N} \mathrm{d}^n \theta_{b_N} \, F_B(x_{b_1}, \theta_{b_1}, \dots x_{b_N}, \theta_{b_N}) = w, \quad (45)$$

(ii)

$$\begin{aligned} \int_{\mathbb{R}^m}^{S} \mathrm{d}^m x_{b_N} \mathrm{d}^n \theta_{b_N} \, F_B(x_{b_1}, \theta_{b_1}, \dots, x_{b_N}, \theta_{b_N}) & \\ = F_{B'}(x_{b_1}, \theta_{b_1}, \dots, x_{b_N}, \theta_{b_{N-1}}). & \quad (46) \end{aligned}$$

Such a space will be denoted $\left((\mathbb{R}_S^{m,n})^A, \{F_B\} \right)$.

Because of the presence of anticommuting variables this 'super' probability measure is not true probability measure, so that a specific definition of random variables is required. As with the definition of Grassmann random variables, Definition 4.2, an explicit limiting process is used.

Definition 5.2 Suppose that $\mathcal{F}$ is a class of functions on $\mathbb{R}_S^{r,s}$. Then an (r, s)-dimensional super random variable of class $\mathcal{F}$ on on a super probability space $\left((\mathbb{R}_S^{m,n})^A, \{F_B\} \right)$ consists of

(a) a sequence $B_1 = \{b_{1,1}, \dots, b_{1,|B_1|}\}, B_2 = \{b_{2,1}, \dots, b_{2,|B_2|}\}, \dots$ of finite subsets of A;

(b) a sequence $G_1, G_2, \dots$ of supersmooth functions, with $G_k : (\mathbb{R}_S^{m,n})^{B_k} \to \mathbb{R}_S^{r,s}$ such that for each function H in $\mathcal{F}$ the sequence

$$\begin{aligned} I_k H = \int_{\mathrm{d}}^{S\, m} & x_{b_{k,1}} \mathrm{d}^n \theta_{b_{k,1}} \dots \mathrm{d}^m x_{b_{k,|B_k|}} \mathrm{d}^n \theta_{b_{k,|B_k|}} \\ & F_B(x_{b_{k,1}}, \theta_{b_{k,1}}, \dots x_{b_{k,|B_k|}}, \theta_{b_{k,|B_k|}}) \\ & \times H(G_k(x_{b_{k,1}}, \theta_{b_{k,1}}, \dots x_{b_{k,|B_k|}}, \theta_{b_{k,|B_k|}})) \quad (47) \end{aligned}$$

tends to a limit in $\mathbb{R}_S$ as $k \to \infty$. (The notion of convergence in $\mathbb{R}_S$ is given in Definition 2.3.)

The limit of this sequence is the *super expectation* of $H(G)$, and is denoted $\mathbb{E}_{\mathcal{S}}[H(G)]$.

It is also useful to introduce the notion of stochastic process.

Definition 5.3 Let I be an interval of the real line. Then a collection $\{w_t | t \in I\}$ of (r,s)-dimensional super random variables on a super probability space is called an (r,s)-dimensional *stochastic process* if each each finite subcollection is a super random variable of the appropriate dimension.

The notion of different versions of a process, and of Gaussian process, correspond directly to the conventional objects. The stochastic process which will play a major role in the geometric applications defined below is Brownian motion in superspace, which is obtained by combining the fermionic Brownian paths constructed in section 4 with conventional paths. These paths are obtained from super Wiener measure, which is essentially the product of standard Wiener measure and the fermionic Wiener measure of Definition 4.3.

Definition 5.4 (a) $(m, 2n)$-dimensional Super Wiener space is the super probability space $((\mathbb{R}_S^{m,n})^A, \{F_B\})$ with A the interval $[0, \infty)$ and, given $B = \{t_1, \ldots, t_N\} \subset A$ with $0 \leq t_1 < t_2 < \ldots < t_N$,

$$\begin{aligned} F_B(b_1, \theta_1, \rho_1, \ldots, b_N, \theta_N, \rho_N) &= P_{t_1}(b_1, \theta_1, \rho_1, 0, 0) P_{t_2 - t_1}(b_2, \theta_2, \rho_2, b_1, \theta_1) \\ \ldots & \quad P_{t_N - t_{N-1}}(b_N, \theta_N, \rho_N, b_{N-1}, \theta_{N-1}) \end{aligned} \tag{48}$$

where

$$P_t(b, \theta, \rho, b', \theta') = \frac{1}{(2\pi t)^{\frac{m}{2}}} \exp\left(-\frac{(b - b')^2}{2t}\right) \exp i\left(-\rho.(\theta - \theta')\right). \tag{49}$$

(b) The corresponding $(m, 2n)$-dimensional process (b_t, θ_t, ρ_t) is called *super Brownian motion.*

The motivation for constructing this super stochastic process is to give a rigorous mathematical formulation of the combined bosonic and fermionic path integrals which play a key role in many of the applications of supersymmetric quantum mechanics to geometry. Its use in solving diffusion equations on certain geometric bundles is described section 7 and applied in Sections 8, 9 and 11.

6 Stochastic calculus of super Brownian paths

In this section some modest generalisations of stochastic calculus are introduced, so that a useful notion can be developed of Brownian path on the various supermanifolds we have introduced, and applied to various geometric diffusions. We begin by defining the concepts of adapted process and stochastic integrals in the following way:

Definition 6.1 A stochastic process $\{F_s : s \in [0,t]\}$ on (m,n)-dimensional super Wiener space such that for each s in $[0,t]$ the random variable F_s is a function of $\{b_{s'}, \theta_{s'}, \rho_{s'} | 0 \leq s' \leq s\}$ is said to be $[0,t]$-*adapted.* (The time interval, $[0,t]$, may be omitted when the context makes it clear.)

As in the classical case, it can then be shown by direct calculation that if F_s is a $[0,t]$-adapted process and $0 \leq s_1 \leq s_2 \leq s_3 \leq t$ then

$$\begin{aligned} \mathbb{E}_{\mathcal{S}}\left[F_{s_1}\left(b^i_{s_3} - b^i_{s_2}\right)\right] &= 0 \\ \text{and} \qquad \mathbb{E}_{\mathcal{S}}\left[F_{s_1}\left(b^i_{s_3} - b^i_{s_2}\right)\left(b^j_{s_3} - b^j_{s_2}\right)\right] &= \tfrac{1}{2}\mathbb{E}_{\mathcal{S}}[F_{s_1}](s_3 - s_2), \end{aligned} \tag{50}$$

so that the commuting part of the super Brownian motion will contribute to results in stochastic calculus in the normal way. There is no analogue of Itô integration along odd Brownian paths because the measure does not lead to increments which are of order any positive power of δt; however, when applying super Brownian paths to diffusions, the use of $d\theta$ is unnecessary, since the presence of the paths ρ_t enables one to obtain a Feynman-Kac formula for fermionic differential operators directly rather than by application of Itô calculus.

As a result, it is sufficient to consider integrals with respect to time and with respect to even Brownian paths. The definitions, which are rather cumbersome, are given here for completeness. It will emerge below that they do lead to natural analogues of the results in stochastic calculus which are important in applications to diffusions. The integral $\int_0^t F_s ds$ of an adapted process F_s with respect to time will now be defined.

Definition 6.2 Let F_s be a $[0,t]$-adapted process on $(m,2n)$-dimensional super Wiener space. Suppose that for each s in $[0,t]$, F_s corresponds, as in definition 5.2, to the sequence of pairs of subsets and functions $(J_{s,M}, \mathcal{F}_{s,M})$, $M = 1, 2, \ldots$ with $J_M = \cup_{r=1}^{2^M-1} J_{t_r,M}$ where, for $r = 1, \ldots 2^M - 1$, $t_r = \frac{rt}{2^M}$. Also define functions K_M on $(\mathbb{R}_S^{m,n})^{J_M}$ by

$$K_M = \sum_{r=1}^{2^M-1} \mathcal{F}_{t_r,M} \frac{t}{2^M}. \tag{51}$$

Then, if the appropriate limits exist as M tends to infinity, the sequence (J_M, K_M) defines a super random variable which is denoted $\int_0^t \mathcal{F}_s \, ds$.

The other stochastic integral needed is the integral along even Brownian paths:

Definition 6.3 With the notation of previous definition, and with $\mathcal{F}_s^a$ an m-dimensional super random variable, for each $M = 1, 2, \ldots$, define the functions $\mathcal{G}_M$ on $(\mathbb{R}_S^{m,2n})^{J_M}$ by

$$\mathcal{G}_M = \sum_{a=1}^{m} \sum_{r=1}^{2^M-1} \mathcal{F}_{t_r,M}^a (x^{r+1,a} - x^{r,a}). \tag{52}$$

Then, if the appropriate limits exist as M tends to infinity, the sequence $(J_M, \mathcal{G}_M)$ defines a super random variable which is denoted $\int_0^t \mathcal{F}_s^a \, db_s^a$.

A key result in the application of super Brownian paths to the study of diffusions on supermanifolds is the following Itô formula:

Theorem 6.4 *Suppose that a_t is a p-dimensional stochastic integral on m-dimensional Wiener space with*

$$\begin{aligned} x_t^i - x_0^i = &\int_0^t \sum_{\mu \in M_n, \nu \in M_n} g_{\mu\nu,s}^i \theta_s^\mu \rho_s^\nu \, ds \\ &+ \int_0^t \sum_{a=1}^{m} \sum_{\mu \in M_n, \nu \in M_n} g_{\mu\nu,s}^i h_{\mu\nu\, a\, s}^i \theta_s^\mu \rho_s^\nu \, db_s^a, \; i = 1, \ldots, p \end{aligned} \tag{53}$$

where for $i =, \ldots, p$ and $a = 1, \ldots m$ the processes $g_{\mu\nu,t}^i$ and $h_{\mu\nu\, a\, t}^i$ are adapted processes on m-dimensional Wiener space with

$$\mathbb{E}_c \left[\int_0^t |h_{\mu\nu s}^i|^2 ds \right] < \infty \quad \text{and} \quad \mathbb{E}_c \left[\int_0^t |g_{\mu\nu\, a\, s}^i| ds \right] < \infty.$$

(Here $\mathbb{E}_c$ denotes expectation with respect to classical Wiener measure.) Also suppose that $F : \mathbb{R}_S^{p,2n} \to \mathbb{R}_S$ is supersmooth with compact support. Let $\partial_i, i = 1, \ldots, p$ denote differentiation with respect to the i^{th} even variable. Then if $0 < t' < t < \infty$,

$$F(x_t, \theta_t, \rho_t) - F(x_{t'}, \theta_{t'}, \rho_{t'}) =_{\mathbb{E}} \int_0^t \sum_{i=1}^p \partial_i F(x_s, \theta_s, \rho_s)(h_s^i\, ds + \sum_{a=1}^m g^i_{a\,s}\, db_s^a) + \int_0^t \sum_{i=1}^p \sum_{j=1}^p \sum_{a=1}^m \tfrac{1}{2} \partial_i \partial_j F(x_s, \theta_s, \rho_s) g^i_{a\,s} g^j_{a\,s}\, ds. \tag{54}$$

Outline of proof

$$F(x_t, \theta_t, \rho_t) - F(x_{t'}, \theta_{t'}, \rho_{t'}) = \sum_{r=0}^N \Delta_r F \tag{55}$$

where $\Delta_r F = F(x_{t_r}, \theta_{t_r}, \rho_{t_r}) - F(x_{t_{r-1}}, \theta_{t_{r-1}}, \rho_{t_{r-1}})$ with $t_r = rt/N$, $r = 0, \ldots, N$. Now

$$\begin{aligned} &F(x_{t_r}, \theta_{t_r}, \rho_{t_r}) - F(x_{t_{r-1}}, \theta_{t_{r-1}}, \rho_{t_{r-1}}) \\ &= F(x_{t_r}, \theta_{t_r}, \rho_{t_r}) - F(x_{t_{r-1}}, \theta_{t_r}, \rho_{t_r}) \\ &\quad + F(x_{t_{r-1}}, \theta_{t_r}, \rho_{t_r}) - F(x_{t_{r-1}}, \theta_{t_{r-1}}, \rho_{t_{r-1}}) \end{aligned} \tag{56}$$

and hence, since

$$\mathbb{E}_{\mathcal{S}} \left[F(x_{t_{r-1}}, \theta_{t_r}, \rho_{t_r}) - F(x_{t_{r-1}}, \theta_{t_{r-1}}, \rho_{t_{r-1}}) \right] = 0, \tag{57}$$

as may be seen directly from the definition of fermionic Wiener measure,

$$F(x_{t_r}, \theta_{t_r}, \rho_{t_r}) - F(x_{t_{r-1}}, \theta_{t_{r-1}}, \rho_{t_{r-1}}) =_{\mathbb{E}} F(x_{t_r}, \theta_{t_r}, \rho_{t_r}) - F(x_{t_{r-1}}, \theta_{t_r}, \rho_{t_r}). \tag{58}$$

The proof may then be completed in the same way as the classical Itô theorem, making use of 4.5. ■

7 Brownian paths and a Feynman-Kac-Itô formula for supermanifolds

In this section we develop a notion of Brownian paths on supermanifolds, by extending to odd directions the constructions of Ikeda and Watanabe [19] and Elworthy [23, 18]. These Brownian paths on supermanifolds will lead to a Feynman-Kac-Itô formula for the Laplace-Beltrami operator on forms on M twisted by the bundle E.

Before introducing the twisting by the vector bundle, we consider the simpler case of untwisted forms, and investigate paths on the supermanifold $\mathcal{S}(O(M), T(M))$. For the commuting, even components of the paths, we use the construction referred to above, that is we consider paths $x_t^i, e_{a,t}^i$ in the bundle of orthonormal frames $O(M)$ of the manifold M (with metric g) starting from a point (x^i, e_a^i) in $O(M)$ and satisfying the stochastic differential equations

$$x_t^i \;=\; x^i + \int_0^t e_{a,s}^i \circ db_s^a \qquad e_{a,t}^i = e_a^i - \int_0^t e_{a,s}^l {\Gamma_{kl}}^i(x_s) e_{b,s}^k \circ db_s^b \tag{59}$$

where $\circ$ denotes the Stratonovich product. A patching construction allows a global solution to be constructed, since they change covariantly under change of coordinate on $O(M)$. (Note that almost surely $g^{ij}(x_t) = \sum_{a=1}^m e_{a\,t}^i e_{a\,t}^j$.) The odd, anticommuting components of the paths $\xi_t^i, \pi_{i\,t}, i = 1, \ldots, m$ are obtained by rotating the flat fermionic paths θ_t, ρ_t:

$$\xi_t^i = \xi^i + \sum_{a=1}^m e_{a\,t}^i \theta_t^a, \qquad \pi_{i\,t} = \sum_{j=1}^n \sum_{a=1}^m e_{a\,t}^j g_{ij}(x_t) \rho_t^a. \tag{60}$$

In order to establish a Feynman-Kac-Itô formula for the Laplace-Beltrami operator $(d+\delta)^2$ on M, vector fields W_a, $a = 1, \ldots, m$, must be defined which correspond to horizontal vector fields on $\mathcal{S}(O(M), T(M))$ regarded as a bundle over M with connection Γ the Levi-Cevita connection for the metric g . In a local coordinate system (x^i, e_a^i, ξ^i) on $\mathcal{S}(O(M), T(M))$ these vector fields take the form

$$W_a = e_a^i \tfrac{\partial}{\partial x^i} - e_a^j e_b^k {\Gamma_{jk}}^i(x) \tfrac{\partial}{\partial e_b^i} - e_a^j \xi^k {\Gamma_{jk}}^i(x) \tfrac{\partial}{\partial \xi^i}. \tag{61}$$

The key property of the vector fields W_a is that, when acting on functions on $\mathcal{S}(O(M), T(M))$ which are independent of the e_a^i (that is, on functions of the form $f = g \circ \pi$ where π is the canonical projection from $\mathcal{S}(O(M), T(M))$ onto M) they are related to the Laplace-Beltrami operator L by

$$L = -\frac{1}{2}\left(W_a W_a - R_i^j(x) \xi^i \tfrac{\partial}{\partial \xi^j} - \frac{1}{2} {R_{ki}}^{jl}(x) \xi^i \xi^k \tfrac{\partial}{\partial \xi^j} \tfrac{\partial}{\partial \xi^l} \right) \tag{62}$$

as may be seen from the Weitzenbock formula Lemma 33 (ignoring the term involving F which does not occur in the untwisted case). This leads to the following Feynman-Kac-Itô formula:

Theorem 7.1 *Let $(x_t^i, e^i_{a\,t}, \xi^i_t)$ be the paths defined by (59) and (60). Then*

$$\begin{aligned}&\exp(-Lt)g(x,\xi,\eta)\\ &\quad = \mathbb{E}_{\mathcal{S}}\left[e^{-\int_0^t \frac{1}{2}R_i^j(x_s)\xi_s^i\pi_{j\,s}+\frac{1}{4}R_{ki}{}^{jl}\xi_s^i\xi_s^k\pi_{l\,s}\pi_{j\,s}ds}g(x_t,\xi_t,\eta_t)\right]\end{aligned}\tag{63}$$

where $L=(d+\delta)^2$ is the Laplace-Beltrami operator acting on supersmooth functions on $\mathcal{S}(M,TM)$.

Outline of proof Using the Itô formula (54) and equation (62),

$$\begin{aligned}&\mathbb{E}_{\mathcal{S}}[g(x_t,\xi_t,\eta_t)-g(x,\xi,\eta)]\\ &\quad = \mathbb{E}_{\mathcal{S}}\left[\int_0^t \tfrac{1}{2}\left(W_aW_a - R_i^j(x)\xi^i\tfrac{\partial}{\partial\xi^j} - \tfrac{1}{2}R_{ki}{}^{jl}(x)\xi^i\xi^k\tfrac{\partial}{\partial\xi^j}\tfrac{\partial}{\partial\xi^l}\right)g(x_s,\xi_s,\eta_s)\right]ds\\ &\quad = \mathbb{E}_{\mathcal{S}}\left[\int_0^t -Lg(x_s,\xi_s,\eta_s)ds\right].\end{aligned}\tag{64}$$

Hence, if $f(x,\xi,\eta,t)=\mathbb{E}_{\mathcal{S}}[g(x_t,\xi_t,\eta_t)]$,

$$f(x,\xi,\eta,t)-f(x,\xi,\eta,0)=\int_0^t -Lf(x,\xi,\eta,s)\,ds\tag{65}$$

and the result follows. ■

To extend this Feynman-Kac-Itô formula for the Laplace-Beltrami operator to the twisted case, we use vector fields $\widetilde{W}_a$, $a=1,\ldots,m$, which correspond to horizontal vector fields on $\mathcal{S}(O(M),T(M)\otimes E)$ regarded as a bundle over M with connection (Γ, A). In a local coordinate system $(x^i, e^i_a, \xi^i, \eta^p)$ on $\mathcal{S}(O(M),T(M)\otimes E)$ these vector fields take the form

$$\widetilde{W}_a = e^i_a\tfrac{\partial}{\partial x^i} - e^j_a e^k_b\Gamma_{jk}{}^i(x)\tfrac{\partial}{\partial e^i_b} - e^j_a\xi^k\Gamma_{jk}{}^i(x)\tfrac{\partial}{\partial\xi^i} - e^j_a\eta^r A_{j\,r}{}^p(x)\tfrac{\partial}{\partial\eta^p}.\tag{66}$$

In this case we have

$$\begin{aligned}L \;=\; -\frac{1}{2}\Big(&\widetilde{W}_a\widetilde{W}_a - R_i^j(x)\xi^i\tfrac{\partial}{\partial\xi^j} - \tfrac{1}{2}R_{ki}{}^{jl}(x)\xi^i\xi^k\tfrac{\partial}{\partial\xi^j}\tfrac{\partial}{\partial\xi^l}\\ &+\tfrac{1}{4}[\psi^i,\psi^j]F_{ij\,r}{}^s(x)\eta^r\tfrac{\partial}{\partial\eta^s}\Big)\end{aligned}\tag{67}$$

as may again be seen from the Weitzenbock formula Lemma 33.

In order to construct odd paths on $\mathcal{S}(O(M), T(M) \otimes E)$ we need an extended set of commuting paths. These are defined on the principle bundle $O(M) \otimes E$, where E is the principle bundle corresponding to the vector bundle E, by the stochastic differential equations:

$$\begin{aligned} x_t^i &= x^i + \int_0^t e_{a,s}^i \circ db_s^a \qquad e_{a,t}^i = e_a^i - \int_0^t e_{a,s}^l {\Gamma_{kl}}^i(x_s) e_{b,s}^k \circ db_s^b \\ h_{q\,t}^p &= h_q^p + \int_0^t h_{q\,s}^r {A_{i\,r}}^p(x_s) e_{a\,s}^i \circ db_s^a. \end{aligned} \tag{68}$$

The full set of odd paths is now obtained from fermionic Brownian paths of dimension $m+n$, setting $\xi_t^i = \xi^i + \sum_{a=1}^m e_{a\,t}^i \theta_t^a$ and $\pi_{i\,t} = \sum_{j=1}^m \sum_{a=1}^m e_{a\,t}^j g_{ij}(x_t) \rho_t^a$ for $i = 1, \ldots m$ as before, and taking

$$\eta_t^p = \eta^p + \sum_{q=1}^n h_{q\,t}^p \theta_t^{m+q}, \quad \phi_{p\,t} = \sum_{q=1}^n (h^{-1})_{p\,t}^q \rho_{m+q\,t}. \tag{69}$$

Similar arguments to those given above lead to the Feynman-Kac-Itô formula

Theorem 7.2

$$\begin{aligned} &\exp(-Lt) g(x, \xi, \eta) \\ = \ &\mathbb{E}_{\mathcal{S}}\left[e^{\left[-\int_0^t \frac{1}{2} R_i^j(x_s) \xi_s^i \pi_{j\,s} + \frac{1}{4} {R_{ki}}^{jl} \xi_s^i \xi_s^k \pi_{l\,s} \pi_{j\,s} + \frac{1}{4} \psi_s^i \psi_s^j {F_{ij\,p}}^q(x_s) \eta_s^p \phi_{q\,s} ds\right]} \times g(x_t, \xi_t, \eta_t)\right] \end{aligned} \tag{70}$$

where $L = (d+\delta)^2$ is the Laplace-Beltrami operator acting on supersmooth functions on $\mathcal{S}(M, E)$.

8 Supersymmetric paths and the Atiyah-Singer Index Theorem

The Feynman-Kac-Itô formula established in the previous section will now be used to establish the Atiyah-Singer index theorem. This is achieved by establishing a stronger, local version of the theorem as in the paper of Atiyah, Bott and Patodi [24] for the case of the twisted Hirzebruch signature complex.

(In [24] it is shown by K-theoretic arguments that this establishes the full theorem.)

The starting point is the formula of McKean and Singer [25] which expresses the Hirzebruch signature of the complex vector bundle E over the Riemann manifold (M,g) as the supertrace of heat kernel of the Laplacian in the following way:

$$\mathrm{Index}\,(d+\delta) = \mathrm{Supertrace}\,(\exp(-Lt))\,. \tag{71}$$

In the course of proving this theorem it emerges that the right hand side is independent of t. The significance of this formula in the context of supersymmetry was first appreciated by Witten [26]. Using the identification of the space of twisted forms on M with the space of supersmooth functions on $\mathcal{S}(M,E)$, the supertrace is defined in terms of the involution τ on the space of supersmooth functions whose action on F with $F(x,\xi,\eta) = \sum_{\mu\in M_m}\sum_{r=1}^{n} F_{\mu r}(x)\xi^\mu\eta^r$ is given by

$$\begin{aligned}&\tau\,(F)\,(x,\xi,\eta)\\ &\quad= \sum_{\mu\in M_m}\sum_{r=1}^{n}\left(\int_{\mathcal{B}} d^m\rho\,(\det(g_{ij}(x)))^{-\frac{1}{2}}\exp\left(i\rho^i\xi^j g_{ij}(x)\right) F_{\mu r}(x)\rho^\mu\eta^r\right).\end{aligned} \tag{72}$$

This involution is essentially the Hodge dual.

The supertrace of an operator K on the space of supersmooth functions on $\mathcal{S}(M,E)$ is then defined (when it exists) by the formula

$$\mathrm{Supertrace}\,K = \mathrm{Trace}\,(\tau K). \tag{73}$$

The Atiyah-Singer index theorem for the Hirzebruch signature complex can now be stated.

Theorem 8.1

$$\mathrm{Index}(d+\delta) = \int_M \left\{\mathrm{Trace}\,\exp\left(\frac{-F}{2\pi}\right)\det\left(\frac{i\Omega/2\pi}{\tanh i\Omega/2\pi}\right)^{\frac{1}{2}}\right\}_m \tag{74}$$

where as before F is the curvature of the connection A on the bundle E while Ω is the curvature 2-form of the Levi-Cevita connection on (M,g), and the brackets $\{\}_m$ indicate projection onto the m-form component of the integrand.

Using the Mckean and and Singer formula 71, we see that an equivalent statement of the theorem is

$$\text{Supertrace}(\exp(-Lt)) = \int_M \left\{ \text{Trace}\exp\left(\frac{-F}{2\pi}\right) \det\left(\frac{i\Omega/2\pi}{\tanh i\Omega/2\pi}\right)^{\frac{1}{2}} \right\}_m \tag{75}$$

and thus that the theorem certainly holds if the following stronger, local theorem holds.

Theorem 8.2 *At each point x in M*

$$\begin{aligned} &\lim_{t\to 0} \text{str}(\exp(-Lt))(x,x)d\text{vol} \\ &= \left\{ \text{Trace}\exp\left(\frac{-F}{2\pi}\right) \det\left(\frac{i\Omega/2\pi}{\tanh i\Omega/2\pi}\right)^{\frac{1}{2}} \right\}_m \Bigg|_x \end{aligned} \tag{76}$$

where str *denotes the* $(2^m n \times 2^m n)$ *matrix supertrace.*

The effect of the matrix supertrace is that $\text{str}(\exp(-Lt))(x,y)$ is the kernel of an operator on smooth functions on M. The proof of this theorem makes use of the Feynman-Kac-Itô formula of theorem 7.2 to analyse $\exp(-Lt)$, and then (following Getzler [27]), employs Duhamel's formula to extract information about the heat kernel and show that in the limit as t tends to zero only the required term survives.

Proof of Theorem 8.2

Using (20) we see that if K is a differential operator on the space of supersmooth functions on $\mathcal{S}(M,E)$ then (with tr denoting the matrix trace)

$$\begin{aligned} &\text{str}K(x,y) = \text{tr}(\tau K(x,y)) \\ &= \int_B d^m\rho d^m\xi d^n\eta\, K(x,y,\rho,-\xi,\eta,-\eta)\left(\det(g_{ij}(x))\right)^{-\frac{1}{2}} \exp\left(i\rho^i\xi^j g_{ij}(x)\right). \end{aligned} \tag{77}$$

Now the theorem is local, and it is shown by Cycon, Froese, Kirsch and Simon in [20] that at any particular point x of M the manifold can be replaced by $\mathbb{R}^m$ and the n-dimensional vector bundle by $\mathbb{R}^m \times \mathbb{C}^n$, and a metric and connection chosen, so that in the limit as t tends to zero calculation using the Hamiltonian on $\mathbb{R}^m$ gives the same matrix supertrace as that on M. This is done by first choosing a neighbourhood W of x which has compact closure and is both a coordinate neighbourhood for M and a local trivialization neighbourhood for E. Now suppose that U is also an neighbourhood of x and that $\overline{U} \subset W$. Let $\phi : W \to \mathbb{R}^m$ be a system of normal coordinates on W about x which satisfy $\det(g_{ij}) = 1$ throughout W. (The existence of such coordinates are established in [20].) Additionally a local trivialization of E is chosen such that $A_{i\,r}{}^p(0) = 0$. (For simplicity we identify points in W by their coordinates; in particular x becomes 0.)

The standard Taylor expansion in normal coordinates about 0 gives [24]

$$\begin{aligned} g_{ij}(y) &= \delta_{ij} - \frac{1}{3} y^k y^l R_{ijkl}(0) + \dots \\ \Gamma_{ij}{}^k(y) &= \frac{1}{3} y^l \left(R_{lji}{}^k(0) + R_{lij}{}^k(0) \right) + \dots \\ A_{i\,r}{}^p(y) &= -\frac{1}{2} y^j F_{ij\,r}{}^p + \dots . \end{aligned} \tag{78}$$

Now let L_0 be the flat Laplacian $\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}$ on $\mathbb{R}^m$ and let $K_t(y, y', \xi, \xi', \eta, \eta')$ and $K_t^0(y, y', \xi, \xi', \eta, \eta')$ be the heat kernels of L and L_0 respectively. From Duhamel's formula (as quoted by Getzler in [27]) we learn that

$$\begin{aligned} & K_t(y, y', \xi, \xi', \eta, \eta') - K_t^0(y, y', \xi, \xi', \eta, \eta') \\ &= \int_0^t ds\, e^{-(t-s)L} (L - L_0) K_s^0(y, y', \xi, \xi', \eta, \eta') \end{aligned} \tag{79}$$

where the differential operators L and L_0 act with respect to the unprimed arguments. Now

$$\begin{aligned} & K_s^0(y, y', \xi, \xi', \eta, \eta') \\ &= \int_B d^m\rho\, d^n\kappa\, (2\pi s)^{\frac{m}{2}} \exp\left(-(y - y')^2/2s - i\rho(\xi - \xi') - i\kappa(\eta - \eta') \right) . \end{aligned} \tag{80}$$

From this we may deduce by direct calculation that $\mathrm{str} K_t^0(0, 0) = 0$ and, using (19),

$$\begin{aligned} & \mathrm{str} K_t(0, 0) = \\ & \int_B d^m\xi\, d^m\xi'\, d^n\eta\, \exp -i\xi\xi' \times \int_0^t ds \Big[\left(e^{-(t-s)L} (L - L_0) K_s^0(0, 0, \xi, \xi', \eta, \eta) \right) \Big] . \end{aligned} \tag{81}$$

The Feynman-Kac-Itô formula theorem 7.2 can then be used to evaluate this expression. Rather than proceeding directly, we now construct a simplified Hamiltonian L_1 on $\mathbb{R}^m \times \mathbb{C}^n$ with heat kernel K_t^1 which has the property

$$\lim_{t\to 0} \mathrm{str} K_t^1(0,0) = \lim_{t\to 0} \mathrm{str} K_t(0,0)$$

so that the required supertrace can be calculated.

The modified Laplacian is built from the approximate metric and connections of (78), working to first order in y, and with Euclidean metric. Writing $R_{ijkl}{}^l$ for $R_{ijkl}{}^l(0)$ and $F_{ij\,p}{}^q$ for $F_{ij\,p}{}^q(0)$ it is

$$\begin{aligned} L_1 = -\Big[& \frac{1}{2}\tfrac{\partial}{\partial x^i}\tfrac{\partial}{\partial x^i} - \frac{1}{4}R_{ij}{}^{kl}\theta^i\theta^j\tfrac{\partial}{\partial\theta^l}\tfrac{\partial}{\partial\theta^k} + \frac{1}{3}R^k_j\theta^j\tfrac{\partial}{\partial\theta^k} \\ & +\frac{1}{4}\psi^i\psi^j\eta^p F_{ij\,p}{}^q\tfrac{\partial}{\partial\eta^q} - \frac{1}{3}\theta^j x^l\left(R_l{}^k{}_j{}^i + R^{ki}_{lj}\right)\tfrac{\partial}{\partial\theta^i}\tfrac{\partial}{\partial x^k} \\ & -\frac{1}{18}\theta^{k'}\theta^{l'}x^k x^l\left(R_{kj'k'}{}^i + R_{kk'j'}{}^i\right)\left(R_l{}^{j'}{}_{l'}{}^j + R_{ll'}{}^{j'j}\right)\tfrac{\partial}{\partial\theta^i}\tfrac{\partial}{\partial\theta^j}\Big]. \end{aligned} \tag{82}$$

The brownian paths which lead to a Feynman-Kac-Itô formula as in 7.2 are

$$x^{1}{}^i_t = b^i_t, \qquad \xi^{1}{}^i_t = \xi^i + \theta^i_t, \qquad \eta^{1}{}^p_s = \eta^p + \theta^{m+p}_t. \tag{83}$$

Using these paths in the Feynman-Kac-Itô formula, it can then be shown (the details are in [28]) that

$$lim_{t\to 0}K_t(0,0) = lim_{t\to 0}K_t^1(0,0). \tag{84}$$

By evaluating the supertrace of $\exp(-L_1 t)$ (using flat space path integration techniques for both fermionic and bosonic paths)the local form of the Atiyah-Singer index theorem 8.2 will be established.

Using Duhamel's formula and changing variable $\phi = \sqrt{2\pi t}\,\xi$ gives

$$\begin{aligned} & K_t^1(0,0) \\ & = \mathbb{E}_{\mathcal{S}}\Big[\int_{\mathcal{B}} d^m\phi d^n\eta d^n\kappa d^m\xi (2\pi t)^{\frac{m}{2}} \int_0^t ds (2\pi s)^{-\frac{m}{2}} \\ & \quad \times\Big\{ \exp\left(-L_1(t-s)\right)(L_1 - L_0) G_s(0, \frac{\phi}{\sqrt{2\pi t}}, \xi', \eta, -\eta) \\ & \quad \times \exp\left(-i\kappa\eta - i\frac{\phi}{\sqrt{2\pi t}}\xi'\right)\Big\}\Big] \end{aligned} \tag{85}$$

where

$$G_s(x,\xi,\xi',\eta,\kappa) = \exp -(x^2/2s + i\xi.\xi' + i\kappa.\eta) \tag{86}$$

so that

$$\int_{\mathcal{B}} d^n\kappa G_s(x,\xi,\xi',\eta,\kappa) = \tau K_s^0(x,0,\xi,\xi',\eta,0). \tag{87}$$

Hence, using the fact that (for small t) $x_t \sim \sqrt{t}$ while the fermionic paths are of order 1 [28] it can be shown that

$$\begin{aligned} &\lim_{t\to 0} K_t^1(0,0) \\ &= \mathbb{E}_{\mathcal{S}}\Bigg[\int_{\mathcal{B}} d^m\phi d^n\eta d^n\kappa d^m\xi (2\pi t)^{\frac{m}{2}} \int_0^t ds (2\pi s)^{-\frac{m}{2}} \\ &\quad \times\Big\{ (L_2 - L_0)\, G_s(b_{t-s}^i, \theta_{t-s}^i, \frac{\phi^i}{\sqrt{2\pi t}}, \theta_{t-s}^{m+p}, \kappa^p) \exp i\kappa.\eta\Big\}\Bigg] \end{aligned} \tag{88}$$

where $L_2 = L_{2x} + L_{2\xi} + L_{2\eta}$ with

$$\begin{aligned} L_{2x} &= -\Big(\frac{1}{2}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i} + \frac{1}{2}x^l\frac{\phi^j\phi^i}{2\pi t}R_{jil}{}^k\frac{\partial}{\partial x^k} + \frac{1}{8}x^kx^l\frac{\phi^i\phi^j\phi^{i'}\phi^{j'}}{(2\pi t)^2}R_{i'ikk'}R_{j'jl}{}^{k'}\Big) \\ L_{2\xi} &= -\frac{1}{4}R_{ijkl}\frac{\phi^j\phi^i}{2\pi t}\psi^k\psi^l \\ L_{2\eta} &= -\frac{\phi^j\phi^i}{2\pi t}F_{ij\,p}{}^q\eta^p\frac{\partial}{\partial\eta^q}. \end{aligned} \tag{89}$$

Now $\exp\left(-L_{2x}t(0,0)\right)$ can be evaluated using the standard $\mathbb{R}^2$ result [29] that if

$$L = -\frac{1}{2}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i} + \frac{iB}{2}\left(x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}\right) + \frac{1}{8}B^2\left((x^1)^2 + (x^2)^2\right)$$

then

$$\exp\left(-Lt(0,0)\right) = \frac{B}{4\pi\sinh(\frac{1}{2}Bt)}. \tag{90}$$

Skew-diagonalising the $\Omega_k{}^l = \frac{1}{2}\phi^i\phi^jR_{ijk}{}^l$ by 2 by 2 blocks $\begin{pmatrix} 0 & \Omega_k \\ -\Omega_k & 0 \end{pmatrix}$, $k = 1,\dots\frac{1}{2}m$ down the leading diagonal we see that

$$\exp\left(-L_{2x}t(0,0)\right) = \prod_{k=1}^{m/2}\frac{i\Omega_k}{2\pi t}\,\frac{1}{\sinh(i\Omega_k/2\pi)}. \tag{91}$$

Also, using flat fermionic paths or direct calculation we see that

$$\exp\left(-L_{2\xi}t(\xi,\xi')\right) = \int_{\mathcal{B}} d^m\rho \Big\{ \exp\left(-i\rho(\xi-\xi')\right)$$
$$\times \prod_{k=1}^{m/2} \Big(\cosh\frac{i\Omega_k}{2\pi} + (\xi^{2k-1}+i\rho^{2k-1})(\xi^{2k}+i\rho^{2k}) \sinh\frac{i\Omega_k}{2\pi} \Big) \Big\}. \tag{92}$$

This leads to

$$\operatorname{str}\exp\left(-L_2 t(0,0)\right) =$$
$$= \int_{\mathcal{B}} d^m\phi \prod_{k=1}^{m/2} \left(\frac{i\Omega_k}{2\pi t} \frac{1}{\sinh(i\Omega_k/2\pi)} \cosh\frac{i\Omega_k}{2\pi} \right) \operatorname{tr}\left(\exp\left(-\phi^i\phi^j \frac{F_{ij}}{2\pi} \right) \right). \tag{93}$$

so that

$$\operatorname{str}\exp\left(-Lt(0,0)\right)\, d\mathrm{vol} = \left\{ \operatorname{tr}\left(\exp\left(-\frac{F}{2\pi} \right) \right) \det\left(\frac{i\Omega/2\pi}{\tanh(i\Omega/2\pi)} \right)^{\frac{1}{2}} \right\}_m . \tag{94}$$

as required. ■

9 Superpaths and the index of the Dirac operator

In this section we introduce super Brownian paths on the spin supermanifold $\mathcal{S}(M, E_{O(M)})$, parametrised by a commuting parameter t and an anticommuting parameter τ. An Itô theorem is established for these paths, which is applied to give a path integral formula for the kernel of the super evolution operator $\exp(\not{D}^2 t + \not{D}\tau)$.

Suppose that p is a point on the spin supermanifold $\mathcal{S}(M, E_{O(M)})$ with coordinates $(x;\xi)$. The superpath based at p is constructed from solutions to the following system of purely bosonic stochastic differential equations

$$\begin{aligned} x_t^i &= x^i + \int_0^t M_{a\,s}^b e_b^i(x_s) \circ db_s^a, \quad i=1,\ldots,m \\ M_{a\,t}^b &= \delta_a^b + \int_0^t M_{e\,s}^d M_{a\,s}^c \Gamma_{dc}{}^b(x_s) \circ db_s^e - \int_0^t \tfrac{1}{2} M_{a\,s}^c R_c^b\, ds, \quad a,b=1,\ldots,m. \end{aligned} \tag{95}$$

The superpath $X_{(t;\tau)} = (x_{(t;\tau)}; \xi_{(t;\tau)})$ is then defined by

$$
\begin{aligned}
x^i{}_{(t;\tau)} &= x^i_t + \tfrac{i}{\sqrt{2}}\tau\xi^a_t e^i_a(x_t), \qquad i = 1,\dots,m \\
\xi^a{}_{(t;\tau)} &= \xi^a_t + \tfrac{i}{2\sqrt{2}}\tau\Gamma^a{}_{bc}\xi^b_t\xi^c_t, \qquad a = 1,\dots,m
\end{aligned}
\tag{96}
$$

where $\xi^a_t = \omega^b_t M^a_{b\,t}$. The Itô formula for these paths, generalising the super integration formula (28), is the content of the following theorem:

Theorem 9.1 *Suppose that F is supersmooth on $\mathcal{S}(M, E_{O(M)})$. Then*

$$
f(X_{(t;\tau)}) - f(X_0) =_{\mathbb{E}} \int_0^\tau \int_0^t Dx^i \partial_i f(X_{(s;\sigma)}) + D\xi^a \partial_{m+a} f(X_{(s;\sigma)}) \tag{97}
$$

where

$$
\begin{aligned}
DX^i &= D[s;\sigma]\, \tfrac{i}{\sqrt{2}} e^i_a(x_s)\xi^a_{s^-}, \\
D\xi^a &= D[s;\sigma]\, \tfrac{i}{2\sqrt{2}}\Gamma^a{}_{bc}\xi^b_{s^-}\xi^c_{s^-} + d\sigma\, \sigma d\xi^a_{s^-}.
\end{aligned}
\tag{98}
$$

(Here the symbol s^- indicates that the path ξ^a_s is to be taken at a time infinitesimally before s.)

This theorem may be proved by expanding in powers of τ. It leads immediately to the supersymmetric Feynman-Kac-Itô formula for the Dirac operator.

Theorem 9.2

$$
\exp\left(-\not{D}^2 t - \not{D}\tau\right) f(p) = \mathbb{E}_{\mathcal{S}}\left[f(X_{(t;\tau)})\right]. \tag{99}
$$

Outline of proof Expanding the right hand side of (97) gives

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{S}}\left[\int_0^\tau \int_0^t Dx^i \partial_i f(X_{(s;\sigma)}) + D\xi^a \partial_{m+a} f(X_{(s;\sigma)})\right] \\
&= \mathbb{E}_{\mathcal{S}}\left[\int_0^\tau \int_0^t D[s;\sigma]\, \not{D} f(X_{(s;\sigma)}) - 2\sigma\, \not{D}^2 f(X_{(s;\sigma)})\right].
\end{aligned}
\tag{100}
$$

If we now define the operator $U_{(t;\tau)}$ by

$$
U_{(t;\tau)} f(p) = (Pg)(p) \quad \text{where} \quad g(p) = \mathbb{E}_{\mathcal{S}}\left[f(X_{(t;\tau)})\right] \tag{101}
$$

we find that

$$U_{(t;\tau)}f - U_{0;0}f = \int_0^\tau \int_0^t D[s;\sigma] U_{(s;\sigma)}(\not{D}f - 2\sigma\, \not{D}^2 f) \tag{102}$$

so that

$$D_{(t;\tau)} U_{(t;\tau)} f(x;\xi) = U_{(t;\tau)}(\not{D} - 2\tau\, \not{D}^2) f(x;\xi). \tag{103}$$

Using the fact that

$$D_{(t;\tau)}\left(\exp\left(-\not{D}^2 t - \not{D}\tau\right)\right) = \exp\left(-\not{D}^2 t - \not{D}\tau\right)(\not{D} - 2\tau\, \not{D}^2), \tag{104}$$

we see that $U_{(t;\tau)} = \exp\left(-\not{D}^2 t - \not{D}\tau\right)$ as required. ■

10 Ghosts and the quantization of systems with symmetry

When using functional integrals to quantize a system with gauge symmetries, some mechanism must be found to remove the redundancy which arises from gauge equivalence. A widely-used method to achieve this uses a Lagrangian modified by gauge-fixing and Faddeev-Popov ghost terms. The canonical quantization of such theories was described by Batalin, Fradkin, Fradkina and Vilkovisky in a seies of papers [30, 31, 32, 33, 34]. (The methods developed in these papers will be refereed to as the BFV approach.) An elegant interpretation of this work, in particular an explanation of the rather complex gauge-fixing terms, was given by Henneaux in terms of BRST cohomology in [35]. A key idea in these methods is that the original Hamiltonian of the theory must be adjusted by a term which is the anticommutator of two operators (both odd), the BRST charge Ω of the theory (which determines the BRST cohomology of the theory) and the gauge-fixing fermion χ. The rôle of the gauge-fixing fermion in ensuring the validity of the gauge-fixing procedure in the path integral was clarified by the author in [36].

In the canonical approach symmetries of a system appear as constraints on the canonical positions and momenta. The simplest situation is that where the constraints are 'first class', that is, in involution under the Poisson bracket of the system and when acting on the Hamiltonian. For definiteness suppose that we have an unconstrained $2n$-dimensional phase space $\mathbb{R}^{2n}$, (with local coordinates (p_i, q^i), where $i = 1, \ldots, n$) together with a set of m first class constraints $T_a(p,q)=0, a=1,\ldots,m$ (with $m<n$) and first class Hamiltonian $H_c(p,q)$.

The true (reduced) phase space of this system is then the space $B=C/G$ where C is the submanifold of $\mathbb{R}^{2n}$ on which the constraints hold and G denotes the group generated by the constraints (which acts naturally on C), with symplectic structure given by the Dirac bracket. A set of gauge-fixing functions $X^a, a=1,\ldots,m$ are also introduced, whose essential properties are described below.

Now, as is shown in [37], the generating functional of the theory is given (in the more informal language of path integrals used in physics) by the Faddeev formula

$$\begin{aligned} Z = & \int \mathcal{D}p\mathcal{D}q\Big[\prod_t \left(\prod_{a=1}^m \left[\delta\left(T_a(p(t),q(t))\right)\delta\left(X^a(p(t),q(t))\right)\right]\right) \\ & \times \det\left(\{T_a(p(t),q(t)),X^b(p(t),q(t))\}\right) \\ & \times \exp\left(i\int_0^t p_a(t)\dot{q}^a(t)-H_c(p(t),q(t))dt\right)\Big], \end{aligned} \tag{105}$$

where the integration is over paths $p(t),q(t)$ in the unconstrained phase space which begin and end at the same point. (Standard quantization of the unconstrained phase space in the Schrödinger picture is used.) In establishing this expression, further assumptions have to be made about the gauge-fixing functions X^a, including the requirement that the matrix $(\{T_a,X^b\})$ (where $\{\ ,\ \}$ denotes the Poisson bracket) must satisfy the condition

$$\det(\{T_a,X^b\})\neq 0 \tag{106}$$

at all points (p,q) in $\mathbb{R}^{2n}$. (In the approach described below alternative, wider, possibilities for gauge-fixing are described [36].)

In the BFV approach the phase space is extended by including Lagrange multipliers $l^a, a=1,\ldots,m$ for the constraints together with their canonically conjugate momenta k_a, a set of m ghosts, η^a together with their conjugate momenta π_a and a set of m antighosts and corresponding momenta ϕ_a and θ^a. An extended Hamiltonian $H_{ext}=H_c+H_g$ is then defined, with H_g taking a form prescribed by the constraints, their commutators, and the gauge-fixing functions. The extended Hamiltonian is quite complicated, but it is shown in [34], by clever manipulation of path integrals, both that the corresponding generating functional

$$\begin{aligned} Z = & \int \mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}l\mathcal{D}\pi\mathcal{D}\eta\mathcal{D}\phi\mathcal{D}\theta \\ & \times\Bigg[\exp\Big(i\int_0^t p(t)\dot{q}(t)+k(t)\dot{l}(t)+\pi(t)\dot{\eta}(t)+\phi(t)\dot{\theta}(t) \\ & \qquad -H_{ext}(p(t),q(t),k(t),l(t),\pi(t),\eta(t),\phi(t),\theta(t))dt\Big)\Bigg] \end{aligned} \tag{107}$$

is independent of the choice of gauge-fixing functions and that it is equal to the Faddeev formula (105) for the generating functional for the original Hamiltonian on the reduced phase space. This result is known as the *Fradkin-Vilkovisky Theorem*. A significant result given by Henneaux [35] was an interpretation of the BFV Hamiltonian $H_{ext} = H_c + H_g$ in terms of the cohomology of the BRST operator Ω corresponding to the $2(m+n)$-dimensional phase space (with typical point (p, q, l, k)) subject to the constraints $T_a(p, q) = 0$, $k_a = 0$. Henneaux showed that the term H_g could be expressed as $i[\Omega, \chi]$ with a gauge-fixing fermion χ constructed from the gauge-fixing functions $X^a, a = 1, \ldots, m$ in a prescribed way. Henneaux also demonstrated the correspondence between observables on the reduced phase space and operators which commute with Ω, and the related correspondence between states for the reduced system and Ω-cohomology classes.

In [36] it is shown that the role of the anticommutator $i[\Omega, \chi]$ is to regulate the generating functional of the theory, ensuring that the summation involved is absolutely convergent so that the necessary cancellations which occur, so that the trace projects onto physical states only. This will occur if χ is chosen so that on each ghost and H_{ext} sector the eigenvalues of $[\Omega, \chi]$ tend to infinity and H_{ext} has a finite trace on the space of zeroes of $[\Omega, \chi]$ at each ghost number. The key steps in the argument are first to recall the argument of Schwarz [38] showing that the supertrace of an observable on the space of states of the theory is equal to the alternating sum of the traces over BRST cohomology classes, and then to observe that the path integral will give a supertrace. An example of this mechanism is described in the following section. The method is also valid when the phase space is extended merely by including ghosts η^a and their conjugate momenta π_a, provided that a gauge-fixing fermion can be found with the necessary properties.

11 Morse theory and the topological particle

This section concerns the quantization of a highly symmetric quantum mechanical system, the topological particle. Starting from the classical action, which involves a single function h on a manifold M, the canonical constraints are derived and the BRST Hamiltonian constructed. This Hamiltonian is the supersymmetric Hamiltonian given by Witten in is analysis of Morse theory [3]. The path integral methods developed earlier in this article are used to calculate the heat kernel (or matrix elements of the evolution operator) between critical points of h, which are central to Witten's arguments. The key technique is to define paths by a stochastic differential equation which encodes fluctuations about classical trajectories and thus gives a rigorous version of the phase space WKB method (described by Blau, Keski-Vakkuri and Niemi in [39]).

The topological particle is a model which was first described by Baulieu and Singer [40]. The field of the theory is a smooth map $x : [0, t] \to M$ (where as before M is a smooth, compact manifold of dimension m), and the action is

$$S\,(x(.)) = \int_0^t \sum_{j=1}^{m} v_j(x(t'))\,\dot{x}^j(t')\,dt'. \tag{108}$$

where $v = dh$ is an exact 1-form with $h : M \to \mathbb{R}$ a Morse function. The components v_j relative to local coordinates x^j on M are as usual defined by $v = \sum_{j=1}^{m} v_j dx^j$ so that $v_j = \frac{\partial x^j}{\partial h}$. This action can of course be expressed in the much simpler form $S\,(x(.)) = h(x(t)) - h(x(0))$, and this form makes it clear that the action is invariant under arbitrary changes of the field x provided that the endpoints remain fixed. This is the first indication that the theory is a so-called topological theory, a theory with no independent degrees of freedom and whose equations of motion are identically satisfied. One might assume such a theory was of no interest, but in fact, although at the linearized level there is no dynamics, on quantization topological information is captured.

The first step in quantization is to investigate the Hamiltonian dynamics of the model. Using the action (108) we see that the Lagrangian is

$$\mathcal{L}(x, \dot{x}) = v_j(x)\dot{x^j} \tag{109}$$

so that, using the Legendre transformation to the phase space $T^*(M)$ the canonical momentum is

$$p_j = i\frac{\delta\mathcal{L}(x,\dot{x})}{\delta\dot{x}^j} = iv_j(x) \tag{110}$$

where we are using Euclidean time, which differs from the usual physical time of quantum mechanics by a factor i.

The symmetries of the system now reveal themselves as m constraints on the canonical position and momenta x^j, p_j of the form

$$T_j \equiv -p_j + iv_j(x) = 0, \qquad j = 1,\ldots,m. \tag{111}$$

The Poisson brackets on the phase space $T*(M)$ are obtained from the standard symplectic form $\omega = dp_i \wedge dx^j$, leading to

$$\{T_j, T_k\} = 0, \qquad j,k = 1,\ldots,m, \tag{112}$$

where we have used the fact that v is closed.

The standard prescription determines the Hamiltonian to be

$$H = ip_jx^j + \mathcal{L}(x,\dot{x}) = 0 \tag{113}$$

so that the constraints are first class and abelian. The number of constraints equals the number of fields, which is a further indication of the topological nature of the theory; by simple counting one would expect all field configurations to be equivalent. The first hint that something more survives comes from the gauge-fixing process; a natural choice of gauge-fixing conditions is

$$X^j = g^{ij}(-p_{ij} - iv_i), \tag{114}$$

where g is a Riemannian metric on M. (Justification for this choice, and the independence of its consequences on the particular Riemannian metric used, will be made when the model is quantized below.)

To implement the constraints and gauge fixing in the quantum theory we use the BRST approach, introducing ghosts η^j and their canonical momenta $\pi_j, j = 1,\ldots m$. The configuration space for the BRST theory is the supermanifold $\mathcal{S}(M, T(M))$ with coordinates x^j, η^j corresponding to the original fields of the theory and the ghosts. The phase space is $\mathcal{S}(T^*(M), T(M) \times T^*(M))$, where $T(M)$ and $T^*(M)$ are regarded as bundles over $T^*(M)$ by pulling back from M using the projection map $T^*(M) \to M$. The even coordinates on phase space are x^j, p_j while the odd coordinates are η^j, π_j which correspond to the ghosts and antighosts respectively. The 'classical' BRST fields are maps from $[0,t]$ into the configuration supermanifold. The simplest choice of symplectic form on the super phase space is

$$\omega_s = dp_j \wedge dx^j + D\pi_j \wedge D\eta^j - \tfrac{1}{2} R_{ijk}{}^l \pi_l \eta^k dx^i \wedge dx^j \tag{115}$$

where R is the Riemann curvature of g and D denotes covariant differentiation using the Levi-Civita connection so that

$$D\eta^j = d\eta^j + \Gamma_{ik}{}^j \eta^k dx^i \qquad \text{and} \qquad D\pi_k = d\pi_k - \Gamma_{ik}{}^j \pi_j dx^i. \tag{116}$$

The corresponding Poisson brackets are

$$\begin{array}{llll} \{p_j, x^i\} & = \delta^i_j & \{p_i, p_j\} & = R_{ijk}{}^l \pi_l \eta^k \\ \{p_j, \eta^i\} & = \Gamma_{jk}{}^i \eta^k & \{p_j, \pi_i\} & = -\Gamma_{ji}{}^k \pi_k \\ \text{and} & & \{\pi_j, \eta^i\} & = \delta^i_j, \end{array} \tag{117}$$

with others being zero.

To quantize this system in the Schrödinger picture we use states which are supersmooth functions on $\mathcal{S}(M, T(M))$, or, in more physical language, wave functions $\psi(x, \eta)$. The position observables x and η are represented by multiplication while the momentum observables are represented by covariant differentiation:

$$p_j = -iD_j \equiv -i\left(\tfrac{\partial}{\partial x^j} + \eta^i \Gamma_{ij}{}^k \tfrac{\partial}{\partial \eta^k}\right) \quad \text{and} \quad \pi_j = -i\tfrac{\partial}{\partial \eta^j}. \tag{118}$$

The BRST operator is (following the standard procedure)

$$\Omega = \eta^j T_j = i\eta^j \left(\tfrac{\partial}{\partial \eta^j} + v_j(x)\right), \tag{119}$$

while the gauge-fixing fermion is

$$\chi = \pi_j X^j = i g^{jk} \pi_j \left(D_k - v_k(x)\right). \tag{120}$$

Identifying functions on configuration space with forms on M, we see that

$$\Omega = ie^{-h} de^h \qquad \text{and} \qquad \chi = e^h \delta e^{-h} \tag{121}$$

where d is exterior differentiation and $\delta = *d*$ is the adjoint of d. Thus we see that Ω and χ are the supersymmetry operators introduced by Witten in his study of supersymmetry and Morse theory [3].

The BRST Hamiltonian for the theory is simply $[\Omega, \chi]$, since the classical Hamiltonian is zero. It is thus equal to the Hamiltonian used by Witten in [3], and has the explicit form

$$H = \tfrac{1}{2}(d+\delta)^2 + \tfrac{1}{2}g^{jk}\frac{\partial h}{\partial x^j}\frac{\partial h}{\partial x^k} + \tfrac{1}{2}g^{jk}(\eta^l \pi_j - \pi_j \eta^l)\frac{D^2 h}{Dx^k Dx^l}. \tag{122}$$

Using the mapping $\psi \mapsto e^{-h}\psi$, Witten shows that the cohomology of Ω is isomorphic to the De Rham cohomology of M, and that forms with zero H eigenvalues give exactly one representative of each Ω cohomology class. Thus the gauge-fixing used is a good one, and leads to results which are independent of the choice of metric.

To describe how the superspace path integral techniques developed in earlier sections can be applied to this section, we begin for simplicity by working in flat space, with the Euclidean metric. Even in this situation the key new idea can be illustrated; this is to use a stochastic differential equation which leads to paths which encode directly Brownian fluctuations about the classical trajectories, giving a stochastic version of the WKB method. In the standard WKB approach only second order fluctuations are considered, first order fluctuations vanishing because they are taken about classical trajectories while higher order fluctuations are ignored. The stochastic formula we present here is exact.

Consider first the stochastic differential equation

$$dx_t^j = db_t^j - v_j(x_t)dt, \quad x_0 = x \tag{123}$$

which is the equation for the classical trajectories augmented by stochastic fluctuations; this corresponds to the Nicolai map of the theory. For positive t we consider the operator U_t acting on functions ψ on configuration space given by

$$U_t\psi(x,\eta) = \mathbb{E}_{\mathcal{S}}\left[e^{\int_0^t (\partial_j h(x_s)dx_s^j + \partial_j\partial_k h(x_s)\left(i\theta_s^j \rho_{s\,k} + \frac{1}{2}\delta_k^j\right)ds)}\psi(x_t,\eta_t)\right]. \tag{124}$$

By Itô calculus we see that

$$\left[\exp\left(\int_0^t (\partial_j h(x_s)dx_s^j + \partial_j\partial_k h(x_s)\left(i\theta_s^j \rho_{s\,k} + \tfrac{1}{2}\delta_k^j\right)ds)\psi(x_t,\eta_t)\right)\right]$$

$$= \left[e^{\int_0^t (\partial_j h(x_s) dx_s^j + \partial_j \partial_k h(x_s) \left(i\theta_s^j \rho_{s\,k} + \frac{1}{2}\delta_k^j \right) ds)} (-H)\psi(x_t, \eta_t) \right]$$
$$+ \quad \text{terms of zero measure,} \tag{125}$$

so that

$$\frac{\partial U_t \psi(x,\eta)}{\partial t} = -U_t H \psi(x,\eta) \tag{126}$$

and so we may conclude that

$$U_t = \exp -tH. \tag{127}$$

Making further use of Itô calculus we can simplify the expression for U_t: we have

$$\int_0^t \left(\partial_j h(x_s) dx_s^j + \tfrac{1}{2} \partial_j \partial_j h(x_s) ds \right) = h(x_s) - h(x) \tag{128}$$

so that equation (124) can be rewritten

$$\begin{aligned} &\exp -tH\psi(x,\eta) \\ &= \mathbb{E}_{\mathcal{S}} \left[\exp\left(-(h(x) - h(x_t))\right) \exp\left(\int_0^t i\partial_j \partial_k h(x_s) \theta_s^j \rho_{s\,k} ds \right) \psi(x_t, \theta_t) \right]. \end{aligned} \tag{129}$$

This expression shows how the WKB factor $\exp -\Delta h$ (which corresponds directly to the original topological action (108)) appears in the path integral. This formula is readily adapted to curved space: in place of 123 one simply uses the covariant stochastic differential equation

$$d\tilde{x}_t^j = dx_t^j - g^{jk}(\tilde{x}_t) v_k(\tilde{x}) dt \tag{130}$$

with x_t the curved space Brownian paths of (59), and then obtains the Feynman-Kac-Itô formula

$$\begin{aligned} \exp -tH_g \psi(x,\eta) &= \int d\mu \exp(-(h(x) - h(x_t))) \\ &\quad \exp\Big(\int_0^t (D_j D_k h(x_s) i g^{lk}(x_s^j) \theta_s^j \rho_{s\,l} \\ &+ R_j^k(x_s) \theta_s^j \rho_{k\,s} + \tfrac{1}{2} R_{jk}^{li}(x_s) \theta_s^j \theta_s^k \rho_{l\,s} \rho_{i\,s}) ds \Big) \psi(x_t, \theta_t). \end{aligned} \tag{131}$$

much as before.

In [3] Witten rescales the Morse function h by a factor (here denoted by u) and by taking the large u limit distils out information about the topology of the manifold, explicitly building the cohomology of the manifold in terms of the critical points of h. The stochastic methods developed here allow a rigorous mathematical version of Witten's calculations to be derived in the spirit of the physicist's approach. Some of Witten's work is derived in a mathematically rigorous way by Cycon, Froese Kirsch and Simon in [20], we make use of their results here. A full mathematical derivation of Witten's result is given by Helffer and Sjörstrand in [41], using semiclassical techniques.

It is useful at this stage to introduce some standard terminology and results from Morse theory: a critical point a of the function h is said to have index p if the Hessian matrix $(D_{x^j}D_{x^k h})$ has exactly p negative eigenvalues. Each critical point a has a neighbourhood N_a on which a special coordinate system can be chosen in which the Morse function h takes the standard form

$$h(x) = h(a) + \tfrac{1}{2}\sum_{j=1}^{m} \sigma_j (x^j - a^j)^2 \tag{132}$$

where $\sigma_j = +1$ for $j = 1, \ldots, m-p$ and $\sigma_j = -1$ for $j = m-p+1, \ldots, m$. A metric g on M satisfies the Smayle transversality condition for h if the solution curves Γ_{ab} to the 'steepest descent' differential equation

$$\frac{dx^\mu(t)}{dt} = -g^{\mu j}\frac{\partial h}{\partial x^j} \tag{133}$$

which start from a critical point b and end at a critical point a (with $h(a)$ necessarily less than $h(b)$) are discrete (and finite in number).

In [3] Witten first shows that, if H_u is the Hamiltonian obtained when h is scaled to uh, then corresponding to each critical point a of h of degree p there is exactly one eigenstate $\psi_a(x, \eta)$ of H_u with the following properties: it has low eigenvalue, in the sense that it does not tend to infinity with u, it is a form of degree p (that is, it is of degree p in η) and it is concentrated near p. Furthermore no there are no other low-eigenvalue eigenstates of H_u. The simplest way to see this is to observe that in Morse coordinates and using the free choice of metric to pick a metric which is Euclidean on N_a the Hamiltonian H_u splits into a bosonic part which is simply an oscillator Hamiltonian and a fermionic part which reflects the index p and is easily seen to have p-form eigenfunctions. A fully rigorous proof of the result is given by Simon et al in [20].

Now H_u and $d_u \equiv e^{-hu} d e^{hu}$ commute, and so if ψ is a low eigenvalue eigenstate of H_u, then $d_u \psi$ is also a low eigenvalue eigenstate. Hence for each critical point a of index p

$$d_u \psi_a = \sum_{b \text{ critical of index } p+1} C_{ab} \psi_b \tag{134}$$

for some real numbers C_{ab} (which will in due course be determined). We thus see that the real cohomology of M can be modelled by p-cochains σ taking the form

$$\sigma = \sum_{a \text{ critical}} \sigma_a \psi_a \tag{135}$$

where the coefficients σ_a are real numbers, and the coboundary operator is d_u.

The coefficients C_{ab} which determine the action of d_u may be determined from the matrix elements $d_{u\,2} \exp -H_u t(A, B)$ in the large u limit. (The subscript $_2$ indicates that the derivative is taken with respect to the second argument.) We may take an orthonormal basis of eigenstates of H_u consisting of the low eigenvalue states $\psi_{u\,a}$ with eigenvalues $\lambda_{u\,a}$ for each critical point a and further eigenstates $\psi_{u\,n}, n = 1, \ldots$ with eigenvalues $\lambda_{u\,n}$ all of which tend to infinity with u and, for fixed u, tend to infinity with n. Then we can express the kernel of the evolution operator as

$$\exp -H_u t(X, Y) = \sum_{a \text{ critical}} e^{-\lambda_{u\,a} t} * \psi_{u\,a}(X) \psi_{u\,a}(Y) + \sum_n e^{-\lambda_n t} * \psi_n(X) \psi_n(Y). \tag{136}$$

For large u at leading order this becomes

$$\exp -H_u t(X, Y) = \sum_{a \text{ critical}} * \psi_{u\,a}(X) \psi_{u\,a}(Y) \tag{137}$$

so that at leading order for large u

$$d_{u\,2} \exp -H_u t(A, B) = C_{ab} * \psi_a(A) \psi_b(B) \tag{138}$$

where for notational simplicity we have omitted the subscripts u.

To evaluate this expression we use the Feynman-Kac-Itô formula (131), together with the explicit form (132) of h in the neighbourhood of a critical point of h. We choose a metric which globally satisfies the Smale transversality condition (so that classical curves Γ_{ab} of steepest descent from the critical point a to the critical point b are finite in number and discrete) and also one which is Euclidean in a set of Morse coordinates around each critical p

Before proceeding further it is useful to introduce some specific coordinate systems. For each critical point c in M we will choose on N_c a fiducial set of Morse coordinates $x^{\mu}_{[c]}$ and fermionic partners $\eta^{\mu}_{[c]}$. Additionally for each steepest descent curve Γ_{ab} joining the pair of critical points a and b with indices p and $p+1$ respectively we will choose a coordinate neighbourhood $U_{\Gamma_{ab}}$ which contains $N_a \cup N_b \cup U_{\Gamma_{ab}}$ with coordinates $x_{\Gamma_{ab}}, \eta_{\Gamma_{ab}}$ such that Γ_{ab} lies along $x^{n-p}_{\Gamma_{ab}}$ while $x_{\Gamma_{ab}}, \eta_{\Gamma_{ab}}$ match $x_{[b]}, \eta_{[b]}$ on N_b apart from possible rotations, and also match $x_{[a]}$ on N_a apart from possible rotations and (necessarily) a translation in the x^{n-p} coordinate with ${x^{n-p}}_{\Gamma_{ab}} = {x_{[a]}}^{n-p} + k_a$ for some positive constant k_a. Eventually, when the coordinate systems are reconciled, sign factors will be produced.

Within N_a the Hamiltonian then has the form

$$H_u = \sum_{i=1}^{n} \left[\tfrac{1}{2}\Big(- \frac{\partial^2}{\partial x^i_{\Gamma_{ab}} \partial x^i_{\Gamma_{ab}}} + u^2 (x^i_{\Gamma_{ab}} - a^i_{\Gamma_{ab}})(x^i_{\Gamma_{ab}} - a^i_{\Gamma_{ab}}) \Big) \right.$$
$$\left. + \quad \tfrac{i}{2} u \sigma_i (\eta^i_{\Gamma_{ab}} \pi_{\Gamma_{ab} i} - \pi_{\Gamma_{ab} i} \eta^i_{\Gamma_{ab}}) \right]. \tag{139}$$

The bosonic and fermionic parts commute so that their heat kernels may be considered separately; the bosonic part is the Harmonic oscillator Hamiltonian whose heat kernel is given by Mehler's formula [29], while the fermionic part is (apart from sign factors σ_i) the fermionic oscillator whose heat kernel is given in [22]. If x is near a and in N_a then at leading order for large u

$$\begin{aligned} &\exp -H_u t(A,X) =_{\mathrm{def}} M(A,X) \\ &= \left(\frac{u}{\pi}\right)^{n/2} \exp\left(-\tfrac{1}{2} u (x_{\Gamma_{ab}} - k_a)^2\right) \prod_{i=1}^{n-p} (-\alpha^i_{\Gamma_{ab}}) \prod_{j=n-p+1}^{n} \eta^j_{\Gamma_{ab}} \end{aligned} \tag{140}$$

where X is a point over x in N_a, with coordinates $(x_{\Gamma_{ab}}, \eta_{\Gamma_{ab}})$.

Next we calculate $\exp -H_u t(A,X)$ for x near Γ_{ab} using the Feynman-Kac-Itô formula (131). In this case the steepest descent curve (satisfying (133)) from x approaches a very fast. Thus after very small time δt the path $\tilde{x}_{\delta t}$ is almost certainly near a, so that to leading order in u we have a contribution from Γ_{ab} of

$$\begin{aligned}
&\exp -H_u t(A,X)_{\Gamma_{ab}} = \exp -H_u \delta t \exp -H_u(t-\delta t)(A,X) \\
&= \mathbb{E}_{\mathcal{S}}\Bigg[\exp\left(-u(h(x)-h(\tilde{x}_{\delta t}))\right) \\
\times \quad &\exp\Bigg(\int_0^{\delta t} u D_i D_j h(\tilde{x}_s) i g^{kj}(\tilde{x}_s)\tilde{\theta}^i_s \tilde{\rho}_{s\,k} \\
&\quad + R_i{}^j(\tilde{x}_s)\tilde{\theta}^i_s\tilde{\rho}_{j\,s} + \frac{1}{2}R^{kj}_{ik}(\tilde{x}_s)\tilde{\theta}^i_s\tilde{\theta}^k_s\tilde{\rho}_{k\,s}\tilde{\rho}_{j\,s}\,ds\Bigg) M(a_{\Gamma_{ab}},\alpha_{\Gamma_{ab}},\tilde{x}_{\delta t},\tilde{\theta}_{\delta t})\Bigg] \\
&= \left(\frac{u}{\pi}\right)^{n/2} \exp\left(-u(h(x)-h(a))\right)\prod_{i=1}^{n-p}(-\alpha^i_{\Gamma_{ab}})\prod_{j=n-p+1}^{n}\eta^j_{\Gamma_{ab}}.
\end{aligned} \tag{141}$$

Here we have used the fact that the operator $\eta^i\pi_i$ which corresponds to the term $g^{kj}(\tilde{x}_s)\tilde{\theta}^i_s\tilde{\rho}_{s\,k}$ in the path integral has zero eigenvalue on the kernel $\exp -H_u t(A,\tilde{x}_{\delta t},\tilde{\theta}_{\delta t})$ when x lies on Γ_{ab}.

To calculate $d_{u\,(2)}\exp -H_u t(A,B)$ we cannot take the derivative of the separate contributions from each Γ_{ab} using (141) because as we vary x around b we will jump from one Γ_{ab} to another. To avoid this difficulty we note that

$$\begin{aligned}
&d_{u\,(2)}\exp -H_u t(A,B) \\
&\quad = d_{u\,(2)}\exp -H_u s\,\exp -H_u(t-s)(A,B) \\
&\quad = \int_M d^n x\, d^n\eta \exp -H_u(t-s)(A,X)\, d_{u\,(2)}\exp -H_u s(X,B).
\end{aligned} \tag{142}$$

Because of the concentration of $d_{u\,(2)}\exp -Hs(X,B)$ near b we can integrate over $\mathbb{R}^n$ rather than M using the form of $\exp -Hs(X,B)$ which is approximately true for large u on N_b; although ultimately we will obtain a result independent of s and t, at this stage we must use Mehler's formula in full (including terms of order e^{-us} whose equivalent we could neglect near a for our purposes) because it is not the zero mode of H_u which will contribute to

$d\psi_a(b)$ at leading order. Thus for x and y near b we use

$$\exp -H_u s(X,Y) = \left(\frac{u}{\pi}\right)^{n/2} \exp\left(-\tfrac{1}{2}u\left(x^2_{\Gamma_{ab}}\frac{\cosh us}{\sinh us}\right) + u\,\frac{x_{\Gamma_{ab}}y_{\Gamma_{ab}}}{\sinh us}\right)$$
$$\times \prod_{i=1}^{n-p-1}\left(\phi^i_{\Gamma_{ab}}e^{-us} - \eta^i_{\Gamma_{ab}}\right)\prod_{j=n-p}^{n}\left(\phi^j_{\Gamma_{ab}} - \eta^j_{\Gamma_{ab}}e^{-us}\right). \tag{143}$$

where and $(x_{\Gamma_{ab}}, \eta_{\Gamma_{ab}})$, $(y_{\Gamma_{ab}}, \phi_{\Gamma_{ab}})$ are the coordinates of X and Y respectively, so that the term of $d_{u\,(2)} \exp -H_u s(X,B)$ of leading order in u is

$$\left(\frac{u}{\pi}\right)^{n/2} u\,\frac{x^{n-p}}{\sinh us}\exp\left(-\tfrac{1}{2}u\left(x^2_{\Gamma_{ab}}\frac{\cosh us}{\sinh us}\right)\right)\quad \prod_{i=1}^{n-p}\eta^i_{\Gamma_{ab}}e^{-us}\prod_{j=n-p}^{n}\beta^j_{\Gamma_{ab}}. \tag{144}$$

Using (142) gives

$$\begin{aligned}
&d_{u\,(2)} \exp -H_u t(A,B)\\
&= \int_{\mathbb{R}^n} d^n x_{\Gamma_{ab}}\ \left(\frac{u}{\pi}\right)^n \theta(x^{n-p}_{\Gamma_{ab}})u\,\frac{e^{-us}}{\sinh us}x^{n-p}_{\Gamma_{ab}}\prod_{i=1}^{n-p}\left(-\alpha^i_{\Gamma_{ab}}\right)\prod_{j=n-p}^{n}\beta^j_{\Gamma_{ab}}\\
&\quad\times \exp\left(-\tfrac{1}{2}ux^2_{\Gamma_{ab}}\frac{\cosh us}{\sinh us}\right)\exp -u(h(x)-h(a))\\
&= \int_0^\infty dx^{n-p}_{\Gamma_{ab}}\left(\frac{u}{\pi}\right)^{(n+1)/2} u\,\frac{e^{-us}}{\sinh us}x^{n-p}_{\Gamma_{ab}}\prod_{i=1}^{n-p}\left(-\alpha^i_{\Gamma_{ab}}\right)\prod_{j=n-p}^{n}\beta^j_{\Gamma_{ab}}\\
&\quad\times \exp\left(-\tfrac{1}{2}ux^2_{\Gamma_{ab}}\left(\frac{\cosh us}{\sinh us}-1\right)\right)\exp -u(h(b)-h(a))\\
&= \left(\frac{u}{\pi}\right)^{(n+1)/2}\prod_{i=1}^{n-p}\left(-\alpha^i_{\Gamma_{ab}}\right)\prod_{j=n-p}^{n}\beta^j_{\Gamma_{ab}}\exp -u(h(b)-h(a))
\end{aligned}\tag{145}$$

at leading order in u. Here the θ-function occurs because the contribution from Γ_{ab} to $\exp -H_u(t-s)$ is zero on the side of b away from a. Now using equation (138) and the fact that

$$^*\psi_a(A) = \left(\frac{u}{\pi}\right)^{n/4}\prod_{i=1}^{n-p}\alpha_{[a]}{}^j, \quad \psi_b(B) = \left(\frac{u}{\pi}\right)^{n/4}\prod_{j=n-p}^{n}\beta^j_{[b]} \tag{146}$$

we see that

$$c_{ab} = \left(\frac{u}{\pi}\right)^{1/2} \exp -u(h(b) - h(a)) \sum_{\Gamma_{ab}} (-1)^{\sigma_{\Gamma_{ab}}} \tag{147}$$

where $(-1)^{\sigma_{\Gamma_{ab}}}$ is a sign factor which comes from transforming between the $[a]$ and $[b]$ coordinates and the $[\Gamma_{ab}]$ coordinates.
If (again following Witten [3]) we rescale each ψ_c to $\tilde{\psi}_c = e^{-uh(c)}\psi$, and also define $\tilde{d_u} = \sqrt{\frac{\pi}{u}} d_u$, we obtain

$$\tilde{d}_u \tilde{\psi}_a = \sum_{\Gamma_{ab}} (-1)^{\sigma_{\Gamma_{ab}}} \tilde{\psi}_b \tag{148}$$

which coincides with the geometrical approach using ascending and descending spheres.

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