Convergence to time-periodic solutions in Hamilton-Jacobi equations on the circle

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ABSTRACT. – . The goal of this paper is to give a simple proof of the convergence to time-periodic states of the solutions of time-periodic Hamilton-Jacobi equations on the circle with convex Hamiltonian. Note that the period of limiting solutions may be greater than the period of the Hamiltonian.

RESUMÉ. – . On donne une preuve simple de la convergence vers des états périodiques en temps pour les solutions d'équations de Hamilton-Jacobi sur le cercle avec un Hamiltonien convexe et périodique en temps. Il est à noter que la périodes des solutions limites peut être plus grande que la période du Hamiltonien.

1. Introduction

We consider the Hamilton-Jacobi equation

$$u_t + H(t, x, u_x) = 0, \quad x \in \mathbf{T} \tag{1.1}$$

where $\mathbf{T}$ is the unit circle. The Hamiltonian $H(t, x, p) : \mathbb{R} \times \mathbf{T} \times \mathbb{R} \mapsto \mathbb{R}$ is C^2, 1-periodic in t, and satisfies the following classical hypotheses:

- Strict convexity: $H_{pp}(t, x, p) > 0$ for all $(t, x, p) \in \mathbb{R} \times \mathbf{T} \times \mathbb{R}$.
- Super-linearity: $H(t, x, p)/p \longrightarrow \infty$ as $|p| \longrightarrow \infty$ for each $(t, x) \in \mathbb{R} \times \mathbf{T}$.
- Completeness: The Hamiltonian vector-field

  $$X(t, x, p) = (H_p(t, x, p), -H_x(t, x, p))$$

  is complete, *i.e.* for all (t_0, x_0, p_0), there exists a C^2 curve $\gamma(t) = (x(t), p(t)) : \mathbb{R} \longrightarrow \mathbf{T} \times \mathbb{R}$ such that $(x(t_0), p(t_0)) = (x_0, p_0)$ and $\dot\gamma(t) = X(t, \gamma(t))$ for all $t \in \mathbb{R}$.

The first two assumptions are classical in the viscosity solutions theory; see [14]. The last one is introduced in Mather [12]; note that it is satisfied if there exists a constant C such that $|H_t| \leq C(1+H)$.

Under the above three assumptions, the Cauchy Problem for (1.1) is well posed in the viscosity sense: given a time $s \in I\!R$ and a continuous function $u_0 : \mathbf{T} \to I\!R$, equation (1.1) has a unique viscosity solution $u(t,x) : [s, +\infty[\times \mathbf{T} \longrightarrow I\!R$, such that $u(s,.) = u_0$. It will be denoted by $\mathcal{T}(s,t)u_0$. See [14], for instance.

It is known - and this is not specific to the one-dimensional setting - that there exists a real number λ such that $u(t,x) - \lambda t$ is bounded for all viscosity solution $u : [s, +\infty[\times \mathbf{T} \longrightarrow I\!R$ of (1.1). The real number λ has at least three different names: It is the critical value of Mañe, see [13], [10] or the value $\alpha(0)$, where $\alpha : I\!R \longrightarrow I\!R$ is the Mather function, see [12], or the averaged Hamiltonian [15]. We are interested in proving the following result.

Theorem 1.1 *Let $u(t,x) : [s, +\infty[\times \mathbf{T} \longrightarrow I\!R$ be a viscosity solution of (1.1). There exist an integer T and a viscosity solution $l(t,x) = \lambda t + \phi(t,x) : I\!R \times \mathbf{T} \longrightarrow I\!R$ such that ϕ is T-periodic in t and*

$$\lim_{t \longrightarrow \infty} \|u(t,.) - l(t,.)\|_\infty = 0.$$

In the following, we will always assume that $\lambda = 0$, which can be obtained by replacing the Hamiltonian H by $H - \lambda$.

It is known that there always exist a viscosity solution of (1.1) which is 1-periodic in time. However, it is not hard to build examples of viscosity solutions of equations of the form (1.1) which do not converge to 1-periodic solutions, see [3] and [9]. More precisely, one can build solutions which are periodic in time, but of minimal period greater than one. Hence one cannot expect to have always $T = 1$ in the theorem.

For time-independent Hamiltonians, convergence to steady states is known: a particular nontrivial multidimensional case is studied in [16], and the general result in one dimension is given in [19]. The general multidimensional result is due to Fathi [7], and two different proofs are available in [18] and [3].

The situation is not so clear when the Hamiltonian is time-periodic. In order to be more precise, it is useful to recall that one can associate to the equation (1.1) a rotation number $\rho \in I\!R$, see section 4., which is the rotation number of extremals. We then have the following

Addendum *The period T in the theorem is* 1 *if the rotation number ρ is irrational and is not greater than q if ρ is a rational p/q.*

The theorem and its addendum have been proved in [4] from the dynamical system point of view, that is from the study of extremals.

The result in the case of a rational rotation number had been previously obtained by a method relying on the dynamic programming principle in [18]. It turns out that the ideas of this paper can be exploited further to provide a simpler proof of the general case. It is our aim to present this proof here.

We shall first recall the general properties of viscosity solutions, in Section 2.. In Section 3., we introduce some dynamics, define some Aubry-Mather sets, and recall specific observations concerning ω-limit solutions, mostly taken from [18]. All the results in these sections are general, and remain true if one considers equation (1.1) on any compact manifold. We complement these general observations by specific one-dimensional arguments in Section 4. to conclude the proof.

2. General properties and a large time behaviour candidate

Let us start with some well understood properties of viscosity solutions without proof, see [14], [6] or [9]. We denote $\mathcal{T}(s,t) : C(\mathbf{T}, I\!R) \longrightarrow C(\mathbf{T}, I\!R)$ the mapping which associates to each function $u_0 \in C(\mathbf{T}, I\!R)$ the function $u(t,.)$, where $u(t,x) \in C([s,+\infty[\times \mathbf{T}, I\!R)$ is the viscosity solution of (1.1) such that $u(s,.) = u_0$. We have the Markov property

$$\mathcal{T}(t,t') \circ \mathcal{T}(s,t) = \mathcal{T}(s,t')$$

for $s \leq t \leq t'$, hence the mappings $\mathcal{T}(0,n) = \mathcal{T}(0,1)^n$, $n \in I\!N$ form a discrete semi-group. We will note $\mathcal{T}$ for $\mathcal{T}(0,1)$ for simplicity.

The mappings $\mathcal{T}(s,t)$ are contractions,

$$\|\mathcal{T}(s,t)u - \mathcal{T}(s,t)v\|_\infty \leq \|u - v\|_\infty.$$

The mappings $\mathcal{T}(s,t)$ are compact. More precisely, given a bounded set $B \subset C(\mathbf{T}, I\!R)$, there exists a positive nondecreasing function $K(\epsilon) :]0,\infty[\longrightarrow]0,\infty[$ such that $\mathcal{T}(s,t)u$ is $K(\epsilon)$-Lipschitz for all $u \in B$ and all $t \geq s + \epsilon$.

The mappings $\mathcal{T}(s,t)$ are order-preserving and satisfy $\mathcal{T}(s,t)(u+c) = c + \mathcal{T}(s,t)(u)$ for all real c. Let us end this list by recalling the Hopf-Lax-Oleinik formula:

$$\mathcal{T}(s,t)u_0 = \inf_\gamma \Big(u_0(\gamma(s)) + \int_s^t L(\sigma, \gamma, \dot\gamma)\, d\sigma \Big) \tag{2.1}$$

where the infimum is taken on the set of piecewise C^1 curves with values in $\mathbf{T}$, such that $\gamma(t) = x$. In the above, the Lagrangian $L(t,x,v)$ is defined as the Legendre transform of H:

$$L(t,x,v) = \max_{p \in I\!R} \Big(pv - H(t,x,p) \Big).$$

It is a classical result from the calculus of variations that, under our assumptions, the infimum in (2.1) is reached by C^2 curves.

The main point of this paragraph is the following simple proposition below. Note that it also implies the existence of a fixed point of the semi-group generated by $\mathcal{T}$, *i.e.* the existence of a 1-periodic viscosity solution of (1.1).

Proposition 2.1 *Let $u(t,x) \in C([s,\infty[\times \mathbf{T}, \mathbb{R})$ be a viscosity solution of (1.1); recall that it is bounded on $[s,\infty[\times \mathbf{T}$. Then the 1-periodic function*

$$\phi(t,x) = \liminf_{n\to+\infty} u(t+n,x)$$

is a viscosity solution of (1.1).

Proof. We have to prove that $\mathcal{T}(s,t)\phi(s,.) = \phi(t,.)$ for all $s \leq t$.

It follows from the Barles-Perthame Lemma [1] that ϕ is a viscosity super-solution of (1.1), *i.e.* that $\mathcal{T}(s,t)\phi(s,.) \leq \phi(t,.)$. This fact, which is a general feature of viscosity solutions of first or second order equations, can be easily seen on (2.1). In order to do so, we fix (t,x) and consider an increasing sequence n_k of integers such that $u(t+n_k,x) \longrightarrow \phi(t,x)$. There exists a sequence of curves $\gamma_k : [s,t] \longrightarrow \mathbf{T}$ such that

$$u(t+n_k,x) = u(s+n_k,\gamma_k(0)) + \int_s^t L(\sigma,\gamma_k(\sigma+t),\dot{\gamma}_k(\sigma+t))\, d\sigma.$$

The sequence γ_k is compact for the C^1 topology, and we will assume by possibly taking a subsequence in n_k that it is convergent, and note γ the limit. Taking the $\liminf$ in the equality above gives

$$\phi(t,x) \geq \phi(s,\gamma(0)) + \int_s^t L(\sigma,\gamma(\sigma+t),\dot{\gamma}(\sigma+t))\, d\sigma \geq \mathcal{T}(s,t)\phi(s,.)(x).$$

We have used that the functions $u(s+n_k,.)$ have a common Lipschitz constant to conclude that $\liminf u(s+n_k,\gamma_k(0)) = \liminf u(s+n_k,\gamma(0)) \geq \phi(s,\gamma(0))$.

The reverse inequality is specific to Hamilton-Jacobi equations, and explicitely relies on (2.1). Note that for all curve $\gamma : [s,t] \longrightarrow \mathbf{T}$, we have

$$u(t+n,x) \leq u(s+n,\gamma(0)) + \int_s^t L(\sigma,\gamma(\sigma+t),\dot{\gamma}(\sigma+t))\, d\sigma.$$

Taking the $\liminf$, we obtain

$$\phi(t,x) \leq \phi(s,\gamma(0)) + \int_s^t L(\sigma,\gamma(\sigma+t),\dot{\gamma}(\sigma+t))\, d\sigma$$

for each curve γ, hence $\phi(t,.) \leq \mathcal{T}(s,t)\phi(s,.)$, which is the desired inequality. $\square$

The basic objects to understand in order to study the asymptotic behaviour of solutions of (1.1) are the ω-limit solutions. Recall that a solution $u(t,x) : I\!\!R \times \mathbf{T} \longrightarrow I\!\!R$ is called an ω-limit solution if there exists a solution $v : [s,\infty[\times\mathbf{T} \longrightarrow I\!\!R$ and an increasing sequence n_k of integers such that

$$u(t,x) = \lim_{k\longrightarrow\infty} v(t+n_k,x).$$

In other words, ω-limit solutions are solutions whose initial value $u(0,.)$ is an ω-limit of the semi-group $\mathcal{T}$.

3. Calibrated curves and Uniqueness set

In this section, we give - without many proofs, for they are already stated in [6] or [18] - some salient features of ω-limit solutions that do not depend of the dimension of the ambient space.

Let $u : [s,\infty[\times\mathbf{T} \longrightarrow I\!\!R$ be a viscosity solution of (1.1). A curve $\gamma : [s,\infty[\supset [t,t'] \longrightarrow \mathbf{T}$ is said calibrated by u if

$$u(t',\gamma(t')) = u(t,\gamma(t)) + \int_t^{t'} L(\sigma,\gamma,\dot\gamma)\ d\sigma.$$

From Fathi [6], if $\gamma : [t,t'] \longrightarrow \mathbf{T}$ is calibrated by u, then u_x exists at each point $(s,\gamma(s)), s \in [t,t'[$ and satisfies

$$u_x(s,\gamma(s)) = L_v(s,\gamma(s),\dot\gamma(s)) \Longleftrightarrow \dot\gamma(s) = H_p(s,\gamma(s),u_x(s,\gamma(s))).$$

Such a property was already understood in [17] for the equation $|\nabla u| = f(x)$ on the sphere, f nonnegative with nonempty zero set. It was not, however, made that systematic.

We now choose once and for all a 1-periodic solution $\phi(t,x)$ of (1.1). First, let us note that a classical compactness argument gives the existence of curves $\gamma : I\!\!R \longrightarrow \mathbf{T}$ which are calibrated by ϕ on all compact interval.

It is a consequence of [6], or Mather's shortening lemma, see [12] that two such curves cannnot intersect. More precisely if γ_1 and $\gamma_2 : I\!\!R \longrightarrow \mathbf{T}$ are calibrated by ϕ, and if there exists a t such that $\gamma_1(t) = \gamma_2(t)$, then $\gamma_1 = \gamma_2$.

Let

$$\mathcal{D} \subset I\!\!R \times \mathbf{T}$$

be the union of the graphs of these orbits, and $\mathcal{D}_0 \subset \mathbf{T}$ be the set of points $\gamma(0)$, where $\gamma : I\!\!R \longrightarrow \mathbf{T}$ is calibrated. This is a nonempty compact set. For each t, we define the mapping $S^t : \mathcal{D}_0 \longrightarrow \mathbf{T}$ which associates to each $x \in \mathcal{D}_0$, the value

$\gamma(t)$, where $\gamma : I\!\!R \longrightarrow \mathbf{T}$ is the unique calibrated curve satisfying $\gamma(0) = x$. It is a bi-Lipschitz homeomorphism onto its image.

Clearly, S^1 is a homeomorphism of $\mathcal{D}_0$. Let us note $\mathcal{M}_0$ its ω-limit. This is the closure in $\mathbf{T}$ the set of points $x \in \mathcal{D}_0$ which are the limit of a sequence $S^{n_k}(y)$ with $y \in \mathcal{D}_0$ and n_k an increasing sequence of integers. The set $\mathcal{M}_0$ is non-empty and compact. We call $\mathcal{M}$ the union, in $I\!\!R \times \mathbf{T}$, of the graphs of curves $S^t(x), x \in \mathcal{M}_0$.

The following remark, noticed in [18], is the key point to the convergence proof:

Lemma 3.1 *Let $u(t,x) : [s, \infty[\times \mathbf{T} \longrightarrow I\!\!R$ be a viscosity solution of (1.1), let $x \in \mathcal{D}_0$. Then the function $t \mapsto u(t, S^t(x)) - \phi(t, S^t(x))$ is non-increasing.*

The important consequence below is also proved in [18]:

Corollary 3.1 *Let $u(t,x)$ be an ω-limit viscosity solution of (1.1), and let $x \in \mathcal{M}_0$, then the function $t \mapsto u(t, S^t(x)) - \phi(t, S^t(x))$ is constant.*

It follows that the curve $S^t(x)$ is calibrated by u; hence u and ϕ are differentiable on $\mathcal{M}$ by the above-stated regularity results of Fathi.

Corollary 3.2 *Let $u(t,x)$ be an ω-limit viscosity solution of (1.1). Then the functions $u(t,x)$ and $\phi(t,x)$ and differentiable on $\mathcal{M}$, and we have*

$$\partial_t(u - \phi)(t,x) = \partial_x(u - \phi)(t,x) = 0$$

for all $(t,x) \in \mathcal{M}$.

The next step is to define a uniqueness set, i.e. a set such that two global solutions of (1.1) coinciding on this set coincide everywhere. We formulate the results in the general, non-autonomous, setting.

Proposition 3.2 *Let $u(t,x) : I\!\!R \times \mathbf{T} \longrightarrow I\!\!R$ be a global and bounded viscosity solution of (1.1) such that $u = \phi$ on $\mathcal{M}$, then $u = \phi$.*

This proposition easily follows from the following lemma:

Lemma 3.2 *Let $u(t,x) : I\!\!R \times \mathbf{T} \longrightarrow I\!\!R$ be a global and bounded viscosity solution of (1.1) and $\gamma(t) :] - \infty, s] \longrightarrow \mathbf{T}$ be a curve calibrated by u. Then there exists an increasing sequence n_k of integers such that $\gamma(-n_k) \longrightarrow x \in \mathcal{M}_0$.*

Proof. In view of Lemma 3.1, the function $t \longmapsto u(t, \gamma(t)) - \phi(t, \gamma(t))$ is non-increasing, and bounded. Hence this function has a limit as $t \longrightarrow -\infty$. Let us choose an increasing sequence n_k of integers such that the curves $\gamma(t - n_k)$ are converging uniformly on compact sets to a limit $\gamma_\infty : I\!R \longrightarrow \mathbf{T}$. The following calculations show that this curve is calibrated by ϕ:

$$\phi(t', \gamma_\infty(t')) - \phi(t, \gamma_\infty(t)) = \lim \Big(\phi(t', \gamma(t'-n_k)) - \phi(t, \gamma(t-n_k))\Big)$$

$$= \lim \Big(\phi(t'-n_k, \gamma(t'-n_k)) - \phi(t-n_k, \gamma(t-n_k))\Big)$$

$$= \lim \Big(u(t'-n_k, \gamma(t'-n_k)) - u(t-n_k, \gamma(t-n_k))\Big)$$

$$= \lim \int_t^{t'} L(\sigma, \gamma(\sigma-n_k), \dot\gamma(\sigma-n_k))d\sigma = \int_t^{t'} L(\sigma, \gamma_\infty(\sigma), \dot\gamma_\infty(\sigma))d\sigma.$$

As a consequence, the curve $t \longrightarrow (t, \gamma_\infty(t))$ is asymptotic to $\mathcal{M}$, so that there exists an increasing sequences m_k of integers such that that $\gamma_\infty(m_k) \longrightarrow x \in \mathcal{M}_0$. Possibly taking a subsequence of n_k, we obtain that $\gamma(m_k - n_k) \longrightarrow x \in \mathcal{M}$. $\square$

In the autonomous case, where the Hamiltonian H does not depend on the variable t, the above remarks imply that any ω-limit viscosity solution u is independent of t on $\mathcal{M} = I\!R \times \mathcal{M}_0$. One can then conclude that the solution u is independent of t.

The time-periodic case however is more complicated, and we are not able to give a description of ω-limit orbits without using some specific features of the low dimension. This will be done in the next section.

Before we continue, let us give an important remark. All the objects constructed in this section, the sets $\mathcal{D}$ and $\mathcal{M}$ and the mappings S^t, depend on the periodic solution ϕ that was chosen in the beginning. Let us note $\mathcal{D}_\phi$ and $\mathcal{M}_\phi$ and S^t_ϕ in order to emphasize this dependence. If ψ is another 1-periodic viscosity solution, then we see from Corollary 3.1 that the orbits of $\mathcal{M}_\psi$ are calibrated by ϕ. It follows that the set

$$\mathcal{A} = \bigcap_\phi \mathcal{D}_\phi,$$

is not empty, where the intersection is taken on the set of 1-periodic viscosity solutions. This set is usually called the Aubry set. The mappings $S^t_{\phi|\mathcal{A}_0}$ do not depend on ϕ, and for all ϕ, we have $\mathcal{M}_\phi \subset \mathcal{A}$.

4. Rotation number and convergence

In this section, we shall take advantage of the low dimension. More precisely, we shall make use of Poincaré theory of homeomorphisms of the circle, see [11] for example. We have constructed in the previous section a closed subset $\mathcal{A}$ of $I\!R \times \mathbf{T}$, which is the disjoint union of graphs of calibrated curves. These curves have a well defined rotation number $\rho \in I\!R$. Recall that this rotation number can be defined by lifting the calibrated curves to the universal cover $I\!R$ of $\mathbf{T}$. If $\tilde{\gamma} : I\!R \longrightarrow I\!R$ is one of these lifted curves, we define $\rho = \lim \tilde{\gamma}(t)/t$. This number does not depend on the curve chosen, hence it depends only on the Hamiltonian H.

Proof of the theorem. Let $u(t,x)$ be an ω-limit viscosity solution. Let $\phi(t,x) = \liminf u(t+n,x)$ be as in Proposition 2.1. We will prove that $u = \phi$ in the two cases $\rho = 0$ and ρ irrational. The general case of a rational rotation number $\rho = a/b$, $a \neq 0$ can be reduced to the case $\rho = 0$ by considering the Hamiltonian

$$\tilde{H}(t,x,p) = aH(at, bx - at, \frac{p}{b})$$

and noticing that the function $u(t,x)$ is a solution of the equation (1.1) with Hamiltonian H if and only if the function $\tilde{u}(t,x) = u(at, bx - at)$ is a solution of the equation (1.1) with Hamiltonian $\tilde{H}$, and that the rotation number associated to this second equation is $\rho = 0$.

In view of Lemma 3.2, it is enough to prove equality on $\mathcal{M}_\phi$. Let us note $d = u - \phi$. This function is differentiable on $\mathcal{M}_\phi$ and satisfies

$$\partial_t d = \partial_x d = 0.$$

Recall that there exists a constant K such that the function $d(t,.)$ is K-Lipschitz for each t.

Case 1. $\rho = 0$. In this case, $\mathcal{M}_0$ is a union of fixed points of S^1, or equivalently the orbits in $\mathcal{M}$ are 1-periodic. The function d is constant on each of these periodic orbits, so that $d(n,x) = d(0,x)$ for each $x \in \mathcal{M}_0$. It follows that $d(0,x) = 0$ since $\liminf d(n,x) = 0$. as a consequence, $d = 0$ on $\mathcal{M}$.

Case 2. ρ is irrational. The proof that $d|_{\mathcal{M}} = 0$ is similar to the proof of Proposition 6.5. in [4]. Let us lift the set $\mathcal{M}$ to the universal cover $I\!R \times I\!R$, as well as the function d. The set $\mathcal{M}$ is a disjoint union of graphs of curves $\tilde{\gamma}(t) : I\!R \longrightarrow I\!R$, which are the liftings of calibrated curves in $\mathbf{T}$. The function d is constant on each of these graphs. Let us set $d_0 = d(0,.) : I\!R \longrightarrow I\!R$. We want to prove that $d_{0|\mathcal{M}_0}$ is constant. In order to do so, let us consider a connected component $]x,y[$ of the complement of $\mathcal{M}_0$ in $I\!R$, and let $\tilde{\gamma}_x$ and $\tilde{\gamma}_y : I\!R \longrightarrow I\!R$ be the liftings of the calibrated curves such that $\tilde{\gamma}_x(0) = x$ and $\tilde{\gamma}_y(0) = y$. Since $\inf |\tilde{\gamma}_x(t) - \tilde{\gamma}_y(t)| = 0$, and since

$$|d_0(y) - d_0(x)| = |d(t, \tilde{\gamma}_y(t)) - d(t, \tilde{\gamma}_x(t))| \leq K |\tilde{\gamma}_x(t) - \tilde{\gamma}_y(t)|$$

for each t, we have $d_0(y) = d_0(x)$. Hence there exists a continuous function $f : I\!R \longrightarrow I\!R$ which is equal to d_0 on $\mathcal{M}_0$, and which is constant on all connected components of the complement of $\mathcal{M}_0$. This function is differentiable and satisfies $f' = 0$ on each point of the complement of $\mathcal{M}_0$, but also on each point of

$\mathcal{M}_0$ since $d_0' = 0$ there. As a consequence, it is constant, hence $d_{0|\mathcal{M}_0}$ is constant, hence $d_{|\mathbf{T}}$ is constant, this constant has to be zero since $\liminf d(t+n, x) = 0$. $\square$

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