Construction of a Family of Quantum Ornstein-Uhlenbeck Semigroups

Chul Ki Ko[1] and Yong Moon Park[2]

Department of Mathematics, Yonsei University, Seoul 120-749, Korea

Abstract

For a given quasi-free state on the CCR algebra over one dimensional Hilbert space, a family of Markovian semigroups which leave the quasi-free state invariant is constructed by means of noncommutative elliptic operators and Dirichlet forms on von Neumann algebras. The generators (Dirichlet operators) of the semigroups are analyzed and the spectrums together with eigenspaces are found. When restricted to a maximal abelian subalgebra, the semigroups are reduced to a unique Markovian semigroup of classical Ornstein-Uhlenbeck process.

Key words : Quantum Ornstein-Uhlenbeck Semigroups, noncommutative elliptic operators, standard forms, Dirichlet forms, CCR algebras, quasi-free states.

Mathematical subject classification (2000) : 46L55, 46L57.

1. Introduction

The purpose of this paper is to find the explicit expressions of the generators of Markovian semigroups on the CCR algebra $\mathcal{A}$ over one dimensional Hilbert space, which leave a given (fixed) quasi-free state ω on $\mathcal{A}$ invariant. By means of noncommutative elliptic operators [BP] and Dirichlet forms on standard forms of von Neumann algebras [Cip1], we construct a family of symmetric Markovian semigroups on the natural standard form associated to the GNS representation of the pair $(\mathcal{A}, \omega)$. If one restricts the semigroups to a maximal abelian subalgebra, then the semigroups are reduced to a unique Markovian

[1]E-mail: kochulki@hotmail.com
[2]E-mail: ympark@yonsei.ac.kr

semigroup of classical Ornstein-Uhlenbeck (O-U) process. Each generator of the semigroups has a discrete spectrum and a simple lowest eigenvalue. Thus the semigroups are ergodic and tend to the equilibrium exponentially fast.

The need to construct Markovian semigroups on von Neumann algebras, which are symmetric with respect to a non-tracial state, is clear for various applications to open systems [Dav], quantum statistical mechanics [BR], and quantum probability theory [Acc, AFL, Part]. Although on the abstract level we have quite a well-developed theory [Cip1, GL1, GL2], the progress in concrete applications is very slow. We would like to mention a few works in this direction. The completely positive Hamiltonian semigroup for quantum spin chains in the ground state representation has been considered in [Mat]. In [MZ1, MZ2], the generalized conditional expectation is used to construct generators of spin-flip type dynamics for quantum spin systems. In [Par], one of authors gave a general construction method of Dirichlet forms on standard forms of von Neumann algebras and applied the method to construct translation invariant Markovian semigroups for quantum spin systems. The method of [Par] has been extended to construct symmetric Markovian semigroups on the CCR and CAR algebras with respect to quasi-free states [BKP1, BKP2], and on quantum mechanical systems [BK]. In [CFL], quantum O-U semigroups were constructed by means of noncommutative Dirichlet forms. It may be worth to mention that for a given quasi-free state ω on the CCR algebra $\mathcal{A}$, the O-U semigroup constructed in [CFL] belongs to the family of semigroups we constructed in this paper. See Section 3 for the details.

Let us describe the content of this paper briefly. Let $\mathcal{A}$ be the C^*-algebra generated by the Weyl operators $W(z)$, $z \in \mathbb{C}$. See Section 5.2 of [BR] for the details. The quasi-free state ω on $\mathcal{A}$ is given by

$$\omega(W(z)) = \exp\big\{ -\frac{1}{4}|z|^2(1+e^{-\beta})(1-e^{-\beta})^{-1}\big\}, \ \ z \in \mathbb{C}, \tag{1.1}$$

where $\beta \in (0,\infty)$ is the inverse temperature, which is fixed throughout in this paper.

Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS representation [BR] of $(\mathcal{A},\omega)$ and let $\mathcal{M} = \pi_\omega(\mathcal{A})''$. For each $z \in \mathbb{C}$, we denote by $\Phi_\omega(z)$ the infinitesimal generator of the strongly continuous unitary group $\pi_\omega(W(tz)), t \in \mathbb{R}$:

$$\pi_\omega(W(tz)) = \exp\big(it\Phi_\omega(z)\big). \tag{1.2}$$

Let Q and P be the position and momentum operators given by

$$Q = \Phi_\omega(1), \quad P = \Phi_\omega(i). \tag{1.3}$$

From now on we write $\mathcal{H} = \mathcal{H}_\omega$, $\xi_0 = \Omega_\omega$ and $W(z) = \pi_\omega(W(z))$, $\forall z \in \mathbb{C}$. Let σ_t be the group of automorphisms on $\mathcal{M}$ defined by

$$\sigma_t(W(z)) = W(e^{-i\beta t} z), \ \forall z \in \mathbb{C}. \tag{1.4}$$

The state ω satisfies the σ-KMS condition [BR] and thus σ_t, $t \in \mathbb{R}$, is the group of the modular automorphisms. Let Δ and J be the modular operator and modular conjugation respectively associated to $(\mathcal{M}, \xi_0)$. Then $\sigma_t(A) = \Delta^{it} A \Delta^{-it}$, $A \in \mathcal{M}$. Let $\mathcal{M}'$ be the commutant of $\mathcal{M}$. The map $j : \mathcal{M} \to \mathcal{M}'$ is the anti-linear $*$-isomorphism defined by $j(A) = JAJ$, $A \in \mathcal{M}$. The natural positive cone $\mathcal{P}$ associated with $(\mathcal{M}, \xi_0)$ is the closure of the set $\{Aj(A)\xi_0 : A \in \mathcal{M}\}$. Then $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ is the natural standard form associated to $(\mathcal{M}, \xi_0)$.

Next let us describe the explicit forms of the generators of Markovian semigroups on $\mathcal{M}$ which leave ω invariant. In [BP], the authors constructed Markovian semigroups on von Neumann algebras by the quantum Feynman-Kac formula. Let $\mathcal{N}$ be a von Neumann algebra and $b = b^* \in \mathcal{N}$. Let δ be a densely defined *-derivation on $\mathcal{N}$ such that $\mathbf{1} \in D(\delta)$ and $\delta(\mathbf{1}) = 0$. Let G^b be an elliptic operator on $\mathcal{N}$ defined by

$$\begin{aligned} D(G^b) &= D(\delta^2), \\ G^b(A) &= -\frac{1}{2}\delta^2(A) - b\delta(A) - \delta(A)b + \frac{1}{2}[b, [b, A]], \ A \in D(G^b), \end{aligned} \tag{1.5}$$

where $[A, B] = AB - BA$, $A, B \in \mathcal{N}$. It was shown that the closed extension of G^b generates a Markovian semigroup on $\mathcal{N}$ [BP].

We are searching for elliptic operators on $\mathcal{M}$ which generate Markovian semigroup leaving ω invariant. Let α_1, α_2, $\alpha_3 \in \mathbb{R}$ be real parameters. In (1.5), we choose $\delta(A) = i[P, A]$ and $b = \alpha_1 Q$, and let

$$G_1(A) := \frac{1}{2}[P, [P, A]] - i\alpha_1\big(Q[P, A] + [P, A]Q\big) + \frac{1}{2}\alpha_1^2[Q, [Q, A]].$$

Next, choose $\delta(A) = i\alpha_3[Q, A]$ and $b = \alpha_2 P$, and let

$$G_2(A) := \frac{1}{2}\alpha_3^2[Q, [Q, A]] - i\alpha_2\alpha_3\big(P[Q, A] + [Q, A]P\big) + \frac{1}{2}\alpha_2^2[P, [P, A]].$$

Let G be the elliptic operator on $\mathcal{M}$ given by

$$\begin{aligned} G(A) &:= \gamma\big\{G_1(A) + G_2(A)\big\} \\ &= \frac{\gamma}{2}(1 + \alpha_2^2)[P, [P, A]] + \frac{\gamma}{2}(\alpha_1^2 + \alpha_3^2)[Q, [Q, A]] \\ &\quad -i\gamma\alpha_1\big(Q[P, A] + [P, A]Q\big) - i\gamma\alpha_2\alpha_3\big(P[Q, A] + [Q, A]P\big), \end{aligned} \tag{1.6}$$

where γ is the normalized constant which will be chosen as $\gamma = (1+\alpha_2^2)^{-1}$. Since P and Q are unbounded operators affiliated to $\mathcal{M}$, the above is a formal expression.

Because of the well-developed theory [Cip1], it is convenient to construct Markovian semigroups on the standard form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$. Consider the symmetric embedding [Cip1]:

$$\begin{aligned} i_0 \, : \, \mathcal{M} \; &\rightarrow \; \mathcal{H} \\ i_0(A) = \Delta^{1/4} A \xi_0 &= \sigma_{-i/4}(A)\xi_0, \end{aligned} \tag{1.7}$$

and define the operator H on $\mathcal{H}$ by

$$H\sigma_{-i/4}(A)\xi_0 = \sigma_{-i/4}(G(A))\xi_0. \tag{1.8}$$

It follows from (1.8) that

$$\begin{aligned} H \;=\; & \frac{\gamma}{2}(1+\alpha_2^2)\big(\sigma_{-i/4}(P) - j(\sigma_{-i/4}(P))\big)^2 \\ & +\frac{\gamma}{2}(\alpha_1^2+\alpha_3^2)\big(\sigma_{-i/4}(Q) - j(\sigma_{-i/4}(Q))\big)^2 \\ & -i\gamma\alpha_1\big(\sigma_{-i/4}(Q) + j(\sigma_{-i/4}(Q))\big)\big(\sigma_{-i/4}(P) - j(\sigma_{-i/4}(P))\big) \\ & -i\gamma\alpha_2\alpha_3\big(\sigma_{-i/4}(P) + j(\sigma_{-i/4}(P))\big)\big(\sigma_{-i/4}(Q) - j(\sigma_{-i/4}(Q))\big). \end{aligned} \tag{1.9}$$

Let the parameters α_1, α_2, $\alpha_3 \in \mathbb{R}$ satisfy the following relations

$$\begin{aligned} \frac{1}{2}(1+\alpha_2^2)\sinh(\beta/2) &= -\alpha_1\cosh(\beta/2), \\ \frac{1}{2}(\alpha_1^2+\alpha_3^2)\sinh(\beta/2) &= \alpha_2\alpha_3\cosh(\beta/2). \end{aligned} \tag{1.10}$$

Then the operator H can be written as

$$\begin{aligned} H \;=\; & \gamma_1\big(\sigma_{-i/4}(P) - j(\sigma_{-i/4}(P))\big)^*\big(\sigma_{-i/4}(P) - j(\sigma_{-i/4}(P))\big) \\ & +\gamma_2\big(\sigma_{-i/4}(Q) - j(\sigma_{-i/4}(Q))\big)^*\big(\sigma_{-i/4}(Q) - j(\sigma_{-i/4}(Q))\big), \end{aligned} \tag{1.11}$$

where $\gamma_1 = -\gamma\alpha_1/\sinh(\beta/2)$, $\gamma_2 = \gamma\alpha_2\alpha_3/\sinh(\beta/2)$. Recall that $\gamma = (1+\alpha_2^2)^{-1}$. See the proof of Proposition 3.1 in Section 4.

Notice that formally the operator H in (1.11) is symmetric and positive. By means of Dirichlet forms on von Neumann algebras [Cip1] and the approximation procedure used in [BKP1], we will show that H becomes a positive self-adjoint operator on a dense domain in $\mathcal{H}$, which generates a strongly continuous symmetric Markovian semigroup $\{T_t\}_{t\geq 0}$ on $\mathcal{H}$. Define the map S_t on $\mathcal{M}$ by

$$S_t : \mathcal{M} \to \mathcal{M}, \quad i_0 \circ S_t = T_t \circ i_0.$$

Then $\{S_t\}_{t\geq 0}$ is a weakly continuous Markovian semigroup on $\mathcal{M}$ which leaves ω invariant. See Remark 3.1.

Let φ be a bounded smooth function on $\mathbb{R}$. It follows from (1.6) and the commutation relation $[P, Q] = -i\mathbf{1}$ that

$$\begin{aligned} G(\varphi(Q)) &= -\frac{\gamma}{2}(1+\alpha_2^2)\varphi''(Q) - 2\gamma\alpha_1 Q\varphi'(Q) \\ &= -\frac{1}{2}\varphi''(Q) + \tanh(\beta/2)Q\varphi'(Q). \end{aligned} \tag{1.12}$$

Thus for any α_1, α_2 and α_3 satisfying (1.10), the restriction of S_t to the maximal abelian subalgebra $\mathcal{N} \subset \mathcal{M}$ generated by $\exp(ikQ)$, $k \in \mathbb{R}$, is reduced to a unique classical O-U semigroup.

The spectrum of H will be analyzed by a method similar to that used in [BKP1]. It turns out that H has discrete spectrum and H has a simple eigenvalue zero with eigenvector ξ_0.

The paper is organized as follows: In Section 2, we review the theory of noncommutative Dirichlet forms on von Neumann algebras [Cip1]. In Section 3, we give explicit expressions of Dirichlet forms and state the main results. Section 4 is devoted to the proof of the Markov property of semigroups by means of the theory of noncommutative Dirichlet forms and the approximation procedure used in [BKP1]. In Section 5, we decompose $\mathcal{H}$ to a direct sum of eigenspaces of Direchlet operators and analyze the spectrum of generators of the semigroups.

2. Review on the noncommutative Dirichlet forms

In this section we give a brief review on the theory of Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras. For details, we refer the reader to [Cip1].

Let $\mathcal{M}$ be a σ-finite von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$. A *self-dual cone* $\mathcal{P}$ in $\mathcal{H}$ is a subset satisfying the property

$$\{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \geq 0, \ \forall \eta \in \mathcal{P}\} = \mathcal{P}.$$

Then $\mathcal{P}$ is a closed convex cone and $\mathcal{H}$ is the complexification of the real subspace $\mathcal{H}^J := \{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \in \mathbb{R}, \forall \eta \in \mathcal{P}\}$, whose elements are called *J-real*: $\mathcal{H} = \mathcal{H}^J \oplus i\mathcal{H}^J$.

For any two elements $\xi, \eta \in \mathcal{H}^J$, $\xi \leq \eta$ means $\eta - \xi \in \mathcal{P}$, thus $\mathcal{H}^J$ has an ordered structure. An anti-unitary involution J on $\mathcal{H}$ by $J(\xi + i\eta) := \xi - i\eta$, $\forall\, \xi, \eta \in \mathcal{H}^J$, preserves $\mathcal{P}$ and $\mathcal{H}^J$. Any J-real element ξ can be decomposed uniquely as a difference, $\xi = \xi_+ - \xi_-$, of two orthogonal, positive elements, called the positive and the negative part of ξ: $\xi_+, \xi_- \in \mathcal{P}$, $\langle \xi_+, \xi_- \rangle = 0$.

A *standard form* $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ of the von Neumann algebra $\mathcal{M}$ acting faithfully on the Hilbert space $\mathcal{H}$ consists of self-dual, closed, convex cone $\mathcal{P}$ in $\mathcal{H}$ and the anti-unitary involution J satisfying the following properties:

(a) $J\mathcal{M}J = \mathcal{M}'$,

(b) $JAJ = A^*, \quad \forall A \in \mathcal{M} \cap \mathcal{M}'$,

(c) $J\xi = \xi, \quad \forall \xi \in \mathcal{P}$,

(d) $AJAJ(\mathcal{P}) \subset \mathcal{P}, \quad \forall A \in \mathcal{M}$,

where $\mathcal{M}'$ is the commutant of $\mathcal{M}$.

A bounded operator A on $\mathcal{H}$ is called *J-real* if $AJ = JA$ and *positive preserving* if $A\mathcal{P} \subset \mathcal{P}$. The semigroup $\{T_t\}_{t\geq 0}$ is said to be *J-real* if T_t is J-real for any $t \geq 0$ and it is called *positive preserving* if T_t is positive preserving for any $t \geq 0$.

Let us fix a cyclic and separating vector ξ_0 in $\mathcal{P}$. A bounded operator $A : \mathcal{H} \to \mathcal{H}$ is called *sub-Markovian* (with respect to ξ_0) if $0 \leq \xi \leq \xi_0$ implies $0 \leq A\xi \leq \xi_0$. A is called *Markovian* if it is sub-Markovian and also $A\xi_0 = \xi_0$. A semigroup $\{T_t\}_{t\geq 0}$ is said to be *sub-Markovian* (with respect to ξ_0) if T_t is sub-Markovian for every $t \geq 0$. The semigroup $\{T_t\}_{t\geq 0}$ is called *Markovian* if T_t is Markovian for every $t \geq 0$.

Next, we consider a sesquilinear form on some linear manifold of $\mathcal{H}$: $\mathcal{E}(\cdot, \cdot) : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{C}$. We also consider the associated quadratic form: $\mathcal{E}[\cdot] : D(\mathcal{E}) \to \mathbb{C}$, $\mathcal{E}[\xi] := \mathcal{E}(\xi, \xi)$. A real valued quadratic form $\mathcal{E}[\cdot]$ is said to be *semi-bounded* if $\inf\{\mathcal{E}[\xi] : \xi \in D(\mathcal{E}),\ ||\xi|| = 1\} = -b > -\infty$. A quadratic form $(\mathcal{E}, D(\mathcal{E}))$ is said to be *J-real* if $JD(\mathcal{E}) \subset D(\mathcal{E})$ and $\mathcal{E}[J\xi] = \overline{\mathcal{E}[\xi]}$ for any $\xi \in D(\mathcal{E})$. For a given semi-bounded quadratic form $\mathcal{E}$, one considers the inner product given by $\langle \xi, \eta \rangle_\lambda := \mathcal{E}(\xi, \eta) + \lambda \langle \xi, \eta \rangle$, for $\lambda > b$. The form $\mathcal{E}$ is *closed* if $D(\mathcal{E})$ is a Hilbert space for some of the above norms. The form $\mathcal{E}$ is called *closable* if it admits a closed extension.

Associated to a semi-bounded closed form $\mathcal{E}$, there are a self-adjoint operator $(H, D(H))$ and a strongly continuous, symmetric semigroup $\{T_t\}_{t\geq 0}$. Each of the above objects determines uniquely the others according to well known relations (see Section 3.1 of [BR]).

From now on we will consider only J-real, real-valued, semi-bounded, densely defined quadratic forms. It is easy to check that these forms satisfy the relation: $\mathcal{E}[\xi + i\eta] = \mathcal{E}[\xi] + \mathcal{E}[\eta]$ for all $\xi + i\eta \in D(\mathcal{E})^J + iD(\mathcal{E})^J = D(\mathcal{E})$ where $D(\mathcal{E})^J := D(\mathcal{E}) \cap H^J$.

Let us denote by $\mathrm{Proj}(\xi, \mathcal{Q})$ the projection of the vector $\xi \in \mathcal{H}^J$ onto the closed, convex cone $\mathcal{Q} \subset \mathcal{H}^J$. For $\xi, \eta \in \mathcal{H}^J$, define

$$\begin{aligned} \xi \vee \eta &:= \mathrm{Proj}(\xi, \eta + \mathcal{P}), \\ \xi \wedge \eta &:= \mathrm{Proj}(\xi, \eta - \mathcal{P}). \end{aligned}$$

Definition 2.1. *A J-real, real-valued, densely defined quadratic form $(\mathcal{E}, D(\mathcal{E}))$ is called Markovian with respect to $\xi_0 \in \mathcal{P}$ if*

$$\xi \in D(\mathcal{E})^J \textit{ implies } \xi \wedge \xi_0 \in D(\mathcal{E}) \textit{ and } \mathcal{E}[\xi \wedge \xi_0] \le \mathcal{E}[\xi].$$

A closed Markovian form is called a Dirichlet form.

Next, we state main results which will be used in the sequel. For the proofs we refer to [Cip1].

Proposition 2.1. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a J-real, real valued, densely defined closed form. Assume that the following properties hold:*

(a) $\xi_0 \in D(\mathcal{E})$,

(b) $\mathcal{E}(\xi, \xi) \ge 0$ for $\xi \in D(\mathcal{E})$,

(c) $\xi \in D(\mathcal{E})^J$ implies $\xi_\pm \in D(\mathcal{E})$ and $\mathcal{E}(\xi_+, \xi_-) \le 0$.

Then $\mathcal{E}$ is a Dirichlet form if and only if $\mathcal{E}(\xi, \xi_0) \ge 0$ for all $\xi \in D(\mathcal{E}) \cap \mathcal{P}$.

The above result follows from Proposition 4.5 (b) of [Cip1] and Proposition 4.10 (ii) of [Cip1].

Theorem 2.1.*(Theorem 4.11 of [Cip1]) Let $\{T_t\}_{t\ge0}$ be a J-real, strongly continuous, symmetric semigroup on $\mathcal{H}$ and let $(\mathcal{E}, D(\mathcal{E}))$ be the associated densely defined J-real, real valued quadratic form. Then the followings are equivalent:*

(a) $\{T_t\}_{t\ge0}$ is sub-Markovian.

(b) $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form .

3. Dirichlet forms : Preliminaries and main results

Let $\mathcal{A}$ be the C^*-algebra generated by the Weyl operators $W(z)$, $z \in \mathbb{C}$, satisfying

$$\begin{aligned} W(z)^* &= W(-z), \\ W(z)W(z') &= e^{-\frac{i}{2}\mathrm{Im}(\bar{z}z')}W(z+z'), \ \forall z, z' \in \mathbb{C}, \end{aligned} \tag{3.1}$$

where $\bar{z}$ is a complex conjugate of z. $\mathcal{A}$ is the CCR algebra over one dimensional Hilbert space. Let ω be a quasi-free state on $\mathcal{A}$ defined by

$$\omega(W(z)) = \exp\big\{ -\frac{1}{4}|z|^2(1+e^{-\beta})(1-e^{-\beta})^{-1}\big\}, \ \forall z \in \mathbb{C}, \tag{3.2}$$

where $\beta > 0$ is the inverse temperature.

Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS representation [BR] of $(\mathcal{A}, \omega)$, and let $\mathcal{M} = \pi_\omega(\mathcal{A})''$. Denote by $\Phi_\omega(z)$ the infinitesimal generator of the strongly continuous unitary group $\pi_\omega(W(tz))$ for all $z \in \mathbb{C}$:

$$\pi_\omega(W(tz)) = \exp(it\Phi_\omega(z)).$$

The position and momentum operators are defined by

$$Q := \Phi_\omega(1), \quad P := \Phi_\omega(i), \tag{3.3}$$

respectively. Since ω is an entire analytic state [BR] by (3.2), the cyclic vector Ω_ω is an entire analytic vector for P and Q. The annihilation and creation operators given by

$$\begin{gathered} D(a_\omega) = D(P) \cap D(Q) = D(a_\omega^*), \\ a_\omega := 2^{-1/2}(Q+iP), \quad a_\omega^* := 2^{-1/2}(Q-iP), \end{gathered} \tag{3.4}$$

are densely defined, closed and $(a_\omega)^* = a_\omega^*$, and satisfy the canonical commutation relation(CCR): $[a_\omega, a_\omega^*] = \mathbf{1}$. See Section 5.2.3 of [BR] for the details.

In the rest of this paper, we write that $\mathcal{H} = \mathcal{H}_\omega$, $\xi_0 = \Omega_\omega$, $W(z) = \pi_\omega(W(z))$, $\Phi(z) = \Phi_\omega(z)$, $\forall z \in \mathbb{C}$, $a = a_\omega$ and $a^* = a_\omega^*$. Let $\sigma_t : \mathcal{M} \to \mathcal{M}$ be the group of automorphisms on $\mathcal{M}$ defined by

$$\sigma_t(W(z)) = W(e^{-i\beta t}z), \ \forall z \in \mathbb{C}.$$

The state ω in (3.2) satisfies the σ-KMS condtion [BR] and thus σ_t, $t \in \mathbb{R}$, is the modular automorphism by Theorem 5.3.10 of [BR]. Let Δ and J be the modular operator and modular conjugation respectively associated to the pair $(\mathcal{M}, \xi_0)$ [BR]. Then $\sigma_t(A) = \Delta^{it}A\Delta^{-it}$, $A \in \mathcal{M}$. Let $\mathcal{M}'$ be the commutant of $\mathcal{M}$. The map $j : \mathcal{M} \to \mathcal{M}'$ is the anti-linear $*$-isomorphism defined by $j(A) = JAJ$, $A \in \mathcal{M}$. The natural positive cone $\mathcal{P}$ associated with $(\mathcal{M}, \xi_0)$ is the closure of the set $\{Aj(A)\xi_0 : A \in \mathcal{M}\}$. Then $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ is the natural standard form associated to $(\mathcal{M}, \xi_0)$.

Next, we introduce two dense manifolds of $\mathcal{H}$. For any $A \in \mathcal{M}$, define

$$A_n = \Big(\frac{n}{\pi}\Big)^{1/2} \int \sigma_t(A) e^{-nt^2} dt, \quad n \in \mathbb{N}. \tag{3.5}$$

Then A_n is an entire analytic element for σ_t, $\|A_n\| \le \|A\|$ $\forall n \in \mathbb{N}$ and $A_n \to A$ strongly. See the proof of Proposition 2.5.22 in [BR]. Put

$$\mathcal{M}_0 := \text{ the algebra generated by } A_n,\ A \in \mathcal{M},\ n \in \mathbb{N}, \tag{3.6}$$

and denote by $\mathcal{H}_{fin}$ the subset of finite linear combinations of the vectors of the following type:

$$\psi_m^n = P^m Q^n \xi_0, \ \ m, n = 0, 1, 2, \cdots.$$

Clearly $\mathcal{M}_0\xi_0$ and $\mathcal{H}_{fin}$ are dense in $\mathcal{H}$, and P, Q and $j(P)$, $j(Q)$ are affiliated with $\mathcal{M}$ and $\mathcal{M}'$, respectively (see Lemma 4.1).

For any $z \in \mathbb{C}$, we write

$$\sigma_z(a) := e^{i\beta z} a, \quad \sigma_z(a^*) := e^{-i\beta z} a^*, \tag{3.7}$$

and

$$\sigma_z(P) := -i 2^{-1/2}\big(e^{i\beta z} a - e^{-i\beta z} a^*\big), \quad \sigma_z(Q) := 2^{-1/2}\big(e^{i\beta z} a + e^{-i\beta z} a^*\big). \tag{3.8}$$

Denote by Φ either P or Q. In fact, one may be able to show that for $\xi \in \mathcal{M}_0\xi_0$ the function $t \mapsto \sigma_t(\Phi)\xi$ has an analytic extension on $\mathbb{C}$, which is denoted by $\sigma_z(\Phi)\xi$, and that $\sigma_z(\Phi)$ is equal to that of (3.8) on $\mathcal{M}_0\xi_0$. See Lemma 4.1 for $D(\Phi)$.

As discussed in Introduction, for any parameters α_1, α_2, α_3, we introduce a sesquilinear form corresponding to the operator H given in (1.9) by

$$\begin{aligned}
D(\mathcal{E}) &= \mathcal{M}_0\xi_0, \qquad\qquad (3.9)\\
\mathcal{E}(\eta, \xi) &= \frac{\gamma}{2}(1+\alpha_2^2)\langle\big(\sigma_{-i/4}(P) - j(\sigma_{-i/4}(P))\big)^*\eta, \big(\sigma_{-i/4}(P) - j\,(\sigma_{-i/4}(P))\big)\xi\,\rangle \\
&\quad +\frac{\gamma}{2}(\alpha_1^2+\alpha_3^2)\langle\big(\sigma_{-i/4}(Q) - j(\sigma_{-i/4}(Q))\big)^*\eta, \big(\sigma_{-i/4}(Q) - j\,(\sigma_{-i/4}(Q))\big)\xi\,\rangle \\
&\quad -i\gamma\alpha_1\langle\big(\sigma_{-i/4}(Q) + j(\sigma_{-i/4}(Q))\big)^*\eta, \big(\sigma_{-i/4}(P) - j\,(\sigma_{-i/4}(P))\big)\xi\,\rangle \\
&\quad -i\gamma\alpha_2\alpha_3\langle\big(\sigma_{-i/4}(P) + j(\sigma_{-i/4}(P))\big)^*\eta, \big(\sigma_{-i/4}(Q) - j\,(\sigma_{-i/4}(Q))\big)\xi\rangle
\end{aligned}$$

for η, $\xi \in \mathcal{M}_0\xi_0$, where $\gamma = (1+\alpha_2^2)^{-1}$. Notice that $\mathcal{E}$ is well defined on $\mathcal{M}_0\xi_0$ by Lemma 4.1.

Proposition 3.1. *Let the parameters α_1, α_2, $\alpha_3 \in \mathbb{R}$ satisfy the following relations:*

$$\frac{1}{2}(1+\alpha_2^2)\sinh(\beta/2) = -\alpha_1 \cosh(\beta/2), \tag{3.10}$$
$$\frac{1}{2}(\alpha_1^2+\alpha_3^2)\sinh(\beta/2) = \alpha_2\alpha_3 \cosh(\beta/2).$$

Then the sesquilinear form given by (3.9) equals to

$$\begin{aligned} D(\mathcal{E}) &= \mathcal{M}_0\xi_0, \\ \mathcal{E}(\eta,\xi) &= \gamma_1\langle\big(\sigma_{-i/4}(P) - j(\sigma_{-i/4}(P))\big)\eta, \big(\sigma_{-i/4}(P) - j\,(\sigma_{-i/4}(P))\big)\xi\ \rangle \\ &\quad +\gamma_2\langle\big(\sigma_{-i/4}(Q) - j(\sigma_{-i/4}(Q))\big)\eta, \big(\sigma_{-i/4}(Q) - j\,(\sigma_{-i/4}(Q))\big)\xi\rangle \end{aligned} \tag{3.11}$$

where

$$\gamma_1 = -\gamma\alpha_1/\sinh(\beta/2), \quad \gamma_2 = \gamma\alpha_2\alpha_3/\sinh(\beta/2). \tag{3.12}$$

Thus by (3.11) $\mathcal{E}$ is a positive form.

In the rest of the paper, *we assume that the relations (3.10) hold.* We state the main results.

Theorem 3.1. *The form $(\mathcal{E}, D(\mathcal{E}))$ defined as in (3.9) (and so as in (3.11)) is closable. Denote by $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ the closure of $(\mathcal{E}, D(\mathcal{E}))$ and by H the positive self-adjoint operator associated to $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$. Then the following properties hold:*

(a) $\xi_0 \in D(H)$ and $H\xi_0 = 0$,

(b) $\overline{\mathcal{E}}$ is J-real, that is, $\overline{\mathcal{E}}[J\xi] = \overline{\mathcal{E}}[\xi]$ for $\xi \in D(\overline{\mathcal{E}})$,

(c) $\overline{\mathcal{E}}(\xi_+, \xi_-) \le 0$ for any $\xi \in D(\overline{\mathcal{E}}) \cap \mathcal{H}^J$.

Furthermore the form $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ is a Dirichlet form.

Recall that a symmetric, strongly continuous, positive preserving semigroup $\{T_t\}_{t\ge0}$ on $\mathcal{H}$ is called *ergodic* if for any η, $\xi \in \mathcal{P} - \{0\}$, there exists $t > 0$ such that $\langle\eta, T_t\xi\rangle > 0$. Let H be the self-adjoint generator of T_t: $T_t = e^{-tH}$, $t \ge 0$. An eigenvector of H corresponding to the lowest eigenvalue $\inf\sigma(H)$ will be called a *ground state* for H(or $\{T_t\}_{t\ge0}$). The ergodicity of its semigroup is equivalent to that the lowest eigenvalue of H has multiplicity one and there exists a strictly positive ground state for H (Theorem 4.3 of [Cip2]).

Theorem 3.2. *Let $\{T_t\}_{t\ge0}$ be the symmetric Markovian semigroup associated to the Dirichlet form $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ and let H be the Dirichlet operator, i.e., $T_t = e^{-tH}$. Then the following results hold:*

(a) H is essentially self-adjoint on $\mathcal{H}_{fin}$.

(b) The spectrum of generator H are discrete, i.e., $\sigma(H) = \{m\tau_1 + n\tau_2 \;:\; m, n = 0, 1, 2, ...\}$, where $\tau_1 = 2\gamma_1 \sinh(\beta/2)$, $\tau_2 = 2\gamma_2 \sinh(\beta/2)$.

(c) The zero is a simple eigenvalue of H with eigenvector ξ_0. Thus $\{T_t\}_{t\geq 0}$ is ergodic.

The eigenspaces of H will be given in Theorem 5.2 (b). By the spectral theorem, Theorem 3.2 (b) implies that for any $\xi \in \mathcal{H}$ and $t > 0$

$$\|T_t\xi - \langle \xi_0, \xi\rangle \xi_0\| \leq e^{-\tau t}\|\xi - \langle \xi_0, \xi\rangle \xi_0\|,$$

where $\tau = \min\{\tau_1, \tau_2\}$. Thus $\{T_t\}_{t\geq 0}$ converges to the equilibrium exponentially fast.

Remark 3.1. *Let $\{T_t\}_{t\geq 0}$ be a strongly continuous, symmetric Markovian semigroup on $\mathcal{H}$. Consider the symmetric embedding:*

$$\begin{aligned} i_0 : \mathcal{M} &\longrightarrow \mathcal{H} \\ i_0(A) &= \Delta^{1/4} A \xi_0. \end{aligned}$$

Define the maps S_t on $\mathcal{M}$ by

$$S_t : \mathcal{M} \to \mathcal{M}, \quad i_0 \circ S_t = T_t \circ i_0. \tag{3.13}$$

It follows from Theorem 2.12 of [Cip1] that $\{S_t\}_{t\geq 0}$ is a weakly continuous Markovian semigroup on $\mathcal{M}$.

Remark 3.2. *Let $\mathcal{N}$ be the maximal abelian subalgebra of $\mathcal{M}$ generated by $\exp\{ikQ\}$, $k \in \mathbb{R}$. Let S_t be the Markovian semigroup on $\mathcal{M}$ obtained by (3.13). As mentioned in Introduction, it can be checked that for any α_1, α_2, $\alpha_3 \in \mathbb{R}$ satisfying (3.10) the restriction of S_t to $\mathcal{N}$ gives a unique O-U semigroup whose generation satisfies the equation in (1.12).*

Before closing this section, we show that distinct values of the parameters α_1, α_2, $\alpha_3 \in \mathbb{R}$ satisfying (3.10) give distinct semigroups on $\mathcal{H}$. It follows from Theorem 3.2 (b) that the spectrum of each generator H is characterized by

$$\gamma_1/\gamma_2 = -\alpha_1/(\alpha_2\alpha_3).$$

We first consider two special examples and then describe general case.

Example 3.1. *the O-U semigroup studied in [CFL].* Choose $\alpha_3 = 1$ and $-\alpha_1 = \alpha_2$. It follows from (3.10) and (3.12) that

$$\begin{aligned} \gamma_1 &= \gamma_2, \\ \alpha_2 &= (\cosh(\beta/2) \pm 1)/\sinh(\beta/2) \\ &\equiv \alpha_{2,\pm}(\beta). \end{aligned} \tag{3.14}$$

Using (3.8), we obtain from (3.11) that

$$\mathcal{E}(\xi, \xi) = \gamma\Big(\|\big(\mu a - \lambda j(a^*)\big)\xi\|^2 + \|\big(\lambda a^* - \mu j(a)\big)\xi\|^2\Big), \ \forall \xi \in \mathcal{M}_0\xi_0$$

where $\mu = \lambda^{-1} = e^{\beta/4}$ and $\gamma = \gamma_1 = \gamma_2$. The above is the form studied in [CFL].

Example 3.2. Consider $\alpha_2 = 1$. It follows from (3.10) and (3.12) that

$$\begin{aligned} \alpha_1 &= -\tanh(\beta/2), \\ \alpha_3 &= \coth(\beta/2) \pm \big(\coth^2(\beta/2) - \tanh^2(\beta/2)\big)^{1/2} \\ &= \tanh^2(\beta/2)/\big[\coth(\beta/2) \mp \big(\coth^2(\beta/2) - \tanh^2(\beta/2)\big)^{1/2}\big], \\ \gamma_1/\gamma_2 &= \coth(\beta/2)\big[\coth(\beta/2) \mp \big(\coth^2(\beta/2) - \tanh^2(\beta/2)\big)^{1/2}\big] \neq 1. \end{aligned} \tag{3.15}$$

Next, we consider the general case. Consider α_2 as a parameter. Then we get from (3.10) that

$$\begin{aligned} \alpha_1 &= -\frac{1}{2}(1 + \alpha_2^2)\tanh(\beta/2), \\ \alpha_3 &= \big[\alpha_2\cosh(\beta/2) \pm \big(\alpha_2^2\cosh^2(\beta/2) - \alpha_1^2\sinh^2(\beta/2)\big)^{1/2}\big]/\sinh(\beta/2). \end{aligned} \tag{3.16}$$

For real α_3, one requires that $\alpha_2^2\cosh^2(\beta/2) \geq \alpha_1^2\sinh^2(\beta/2)$. By the first equation in (3.16), positive solutions α_2 exist if and only if

$$2\alpha_2\cosh^2(\beta/2) \geq (1 + \alpha_2^2)\sinh^2(\beta/2).$$

Denote

$$\begin{aligned} \kappa_\pm(\beta) &:= \big[\cosh^2(\beta/2) \pm \big(\cosh^4(\beta/2) - \sinh^4(\beta/2)\big)^{1/2}\big]/\sinh^2(\beta/2), \\ I_\beta &:= [\kappa_-(\beta), \kappa_+(\beta)]. \end{aligned}$$

Thus $\alpha_2 \in I_\beta$. Note that

$$\begin{aligned} \kappa_+(\beta) &= [\cosh^2(\beta/2) + \big(\cosh^2(\beta/2) + \sinh^2(\beta/2)\big)^{1/2}\big]/\sinh^2(\beta/2), \\ &\geq \big[\cosh^2(\beta/2) + \cosh(\beta/2)\big]/\sinh^2(\beta/2), \\ &= \coth(\beta/2)\alpha_{2,+}(\beta), \end{aligned}$$

where $\alpha_{2,+}(\beta)$ has been defined in (3.14). The above implies that $\kappa_+(\beta) > \alpha_{2,+}(\beta)$. An argument similar to that used above yields that $\kappa_-(\beta) < \alpha_{2,-}(\beta)$. Thus we conclude that $1 \in [\alpha_{2,-}(\beta), \alpha_{2,+}(\beta)] \subset I_\beta$.

Define

$$\begin{aligned}
f_\pm(\alpha_2) &:= -\alpha_1(\alpha_2)/\alpha_2\alpha_{3,\pm}(\alpha_2), \\
\alpha_{3,\pm}(\alpha_2) &= \left[\alpha_2 \cosh(\beta/2) \pm \left(\alpha_2^2 \cosh^2(\beta/2) - \alpha_1^2 \sinh^2(\beta/2)\right)^{1/2}\right]/\sinh(\beta/2) \\
&= \alpha_2 \coth(\beta/2) \pm \left(\alpha_2^2 \coth^2(\beta/2) - \frac{1}{4}(1+\alpha_2^2)^2 \tanh^2(\beta)\right)^{1/2}.
\end{aligned}$$

Since $f_\pm$ are continuous on $I_\beta \subset (0,\infty)$, $f_+(I_\beta)$ and $f_-(I_\beta)$ are closed intervals which have common point at $\alpha_2 = \kappa_-(\beta)$ and $\alpha_2 = \kappa_+(\beta)$. Thus $f_+(I_\beta) \cup f_-(I_\beta)$ is a closed interval containing 1 with non-zero length (by (3.15)). Therefore there exist infinitely many different semigroups, which containing the semigroups constructed in [CFL].

4. Markov property of forms : Proofs of Proposition 3.1 and Theorem 3.1

In this section we produce the proofs of Proposition 3.1 and Theorem 3.1. Since P and Q are unbounded operators, we need to employ the approximation procedure used in [BKP1]. In the rest of this paper we denote by Φ either P or Q and by $a^\sharp$ either a or a^*.

Lemma 4.1. *(a) $\mathcal{M}_0\xi_0$ and $\mathcal{H}_{fin}$ are dense in $\mathcal{H}$.*

(b) The inclusions

$$\mathcal{M}_0\xi_0 \subset D(\Phi) \ \text{ and } \ \mathcal{H}_{fin} \subset D(\Phi)$$

hold, moreover the relation

$$\Phi A\xi_0 = j(\sigma_{-i/2}(A^*))\Phi\xi_0$$

holds for any $A \in \mathcal{M}_0$. Also we have $\mathcal{M}_0\xi_0 \subset D(a^\sharp)$, $\mathcal{H}_{fin} \subset D(a^\sharp)$ and $a^\sharp A\xi_0 = j(\sigma_{-i/2}(A^*))a^\sharp\xi_0$, $A \in \mathcal{M}_0$.

Proof: (a) Since for $A \in \mathcal{M}$, A_n in (3.5) converges strongly to A as $n \to \infty$ and since $\mathcal{M}\xi_0$ is dense in $\mathcal{H}$, $\mathcal{M}_0\xi_0$ is dense in $\mathcal{H}$. It follows from (3.2) that ω is an entire analytic state, and so ξ_0 is an entire analytic vector for $\Phi(z)$, $z \in \mathbb{C}$. Thus $\mathcal{H}_{fin}$ is dense. See Section 5.2.3 of [BR] for the details.

(b) Since ξ_0 is an entire analytic vector for Φ, $\mathcal{H}_{fin} \subset D(\Phi)$. Notice that for any $A \in \mathcal{M}_0$, $j(\sigma_{-i/2}(A^*))\xi_0 = A\xi_0$. For $z = 1, i$ and $n \in \mathbb{N}$, we write that

$$\Phi(z,n) = -in\Big(W\big(\frac{1}{n}z\big) - 1\Big). \tag{4.1}$$

Notice that

$$\begin{aligned}\Phi(z,n)A\xi_0 &= \Phi(z,n)j(\sigma_{-i/2}(A^*))\xi_0 \\ &= j(\sigma_{-i/2}(A^*))\Phi(z,n)\xi_0.\end{aligned}$$

Since the sequence $\{\Phi(z,n)\xi_0\}$ converges to $\Phi\xi_0$, it follows that $A\xi_0 \in D(\Phi)$ and the relation

$$\Phi A\xi_0 = j(\sigma_{-i/2}(A^*))\Phi\xi_0$$

hold for $A \in \mathcal{M}_0$. The rest part of (b) follows from (3.4). $\square$

Recall the definitions of $\sigma_z(\Phi)$, $z \in \mathbb{C}$ in (3.8), i.e.,

$$\sigma_z(P) = -i2^{-1/2}\big(e^{i\beta z}a - e^{-i\beta z}a^*\big), \quad \sigma_z(Q) = 2^{-1/2}\big(e^{i\beta z}a + e^{-i\beta z}a^*\big).$$

For any $n \in \mathbb{N}$, we write that

$$\Phi_n := -\Big(\frac{n}{\pi}\Big)^{1/2} \int in\Big\{\sigma_t\Big(\exp\big(\frac{i}{n}\Phi\big) - \mathbf{1}\Big)\Big\}e^{-nt^2}\,dt. \tag{4.2}$$

Notice that for any $n \in \mathbb{N}$, Φ_n is an entire analytic element for σ_t and $\sigma_z(\Phi_n)$ is given by

$$\sigma_z(\Phi_n) := -\Big(\frac{n}{\pi}\Big)^{1/2} \int in\Big\{\sigma_t\Big(\exp\big(\frac{i}{n}\Phi\big) - \mathbf{1}\Big)\Big\}e^{-n(t-z)^2}\,dt \tag{4.3}$$

for any $z \in \mathbb{C}$. See the proof of Proposition 2.5.22(1) in [BR].

Lemma 4.2. *For any $n \in \mathbb{N}$, $z = t + is \in \mathbb{C}$ and $A \in \mathcal{M}_0$, there exists a constant $M(A,s)$ such that the bound*

$$\|\sigma_z(\Phi_n)A\xi_0\| + \|\sigma_z(\Phi)A\xi_0\| \le M(A,s),$$

and moreover $\sigma_z(\Phi)A\xi_0 = \lim_{n\to\infty}\sigma_z(\Phi_n)A\xi_0$ holds.

Proof: By the method used in the proof Lemma 4.1, one can check that

$$\sigma_z(\Phi_n)A\xi_0 = j(\sigma_{-i/2}(A))\sigma_z(\Phi_n)\xi_0,$$

and a similar relation for $\sigma_z(\Phi)$. Since $\sigma_z(\Phi_n)\xi_0$ converges to $\sigma_z(\Phi)\xi_0$, the lemma follows from above relation. $\Box$

We first produce the proof of Proposition 3.1. Note that by (3.7)

$$\begin{aligned} \sigma_{-i/4}(a^*) = e^{-\beta/4}a^*, \qquad \sigma_{-i/4}(a) = e^{\beta/4}a \\ \sigma_{i/4}(a^*) = e^{\beta/4}a^*, \qquad \sigma_{i/4}(a) = e^{-\beta/4}a. \end{aligned} \tag{4.4}$$

Proof of Proposition 3.1: By Lemma 4.1 and Lemma 4.2 the operators $\sigma_{\pm i/4}(P)$ and $\sigma_{\pm i/4}(Q)$ are well defined on $\mathcal{M}_0\xi_0$. We have that

$$\begin{aligned} P &= \sigma_{-i/4}(\sigma_{i/4}(P)) = i\sigma_{-i/4}(e^{\beta/4}a^* - e^{-\beta/4}a)/\sqrt{2} \\ &= ie^{\beta/4}\big(\sigma_{-i/4}(Q) - i\sigma_{-i/4}(P)\big)/2 - ie^{-\beta/4}\big(\sigma_{-i/4}(Q) + i\sigma_{-i/4}(P)\big)/2 \\ &= i\sinh(\beta/4)\sigma_{-i/4}(Q) + \cosh(\beta/4)\sigma_{-i/4}(P). \end{aligned} \tag{4.5}$$

By the method used above, we also have that

$$Q = \cosh(\beta/4)\sigma_{-i/4}(Q) - i\mathrm{sinh}(\beta/4)\sigma_{-i/4}(P). \tag{4.6}$$

It follows from (4.4) and (4.6) that

$$\begin{aligned} & [\sigma_{i/4}(P) \mp j(\sigma_{i/4}(P))] - [\sigma_{-i/4}(P) \mp j(\sigma_{-i/4}(P))] \\ = \;& i[(e^{\beta/4} - e^{-\beta/4})a^* - (e^{-\beta/4} - e^{\beta/4})a]/\sqrt{2} \\ & \mp j\big\{i[(e^{\beta/4} - e^{-\beta/4})a^* - (e^{-\beta/4} - e^{\beta/4})a]\big\}/\sqrt{2} \\ = \;& 2i\sinh(\beta/4)Q \mp j(2i\sinh(\beta/4)Q) \\ = \;& 2i\sinh(\beta/4)(Q \pm j(Q)) \\ = \;& 2i\sinh(\beta/4)\big\{\cosh(\beta/4)\sigma_{-i/4}(Q) - i\mathrm{sinh}(\beta/4)\sigma_{-i/4}(P)\big\} \\ & \pm 2i\sinh(\beta/4)j\big\{\cosh(\beta/4)\sigma_{-i/4}(Q) - i\mathrm{sinh}(\beta/4)\sigma_{-i/4}(P)\big\} \\ = \;& i\sinh(\beta/2)[\sigma_{-i/4}(Q) \pm j(\sigma_{-i/4}(Q))] + 2\sinh^2(\beta/4)[\sigma_{-i/4}(P) \mp j(\sigma_{-i/4}(P))]. \end{aligned} \tag{4.7}$$

Here we have used the equality $2\sinh(\beta/4)\cosh(\beta/4) = \sinh(\beta/2)$. The argument used above yields

$$\begin{aligned} & [\sigma_{i/4}(Q) \mp j(\sigma_{i/4}(Q))] - [\sigma_{-i/4}(Q) \mp j(\sigma_{-i/4}(Q))] \\ = \;& 2\sinh^2(\beta/4)[\sigma_{-i/4}(Q) \mp j(\sigma_{-i/4}(Q))] - i\sinh(\beta/2)[\sigma_{-i/4}(P) \pm j(\sigma_{-i/4}(P))]. \end{aligned} \tag{4.8}$$

Note that $1+2\sinh^2(\beta/4)=\cosh(\beta/2)$. Thus (4.7) and (4.8) imply

$$\begin{aligned}
\sigma_{i/4}(P)\mp j(\sigma_{i/4}(P)) &= i\sinh(\beta/2)[\sigma_{-i/4}(Q)\pm j(\sigma_{-i/4}(Q))] \\
&\quad +\cosh(\beta/2)[\sigma_{-i/4}(P)\mp j(\sigma_{-i/4}(P))], \\
\sigma_{i/4}(Q)\mp j(\sigma_{i/4}(Q)) &= -i\sinh(\beta/2)[\sigma_{-i/4}(P)\pm j(\sigma_{-i/4}(P))] \\
&\quad +\cosh(\beta/2)[\sigma_{-i/4}(Q)\mp j(\sigma_{-i/4}(Q)].
\end{aligned}$$

Notice that $\big(\sigma_{-i/4}(\Phi)\pm j(\sigma_{-i/4}(\Phi))\big)^*=\sigma_{i/4}(\Phi)\pm j(\sigma_{i/4}(\Phi))$ on $\mathcal{M}_0\xi_0$. We substitute the above relations into (3.9) to obtain that for $\eta,\ \xi\in\mathcal{M}_0\xi_0$,

$$\begin{aligned}
\mathcal{E}(\eta,\xi) &= A_1(\alpha_1,\alpha_2,\alpha_3)\langle\big(\sigma_{-i/4}(P)-j(\sigma_{-i/4}(P)\big)\eta,\big(\sigma_{-i/4}(P)-j(\sigma_{-i/4}(P)\big)\xi\rangle \\
&+ A_2(\alpha_1,\alpha_2,\alpha_3)\langle\big(\sigma_{-i/4}(Q)-j(\sigma_{-i/4}(Q)\big)\eta,\big(\sigma_{-i/4}(Q)-j(\sigma_{-i/4}(Q)\big)\xi\rangle \\
&+ A_3(\alpha_1,\alpha_2,\alpha_3)\langle\big(\sigma_{-i/4}(Q)+j(\sigma_{-i/4}(Q)\big)\eta,\big(\sigma_{-i/4}(P)-j(\sigma_{-i/4}(P)\big)\xi\rangle \\
&+ A_4(\alpha_1,\alpha_2,\alpha_3)\langle\big(\sigma_{-i/4}(P)+j(\sigma_{-i/4}(P)\big)\eta,\big(\sigma_{-i/4}(Q)-j(\sigma_{-i/4}(Q)\big)\xi\rangle,
\end{aligned}$$

where

$$\begin{aligned}
A_1(\alpha_1,\alpha_2,\alpha_3) &= \gamma\big[\frac{1}{2}(1+\alpha_2^2)\cosh(\beta/2)+\alpha_1\sinh(\beta/2)\big], \\
A_2(\alpha_1,\alpha_2,\alpha_3) &= \gamma\big[\frac{1}{2}(\alpha_1^2+\alpha_3^2)\cosh(\beta/2)-\alpha_2\alpha_3\sinh(\beta/2)\big], \\
A_3(\alpha_1,\alpha_2,\alpha_3) &= -i\gamma\big[\frac{1}{2}(1+\alpha_2^2)\sinh(\beta/2)+\alpha_1\cosh(\beta/2)\big], \\
A_4(\alpha_1,\alpha_2,\alpha_3) &= i\gamma\big[\frac{1}{2}(\alpha_1^2+\alpha_3^2)\sinh(\beta/2)-\alpha_2\alpha_3\cosh(\beta/2)\big].
\end{aligned}$$

By the relations (3.10), the form in (3.9) equals to that in (3.11). □

In order to prove Theorem 3.1, we employ the approximation procedure similar to that used in [BKP1]. For any $n\in\mathbb{N}$, let $(\mathcal{E}_n,\mathcal{H})$ be the form given by

$$\begin{aligned}
&\mathcal{E}_n(\eta,\xi) \qquad\qquad (4.9)\\
&=\frac{\gamma}{2}(1+\alpha_2^2)\langle\big(\sigma_{-i/4}(P_n)-j(\sigma_{-i/4}(P_n))\big)^*\eta,\big(\sigma_{-i/4}(P_n)-j\,(\sigma_{-i/4}(P_n))\big)\xi\ \rangle \\
&+\frac{\gamma}{2}(\alpha_1^2+\alpha_3^2)\langle\big(\sigma_{-i/4}(Q_n)-j(\sigma_{-i/4}(Q_n))\big)^*\eta,\big(\sigma_{-i/4}(Q_n)-j\,(\sigma_{-i/4}(Q_n))\big)\xi\ \rangle \\
&-i\gamma\alpha_1\langle\big(\sigma_{-i/4}(Q_n)+j(\sigma_{-i/4}(Q_n))\big)^*\eta,\big(\sigma_{-i/4}(P_n)-j\,(\sigma_{-i/4}(P_n))\big)\xi\ \rangle \\
&-i\gamma\alpha_2\alpha_3\langle\big(\sigma_{-i/4}(P_n)+j(\sigma_{-i/4}(P_n))\big)^*\eta,\big(\sigma_{-i/4}(Q_n)-j\,(\sigma_{-i/4}(Q_n))\big)\xi\rangle
\end{aligned}$$

where $\gamma=(1+\alpha_2^2)^{-1}$ and $P_n,\ Q_n$ have been defined as in (4.2). Notice that $\mathcal{E}_n$ is bounded.

Lemma 4.3. *Let $(\mathcal{E},D(\mathcal{E}))$ be the form defined by (3.9). For any $\eta,\ \xi\in\mathcal{M}_0\xi_0$*

$$\mathcal{E}(\eta,\xi)=\lim_{n\to\infty}\mathcal{E}_n(\eta,\xi).$$

Proof: The lemma follows from Lemma 4.2. □

We write that

$$\begin{aligned} \delta_-(A) &:= A - j(\sigma_{-i/2}(A^*)), \\ \delta_+(A) &:= A + j(\sigma_{-i/2}(A^*)) \end{aligned} \tag{4.10}$$

where A is an operator on $\mathcal{H}$. By Lemma 4.1 and Lemma 4.2, $\delta_-(\Phi)$ and $\delta_+(\Phi)$ are also well defined on $\mathcal{M}_0\xi_0$. We are ready to prove Theorem 3.1.

Proof of Theorem 3.1: Two operators $\delta_-(\sigma_{-i/4}(P))$ and $\delta_-(\sigma_{-i/4}(Q))$ are defined on $\mathcal{M}_0\xi_0$ and have the adjoint operators $\delta_-(\sigma_{i/4}(P))$ and $\delta_-(\sigma_{i/4}(Q))$ respectively which are also defined on $\mathcal{M}_0\xi_0$, and so they are closable. Thus by Proposition 3.1 the form $(\mathcal{E}, D(\mathcal{E}))$ is closable. Notice that $(\mathcal{E}, D(\mathcal{E}))$ in (3.11) is positive. Denote by $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ the closure of $(\mathcal{E}, D(\mathcal{E}))$ and by H the positive self-adjoint operator associated to $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$.

(a) Notice that $\xi_0 \in \mathcal{M}_0\xi_0$ and $\delta_-(\sigma_{-i/4}(\Phi))\xi_0 = 0$ implies $\xi_0 \in D(\overline{\mathcal{E}})$, and so $\xi_0 \in D(H^{1/2})$ and $H^{1/2}\xi_0 = 0$. By the spectral theorem one has $\xi_0 \in D(H)$ and $H\xi_0 = 0$.

(b) Note that $JA\xi_0 = \sigma_{-i/2}(A^*)\xi_0 \in \mathcal{M}_0\xi_0$, $A \in \mathcal{M}_0$. Thus $\xi \in \mathcal{M}_0\xi_0$ implies $J\xi \in \mathcal{M}_0\xi_0$. Consider the form $(\mathcal{E}, D(\mathcal{E}))$ in (3.11). The proof of (b) follows from $\delta_-(\sigma_{-i/4}(\Phi))J\xi = -J\delta_-(\sigma_{-i/4}(\Phi))\xi$, $\xi \in \mathcal{M}_0\xi_0$ and the fact that $\mathcal{M}_0\xi_0$ is a form core of $\overline{\mathcal{E}}$.

(c) We first assert that

$$\xi \in \mathcal{M}_0\xi_0 \cap \mathcal{H}^J \Rightarrow \xi_+, \xi_- \in D(\overline{\mathcal{E}}) \quad \text{and} \quad \overline{\mathcal{E}}[\xi_\pm] = \lim_{n\to\infty} \mathcal{E}_n[\xi_\pm]. \tag{4.11}$$

Let us prove our assertion. Let s_+ and s_- be the projections onto the closure of $\mathcal{M}'\xi_+$ and $\mathcal{M}'\xi_-$ respectively, where s_+, $s_- \in \mathcal{M}$ [Ara]. For $\xi = A\xi_0$, $A \in \mathcal{M}_0$, we write that

$$\xi_{n,\pm} = (s_\pm A)_n\xi_0,$$

where

$$(s_\pm A)_n = \left(\frac{n}{\pi}\right)^{1/2} \int \sigma_t(s_\pm A)e^{-nt^2}\, dt.$$

Notice that $\|(s_\pm A)_n\| \le \|A\|$, $n \in \mathbb{N}$, and that for any $\eta \in \mathcal{H}$, $(s_\pm A)_n\eta \to s_\pm A\eta$ as $n \to \infty$. See the proof of Proposition 2.5.22 of [BR]. Since $\Delta^{it}\mathcal{P} \subset \mathcal{P}$ for $t \in \mathbb{R}$, $\xi_{n,\pm} = (s_\pm A)_n\xi_0 = j((s_\pm A)_n)\xi_0$, and so

$$\sigma_{-i/4}(\Phi)\xi_{n,\pm} = j((s_\pm A)_n)\sigma_{-i/4}(\Phi)\xi_0.$$

Since $\sigma_{-i/4}(\Phi)$ is a closed operator, we take n to infinity to conclude that $\xi_\pm \in D(\sigma_{-i/4}(\Phi))$, and $\sigma_{-i/4}(\Phi)\xi_\pm = j(s_\pm A)\sigma_{-i/4}(\Phi)\xi_0$, and so

$$\delta_-(\sigma_{-i/4}(\Phi))\xi_\pm = j(s_\pm A)\sigma_{-i/4}(\Phi)\xi_0 - (s_\pm A)j(\sigma_{-i/4}(\Phi))\xi_0. \tag{4.12}$$

By the method used above, we also have that $\xi_\pm \in D(\sigma_{i/4}(\Phi))$ and

$$\begin{aligned} \delta_-(\sigma_{i/4}(\Phi))\xi_\pm &= j(s_\pm A)\sigma_{i/4}(\Phi)\xi_0 - (s_\pm A)j(\sigma_{i/4}(\Phi))\xi_0, \\ \delta_+(\sigma_{i/4}(\Phi))\xi_\pm &= j(s_\pm A)\sigma_{i/4}(\Phi)\xi_0 + (s_\pm A)j(\sigma_{i/4}(\Phi))\xi_0. \end{aligned} \tag{4.13}$$

The relations analogous to (4.12) and (4.13) for Φ_n hold. Thus, it follows from (3.9) and Lemma 4.2 that $\overline{\mathcal{E}}[\xi_{m,\pm} - \xi_{n,\pm}] \to 0$ as $m, n \to \infty$. Since $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ is closed, $\xi_\pm \in D(\overline{\mathcal{E}})$. Next we will prove that $\mathcal{E}_n[\xi_\pm]$ converges to $\overline{\mathcal{E}}[\xi_\pm]$ as n tends to infinity. By (4.12), (4.13) and Lemma 4.2, we conclude that for any $\xi = A\xi_0$, there exists a positive real number M such that the bounds

$$\begin{aligned} \|\delta_-(\sigma_{-i/4}(\Phi_n))\xi_\pm\| + \|\delta_-(\sigma_{-i/4}(\Phi))\xi_\pm\| &\leq M, \\ \|\delta_-(\sigma_{i/4}(\Phi_n))\xi_\pm\| + \|\delta_-(\sigma_{i/4}(\Phi))\xi_\pm\| &\leq M, \\ \|\delta_+(\sigma_{i/4}(\Phi_n))\xi_\pm\| + \|\delta_+(\sigma_{i/4}(\Phi))\xi_\pm\| &\leq M \end{aligned} \tag{4.14}$$

hold uniformly in $n \in \mathbb{N}$. It follows from (3.9), (4.9), (4.12), (4.13) and the above bounds that

$$\begin{aligned} |\mathcal{E}_n[\xi_\pm] - \overline{\mathcal{E}}[\xi_\pm]| &\leq 4\alpha M \|\big(\delta_-(\sigma_{-i/4}(\Phi_n)) - \delta_-(\sigma_{-i/4}(\Phi))\big)\xi_\pm\| \\ &\quad +2\alpha M \|\big(\delta_-(\sigma_{i/4}(\Phi_n)) - \delta_-(\sigma_{i/4}(\Phi))\big)\xi_\pm\| \\ &\quad +2\alpha M \|\big(\delta_+(\sigma_{i/4}(\Phi_n)) - \delta_+(\sigma_{i/4}(\Phi))\big)\xi_\pm\| \\ &\leq M'\Big(\|\sigma_{-i/4}(\Phi_n)\xi_0 - \sigma_{-i/4}(\Phi)\xi_0\| + \|\sigma_{i/4}(\Phi_n)\xi_0 - \sigma_{i/4}(\Phi)\xi_0\|\Big) \\ &\to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

where $\alpha = \gamma \max\{(1+\alpha_2^2)/2,\ (\alpha_1^2+\alpha_3^2)/2,\ |\alpha_1| + |\alpha_2\alpha_3|\}$. This completes the proof of our assertion (4.11).

Recall the definition of $\mathcal{E}_n$ in (4.9). To show the Dirichlet property, $\mathcal{E}_n(\xi_+, \xi_-)$ in (4.9) can be written as

$$\begin{aligned} \mathcal{E}_n(\xi_+, \xi_-) &= \mathcal{E}_n^{(1)}(\xi_+, \xi_-) + \mathcal{E}_n^{(2)}(\xi_+, \xi_-) \\ &= (I^{(1)} + II^{(1)}) + (I^{(2)} + II^{(2)}), \end{aligned} \tag{4.15}$$

where

$$\begin{aligned}\mathcal{E}_n^{(1)}(\xi_+,\xi_-) &= \frac{\gamma}{2}\langle\delta_-(\sigma_{-i/4}(P_n))^*\xi_+,\delta_-(\sigma_{-i/4}(P_n))\xi_-\rangle\\ &\quad+\frac{\gamma}{2}\alpha_1^2\langle\delta_-(\sigma_{-i/4}(Q_n))^*\xi_+,\delta_-(\sigma_{-i/4}(Q_n))\xi_-\rangle\\ &\quad-i\gamma\alpha_1\langle\delta_+(\sigma_{-i/4}(Q_n))^*\xi_+,\delta_-(\sigma_{-i/4}(P_n))\xi_-\rangle,\end{aligned}$$

$$\begin{aligned}\mathcal{E}_n^{(2)}(\xi_+,\xi_-) &= \frac{\gamma}{2}\alpha_2^2\langle\delta_-(\sigma_{-i/4}(P_n))^*\xi_+,\delta_-(\sigma_{-i/4}(P_n))\xi_-\rangle\\ &\quad+\frac{\gamma}{2}\alpha_3^2\langle\delta_-(\sigma_{-i/4}(Q_n))^*\xi_+,\delta_-(\sigma_{-i/4}(Q_n))\xi_-\rangle\\ &\quad-i\gamma\alpha_2\alpha_3\langle\delta_+(\sigma_{-i/4}(P_n))^*\xi_+,\delta_-(\sigma_{-i/4}(Q_n))\xi_-\rangle,\end{aligned}$$

and

$$\begin{aligned}I^{(1)} &= \frac{\gamma}{2}\big\langle\xi_+,\sigma_{-i/4}\big(P_n^2+\alpha_1^2Q_n^2\big)\xi_-\big\rangle+\frac{\gamma}{2}\big\langle\sigma_{-i/4}(P_n^2+\alpha_1^2Q_n^2\big)\xi_-,\xi_+\big\rangle\\ &\quad-\gamma\big\langle\xi_+,\sigma_{-i/4}\big(i\alpha_1Q_nP_n\big)\xi_-\big\rangle-\gamma\big\langle\big(\sigma_{-i/4}\big(i\alpha_1Q_nP_n\big)\xi_-,\xi_+\big\rangle\\ I^{(2)} &= \frac{\gamma}{2}\big\langle\xi_+,\sigma_{-i/4}\big(\alpha_2^2P_n^2+\alpha_3^2Q_n^2\big)\xi_-\big\rangle+\frac{\gamma}{2}\big\langle\sigma_{-i/4}(\alpha_2^2P_n^2+\alpha_3^2Q_n^2\big)\xi_-,\xi_+\big\rangle\\ &\quad-\gamma\big\langle\xi_+,\sigma_{-i/4}\big(i\alpha_2\alpha_3P_nQ_n\big)\xi_-\big\rangle-\gamma\big\langle j\big(\sigma_{-i/4}\big(i\alpha_2\alpha_3P_nQ_n\big)\big)\xi_-,\xi_+\big\rangle,\\ II^{(1)} &= -\gamma\big\langle\xi_+,\sigma_{-i/4}\big(P_n-i\alpha_1Q_n\big)j\big(\sigma_{-i/4}\big(P_n-i\alpha_1Q_n\big)\big)\xi_-\big\rangle\\ II^{(2)} &= -\gamma\big\langle\xi_+,\sigma_{-i/4}\big(\alpha_2P_n+i\alpha_3Q_n\big)j\big(\sigma_{-i/4}\big(\alpha_2P_n+i\alpha_3Q_n\big)\big)\xi_-\big\rangle.\end{aligned}$$

Here we have used the fact that $\langle\xi_+,j(A)\xi_-\rangle=\langle A\xi_-,\xi_+\rangle$, $A\in\mathcal{M}$ in $I^{(1)}$, $I^{(2)}$. As a consequence of Theorem 4(7) of [Ara], $\mathcal{M}\xi_+\perp\mathcal{M}\xi_-$, which implies $I^{(1)}=0$ and $I^{(2)}=0$. Since $Aj(A)\xi_-\in\mathcal{P}$ for $A\in\mathcal{M}$, we obtain that $II^{(1)}\le 0$ and $II^{(2)}\le 0$. Therefore $\mathcal{E}_n(\xi_+,\xi_-)\le 0$.

We turn to the proof of the property (c). Since $|\xi|=\xi_++\xi_-$, the assertion (4.11) implies that

$$\xi\in\mathcal{M}_0\xi_0\cap\mathcal{H}^J\Rightarrow|\xi|\in D(\overline{\mathcal{E}})\quad\text{and}\quad\overline{\mathcal{E}}[|\xi|]=\lim_{n\to\infty}\mathcal{E}_n[|\xi|].\tag{4.16}$$

Let $\xi\in D(\overline{\mathcal{E}})\cap\mathcal{H}^J$ be given. Choose a sequence $\{\xi_m\}$ in $\mathcal{M}_0\xi_0\cap\mathcal{H}^J$ such that $\xi_m\to\xi$ in $\mathcal{H}$ and $\overline{\mathcal{E}}[\xi_m]\to\overline{\mathcal{E}}[\xi]$ as $m\to\infty$. The inequality $\|\xi_\pm-\eta_\pm\|\le\|\xi-\eta\|$, $\forall\xi,\ \eta\in\mathcal{H}^J$ implies $|\xi_m|\to|\xi|$ as $m\to\infty$. Notice that $\overline{\mathcal{E}}(\xi_+,\xi_-)\le 0$ is equivalent to $\overline{\mathcal{E}}[|\xi|]\le\overline{\mathcal{E}}[\xi]$, and we have shown that each $\mathcal{E}_n$, $n\in\mathbb{N}$, satisfies the property (c). By the lower semi-continuity of $\overline{\mathcal{E}}$ and (4.16), we obtain that

$$\begin{aligned}\overline{\mathcal{E}}[|\xi|] &\le \underline{\lim}_{m\to\infty}\overline{\mathcal{E}}[|\xi_m|]=\underline{\lim}_{m\to\infty}\big(\lim_{n\to\infty}\mathcal{E}_n[|\xi_m|]\big)\\ &\le \underline{\lim}_{m\to\infty}\big(\lim_{n\to\infty}\mathcal{E}_n[\xi_m]\big)=\underline{\lim}_{m\to\infty}\mathcal{E}[\xi_m]=\overline{\mathcal{E}}[\xi].\end{aligned}$$

Thus $|\xi| \in D(\overline{\mathcal{E}})$ and $\overline{\mathcal{E}}[|\xi|] \le \overline{\mathcal{E}}[\xi]$. This completes the proof of the property (c).

Since $\overline{\mathcal{E}}(\xi, \xi_0) = 0$ for any $\xi \in D(\overline{\mathcal{E}})$, it follows from the properties (b) and (c) and Proposition 2.1 that $\overline{\mathcal{E}}$ is a Dirichlet form. $\square$

5. Spectral analysis of Dirichlet operators

In this section we decompose the Hilbert space $\mathcal{H} = \mathcal{H}_\omega$ into direct sum of eigenspaces of Dirichlet operator, and analyze the spectrum of generator(Dirichlet operator) of Markovian semigroup. We also show the ergodicity of its semigroup. The decomposition method we use is essentially same as that in [AW]. See also Example 5.2.20 in [BR].

We write that

$$\begin{aligned} D_1 &:= (2\sinh(\beta/2))^{-1/2}\big(e^{\beta/4}a - e^{-\beta/4}j(a^*)\big), \\ D_2 &:= (2\sinh(\beta/2))^{-1/2}\big(e^{-\beta/4}a^* - e^{\beta/4}j(a)\big). \end{aligned} \tag{5.1}$$

By Lemma 4.1 the operators D_1 and D_2 are well defined on $\mathcal{M}_0\xi_0$ and also on $\mathcal{H}_{fin}$. It follows from (5.1) that

$$\begin{aligned} D_1^* &= (2\sinh(\beta/2))^{-1/2}\big(e^{\beta/4}a^* - e^{-\beta/4}j(a)\big) \\ &= (2\sinh(\beta/2))^{1/2}e^{-\beta/4}a^* + e^{-\beta/2}D_2, \\ D_2^* &= -(2\sinh(\beta/2))^{1/2}e^{\beta/4}a + e^{\beta/2}D_1. \end{aligned} \tag{5.2}$$

Note that $j(a^*)\xi_0 = e^{\beta/2}a\xi_0$ and $j(a)\xi_0 = e^{-\beta/2}a^*\xi_0$. Thus $D_i\xi_0 = 0$, $i = 1, 2$. A direct calculation shows that for i, $j = 1, 2$, the following canonical commutation relations (CCRs)

$$\begin{aligned} &[D_i, D_j] = 0, \quad [D_i^*, D_j^*] = 0, \\ &[D_i, D_j^*] = \delta_{ij}\mathbf{1}, \end{aligned} \tag{5.3}$$

hold on $\mathcal{H}_{fin}$. By (5.1), (5.2) and (3.8), any $P^mQ^n\xi_0$, m, $n = 0, 1, 2, ...$, can be expressed as a finite linear combination of the vectors of the form

$$(D_1^*)^p(D_2^*)^q\xi_0, \ p, q = 0, 1, 2, ...$$

Thus, as a consequence of (5.2) and the CCRs in (5.3), $\mathcal{H}_{fin}$ is spanned by the vectors of the above form, and also the following decomposition holds

$$\mathcal{H} = \oplus_{m,n=0}^{\infty}\mathcal{H}^{(m,n)}$$

where $\mathcal{H}^{(m,n)}$ is the closure of subspace spanned by the vectors of the type $(D_1^*)^m(D_2^*)^n\xi_0$, $m,\, n = 0, 1, 2, \ldots$. See also Section V of [BKP1].

To analyze the spectrum of generator H, we introduce the new operators and decomposition of $\mathcal{H}$. Define

$$A_1 := 2^{-1/2}(D_1 - D_2), \quad A_2 := 2^{-1/2}(D_1 + D_2). \tag{5.4}$$

We first collect some properties of A_i, $i = 1, 2$ which are similar to (5.3).

Proposition 5.1. *(a) $A_i\xi_0 = 0$ for $i = 1, 2$.*

(b) As operators defined on $\mathcal{H}_{fin}$, we have the following CCRs: for $i,\, j = 1, 2$

$$\begin{aligned} &[A_i, A_j] = 0, \quad [A_i^*, A_j^*] = 0, \\ &[A_i, A_j^*] = \delta_{ij}\mathbf{1}. \end{aligned} \tag{5.5}$$

Proof:(a) The result follows from $D_i\xi_0 = 0$, $i = 1, 2$ and (5.4). (b) This follows from (5.4) and the CCRs in (5.3). $\square$

We are ready to get another decomposition of the Hilbert space $\mathcal{H} = \mathcal{H}_\omega$, called quasi-free Hilbert space. According to the CCRs in Proposition 5.1, A_i and A_i^*, $i = 1, 2$, can be thought as annihilation and creation operators respectively. The following decomposition of $\mathcal{H}$ will be used to analyze the spectrum of generator H.

Theorem 5.1. *The following decomposition holds:*

$$\mathcal{H} = \oplus_{m,n=0}^{\infty}\mathcal{H}_{(m,n)}, \tag{5.6}$$

where $\mathcal{H}_{(m,n)}$ is the closure of subspace spanned by the vectors of the type $(A_1^*)^m(A_2^*)^n\xi_0$, $m,\, n = 0, 1, 2, \ldots$.

Proof: Note that

$$D_1 = 2^{-1/2}(A_1 + A_2), \quad D_2 = -2^{-1/2}(A_1 - A_2).$$

Thus any $(D_1^*)^m(D_2^*)^n\xi_0$, $m,\, n = 0, 1, 2, \ldots$ can be expressed as a finite linear combination of the vectors of the form

$$(A_1^*)^p(A_2^*)^q\xi_0, \;\; p, q = 0, 1, 2, \ldots.$$

Since $\mathcal{H}_{fin}$ is dense in $\mathcal{H}$, we conclude that the set of finite linear combinations of the vectors of the above forms is dense in $\mathcal{H}$. The decomposition follows from the CCRs in Proposition 5.1. $\square$

Let l^2 be the space of sequences $\{z_n\}_{n=0}^{\infty}$ of complex numbers satisfying $\sum_{n=0}^{\infty} |z_n|^2 < \infty$ and $\{e_n\}_{n=0}^{\infty}$ be the standard orthonormal basis in l^2. Denote by N, a_1 and a_1^*, the number, annihilation and creation operator on l^2 as follows. The number operator N is the self adjoint multiplication operator, $Ne_n = ne_n$, with maximal domain $D(N) = \{\{z_n\} \in l^2 \mid \sum_{n=0}^{\infty} |nz_n|^2 < \infty\}$. a_1 and a_1^* are given by

$$a_1 e_0 = 0,\ a_1 e_n = n^{1/2} e_{n-1},\ n = 1, 2, ...,$$
$$a_1^* e_n = (n+1)^{1/2} e_{n+1},\ n = 0, 1, 2, ...$$

The operators a_1 and a_1^* are closed and mutually adjoint, $a_1^* a_1 = N$ and $a_1 a_1^* = N + \mathbf{1}$.

Proposition 5.2. *Let U be the operator defined by*

$$U : \mathcal{H} \to l^2 \otimes l^2, \quad (A_1^*)^m (A_2^*)^n \xi_0 \mapsto (a_1^*)^m e_0 \otimes (a_1^*)^n e_0$$

for $m, n = 0, 1, 2, ...$ Then U is unitary.

Proof: Since A_i, A_i^*, $i = 1, 2$ and a_1, a_1^* satisfy the same commutation relation, the unitarity of U is easily checked. $\square$

By (3.8), (4.10) and (5.1) we get that

$$\begin{aligned} \delta_-(\sigma_{-i/4}(P)) &= \sigma_{-i/4}(P) - j(\sigma_{-i/4}(P)) \\ &= -i2^{-1/2}\big\{(e^{\beta/4}a - e^{-\beta/4}a^*) + (e^{\beta/4}j(a) - e^{-\beta/4}j(a^*))\big\} \\ &= -i2^{-1/2}(2\sinh(\beta/2))^{1/2}(D_1 - D_2), \\ &= -i(2\sinh(\beta/2))^{1/2}A_1. \end{aligned} \tag{5.7}$$

The argument used above yields

$$\begin{aligned} \delta_-(\sigma_{-i/4}(Q)) &= 2^{-1/2}(2\sinh(\beta/2))^{1/2}(D_1 + D_2) \\ &= (2\sinh(\beta/2))^{1/2}A_2. \end{aligned} \tag{5.8}$$

Define the operator $\widetilde{H}$ on $\mathcal{H}_{fin}$ by

$$\begin{aligned} \widetilde{H} &= \gamma_1 \delta_-(\sigma_{-i/4}(P))^* \delta_-(\sigma_{-i/4}(P)) + \gamma_2 \delta_-(\sigma_{-i/4}(Q))^* \delta_-(\sigma_{-i/4}(Q)) \\ &= \tau_1 A_1^* A_1 + \tau_2 A_2^* A_2 \\ &\equiv \tau_1 H_1 + \tau_2 H_2, \end{aligned} \tag{5.9}$$

where $\tau_1 = 2\gamma_1 \sinh(\beta/2)$, $\tau_2 = 2\gamma_2 \sinh(\beta/2)$. It follows from Proposition 5.1 that

$$\widetilde{H}(A_1^*)^m(A_2^*)^n\xi_0 = (m\tau_1 + n\tau_2)(A_1^*)^m(A_2^*)^n\xi_0, \ m, n = 0, 1, 2, ... \tag{5.10}$$

Proposition 5.3. *(a)* $\mathcal{H}_{fin} \subset D(\overline{\mathcal{E}})$.

(b) For $\xi \in \mathcal{H}_{fin}$ *the inequality*

$$\begin{aligned} \overline{\mathcal{E}}[\xi] &= \gamma_1\|\delta_-(\sigma_{-i/4}(P))\xi\|^2 + \gamma_2\|\delta_-(\sigma_{-i/4}(Q))\xi\|^2 \\ &= \frac{\tau_1}{2}\|(D_1 - D_2)\xi\|^2 + \frac{\tau_2}{2}\|(D_1 + D_2)\xi\|^2 \end{aligned} \tag{5.11}$$

holds.

(c) For η, $\xi \in \mathcal{H}_{fin}$ *the equality*

$$\overline{\mathcal{E}}(\eta, \xi) = \langle \eta, \widetilde{H}\xi \rangle$$

holds.

Proof: (a) By the method similar to that used in the proof of Lemma 4.1 (b), we get that $W(z)\xi_0 \in D(a^\sharp)$, $z \in \mathbb{C}$, and so $W(z)\xi_0$ belongs to the domain of D_i, $i = 1, 2$. Denote by $\mathcal{W}$ the algebra generated by $W(z), z \in \mathbb{C}$. Clearly $\mathcal{W}\xi_0$ is dense in $\mathcal{H}$. Using CCRs, one can check that the relations for $z \in \mathbb{C}$

$$\begin{aligned} &[a, W(z)] = i2^{-1/2}zW(z), \\ &[a^*, W(z)] = -i2^{-1/2}\overline{z}W(z) \end{aligned} \tag{5.12}$$

hold on $\mathcal{W}\xi_0$. Recall the definition of $W_n(z) := (W(z))_n$, $n \in \mathbb{N}$, $z \in \mathbb{C}$ in (3.5). It can be easily checked that for $z \in \mathbb{C}$

$$\overline{\mathcal{E}}[W_n(z)\xi_0 - W(z)\xi_0] \to 0 \quad \text{as} \quad n \to \infty.$$

See also the proof of Lemma 5.3 of [BKP1]. Since $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ is closed, $W(z)\xi_0 \in D(\overline{\mathcal{E}})$ for any $z \in \mathbb{C}$.

For each $l = 1, ..., m$, $m \in \mathbb{N}$, let z_l be either 1 or i. Recall the definition of $\Phi(z, n)$ in (4.1) for $z = 1, i$ and $n \in \mathbb{N}$. We will use the following abbreviated notations:

$$\begin{aligned} &\Phi_l(n) := \Phi(z_l, n), \ l = 1, ..., m, \\ &\Phi_l := \Phi(z_l), \ l = 1, ..., m, \\ \xi(n) \ &:= \Big(\prod_{l=1}^m \Phi_l(n)\Big)\xi_0 \ \text{ and } \ \xi := \Big(\prod_{l=1}^m \Phi_l\Big)\xi_0. \end{aligned} \tag{5.13}$$

Using the relations in (5.12) and (5.1), we obtain that the relations

$$
\begin{aligned}
[D_1, \Phi_l(n)] &= (2(1-e^{-\beta}))^{-1/2} z_l W\big(\frac{1}{n} z_l\big), \\
[D_1, \Phi_l] &= (2(1-e^{-\beta}))^{-1/2} z_l
\end{aligned}
\tag{5.14}
$$

hold on $\mathcal{W}\xi_0$. Notice that $\xi(n) \in D(\overline{\mathcal{E}})$, $n \in \mathbb{N}$ and $\xi \in \mathcal{H}_{fin}$. It is not hard to show that $\xi(n) \to \xi$ as $n \to \infty$. See the method used in the below. We will show that $\overline{\mathcal{E}}[\xi(n)-\xi] \to 0$ as $n \to \infty$. Since $\overline{\mathcal{E}}$ is closed, this implies that $\mathcal{H}_{fin} \subset D(\overline{\mathcal{E}})$ and

$$
\overline{\mathcal{E}}[\xi] = \lim_{n\to\infty} \overline{\mathcal{E}}[\xi(n)]. \tag{5.15}
$$

Let ξ and $\xi(n)$, $n \in \mathbb{N}$ be defined as in (5.13). Notice that

$$
\xi(n) - \xi = \sum_{p=1}^{m} \big(\prod_{l=1}^{p-1} \Phi_l(n)\big)(\Phi_p(n) - \Phi_p)\big(\prod_{l=p+1}^{m} \Phi_l\big)\xi_0.
$$

We use the Schwarz inequality twice to obtain that

$$
\|D_1\big(\xi(n)-\xi\big)\|^2 \le m \sum_{p=1}^{m} \|D_1\big(\prod_{l=1}^{p-1} \Phi_l(n)\big)(\Phi_p(n) - \Phi_p)\big(\prod_{l=p+1}^{m} \Phi_l\big)\xi_0\|^2. \tag{5.16}
$$

It follows from (5.14) and the definition of D_1 that

$$
D_1\big(\prod_{l=1}^{p-1} \Phi_l(n)\big)(\Phi_p(n) - \Phi_p)\big(\prod_{l=p+1}^{m} \Phi_l\big)\xi_0 = (2(1-e^{-\beta}))^{-1/2} \sum_{q=1}^{m} z_q \Psi(p,q;n) \tag{5.17}
$$

where

$$
\begin{aligned}
\Psi(p,q;n) &:= \big(\prod_{l=1}^{q-1} \Phi_l(n)\big) W\big(\frac{1}{n} z_q\big)\big(\prod_{l=q+1}^{p-1} \Phi_l(n)\big)(\Phi_p(n) - \Phi_p)\big(\prod_{l=p+1}^{m} \Phi_l\big)\xi_0,\ 1 \le q \le p-1, \\
\Psi(p,q;n) &:= \big(\prod_{l=1}^{p-1} \Phi_l(n)\big)\big(W\big(\frac{1}{n} z_p\big) - \mathbf{1}\big)\big(\prod_{l=p+1}^{m} \Phi_l\big)\xi_0,\ q = p, \\
\Psi(p,q;n) &:= \big(\prod_{l=1}^{p-1} \Phi_l(n)\big)(\Phi_p(n) - \Phi_p)\big(\prod_{l=p+1}^{q-1} \Phi_l(n)\big)\Big(\prod_{l=q+1}^{m} \Phi_l\big)\xi_0,\ p+1 \le q \le m.
\end{aligned}
$$

We use the Schwarz inequality twice again to (5.17) and substitute the result into (5.16) to conclude that

$$
\|D_1(\xi(n)-\xi)\|^2 \le \frac{1}{2} m^2 (1-e^{-\beta})^{-1} \sum_{p=1}^{m} \sum_{q=1}^{m} \|\Psi(p,q;n)\|^2.
$$

One notes that each $\Psi(p,q;n)$ contains either $(\Phi_p(n)-\Phi_p)$ or else $(W(\frac{1}{n}z_p)-\mathbf{1})$, which implies that

$$\|\Psi(p,q;n)\|^2 \to 0 \ \text{ as } \ n\to\infty$$

for any p, q, and so

$$\|D_1(\xi(n)-\xi)\|^2 \to 0 \ \text{ as } \ n\to\infty.$$

The method similar to that used in the above implies that

$$\|D_2(\xi(n)-\xi)\|^2 \to 0 \quad \text{as } \ n\to\infty.$$

By (5.7), (5.8) and (3.11), this proved the part (a) of this proposition.

(b) This follows from Theorem 5.1, (3.11) and (5.9).

(c) This follows from (5.4) and the part (b) of the proposition. □

Theorem 5.2. *(a) The operators $\widetilde{H}$ and H_i, $i=1,2$ are essential self-adjoint and the self-adjoint extension of $\widetilde{H}$ equals to the Dirichlet operator H given in Theorem 3.1.*

(b) For each m, n = 0, 1, 2, ..., $\mathcal{H}_{(m,n)}$ is an eigenspace of the Dirichlet operator H corresponding to eigenvalue $\tau_1 m+\tau_2 n$.

Proof: (a) Clearly $\widetilde{H}$ and H_i, $i=1,2$ are symmetric on $\mathcal{H}_{fin}$. It follows from (5.5) that the following relations holds: for $\xi=(A_1^*)^m(A_2^*)^n\xi_0,\ m,n=0,1,2,...,$

$$H_1\xi=m\xi,\quad H_2\xi=n\xi. \tag{5.18}$$

By (5.18) any $\xi\in\mathcal{H}_{fin}$ is an analytic vector for $\widetilde{H}$ and H_i, $i=1,2$, and $\widetilde{H}\mathcal{H}_{fin}\subset\mathcal{H}_{fin}$, $H_i\mathcal{H}_{fin}\subset\mathcal{H}_{fin},\ i=1,2$. It follows from Corollary 2 of Theorem X. 39 in [RS] that $\widetilde{H}$ and H_i, $i=1,2$ are essentially self-adjoint on $\mathcal{H}_{fin}$. By Proposition 5.3 (c), $H=\widetilde{H}$ on $\mathcal{H}_{fin}$, which implies $H=\widetilde{H}$.

(b) By Proposition 5.2 and (5.18), we have that

$$UHU^{-1}=\tau_1 N\otimes\mathbf{1}+\mathbf{1}\otimes\tau_2 N,$$

where N is the number operator on l^2. Thus this completes the proof of (b). □

Finally we are able to produce the proof of Theorem 3.2.

Proof of Theorem 3.2: (a) and (b) follow from Theorem 5.2.

(c) By (b), zero is a simple eigenvalue of H with eigenvector ξ_0. Since ξ_0 is a strictly positive vector, $\{T_t\}_{t\geq 0}$ is ergodic. The proof of Theorem 3.2 is completed. □

Acknowledgement: This work was supported in part by Korea Research Foundation (KRF-2001-005-D00010), Korea Ministry of Education.

References

[Acc] L. Accardi, Topics in quantum probability, *Phys. Rep.* **77**, 169-192 (1981).

[AFL] L. Accaradi, A. Frigerio, and J. T. Lewis, Quantum stochastic processes, *Publ. Res. Inst. Math. Sci.* **18**, 97-133 (1982).

[Ara] H. Araki, Some properties of modular conjugation operator of von Neumann algebras and noncommutative Radon-Nikodym theorem with chain rule, *Pacific J. Math.* **50** (2), 309-354 (1974).

[AW] H. Araki and E. J. Wood, Representations of the canonical commutation relations describing a non-relativistic infinite free Bose gas, *J. Math. Phys.* **4**, 637-662 (1963).

[BK] C. Bahn and C. K. Ko, Construction of unbounded Dirichlet forms on standard forms of von Neumann Algebras, *J. Korean Math. Soc.* **39** (6) 931-951 (2002).

[BKP1] C. Bahn, C. K. Ko and Y. M. Park, Dirichlet forms and symmetric Markovian semigroups on CCR Algebras with quasi-free states, *J. Math. Phys.* **44**, 723-753 (2003).

[BKP2] C. Bahn, C. K. Ko and Y. M. Park, Dirichlet forms and symmetric Markovian semigroups on $\mathbb{Z}_2$-graded von Neumann Algebras, preprint.

[BP] C. Bahn and Y. M. Park, Feynman Kac representation and Markov property of semigroups generated by noncommutative elliptic operators, to be appeared in *Infinite Dimensional Analysis, Quantum Probability and Related Topics*

[BR] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, second edition, Springer-Verlag, New York-Heidelberg-Berlin, vol I (1987), vol. II (1997).

[CFL] F. Cipriani, F. Fagnola and J. M. Lindsay, Spectral analysis and Feller properties for quantum Ornstein-Uhlenbeck semigroups, *Comm. Math. Phys.* **210**, 85-105 (2000).

[Cip1] F. Cipriani, Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras, *J. Funct. Anal.* **147**, 259-300 (1997).

[Cip2] F. Cipriani, Perron theory for positive maps and semigroups on von Neumann algebras, *Canadian Math. Soc. Conference Proceedings* Vol. **29**, 115-123 (2000).

[Dav] E. B. Davies, *Quantum theory of open systems*, Academic Press, London-New York-San Francisco, (1976).

[GL1] S. Goldstein and J. M. Lindsay, Beuring-Deny conditions for KMS-symmetric dynamical semigroups, *C. R. Acad. Sci. Paris. Ser. I* **317**, 1053-1057 (1993).

[GL2] S. Goldstein and J. M. Lindsay, KMS-symmetric Markov semigroups, *Math. Zeit.* **219**, 590-608 (1995).

[MZ1] A. W. Majewski and B. Zegarlinski, Quantum stochastic dynamics I: Spin systems on a lattice, *MPEJ* **1**, Paper 2 (1995).

[MZ2] A. W. Majewski and B. Zegarlinski, Quantum stochastic dynamics II, *Rev. Math. Phys.* **8** (5), 689-713 (1996).

[Mat] T. Matsui, Markov semigroups on UHF algebras, *Rev. Mod. Phys.* **5**, 587-600 (1993).

[Par] Y. M. Park, Construction of Dirichlet forms on standard forms of von Neumann algebras, *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, Vol. 3, No. 1, 1-14 (2000).

[Part] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Birkhäuser, Basel (1992)

[RS] M. Reed and B. Simon, *Method of modern mathmatical physics I, II*, Academic press (1980).