Uniform Lieb-Thirring inequality for the three dimensional Pauli operator with a strong non-homogeneous magnetic field

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Abstract

The Pauli operator describes the energy of a nonrelativistic quantum particle with spin $\frac{1}{2}$ in a magnetic field and an external potential. A new Lieb-Thirring type inequality on the sum of the negative eigenvalues is presented. The main feature compared to earlier results is that in the large field regime the present estimate grows with the optimal (first) power of the strength of the magnetic field. As a byproduct of the method, we also obtain an optimal upper bound on the pointwise density of zero energy eigenfunctions of the Dirac operator. The main technical tools are:

(i) a new localization scheme for the square of the resolvent of a general class of second order elliptic operators;

(ii) a geometric construction of a Dirac operator with a constant magnetic field that approximates the original Dirac operator in a tubular neighborhood of a fixed field line. The errors may depend on the regularity of the magnetic field but they are uniform in the field strength.

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1 Introduction

1.1 Notations

Let $\mathbf{B} \in C^4(\mathbf{R}^3; \mathbf{R}^3)$ be a magnetic field, div $\mathbf{B} = 0$, and $V \in L^1_{loc}(\mathbf{R}^3)$ a real valued potential function. Let $\mathbf{A} : \mathbf{R}^3 \to \mathbf{R}^3$ be a vector potential generating the magnetic field, i.e. $\mathbf{B} = \nabla \times \mathbf{A}$. The 3-dimensional Pauli operator is the following operator acting on the space of $L^2(\mathbf{R}^3; \mathbf{C}^2)$ of spinor-valued functions:

$$H = H(h, \mathbf{A}, V) := [\boldsymbol{\sigma} \cdot (-ih\nabla + \mathbf{A})]^2 + V = (-ih\nabla + \mathbf{A})^2 + V(x) + h\boldsymbol{\sigma} \cdot \mathbf{B}(x) \;, \tag{1.1}$$

where $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ is the vector of the Pauli spin matrices, i.e.,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The spectral properties of H depend only on $\mathbf{B}$ and V and do not depend on the specific choice of $\mathbf{A}$. We shall be concerned only with gauge invariant quantities therefore we can always make the Poincaré gauge choice. In particular, we can always assume that $\mathbf{A}$ is at least as regular as $\mathbf{B}$. The operator $H = H(h, \mathbf{A}, V)$ is defined as the Friedrichs' extension of the corresponding quadratic form from $C_0^\infty(\mathbf{R}^3; \mathbf{C}^2)$.

The Pauli operator describes the motion of a non-relativistic electron, where the electron spin is important because of its interaction with the magnetic field. For simplicity we have not included any physical parameters (i.e., the electron mass, the electron charge, the speed of light, or Planck's constant $\hbar$) in the expressions for the operators. In place of Planck's constant we have the semiclassical parameter h and in most of the paper we also set $h = 1$.

The last identity in (1.1) can easily be checked. If we define the three dimensional Dirac operator

$$\mathcal{D} := \boldsymbol{\sigma} \cdot (-ih\nabla + \mathbf{A}(x)) \ , \tag{1.2}$$

then we recognize the last identity in (1.1) as the Lichnerowicz' formula.

The eigenvalues of H below the essential spectrum are of special interest. They determine the possible bound states of a non-relativistic electron subject to the magnetic field $\mathbf{B}$ and the external potential V. Under very general conditions on V and $\mathbf{B}$ one can show that the bottom of the essential spectrum for the Pauli operator is at zero (see [HNW]). This is in sharp contrast to the case of the spinless magnetic Schrödinger operator, $(-ih\nabla+\mathbf{A})^2+V(x)$, whose essential spectrum is not known in general even for decaying potentials.

Therefore we shall restrict our attention to the negative eigenvalues, $e_1(H) \le e_2(H) \le \ldots \le 0$ of H. It is known that under very general conditions there are infinitely many negative eigenvalues even for constant magnetic field [Sol], [Sob-86], however their sum is typically finite. We recall that the sum of the eigenvalues below the essential spectrum is equal to the ground state energy of the noninteracting fermionic gas subject to H.

The sum of the negative eigenvalues, $\sum_j e_j(H)$, has been extensively studied recently. In order to find the asymptotic behavior of the ground state energy of a large atom with interacting electrons, one needs, among other things, a semiclassical asymptotics for $\sum_j e_j(H)$ as $h \to 0$.

The semiclassical formula for the sum of the negative eigenvalues is given as

$$E_{scl}(h, \mathbf{B}, V) := -h^{-3} \int_{\mathbf{R}^3} P(h|\mathbf{B}(x)|, [V(x)]_-)\mathrm{d}x \tag{1.3}$$

with

$$P(B, W) := \frac{B}{3\pi^2}\left(W^{3/2} + 2\sum_{\nu=1}^{\infty}[2\nu B - W]_-^{3/2}\right) = \frac{2}{3\pi}\sum_{\nu=0}^{\infty} d_\nu B[2\nu B - W]_-^{3/2} \tag{1.4}$$

being the pressure of the three dimensional Landau gas ($B, W \ge 0$). Here $[x]_- = \max\{0, -x\}$ refers to the negative part of x, $d_0 := (2\pi)^{-1}$ and $d_\nu := \pi^{-1}$ if $\nu \ge 1$. Observe that if $\|\mathbf{B}\|_\infty = o(h^{-1})$ then E_{scl} reduces to leading order to the standard Weyl term, $-2(15\pi^2)^{-1}h^{-3}\int_{\mathbf{R}^3}[V]_-^{5/2}$, as $h \to 0$. The main feature of the semiclassical formula is that it behaves linearly with the field strength in the strong field regime.

For the proof that $\sum_j e_j(H)$ is asymptotically equal to E_{scl} as $h \to 0$, first one must establish a non-asymptotic bound on the sum of the negative eigenvalues to control various error terms from the non-semiclassical regions. Such estimates for general Schrödinger type operators are often referred to as Lieb-Thirring (LT) type estimates [LT1]. The bound must behave like the semiclassical formula in all relevant physical parameters; in this case, in particular, it should grow linearly in the field strength. A weaker apriori estimate typically leads to a semiclassical asymptotics that is not uniform in the field strength [Sob-98], [ES-II].

1.2 Summary of previous results

A non-asymptotic LT bound for the Pauli operator has first been established in [LSY-II] for the case of the constant magnetic field, $\mathbf{B} = const.$,

$$\sum_j |e_j(H)| \leq (const.)\Big(\int [V]_-^{5/2} + |\mathbf{B}| \int [V]_-^{3/2} \Big) \tag{1.5}$$

with $h = 1$ and this bound was used to prove that E_{scl} gives the correct asymptotics for the sum of the negative eigenvalues.

The first generalizations of such estimates for non-homogeneous magnetic fields were given in [E-1995]. The first general bound was of the form $(const.)\Big(\int [V]_-^{5/2} + \|\mathbf{B}\|_\infty^{3/2} \int [V]_- \Big)$, then the main focus was to study unbounded fields. It was observed, in particular, that the natural generalization of (1.5),

$$\sum_j |e_j(H)| \leq (const.)\Big(\int [V]_-^{5/2} + \int |\mathbf{B}| [V]_-^{3/2} \Big) , \tag{1.6}$$

cannot hold in general. There are two problems in connection with (1.6).

Firstly, even when $\mathbf{B}$ has constant direction in $\mathbf{R}^3$ (1.6) is correct only if $|\mathbf{B}(x)|$ is replaced by an effective field strength, $B_{\rm eff}(x)$, obtained by averaging $|\mathbf{B}|$ locally on the magnetic lengthscale, $|\mathbf{B}|^{-1/2}$.

Secondly, the existence of the celebrated Loss-Yau zero modes [LY] contradicts (1.6). Indeed, for certain magnetic fields with nonconstant direction the Dirac operator $\mathcal{D}$ has a nontrivial L^2-kernel. In this case a small potential perturbation of $\mathcal{D}^2$ shows that $\sum_j |e_j(H)|$ behaves as $\int n(x)[V(x)]_- {\rm d}x$, i.e. it is linear in V. Here $n(x)$ is the density of zero modes, $n(x) = \sum_j |u_j(x)|^2$, where $\{u_j\}$ is an orthonormal basis in $\mathrm{Ker}\,\mathcal{D}$. Thus an extra term linear in V must be added to (1.6). It turns out that in order to estimate $n(x)$ by the magnetic field it is again important to replace $|\mathbf{B}(x)|$ by an effective field.

The problem of the effective field was first succesfully addressed by Sobolev, [Sob-96], [Sob-97] and later by Bugliaro et. al. [BFFGS] and Shen [Sh]. In particular, the L^2-norm of the effective field, $\|B_{\rm eff}\|_2$, is comparable to $\|\mathbf{B}\|_2$ in [BFFGS], and the same holds for any

L^p-norm in Shen's work. In a very general bound proved in [LLS] the second term in (1.6) is replaced with $\|\mathbf{B}\|_2^{3/2}\|V\|_4$.

In the works [E-1995], [Sob-97], [Sh], [LLS], [BFFGS] on three dimensional magnetic Lieb-Thirring inequalities, the density $n(x)$ is estimated by a function that behaves quantitatively as $|\mathbf{B}(x)|^{3/2}$. In particular, in the strong field regime these estimates are not sufficient to prove semiclassical asymptotics uniformly in the field strength, they typically give results up to $\|\mathbf{B}\|_\infty \le (const.)h^{-1}$ [Sob-98].

We remark that the bounds in [LLS] and [BFFGS] have nevertheless been very useful in the proof of magnetic stability of matter. In this case the magnetic energy, $\int |\mathbf{B}|^2$, is also part of the total energy to be minimized, therefore even the second moment of the magnetic field is controlled. We also remark that if the field has a *constant direction*, then no Loss-Yau zero modes exist, $n(x) \equiv 0$. In this case Lieb-Thirring type bounds that grow linearly with $|\mathbf{B}|$ have been proved in [E-1995] and [Sob-96], [Sob-97]. This problem is technically very similar to the two dimensional case.

Since $n(x)$ scales like $(length)^{-3}$ and $|\mathbf{B}(x)|$ scales like $(length)^{-2}$, a simple dimension counting shows that $n(x)$ cannot be estimated in general by the first power of $|\mathbf{B}(x)|$ or by any smoothed version $B_{\text{eff}}(x)$. However, if an extra lengthscale is introduced, for example certain derivatives of the field are allowed in the estimate, then it is possible to give a bound on the eigenvalue sum that grows slower than $|\mathbf{B}|^{3/2}$ in the large field regime. There are only two results so far in this direction.

The work [BFG] uses a lengthscale on which $\mathbf{B}$ changes. The estimate eventually scales like $b^{17/12}$, if the magnetic field is rescaled as $\mathbf{B}(x) \mapsto b\mathbf{B}(x)$, $b \gg 1$. As far as local regularity is concerned, only $\mathbf{B} \in H^1_{loc}$ is required. However, $n(x)$ is estimated by a quantity that depends *globally* on $\mathbf{B}(x)$ not just in a neighborhood of x. On physical grounds one expects the following *locality property*: the zero modes of $\mathcal{D}$ are supported near the support of the magnetic field, and a configuration with a strong magnetic field separated away from a potential well does not produce many negative eigenvalues if the separation of the supports is large. In particular, for compactly supported potentials the possible growth of $|\mathbf{B}|$ at infinity should be irrelevant for the negative eigenvalues. The result of [BFG] does not give such bound for an important technical reason. In order to produce an effective field strength B_{eff}, the $|\mathbf{B}|$ is averaged out by a convolution function φ that must satisfy $|\nabla\varphi| \le (const.)\varphi$, i.e. φ must have a long tail. For $\mathbf{B} \in L^2$ the effective magnetic field has a comparable L^2-norm, but it is not true for the localized L^2-norms.

Our earlier work [ES-I] had a different approach to reduce the power $3/2$ of $|\mathbf{B}|$ in the estimate of $n(x)$. We introduced two *global* lengthscales, L and ℓ respectively, to measure the variation scale of the field strength $|\mathbf{B}|$ and the unit vector $\mathbf{n} := \mathbf{B}/|\mathbf{B}|$ that determines the geometry of the field lines. This required somewhat more regularity on $\mathbf{B}$ than [BFG] and it also involved the $W^{1,1}$-norm of V. The estimate behaved like $b^{5/4}$ in the large field regime, if we rescaled $\mathbf{B} \mapsto b\mathbf{B}$, $b \gg 1$. For fields with a nearly constant direction, $\ell \gg 1$, the bound was actually better, it behaved like $b + b^{5/4}\ell^{-1/2}$. This indicates that it is only the variation of $\mathbf{n}$ and not that of $\mathbf{B}$ that is responsible for the higher b-power.

Due to the improvement in the b-power from 3/2 to 5/4 in the Lieb-Thirring estimate we could also prove the semiclassical eigenvalue asymptotics in the regime $b \ll h^{-3}$ for potentials in $W^{1,1}$ [ES-II]. This bound turned out to be sufficient to show that the Magnetic Thomas-Fermi theory exactly reproduces the ground state energy of a large atom with nuclear charge Z in the semiclassical regime, i.e. where $b \ll Z^3$, $Z \to \infty$ [ES-II]. The condition $b \ll Z^3$ is optimal as far as the semiclassical theory is applicable as the results of [LSY-I] show for super-strong ($b \geq Z^3$) constant magnetic fields.

Despite the successful application of the bound in [BFG] to the stability of matter with quantized electromagnetic field with an ultraviolet cutoff [BFrG], and despite that the Lieb-Thirring inequality given in [ES-I] fully covered the semiclassical regime of the large atoms, it is still important to establish a uniform Lieb-Thirring type bound with the correct power in the magnetic field. Such bound will likely be the key to generalize the analysis of the super-strong field regime of [LSY-I] to non-homogeneous magnetic fields.

In this paper we present a Lieb-Thirring bound that grows linearly in the field strength, moreover, it has the locality property in the sense discussed above. Our estimate depends only on the magnetic field near the support of the negative part of V and on the variation lengthscale of $\mathbf{B}$. We also state the corresponding semiclassical result in Theorem 3.3 but its details, that are similar to [ES-II], will be published separately.

1.3 Density of zero modes

As a byproduct, we also obtain a bound on the density of the zero modes, $n(x)$, that behaves optimally in the field strength in case of regular fields. Actually, we control the density of all low lying states by giving an estimate for the diagonal element of the spectral projection kernel $\Pi(\mathcal{D}^2 \leq c)(x,x)$ that grows linearly with the strength of the magnetic field for any fixed constant c.

We remark that the zero modes of the Dirac operator for particular classes of magnetic fields are well understood. The surprising first examples were due to Loss-Yau [LY] and later the present authors gave a more systematic geometric construction [ES-III]. This construction, in particular, gives examples that show that the density can grow at least linearly in the magnetic field strength. Other generalizations of the original construction of Loss-Yau are also available [AMN], [El-1]. However, there is no complete understanding of all magnetic fields with zero modes yet.

It is also known that magnetic fields with zero modes form a slim set in the space of all magnetic fields ([BE], [El-2]) but no quantitative result is available in the general case. Our result (Corollary 3.2) is the first general estimate on the density of zero modes that scales optimally (i.e., linearly) in the field strength. This result is formulated as a corollary, since it easily follows from the main theorem, but we shall prove it first on the way to the proof of our main theorem (Theorem 3.1).

It is amusing to note that it takes a considerable effort to show that zero modes exist at all, but it is even more difficult to give an optimal upper bound on their densities for strong regular fields. This is actually the main technical achievement of the present paper.

1.4 Organization of the proof

In Section 2 we introduce what we call the *tempered lengthscale* $L(x)$ of a given magnetic field $\mathbf{B}(x)$ (Proposition 2.3). This is a local variation lengthscale on which the magnetic field does not change substantially. More precisely, this is the case only in the regime where the magnetic field is strong; where the field is weak, the tempered lengthscale is simply chosen to be of the order of the magnetic lengthscale, $|\mathbf{B}|^{-1/2}$.

In Section 3 we formulate our main result on the new Lieb-Thirring inequality (Theorem 3.1) and its corollary on the density of zero modes. We also state a semiclassical result (Theorem 3.3) whose proof will be published separately.

The proof starts in Section 4 with a separation of the contributions from the low and the high energy regimes. The cutoff threshold is space dependent, it is at a level $P(x) \sim L(x)^{-2}$. Technically it is done by inserting $P(x)$ into the resolvent in the Birman-Schwinger kernel and using a resolvent expansion. We will call the two regimes the zero mode regime and the positive energy regime, respectively, because the separation is dictated by the need for a special treatment of the zero modes. The basic estimates on the contribution from these regimes are given in Theorem 4.1. We remark that to ensure ultraviolet convergence in the zero mode regime, squares of resolvents need to be estimated as well ([BFFGS]).

In both regimes we perform a two-scale localization, similarly to [ES-I]. The first localization is isotropic and its lengthscale is determined by $L(x)$. This is constructed in Section 5. The difference between the current isotropic localization and the corresponding one in [ES-I] is that in our earlier paper we assumed a universal positive bound on the tempered lengthscale, therefore we could use a regular grid of congruent cubes. Here we need to use a covering argument to select localization domains of different sizes and with a finite overlap. In domains where the magnetic field is relatively weak ($|\mathbf{B}| \leq (const.)L^{-2}$), we shall neglect all magnetic effects.

In Sections 6 and 7 we show how to localize the eigenvalue estimates onto the isotropic domains. In the positive energy regime we apply a version of the IMS localization formula for the resolvent (Proposition 6.1) that was already used in [BFFGS]. However, the same formula does not hold for the square of the resolvent which is needed in the zero mode regime. A new localization scheme is developed in Proposition 7.1 to localize the square of the resolvent of a second order elliptic operator. The localized versions of the necessary estimates in the positive energy regime and in the zero mode regime are stated in Propositions 6.2 and 7.2, respectively.

Starting from Section 8 a second localization is performed onto curvilinear cylindrical domains with a transversal lengthscale $|\mathbf{B}(x)|^{-1/2}$ along the field lines. The geometry of the cylindrical domains and the coordinate system are explained in Section 8.1, and a new partition of unity subordinated to the cylindrical domains is constructed in Section 8.2.

Within each cylindrical domain the magnetic field is approximated by a field β_c, given as a 2-form, that is constant in the appropriate cylindrical coordinates and after a conformal change of the metric (Definition 9.1). The Dirac operator $\mathcal{D}_c$ with a magnetic field β_c (Definition 9.3) will be used to approximate the original Dirac operator $\mathcal{D}$ in the corresponding cylindrical domain. Section 9 is devoted to the construction of $\mathcal{D}_c$ and it uses the geometric structure behind the Dirac operator on a non-flat manifold outlined, for example, in [ES-III].

The main difference between [ES-I] and the current work lies in the cylindrical localization. In [ES-I] we considered straight cylinders to approximate tubular neighborhoods of magnetic field lines and we approximated the field by a constant one within each cylinder. The curving of the magnetic field was not respected by the approximation hence the error was not uniform in the field strength. This is the main reason why the Lieb-Thirring inequality in [ES-I] does not have the optimal $|\mathbf{B}|$-power.

In the new construction the cylindrical localization domains are curved hence they follow the field lines more accurately. This geometry enables us to control $\mathcal{D}^2$ with errors that are uniform in the field strength although they depend on the tempered lengthscale of $\mathbf{B}$. This eliminates the $|\mathbf{B}|$-dependent error in the large field regime. The near-zero energy states, in particular the zero modes of $\mathcal{D}$, need to be controlled with such a precision in order not to overestimate their contribution to the negative eigenvalues of $\mathcal{D}^2 - V$.

Section 10 completes the proof of the positive energy regime. In this regime errors that are independent of $|\mathbf{B}|$ can be absorbed into the local energy shift $P(x)$. Proposition 10.1 contains the necessary spectral estimate localized onto the cylindrical domains in the original coordinates. We first translate the estimate into cylindrical coordinates. In these coordinates the approximating field is constant and we can use the magnetic localization formula (10.18) from [ES-II]. This method yields a $|\mathbf{B}|$-independent cylindrical localization error. Finally, having constructed the approximating local Dirac operators with constant fields, we can use the Lieb-Thirring inequality for the Pauli operator with a constant field obtained in [LSY-II].

In Section 11 we complete the estimate of the zero mode regime. We need to estimate the density of the near-zero energy states of $\mathcal{D}^2$. This is given by the diagonal kernel of the spectral projection operator, $\Pi(\mathcal{D}^2 \leq P_0)(x,x)$ where P_0 is the typical value of the regularly varying function P around x. This operator can be bounded by the resolvent, but the diagonal element of the resolvent is infinite because of the ultraviolet divergence. Therefore we need to control $\Pi(\mathcal{D}^2 \leq P_0)$ by the square of the resolvent, $(\mathcal{D}^2 + P_0)^{-2}$. For regions with weak magnetic fields the magnetic field can be neglected and we can simply use the diamagnetic inequality. The problem thus can be reduced to estimating the resolvent square of the free Laplacian (Section 11.1).

For regions with a strong field (Section 11.2) we again use the approximating constant field operators. However, the magnetic localization formula (10.18) is not valid for the square of the Pauli operator, so localizing onto cylindrical domains is more complicated. Fortunately, at this stage we do not need operator inequalities, we need to estimate only the diagonal element of the square of the resolvent at each fixed point u. First we transform the problem into the new coordinates associated with the field line through u (estimates (11.6) and (11.7)). Then we use resolvent expansions extensively to approximate $\mathcal{D}$ by $\mathcal{D}_c$. Since we estimate the square of the resolvent, we need to control the offdiagonal elements of the resolvent itself. For a constant field, the offdiagonal decay is Gaussian on a magnetic lengthscale $|\mathbf{B}|^{-1/2}$. A similar feature is proved for the resolvent with a nonconstant field via the constant field approximation.

In the Appendix we collected the proofs of several Propositions and Lemmas which can be skipped at a first reading.

It is amusing to note that the most complicated part of the proof (Sections 7 and 11) controls the possible ultraviolet regime of near zero energy states. On physical grounds this regime should be irrelevant if we knew that low energy eigenstates of $\mathcal{D}^2$ have transversal momentum of order $|\mathbf{B}|^{1/2}$ and parallel momentum independent of the field strength. The main difficulty is to obtain such information on the low lying states.

Convention: Throughout the proof universal constants are denoted by a general c whose value can be different even within the same equation. Constants depending on numbers $a, b, \ldots$ are denoted by $c(a, b, \ldots)$. Integration over $\mathbf{R}^3$ with respect to the Lebesgue measure, $\int_{\mathbf{R}^3} \mathrm{d}x$, is simply denoted by $\int$. We shall say that two positive numbers a, b are **comparable** if $\frac{1}{2} \leq a/b \leq 2$.

2 Lengthscales of the magnetic field

Let $\mathbf{B} \in C^4(\mathbf{R}^3, \mathbf{R}^3)$ be a magnetic field and let $\mathbf{n} := \mathbf{B}/|\mathbf{B}|$ be the unit vectorfield in the direction of the magnetic field at all points where $\mathbf{B}$ does not vanish.

For any $L > 0$ and $x \in \mathbf{R}^3$ we define

$$B_L(x) := \sup\{|\mathbf{B}(y)| \ : \ |x - y| \leq L\} \tag{2.1}$$

and

$$b_L(x) := \inf\{|\mathbf{B}(y)| \ : \ |x-y| \le L\} \tag{2.2}$$

to be the supremum and the infimum of the magnetic field strength on the ball of radius L about x. These functions are continuous in both the L and x variables.

The Pauli operator will be localized on different lengthscales determined by the magnetic field. We now define these scales.

Definition 2.1 (Lengthscales of a magnetic field). *Given a C^4-magnetic field $\mathbf{B}$. We define the* **magnetic lengthscale** *of $\mathbf{B}$ as*

$$L_m(x) := \sup\{L > 0 \ : \ B_L(x) \le L^{-2}\} \ . \tag{2.3}$$

The **variation lengthscale** *of $\mathbf{B}$ is given by*

$$L_v(x) := \min\{L_s(x), L_n(x)\} \ ,$$

where

$$L_s(x) := \sup\Big\{L > 0 \ : \ L^\gamma \sup\Big\{\Big|\nabla^\gamma|\mathbf{B}(y)|\Big| \ : \ |x-y| \le L\Big\} \le b_L(x), \ \gamma = 1,2,3,4\Big\} \tag{2.4}$$

and

$$L_n(x) := \sup\Big\{L > 0 \ : \ L^\gamma \sup\{|\nabla^\gamma \mathbf{n}(y)| \ : \ |x-y| \le L, \ \mathbf{B}(y) \ne 0\} \le 1, \ \gamma = 1,2,3,4\Big\} \tag{2.5}$$

(with the convention that $\sup\emptyset = -\infty$) are called the **lengthscale of the strength** *and the* **lengthscale of the direction** *of the magnetic field $\mathbf{B}$ at x, respectively. Finally we set*

$$L_c(x) := \max\{L_m(x), L_v(x)\} \tag{2.6}$$

to be the **combined lengthscale** *of $\mathbf{B}$ at x.*

A magnetic field $\mathbf{B} : \mathbf{R}^3 \to \mathbf{R}^3$ determines three local lengthscales. The magnetic lengthscale, L_m, is comparable with $|\mathbf{B}|^{-1/2}$. The lengthscale L_s determines the scale on which the strength of the field varies, i.e., it is the variation scale of $\log|\mathbf{B}|$. The field line structure, determined by $\mathbf{n}$, varies on the scale of L_n. The variation lengthscale L_v is the smaller of these last two scales, i.e., it is the scale of variation of the vectorfield $\mathbf{B}$.

For weak magnetic fields the magnetic effects can be neglected in our final eigenvalue estimate, so the variational lengthscale becomes irrelevant. This idea is reflected in the definition of L_c; we will not need to localize on scales shorter than the magnetic scale L_m.

Since localization errors decrease with the localization length, we would optimally like to choose the biggest possible scale, i.e. $L_c(x)$, for our localization scale. However, neighboring localization domains must be comparable in size so that the localization errors could be reallocated. This forces us to require a tempered behavior on the localization scales, which may result in choosing a localization scale smaller than L_c. Proposition 2.3 below shows that this technical requirement can be met at the expense of a factor $\frac{1}{2}$. Before the precise statement we need the following definition:

Definition 2.2. *Let $\varepsilon > 0$ be a positive number. A positive function $f(x)$ on $\mathbf{R}^3$ is called* **ε-tempered** *if*

$$|x-y| \le \varepsilon^{-1} f(x) \Longrightarrow \frac{1}{2} \le \frac{f(y)}{f(x)} \le 2 \qquad \forall x, y \in \mathbf{R}^3 \, . \tag{2.7}$$

If $\varepsilon = 1$, then a 1-tempered function will be simply called **tempered**.

Proposition 2.3 (Existence of tempered lengthscale). *For any not identically constant magnetic field $\mathbf{B} \in C^4(\mathbf{R}^3, \mathbf{R}^3)$ $L(x) := \frac{1}{2} L_c(x)$ is finite and defines a tempered function. This function will be called the* **tempered lengthscale** *of the magnetic field $\mathbf{B}$.*

Moreover, if $B_{L(x)}(x) > L(x)^{-2}$, then $b_{L(x)}(x) > 0$, and for $\gamma = 1, 2, 3, 4$

$$L(x)^\gamma \sup \Big\{ \Big| \nabla^\gamma |\mathbf{B}(y)| \Big| \; : \; |x-y| \le L(x) \Big\} \le b_{L(x)}(x), \tag{2.8}$$

and

$$L(x)^\gamma \sup \{ |\nabla^\gamma \mathbf{n}(y)| \; : \; |x-y| \le L(x) \} \le 1. \tag{2.9}$$

For a constant magnetic field $\mathbf{B} = const$ we have $L_c = \infty$ and we set $L(x) := \infty$.

The proof is given in Section A.1.

3 Main Theorem

We are ready to state our main results.

Theorem 3.1 (Uniform Lieb-Thirring inequality). *We assume that the magnetic field is $\mathbf{B} \in C^4(\mathbf{R}^3, \mathbf{R}^3)$ and let $L : \mathbf{R}^3 \to \mathbf{R}$ be the tempered lengthscale of $\mathbf{B}$. Let $\mathbf{A} \in C^4(\mathbf{R}^3, \mathbf{R}^3)$ be a vector potential, $\nabla \times \mathbf{A} = \mathbf{B}$, and let $\mathcal{D} := \boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A})$ be the free Dirac operator with magnetic field $\mathbf{B}$ on the trivial spinorbundle over $\mathbf{R}^3$, that can be identified with $L^2(\mathbf{R}^3, \mathbf{C}^2)$.*

Let V be a scalar potential. Then the sum of the negative eigenvalues, e_j, of the Pauli operator $H := \mathcal{D}^2 + V = [\boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A})]^2 + V$ satisfies

$$|\mathrm{Tr}\ H_-| = \sum_j |e_j| \le c \int [V]_-^{5/2} + c \int |\mathbf{B}| [V]_-^{3/2} + c \int (|\mathbf{B}| + L^{-2}) L^{-1} [V]_- \tag{3.1}$$

with universal constants.

Notation: For any self-adjoint operator H we let $H_- := \frac{1}{2}[|H| - H]$ denote its negative part.

Corollary 3.2 (Density of zero modes). *Given a magnetic field $\mathbf{B} \in C^4(\mathbf{R}^3, \mathbf{R}^3)$ with a tempered lengthscale L, the density of zero modes of the free Dirac operator $\mathcal{D}$ with magnetic field $\mathbf{B}$ satisfies*

$$n(x) := \sum_j |u_j(x)|^2 \le c(|\mathbf{B}(x)| + L^{-2}(x)) L^{-1}(x) \tag{3.2}$$

with a universal constant, where $\{u_j\}$ is an orthonormal basis in the kernel of $\mathcal{D}$.

Remarks. (i) The density function $n(x)$ was also estimated in [BFG]. In the strong field limit $\mathbf{B} \mapsto b\mathbf{B}$, $b \gg 1$, the estimate behaved as $b^{17/12}$. Moreover, our estimate on $n(x)$ uses only local information on $\mathbf{B}(x)$ unlike [BFG]. Even if $|\mathbf{B}|$ blows up within a finite distance away from x, for example like $|\mathbf{B}(y)| \sim (|x-y| - \ell)^{-2}$, $\ell > 0$, our estimate gives a finite bound on $n(x)$.

(ii) The bound (3.2) is optimal as far as the strength of the field $|\mathbf{B}|$ is concerned. This fact follows from the construction of Dirac operators with kernels of high multiplicity following the method of [ES-III]. For example, the density of Aharonov-Casher zero modes for a constant magnetic field of strength $B \gg 1$ on S^2 is of order B. The geometric procedure of [ES-III] allows one to construct a Dirac operator on $\mathbf{R}^3$ whose zero energy eigenfunctions are obtained from the eigenfunctions on S^2 by an explicit transformation. The density of these states remain comparable to the strength of the magnetic field at least away from infinity.

(iii) Theorem 3.1 and Corollary 3.2 can still be improved in two directions as far as the tempered lengthscale $L(x)$ is concerned. First of all, the effects of the variation scale L_v of $|\mathbf{B}|$ and L_n of $\mathbf{n}$ can be separated as in [ES-I]. A second direction of improvement concerns the regularity of $\mathbf{B}$: the suprema in the definition of L_m, L_s and L_v (Definition 2.1) might be replaced by appropriate local Sobolev norms.

(iv) Notice that the Lieb-Thirring inequality of [LSY-II] for a *constant* field is recovered in Theorem 3.1.

(v) The uniform Lieb-Thirring bound for a *constant direction* field, [Sob-97], [ES-I], does not directly follow from our main theorem as it is stated. On one hand, (3.1) contains a term linear in V that is unnecessary for a constant direction field. On the other hand, we assume high regularity on $\mathbf{B}$. This regularity is needed only to construct the appropriate curvilinear cylindrical localization, which is unnecessary for a field with constant direction.

However, our present technique to estimate squares of the resolvents can improve these results in another aspect. For example, if the support of $\mathbf{B}$ and the support of V are separated, our Lieb-Thirring estimate depends only on the separation distance whereas all previous bounds scale with the magnitude of $\mathbf{B}$. As a byproduct of such a result one can also improve the estimates on the ground state density of the two dimensional Pauli operator given in [E-93].

Armed with a uniform Lieb-Thirring inequality, the following semiclassical asymptotics may be proved by combining the techniques of the current paper and [ES-II]. The details of the proof will be published separately.

Theorem 3.3. *We assume that* $\mathbf{B} \in C^4(\mathbf{R}^3, \mathbf{R}^3)$ *and* $V \in L^{5/2}(\mathbf{R}^3) \cap L^1(\mathbf{R}^3)$*. Then the sum of the negative eigenvalues,* $e_j(b,h)$*, of the Pauli operator* $[\boldsymbol{\sigma} \cdot (-ih\nabla + b\mathbf{A})]^2 + V$ *is asymptotically given by*

$$\lim_{h\to 0} \frac{\sum_j e(b,h)}{E_{scl}(h, b\mathbf{B}, V)} = 1 \tag{3.3}$$

where the limit is uniform in the field strength b*.*

Remark. This result was obtained for a homogeneous magnetic field in [LSY-II]. Analogous results for $d = 2$ were obtained in [ES-II] and [Sob-98]. The latter work also extends the two dimensional analysis to obtain (3.3) for three dimensional magnetic fields with constant direction. For a general three dimensional magnetic field the limit (3.3) is proven up to $b \ll h^{-3}$ for $V \in W^{1,1}$ in [ES-II]. With a different method Sobolev also obtains (3.3) up to $b \le (const.)h^{-1}$ without assumptions on the derivatives of $\mathbf{B}$ and V [Sob-98].

4 Separation into low and high energy regimes

For a constant magnetic field $\mathbf{B}$ the tempered scale has been defined to be infinity for the transparent formulation of our theorem. The estimate (3.1) for this case has been proven in [LSY-II]. It is possible to apply our proof to this case as well, but setting $L = \infty$ directly may require minor remarks along the proof. In order to avoid this inconvenience, we can choose L to be any fixed real number for which the proof goes through without changes and finally let $L \to \infty$ in the final result (3.1).

We introduce a universal constant $0 < \varepsilon < \frac{1}{1000}$ that has to be chosen small enough for the proof to work but we shall not keep track of the exact numerical value needed. We consider it fixed throughout the proof.

Let $L(x)$ be the tempered lengthscale of $\mathbf{B}$. Introduce $\ell(x) := \varepsilon L(x)$, then the properties of $L(x)$ set in Definition 2.1 and Proposition 2.3 are translated into ℓ as follows:

- $\ell(x)$ is ε-tempered.
- For all $x \in \mathbf{R}^3$ such that $\sup\{|\mathbf{B}(y)| \ : \ |x-y| \le \varepsilon^{-1}\ell\} \ge \varepsilon^2 \ell^{-2}$ and for $\gamma = 1, \dots, 4$ we have

$$\ell(x)^\gamma \ \sup\Big\{\Big| \ \nabla^\gamma |\mathbf{B}(y)| \ \Big| \ : \ |x-y| \le \varepsilon^{-1}\ell(x)\Big\} \le \varepsilon^\gamma \inf\Big\{|\mathbf{B}(y)| \ : \ |x-y| \le \varepsilon^{-1}\ell(x)\Big\} \ . \tag{4.1}$$

and

$$\ell(x)^\gamma \ \sup\Big\{ \ \Big|\nabla^\gamma \mathbf{n}(y)\Big| \ : \ |x-y| \le \varepsilon^{-1}\ell(x)\Big\} \le \varepsilon^\gamma \ . \tag{4.2}$$

We define

$$P(x) := \varepsilon^{-5}\ell(x)^{-2}$$

and for any positive function $f > 0$ we introduce the notation

$$R_f = R(f) = (\mathcal{D}^2 + f)^{-1} \ .$$

Proof of Theorem 3.1. Since $\mathcal{D}^2 + V \ge \mathcal{D}^2 - [V]_-$, we may consider only non-positive potentials. For convenience we change the sign and we will work with the operator $H := \mathcal{D}^2 - V$ with $V \ge 0$.

By the Birman-Schwinger principle

$$|\mathrm{Tr}\ H_-| = \int_0^\infty \ n\Big(V^{1/2} R_E V^{1/2}, 1\Big) \mathrm{d}E \tag{4.3}$$

where $n(A, \mu)$ is the number of eigenvalues of the operator A greater than or equal to μ. For any $E > 0$ we have, by the resolvent identity, that

$$R_E = R_{P+E} + R_{P+E} P R_E = R_{P+E} + R_{P+E} P R_{P+E} + R_{P+E} P R_E P R_{P+E} \ .$$

Using that $P \le \mathcal{D}^2 + P + E$ and $R_E \le E^{-1}$, we obtain

$$R_E \le 2R_{P+E} + E^{-1} R_{P+E} P^2 R_{P+E} \ .$$

For any positive operators X_1, X_2,

$$n(X_1 + X_2, e_1 + e_2) \le n(X_1, e_1) + n(X_2, e_2) \ , \tag{4.4}$$

hence (4.3) is estimated as

$$|\mathrm{Tr}\; H_-| \le \int_0^\infty n\Big(V^{1/2} R_{P+E} V^{1/2}, \tfrac{1}{4}\Big)\mathrm{d}E + \int_0^\infty n\Big(2V^{1/2} R_{P+E} P^2 R_{P+E} V^{1/2}, E\Big)\mathrm{d}E \; . \quad (4.5)$$

The second term carries the contribution of the near zero energy eigenfunctions of the free Pauli operator $\mathcal{D}^2$. This will be called the *zero mode regime*.

For the first term we notice that

$$\int_0^\infty n\Big(V^{1/2} R_{P+E} V^{1/2}, \tfrac{1}{4}\Big)\mathrm{d}E = \Big|\mathrm{Tr}(\mathcal{D}^2 + P - 4V)_-\Big| \quad (4.6)$$

by the Birman-Schwinger principle. This term contains the contribution from free eigenfunctions with energy at least $O(P)$ and it will be called the *positive energy regime*.

The following Theorem estimates the two terms in (4.5) and it completes the proof of the Main Theorem by choosing ε sufficiently small. $\square$

Theorem 4.1. *For a sufficiently small universal ε and with the notations above we have*

$$\begin{aligned} \Big|\mathrm{Tr}(\mathcal{D}^2 + P - 4V)_-\Big| &\le c(\varepsilon)\int \Big(V^{5/2} + |\mathbf{B}|V^{3/2}\Big)\;, && (4.7) \\ \int_0^\infty n\Big(2V^{1/2} R_{P+E} P^2 R_{P+E} V^{1/2}, E\Big)\mathrm{d}E &\le c(\varepsilon)\int V P^{1/2}(|\mathbf{B}| + P)\;. && (4.8) \end{aligned}$$

The proof of Theorem 4.1 is given in the rest of the paper.

Convention about operator kernels: If A is a Hilbert-Schmidt operator on a Hilbert space of the form $L^2(\mathrm{d}\mu)\otimes\mathbf{C}^N$, $N\in\mathbf{N}$, we denote by $A(x,y)$ its $N\times N$-matrix valued integral kernel which is L^2 on the product space. If, in addition, A is of trace class, we can even define its diagonal kernel which, by a slight abuse of notation, will be denoted by $A(x,x)$. One possible way to define it is to write A as a product of two Hilbert-Schmidt operators, $A = HK$, and $A(x,x) := \int H(x,y)K(y,x)\mathrm{d}\mu(y)$. This is an L^1 matrix valued function of x and as such it is independent of the choice of H and K.

Convention about traces: We shall denote by Tr the trace on $L^2(\mathrm{d}\mu)\otimes\mathbf{C}^N$ and by tr the trace on $\mathbf{C}^N$. If A is of trace class on $L^2(\mathrm{d}\mu)\otimes\mathbf{C}^N$, then $\mathrm{tr}A(x,x)$ is in $L^1(\mathrm{d}\mu)$.

5 Isotropic geometry of the first localization

In this section we construct the domains for the first localization. The construction is determined by the function $\ell(x)$. We shall construct a discrete set of points $\{x_i\}$. The localization domains will be balls about x_i with radii $\ell(x_i)$ and they will have finite overlap. Moreover, the magnetic field will not change much in each localization ball since $\ell(x)$ determines the local scale of variation of $\mathbf{B}$. Outside of this domain the field will be replaced by a constant field. This procedure will apply to balls with relatively strong fields. On balls where $\mathbf{B}$ is small we neglect magnetic effects and replace the Pauli operator by the free Laplacian.

5.1 Regular fields

Definition 5.1. *Given $\ell, K > 0$ and a ball D of radius ℓ centered at $z_0 \in \mathbf{R}^3$. A magnetic field $\mathbf{B}$ is called D-***strong** *if $|\mathbf{B}(z_0)| \geq \varepsilon^{-2}\ell^{-2}$, otherwise it is called D-***weak**. *A D-strong magnetic field is called (D, K)-***regular** *if for $\gamma = 1, \dots, 4$*

(i) $\Big|\nabla^\gamma |\mathbf{B}|\Big| \leq K\varepsilon^\gamma \ell^{-\gamma} |\mathbf{B}(z_0)|$ on D;

(ii) $|\nabla^\gamma \mathbf{n}| \leq K\varepsilon^\gamma \ell^{-\gamma}$ on D (with $\mathbf{n} := \mathbf{B}/|\mathbf{B}|$);

A (D, K)-regular field $\mathbf{B}$ is called **extended** (D, K)**-regular** *if it is continuous on the whole space and $\mathbf{B}$ is constant outside of D. The value of $\mathbf{B}$ outside of D is denoted by $\mathbf{B}_\infty$ and for $\mathbf{B}_\infty \neq 0$ we set $\mathbf{n}_\infty := \mathbf{B}_\infty / |\mathbf{B}_\infty|$.*

A (D, K)-regular field $\mathbf{B}$ clearly has a small total variation on D:

$$\Big| |\mathbf{B}(x)| - |\mathbf{B}(z_0)| \Big| \leq 2K\varepsilon |\mathbf{B}(z_0)| \tag{5.1}$$

for any $x \in D$. For an extended (D, K)-regular field (5.1) is valid for any $x \in \mathbf{R}^3$, and

$$\|\mathbf{n}(x) - \mathbf{n}_\infty\| \leq K\varepsilon \ . \tag{5.2}$$

The following statement follows from the definitions above:

Lemma 5.2. *Let $\mathbf{B}(x)$ and $\ell(x)$ satisfy the conditions (2.7), (4.1) and (4.2). Let $\widetilde{D}$ be the ball of radius $10\ell(z_0)$ about some $z_0 \in \mathbf{R}^3$.*

(i) If $\mathbf{B}$ is $\widetilde{D}$-strong, then $\mathbf{B}$ is $(\widetilde{D}, 1)$-regular and $|\mathbf{B}(x)| \geq \varepsilon^{-1}\ell(z_0)^{-2}$ for any $x \in \widetilde{D}$.

(ii) If $\mathbf{B}$ is $\widetilde{D}$-weak, then $|\mathbf{B}(x)| \leq \varepsilon^{-2}\ell(z_0)^{-2}$ for any $x \in \widetilde{D}$.

Proof. (i) For $\varepsilon \leq \varepsilon_0$ we obtain that for any $x \in \widetilde{D}$

$$\Big|\ |\mathbf{B}(z_0)| - |\mathbf{B}(x)|\ \Big| \leq 10\varepsilon \inf \Big\{ |\mathbf{B}(y)| \ : \ |x - y| \leq \varepsilon^{-1}\ell(z_0) \Big\} \leq 10\varepsilon |\mathbf{B}(z_0)|$$

using (4.1). In particular, $|\mathbf{B}(x)| \geq (1 - 10\varepsilon)|\mathbf{B}(z_0)| \geq \varepsilon^{-1}\ell(x)^{-2}$ for any $x \in \widetilde{D}$ because $\mathbf{B}$ is $\widetilde{D}$-strong, $|\mathbf{B}(z_0)| \geq \varepsilon^{-2}(10\ell(z_0))^{-2}$, and $\ell(z_0) \leq 2\ell(x)$. Properties (i) and (ii) in Definition 5.1 follow from (4.1) and (4.2).

(ii) Suppose that for some $x \in \widetilde{D}$ we have $|\mathbf{B}(x)| > \varepsilon^{-2}\ell(z_0)^{-2}$. Using that $|z_0 - x| \leq 10\ell(z_0) \leq 20\ell(x) < \varepsilon^{-1}\ell(x)$ if $\varepsilon \leq \frac{1}{20}$, we obtain

$$\Big| \, |\mathbf{B}(z_0)| - |\mathbf{B}(x)| \, \Big| \leq 10\varepsilon \inf \Big\{ |\mathbf{B}(y)| \; : \; |y - x| \leq \varepsilon^{-1}\ell(x) \Big\} \leq 10\varepsilon |\mathbf{B}(z_0)|$$

which contradicts to $|\mathbf{B}(z_0)| < \varepsilon^{-2}(10\ell(z_0))^{-2}$. $\square$

5.2 Covering lemma and cutoff functions

Let $B(x, r)$ denote the closed ball centered at x with radius r. We introduce the following notations for any $x \in \mathbf{R}^3$

$$\widehat{D}_x := B\Big(x, \frac{\ell(x)}{10}\Big), \quad D_x := B\Big(x, \ell(x)\Big), \quad \widetilde{D}_x := B\Big(x, 10\ell(x)\Big) \,.$$

Definition 5.3. *Let $\ell(x)$ be an ε-tempered function and let I be a countable index set. The discrete set of points $\{x_i\}_{i \in I}$ is called an ℓ-**uniform** set of points with intersection constant N if*

(i) $\mathbf{R}^3 \subset \bigcup_{i \in I} \widehat{D}_{x_i}$;

(ii) Any ball $\widetilde{D}_{x_j}$ intersects no more than N other balls from the collection $\{\widetilde{D}_{x_i}\}$.

The proof of the following covering Lemma is given in Section A.2.

Lemma 5.4. *Let $\ell(x)$ be ε-tempered, then there exists an ℓ-uniform set of points $\{x_i\}_{i \in I}$ with some universal intersection constant N.*

In the rest of the proof we fix such a collection of points $\{x_i\}$, determined by the magnetic field via $\ell(x)$. For brevity we shall use $\ell_i := \ell(x_i)$, $\widehat{D}_i := \widehat{D}_{x_i}$, $D_i := D_{x_i}$ and $\widetilde{D}_i := \widetilde{D}_{x_i}$.

Definition 5.5. *An index $i \in I$, the corresponding point x_i and ball D_i are called* **strong** *(***weak***) if $\mathbf{B}$ is D_i-strong (weak).*

The following Lemma is an application of Lemma 5.2:

Lemma 5.6. *Let x_i be a strong point, then $\mathbf{B}$ is $(\widetilde{D}_i, 1)$-regular and $\inf_{\widetilde{D}_i} |\mathbf{B}| \geq \varepsilon^{-1}\ell_i^{-2}$. If x_i is a weak point, then $\sup_{\widetilde{D}_i} |\mathbf{B}| \leq \varepsilon^{-2}\ell_i^{-2}$.*

Given an ε-tempered function $\ell(x)$ and an ℓ-uniform set of points $\{x_i\}_{i \in I}$, for each $i \in I$ we choose smooth functions θ_i, $\widehat{\chi}_i$, χ_i and $\widetilde{\chi}_i$ with values between 0 and 1, such that the following hold:

- $\sum_i \theta_i^2(x) \equiv 1$, $\mathrm{supp}(\theta_i) \subset D_i$ and $\|\nabla\theta_i\|_\infty \le c\ell_i^{-1}$;
- $\widehat{\chi}_i \equiv 1$ on $B(x_i, 3\ell_i)$, $\mathrm{supp}(\widehat{\chi}_i) \subset B(x_i, 4\ell_i)$, $\|\nabla\widehat{\chi}_i\|_\infty \le 2\ell_i^{-1}$;
- $\chi_i \equiv 1$ on $B(x_i, 4\ell_i)$, $\mathrm{supp}(\chi_i) \subset B(x_i, 5\ell_i)$, $\|\nabla\chi_i\|_\infty \le 2\ell_i^{-1}$;
- $\widetilde{\chi}_i \equiv 1$ on $B(x_i, 6\ell_i)$, $\mathrm{supp}(\widetilde{\chi}_i) \subset B(x_i, 7\ell_i)$, $\|\nabla^\gamma\widetilde{\chi}_i\|_\infty \le (2\ell_i)^{-\gamma}$, $\gamma = 1, \dots, 4$.

Such choice is possible since the balls $\widehat{D}_i$ cover. Notice that $\nabla\widehat{\chi}_i$ is supported on the annulus

$$A_i := B(x_i, 4\ell_i) \setminus B(x_i, 3\ell_i) .$$

Finally we choose functions $\{\varphi_i\}_{i\in I}$ such that $\varphi_i \equiv 1$ on A_i, $\mathrm{supp}(\varphi_i) \subset B(x_i, 5\ell_i) \setminus B(x_i, 2\ell_i)$ and $|\nabla\varphi_i| \le 2\ell_i^{-1}$.

5.3 Approximate magnetic fields and Pauli operators

We define approximate vector potentials $\mathbf{A}_i$ and magnetic fields $\mathbf{B}_i := \nabla \times \mathbf{A}_i$, $i = 1, 2, \dots$, subordinated to the balls D_i. The definition is different for weak and strong indices $i \in I$.

If $i \in I$ is a **weak** index, then let $\widehat{\mathbf{A}}_i$ be the Poincaré gauge of $\mathbf{B}$ with base point x_i, in particular $\nabla \times \widehat{\mathbf{A}}_i = \mathbf{B}$ on $\mathbf{R}^3$ and $|\widehat{\mathbf{A}}_i| \le c\ell_i \sup_{\widetilde{D}_i} |\mathbf{B}| \le c\varepsilon^{-2}\ell_i^{-1}$ holds true on $\widetilde{D}_i$ by Lemma 5.6. We define $\mathbf{A}_i := \mathbf{A} - (1 - \widetilde{\chi}_i)\widehat{\mathbf{A}}_i$, then $\mathbf{B}_i = \widetilde{\chi}_i\mathbf{B} + \nabla\widetilde{\chi}_i \times \widehat{\mathbf{A}}_i$. Clearly $\mathbf{A}(x) = \mathbf{A}_i(x)$ for all $x \in B(x_i, 6\ell_i)$ and

$$\|\mathbf{B}_i\|_\infty \le c \sup_{\widetilde{D}_i} |\mathbf{B}| \le c\varepsilon^{-2}\ell_i^{-2} \tag{5.3}$$

with supp $\mathbf{B}_i \subset \widetilde{D}_i$.

If $i \in I$ is a **strong** index, then $\mathbf{A}_i$ is given by the following lemma.

Lemma 5.7 (Choice of the local field on strong balls). *Assume that* $\mathbf{B}$ *is* $(\widetilde{D}_i, 1)$*-regular, then there exists a vector potential* $\mathbf{A}_i$ *such that* $\mathbf{A}_i \equiv \mathbf{A}$ *on* $B(x_i, 6\ell_i)$ *and the magnetic field* $\mathbf{B}_i = \nabla \times \mathbf{A}_i$ *satisfies*

$$\mathbf{B}_i(x) \equiv \mathbf{B}(x), \qquad x \in B(x_i, 6\ell_i) \qquad \text{and} \quad \mathbf{B}_i(x) \equiv \mathbf{B}(x_i) \quad x \in \mathbf{R}^3 \setminus B(x_i, 7\ell_i) . \tag{5.4}$$

Moreover, $\mathbf{B}_i$ *is extended* $(\widetilde{D}_i, 100)$*-regular, in particular for* $\gamma = 1, \dots, 4$

$$\ell_i^\gamma \|\nabla^\gamma |\mathbf{B}_i| \;\|_\infty \le 100\varepsilon^\gamma |\mathbf{B}_i(x_i)| , \tag{5.5}$$

$$\ell_i^\gamma \|\nabla^\gamma \mathbf{n}_i \;\|_\infty \le 100\varepsilon^\gamma , \tag{5.6}$$

$$|\mathbf{B}_i(x) - \mathbf{B}(x_i)| \le 100\varepsilon |\mathbf{B}(x_i)| \tag{5.7}$$

for any $x \in \mathbf{R}^3$.

Armed with these definitions of $\mathbf{A}_i$, we define

$$\mathcal{D}_i := \boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A}_i) \tag{5.8}$$

to be the approximating Dirac operator associated with $\widetilde{D}_i$. The operator $\mathcal{D}_i$ coincides with $\mathcal{D}$ on $B(x_i, 6\ell_i)$ because $\mathbf{A} = \mathbf{A}_i$ in this domain, in particular

$$\mathcal{D}\chi_i = \mathcal{D}_i\chi_i \ . \tag{5.9}$$

Proof of Lemma 5.7. Since $\mathbf{B}$ is $(\widetilde{D}_i, 1)$-regular, from (5.1) we obtain that

$$|\mathbf{B}(x) - \mathbf{B}(x_i)| \le \varepsilon|\mathbf{B}(x_i)|, \qquad x \in \widetilde{D}_i \ . \tag{5.10}$$

Let $\mathbf{A}_i^{\#}$ be the Poincaré gauge on $\widetilde{D}_i$ of the magnetic field $\mathbf{B}-\mathbf{B}(x_i)$, then $\nabla\times\mathbf{A}_i^{\#} = \mathbf{B}-\mathbf{B}(x_i)$ and

$$|\mathbf{A}_i^{\#}(x)| \le \varepsilon\ell_i|\mathbf{B}(x_i)| \tag{5.11}$$

for any $x \in \widetilde{D}_i$. We then define

$$\mathbf{B}_i := \nabla \times (\widetilde{\chi}_i\mathbf{A}_i^{\#}) + \mathbf{B}(x_i) \tag{5.12}$$

Easy calculations show that this field is $(\widetilde{D}_i, 100)$-regular and (5.5)–(5.7) hold.

The gauge $\mathbf{A}_i^{\#} + \frac{1}{2}\mathbf{B}(x_i) \wedge (\cdot - x_i)$ generates $\mathbf{B}$, hence

$$\mathbf{A} = \mathbf{A}_i^{\#} + \tfrac{1}{2}\mathbf{B}(x_i) \wedge (\cdot - x_i) + \nabla\phi_i$$

with some $\phi_i : \mathbf{R}^3 \to \mathbf{R}$. Since $(\widetilde{\chi}_i\mathbf{A}_i^{\#}) + \frac{1}{2}\mathbf{B}(x_i) \wedge (\cdot - x_i)$ generates $\mathbf{B}_i$, we define

$$\mathbf{A}_i := (\widetilde{\chi}_i\mathbf{A}_i^{\#}) + \tfrac{1}{2}\mathbf{B}(x_i) \wedge (\cdot - x_i) + \nabla\phi_i \ . \tag{5.13}$$

Then $\nabla \times \mathbf{A}_i = \mathbf{B}_i$. $\qquad\square$

6 Positive energy regime: proof of (4.7) in Theorem 4.1

We recall the set $\{x_i\}$ constructed in Section 5 and let

$$\ell_i := \ell(x_i), \quad P_i := P(x_i) = \varepsilon^{-5}\ell_i^{-2}, \quad b_i := |\mathbf{B}(x_i)| \ .$$

We also recall that for any positive function f we denote the resolvents by

$$R_f := R[f] := (\mathcal{D}^2 + f)^{-1} \ . \tag{6.1}$$

Note that in general R_f and R_g do not commute. For simplicity we also introduce

$$R_i[f] := (\mathcal{D}_i^2 + f)^{-1}. \tag{6.2}$$

Proposition 6.1 (Pull-up proposition). *Let I be a countable index set and let g_i, $i \in I$, be a family of nonnegative smooth functions such that $0 < \sum_{i\in I} g_i^2(x) < \infty$ for every $x \in \mathbf{R}^3$. Let A_i, $i \in I$ be a family of positive invertible self-adjoint operators on $L^2(\mathbf{R}^3)$. Then*

$$\Big(\sum_{i\in I} g_i^2\Big)\frac{1}{\sum_{i\in I} g_i A_i g_i}\Big(\sum_{i\in I} g_i^2\Big) \le \sum_{i\in I} g_i \, \frac{1}{A_i} \, g_i \ . \tag{6.3}$$

Proof of Proposition 6.1. This proof is basically given in [BFFGS], we repeat it here for completeness. All positive self-adjoint operators below are interpreted as quadratic forms. We start with the operator inequality

$$J^*J\frac{1}{J^*A^{-1}J}J^*J \le J^*AJ \tag{6.4}$$

for any positive self-adjoint operator A and any operator J.

We define a map $J : L^2(\mathbf{R}^3, \mathbf{C}^2) \mapsto \bigoplus_i L^2(\mathbf{R}^3, \mathbf{C}^2) =: \mathcal{H}$ as $J : \psi \mapsto \{g_i\psi\}$. We define an operator $\widetilde{A}$ on $\mathcal{H}$ as $\widetilde{A} : \{\psi_i\} \mapsto \{A_i\psi_i\}$. It is easy to check that

$$J^*\widetilde{A}J = \sum_i g_i A_i g_i \qquad \text{on} \quad L^2(\mathbf{R}^3, \mathbf{C}^2) \ ,$$

$J^*J = \sum_i g_i^2$ and that $(\widetilde{A})^{-1} = \widetilde{A^{-1}}$. Thus

$$\Big(\sum_i g_i^2\Big)\frac{1}{\sum_i g_i A_i g_i}\Big(\sum_i g_i^2\Big) = J^*J\frac{1}{J^*\widetilde{A}J}J^*J \le J^*(\widetilde{A})^{-1}J = \sum_i g_i \frac{1}{A_i} g_i \ . \qquad \square$$

The following proposition is the localized version of (4.7) for strong balls and its proof is given in Section 10.

Proposition 6.2 (Positive energy regime). *Let D be a ball of radius ℓ and let $K > 0$ be a positive number. Let $\mathbf{B}$ be extended (D, K)-regular, let the function $0 \le \chi \le 1$ be supported on D. Then for any positive numbers $M, \mu > 0$ there exists a constant $\varepsilon(M, K, \mu)$ such that for any $\varepsilon \le \varepsilon(M, K, \mu)$ we have*

$$\Big|\mathrm{Tr}(\mathcal{D}^2 + \mu\varepsilon^{-5}\ell^{-2} - M\chi^2 V)_-\Big| \le c(M, K, \varepsilon)\int_D \Big(V^{5/2} + |\mathbf{B}|V^{3/2}\Big) \ . \tag{6.5}$$

Armed with these two Propositions, we can finish the estimate (4.7) in Theorem 4.1. Using the finite overlap property of $\widetilde{D}_i$'s (Lemma 5.4), and that $\theta_i \le \chi_i \le 1$, we see that

$$1 \le \Xi(x) := \sum_{i\in I} \chi_i^2(x) \le N \ .$$

Moreover, by the localization estimate,

$$\int |\mathcal{D}\psi|^2 \geq \tfrac{1}{N}\sum_{i\in I}\int |\chi_i \mathcal{D}\psi|^2 \geq \tfrac{1}{2N}\sum_{i\in I}\int |\mathcal{D}\chi_i\psi|^2 - \tfrac{2}{N}\sum_{i\in I}\langle \psi, (\nabla\chi_i)^2\psi\rangle \ ,$$

hence

$$\mathcal{D}^2 \geq \tfrac{1}{2N}\sum_{i\in I}\chi_i\mathcal{D}^2\chi_i - \tfrac{8}{N}\sum_{i\in I}\ell_i^{-2}\mathbf{1}(\widetilde{D}_i) \ ,$$

where $\mathbf{1}(\cdot)$ is the characteristic function. Using (5.9) we may simply replace $\mathcal{D}^2$ by $\mathcal{D}_i^2$ on the support of χ_i. If ε is sufficiently small, we obtain

$$\mathcal{D}^2 + P + E \geq \tfrac{1}{4N}\sum_{i\in I}\chi_i(\mathcal{D}_i^2 + P_i + E)\chi_i \tag{6.6}$$

using the finite overlap property and that P is comparable to $P_i = \varepsilon^{-5}\ell_i^{-2}$ on $\widetilde{D}_i$. The resolvent can be estimated by

$$R_{P+E} \leq \frac{4N}{\sum_i \chi_i(\mathcal{D}_i^2 + P_i + E)\chi_i} \leq 4N\Xi^{-1}\Big(\sum_{i\in I}\chi_i R_i[P_i+E]\chi_i\Big)\Xi^{-1}$$

using Proposition 6.1. Hence, by the Birman-Schwinger priciple

$$\begin{aligned}\Big|\mathrm{Tr}(\mathcal{D}^2 + P - 4V)_-\Big| &= \int_0^\infty n\Big(V^{1/2}R_{P+E}V^{1/2}, \tfrac{1}{4}\Big)\mathrm{d}E \\ &\leq \int_0^\infty n\Big(\Xi^{-1}V^{1/2}\Big(\sum_{i\in I}\chi_i R_i[P_i+E]\chi_i\Big)V^{1/2}\Xi^{-1}, \tfrac{1}{16N}\Big)\mathrm{d}E \\ &\leq \int_0^\infty n\Big(\sum_{i\in I}V^{1/2}\chi_i R_i[P_i+E]\chi_i V^{1/2}, \tfrac{1}{16N}\Big)\mathrm{d}E \ . \end{aligned} \tag{6.7}$$

Here we used

$$n(ABA, e) = n(B^{1/2}A^2B^{1/2}, e) \tag{6.8}$$

for any nonnegative operator B and arbitrary operator A with the choice $A = \Xi^{-1}$ and we estimated $\Xi^{-2} \leq 1$.

We also use a strengthening of (4.4). If the positive self-adjoint operators A, B are disjointly supported, i.e., there exists an orthogonal projection Π such that $\Pi A \Pi = A$ and $(I-\Pi)B(I-\Pi) = B$, then

$$n(A+B, e) = n(A, e) + n(B, e) \ . \tag{6.9}$$

The proof is trivial.

In order to use Proposition 6.2, we have to pull the summation out in (6.7). We split this sum into a few infinite sums so that each contain disjointly supported terms. Since the balls $\{\widetilde{D}_i\}$ have uniformly finite overlap with constant N (see (ii) of Lemma 5.4), there exists a partition of the index set $I = I_1 \cup I_2 \cup \ldots \cup I_{N+1}$ such that if $j, j' \in I_k$, for any $1 \le k \le N+1$, then $\widetilde{D}_j \cap \widetilde{D}_{j'} = \emptyset$. Such a partition can be obtained by a greedy algorithm. We order the index set I in some way and we put each index one by one into one of the sets. We always put the new index into one of the sets where it has no conflict with the indices already put into this set. A new index j is said to be in conflict with a previously placed index i if $\widetilde{D}_i \cap \widetilde{D}_j \neq \emptyset$. Since every index can have a conflict with at most N other indices, each index can be placed somewhere at each step of the placement.

Hence, using (4.4) first, then (6.9), we have

$$\begin{aligned} n\Big(\sum_{i\in I} V^{1/2}\chi_i\, R_i[P_i+E]\chi_i V^{1/2}, \tfrac{1}{16N}\Big) &= n\Big(\sum_{k=1}^{N+1}\sum_{i\in I_k} V^{1/2}\chi_i\, R_i[P_i+E]\chi_i V^{1/2}, \tfrac{1}{16N}\Big) \\ &\le \sum_{k=1}^{N+1} n\Big(\sum_{i\in I_k} V^{1/2}\chi_i\, R_i[P_i+E]\chi_i V^{1/2}, \tfrac{1}{16N(N+1)}\Big) \\ &= \sum_{k=1}^{N+1}\sum_{i\in I_k} n\Big(V^{1/2}\chi_i\, R_i[P_i+E]\chi_i V^{1/2}, \tfrac{1}{16N(N+1)}\Big) \\ &= \sum_{i\in I} n\Big(V^{1/2}\chi_i\, R_i[P_i+E]\chi_i V^{1/2}, \tfrac{1}{16N(N+1)}\Big), \end{aligned}$$

so combining this estimate with (6.7) and applying the Birman-Schwinger principle in the opposite direction, we obtain

$$\Big|\mathrm{Tr}(\mathcal{D}^2 + P - 4V)_-\Big| \le \sum_{i\in I}\Big|\mathrm{Tr}(\mathcal{D}_i^2 + P_i - M\chi_i^2 V)_-\Big| \tag{6.10}$$

with $M := 16N(N+1)$.

We then apply Proposition 6.2 for each strong index i for the ball $D = \widetilde{D}_i$, radius $\ell = 10\ell_i$, and the magnetic field $\mathbf{B}_i$ that is extended $(\widetilde{D}_i, K = 100)$-regular. For small enough ε we obtain

$$\begin{aligned} \Big|\mathrm{Tr}(\mathcal{D}_i^2 + P_i - M\chi_i^2 V)_-\Big| &\le c(\varepsilon, M)\int_{\widetilde{D}_i}\Big(V^{5/2} + |\mathbf{B}_i|V^{3/2}\Big) \\ &\le c(\varepsilon, M)\int_{\widetilde{D}_i}\Big(V^{5/2} + |\mathbf{B}|V^{3/2}\Big) \qquad \text{for} \quad i \ \text{strong}, \end{aligned}$$

where the last inequality follows from (5.1). For the weak indices i we use $\mathcal{D}_i^2 = (-i\nabla + \mathbf{A}_i)^2 + \boldsymbol{\sigma}\cdot\mathbf{B}_i$ and $\|\boldsymbol{\sigma}\cdot\mathbf{B}_i\| \le c\varepsilon^3 P_i$ (see (5.3)) and we obtain

$$\begin{aligned} \left|\mathrm{Tr}(\mathcal{D}_i^2 + P_i - M\chi_i^2 V)_-\right| &\le \left|\mathrm{Tr}((-i\nabla + \mathbf{A}_i)^2 - M\chi_i^2 V)_-\right| \\ &\le c(M)\int V^{5/2}\chi_i^5 \end{aligned}$$

by the usual Lieb-Thirring inequality for magnetic Schrödinger operators without spin. Summing up these estimates we obtain from (6.10) that

$$\begin{aligned} |\mathrm{Tr}(\mathcal{D}^2 + P - 4V)_-| &\le c(\varepsilon)\sum_{i\in I}\int_{\widetilde{D}_i}\left(V^{5/2} + |\mathbf{B}|V^{3/2}\right) \\ &\le c(\varepsilon)\int\left(V^{5/2} + |\mathbf{B}|V^{3/2}\right), \end{aligned}$$

again by the finite overlap property of $\mathbf{B}$. This completes the estimate (4.7). $\square$.

7 Zero mode regime: proof of (4.8) in Theorem 4.1

The estimate (4.8) essentially involves estimating the square of the resolvent of $\mathcal{D}^2$. However, the analog of Proposition 6.1 does not hold for the square of the resolvent, i.e.

$$\left(\sum g_i^2\right)\Phi\left(\sum g_i A_i g_i\right)\left(\sum g_i^2\right) \le \sum g_i \Phi(A_i) g_i$$

with $\Phi(t) = t^{-2}$ is *not* true in general. Here is a 2 by 2 matrix counterexample with $g_1 = g_2 = 2^{-1/2}$:

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Without such an inequality, we have to use a resolvent expansion. In addition to the square of the localized resolvent, we need to control offdiagonal terms. Such an estimate is given in the following Proposition, which is a general statement about squares of resolvents of second order differential operators. The proof is given in Section A.3.

Proposition 7.1 (Pull-in proposition). *Given an ε-tempered function $\ell(x)$ and a function $F(x) > 0$ satisfying*

$$\frac{1}{2} \le \frac{F(x)}{F(y)} \le 2 \tag{7.1}$$

for all $|x-y| \le \varepsilon^{-1}\ell(x)$. Set $P(x) = \varepsilon^{-5}\ell(x)^{-2}$. Let $A = \mathcal{A}\cdot\nabla + \mathcal{B}$ be a first order differential operator acting on $\bigoplus_k L^2(\mathbf{R}^3)$, $1 \le k < \infty$, with smooth coefficients, i.e. $\mathcal{A}(x)$ is a vector of $k\times k$ matrices, $\mathcal{B}(x)$ is a $k \times k$ matrix, all smoothly depending on x. We assume that

$$\sup_x \|\mathcal{A}(x)\| \le c_0 \, . \tag{7.2}$$

Let $T = A^*A$.

Given an ℓ-uniform set of points $\{x_i\}_{i\in I}$ as in Lemma 5.4. Let $\chi_i, \theta_i, \varphi_i$ be chosen as in Section 5.2. We assume that for every i there exists a first order differential operator $A_i = \mathcal{A}_i \cdot \nabla + \mathcal{B}_i$ on $\bigoplus_k L^2(\mathbf{R}^3)$ such that

$$\mathcal{A} = \mathcal{A}_i, \quad \mathcal{B} = \mathcal{B}_i \quad \textit{on } supp(\chi_i), \tag{7.3}$$

and let $T_i = A_i^*A_i$. Then there exists an ε_0 depending only on c_0 in (7.2) such that for any $\varepsilon \le \varepsilon_0$ and $\mu \ge 0$ we have

$$\frac{1}{T+P+\mu}F^2\frac{1}{T+P+\mu} \le c\sum_{i\in I} F_i^2\theta_i^2\Big(P_i^{-1}\frac{1}{T_i+P_i}A_i^*\varphi_i^2A_i\frac{1}{T_i+P_i} + \frac{1}{(T_i+P_i)^2}\Big)\theta_i^2 \tag{7.4}$$

where $F_i := \sup\{F(x) \ : \ x \in \widetilde{D}_i\}$ and $P_i = P(x_i)$.

Remark. If $\mathcal{A}$, $\mathcal{B}$ are well-behaved, then T looks like an elliptic constant coefficient differential operator on short scales. In this case the estimate localizes the square of the resolvent in such a way that the diagonal element of the operator kernels on the right hand side of (7.4) remain finite. This is clear for the second term on the RHS since the estimate is integrable in the ultraviolet regime (behaves like p^{-4} in the momentum p). The first term behaves only as p^{-2}, but the supports of θ_i and φ_i are well separated, which makes the diagonal element finite.

The diagonal kernels of the localized operators are estimated in the following Proposition whose proof is given in Section 11.

Proposition 7.2 (Zero mode regime). *Let D be a ball of radius $\ell > 0$ with center $z_0 \in \mathbf{R}^3$ and $K > 0$. We assume that either*

(i) $\|\mathbf{B}\|_\infty \le c\varepsilon^{-2}\ell^{-2}$ and $\mathbf{B}$ is supported on the ten times bigger ball $\widetilde{D} = B(z, 10\ell)$; or

(ii) $\mathbf{B}$ is extended (D, K)-regular.

Let $\mathcal{D}$ be any Dirac operator with magnetic field $\mathbf{B}$. Set $P := \varepsilon^{-5}\ell^{-2}$, $R[P] := (\mathcal{D}^2 + P)^{-1}$, let $0 \le \varphi \le 1$ be a function with $dist(z_0, supp\,\varphi) \ge 2\ell$. If $\varepsilon \le \varepsilon(K)$, then the following estimates hold for any $u \in D$

$$\mathrm{tr}\, R[P]^2(u,u) \ \le \ c(|\mathbf{B}(u)|P^{-3/2} + P^{-1/2}) \tag{7.5}$$

$$\mathrm{tr}\Big(R[P]\mathcal{D}\varphi^2\mathcal{D}R[P]\Big)(u,u) \ \le \ c(|\mathbf{B}(u)|P^{-1/2} + P^{1/2}) \, , \tag{7.6}$$

where recall that $\mathrm{tr} := \mathrm{tr}_{\mathbf{C}^2}$ *stands for the trace in the spin space.*

Remark. The diagonal elements in (7.5), (7.6) are gauge invariant, i.e, they do not depend on the choice of the vector potential in the Dirac operator.

Using these Propositions, we can complete the proof of the estimate (4.8) in Theorem 4.1. We use that the function $F(x) := P(x)$ and the operators $A := \mathcal{D}$, $A_i := \mathcal{D}_i$ satisfy the conditions of Proposition 7.1 by using (5.9) . Setting $\mu = E$ in (7.4) we obtain

$$\begin{aligned} &\int_0^\infty n\Big(2V^{1/2}R_{P+E}P^2R_{P+E}V^{1/2}, E\Big)\mathrm{d}E \qquad (7.7)\\ \le\;& 2\int_0^\infty n\Big(cV^{1/2}\sum_{i\in I}P_i^2\theta_i^2\Big(P_i^{-1}R_i[P_i]\mathcal{D}_i\varphi_i^2\mathcal{D}_iR_i[P_i]+R_i^2[P_i]\Big)\theta_i^2V^{1/2}, E\Big)\mathrm{d}E\\ =\;& c\sum_{i\in I}\mathrm{Tr}\;V\theta_i^4\Big(P_iR_i[P_i]\mathcal{D}_i\varphi_i^2\mathcal{D}_iR_i[P_i]+P_i^2R_i^2[P_i]\Big)\end{aligned}$$

using that $\int_0^\infty n(T,E)\mathrm{d}E = \mathrm{Tr}\;T$ for any positive operator T. This sum of traces can be estimated by

$$c\sum_{i\in I}\int V(x)\theta_i^4(x)\Big(|\mathbf{B}_i(x)|P_i^{1/2}+P_i^{3/2}\Big)\mathrm{d}x \le c\int VP^{1/2}(|\mathbf{B}|+P) \qquad (7.8)$$

using Proposition 7.2 with $D = D_i$, $\ell = \ell_i$, $\mathbf{B} = \mathbf{B}_i$, $K = 100$ and for sufficiently small ε. The construction of $\mathbf{B}_i$ for both weak and strong indices in Section 5.3 guarantees that either (i) or (ii) holds true in Proposition 7.2. We also used that $|\mathbf{B}_i| \le c(|\mathbf{B}|+P_i)$ and $P_i \le cP(x)$ on the support of θ_i (see (5.1), (5.3) and (5.7)), moreover that $\sum_i\theta_i^4 \le \sum_i\theta_i^2 = 1$. This completes the proof of (4.8).

8 Cylindrical geometry of the second localization

Throughout this section $\mathbf{B} = (B_1, B_2, B_3)$ is an extended (D,K)-regular magnetic field. Let $\mathbf{B}_\infty$ be the value of $\mathbf{B}$ outside D, we set $b := |\mathbf{B}_\infty|$ and $\mathbf{n}_\infty := \mathbf{B}_\infty/b$. The corresponding magnetic 2-form β is given by

$$\beta := B_3\mathrm{d}x_1\wedge\mathrm{d}x_2 + B_1\mathrm{d}x_2\wedge\mathrm{d}x_3 + B_2\mathrm{d}x_3\wedge\mathrm{d}x_1\;. \qquad (8.1)$$

Let z_0 be the center of D and define the **supporting plane** of $\mathbf{B}$,

$$\mathcal{P} := \{z\in\mathbf{R}^3\;:\;(z-z_0)\cdot\mathbf{n}_\infty = -\ell\}\;,$$

to be the plane that is orthogonal to the parallel field lines outside D. We fix an orthonormal basis p_1, p_2 in $\mathcal{P}$ such that $p_1, p_2, \mathbf{n}_\infty$ is positively oriented. Any point z in $\mathcal{P}$ can be identified with a point $\hat{z} = (\hat{z}_1, \hat{z}_2) \in \mathbf{R}^2$ via $z - z_0 = \hat{z}_1 p_1 + \hat{z}_2 p_2$, i.e. $\hat{z}_i = p_i \cdot (z - z_0)$, $i = 1, 2$. We will use these coordinates to parametrize $\mathcal{P}$.

For any $z \in \mathcal{P}$ we denote by $\varphi_z(\tau)$ the field line through z with arc length parametrization τ, i.e.

$$\dot{\varphi}_z(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\varphi_z(\tau) = \mathbf{n}(\varphi_z(\tau)) , \qquad \varphi_z(0) = z .$$

Since $\mathbf{B}$ is extended (D, K)-regular, $\dot{\varphi}_z$ is constant outside D. If ε is small enough (depending on K), we can assume that the length of the field line within D is at most 4ℓ using (5.2). Therefore $\dot{\varphi}_z(\tau)$ is constant for $|\tau| \geq 4\ell$.

Every field line intersects $\mathcal{P}$ since $\mathbf{n}$ nowhere vanishes and $\mathbf{n} \cdot \mathbf{n}_\infty \geq 1 - \|\mathbf{n} - \mathbf{n}_\infty\|^2 \geq \frac{1}{2}$ if ε is sufficiently small. Therefore the field lines $\{\varphi_z(\tau) \,:\, z \in \mathcal{P}\}$ form a foliation of $\mathbf{R}^3$ and for each $x \in \mathbf{R}^3$ we denote by $\pi(x) \in \mathcal{P}$ the unique point such that $x = \varphi_{\pi(x)}(\tau)$ for some $\tau \in \mathbf{R}$.

8.1 Coordinates and conformal factor

The following lemma will be used to introduce coordinates, $\xi = (\xi_1, \xi_2, \xi_3)$, on $\mathbf{R}^3$ associated with the field line $\varphi_z(\tau)$, $z \in \mathcal{P}$, which will be called the **central field line**. The field line will be characterized by $\xi_1 = \xi_2 = 0$. The point $z \in \mathcal{P}$ will be called the **base** of the coordinate system. The coordinates are functions of $x \in \mathbf{R}^3$ and the inverse function will be denoted by $x(\xi) : \mathbf{R}^3 \to \mathbf{R}^3$. We may also use the notation $\xi^z(x)$ and $x^z(\xi)$ to indicate the dependence on the base. For notational convenience we sometimes use $\xi_\perp := (\xi_1, \xi_2)$.

In order to treat different error terms we introduce a notation similar to the standard "big-oh" notation.

Definition 8.1. *Let k and α be nonnegative integers and let $\ell > 0$ be a real number. We say that a complex function $f(\xi)$ is of class $\mathcal{O}_k^\ell(|\xi_\perp|^\alpha)$ if there exists a constant C such that*

$$|\partial_\xi^{\mathbf{m}} f(\xi)| \leq C\ell^{-|\mathbf{m}|}\Big[\min\Big(\frac{|\xi_\perp|}{\ell}, 1\Big)\Big]^{(\alpha - m_1 - m_2)_+} \tag{8.2}$$

for any multiindex $\mathbf{m} = (m_1, m_2, m_3)$ with $|\mathbf{m}| := m_1 + m_2 + m_3 \leq k$. The definition can be extended to matrix valued functions and to forms by replacing the absolute value with any matrix or form norm on the left hand side of (8.2). For $\ell = 1$ we set $\mathcal{O}_k(|\xi_\perp|^\alpha) := \mathcal{O}_k^1(|\xi_\perp|^\alpha)$, for $k = 0$ we set $\mathcal{O}^\ell(|\xi_\perp|^\alpha) := \mathcal{O}_0^\ell(|\xi_\perp|^\alpha)$ and for $\alpha = 0$ we set $\mathcal{O}_k^\ell(1) := \mathcal{O}_k^\ell(|\xi_\perp|^0)$.

Remark. With a slight abuse of notation $\mathcal{O}_k(|\xi_\perp|^\alpha)$ will be used to denote not only the class of these functions but any element of this class, similarly to the way $\mathcal{O}(1)$ is used.

We also would like the magnetic field to be of constant strength along the central field line which is achieved by a conformal change of metric with a factor Ω. Let $\mathrm{d}s^2$ be the standard Euclidean metric and $\mathrm{d}s_\Omega^2 := \Omega^2 \mathrm{d}s^2$ be a conformally equivalent one. The following lemma describes the necessary information about the new metric and coordinates. The proof is given in Section A.4.

Lemma 8.2 (New metric and coordinates). *Given positive numbers K, ℓ, a ball D of radius ℓ, and center z_0, an extended (D, K)-regular magnetic field $\mathbf{B}$, an orthonormal basis p_1, p_2 in the supporting plane $\mathcal{P}$ such that $p_1, p_2, \mathbf{n}_\infty$ is positively oriented and the coordinate identification $z \in \mathcal{P} \leftrightarrow \hat{z} = (p_1 \cdot (z - z_0), p_2 \cdot (z - z_0)) \in \mathbf{R}^2$. If ε small enough depending on K, then for any $z \in \mathcal{P}$ there exist coordinate functions $\xi = \xi^z(x) = (\xi_1, \xi_2, \xi_3) = (\xi_\perp, \xi_3)$, and positive functions $\Omega(\xi), h(\xi) \in C^2(\mathbf{R}^3)$ with the following properties:*

$$\xi_3^z(x) = 0 \quad \textit{and} \quad \xi_\perp^z(x) = \hat{x} - \hat{z} \qquad \textit{for} \quad x \in \mathcal{P} , \tag{8.3}$$

$$\xi_\perp^z(\varphi_z(\tau)) = 0 \qquad \forall \tau . \tag{8.4}$$

The function $(\hat{z}, x) \in \mathbf{R}^5 \mapsto \xi^z(x) \in \mathbf{R}^3$ and the inverse function $(\hat{z}, \xi) \to x^z(\xi)$ belong to $C^3(\mathbf{R}^5)$. Moreover, if $D\xi$ and Dx denote the Jacobians of these functions, then for $\gamma = 1, 2, 3$ we have

$$\|D^\gamma x\|, \ \|D^\gamma \xi\| \le c(K) \varepsilon^\gamma \ell^{-\gamma+1} . \tag{8.5}$$

The metric $\mathrm{d}s_\Omega^2 := \Omega^2 \mathrm{d}s^2$ can be expressed as

$$\mathrm{d}s_\Omega^2 = \sum_{i,j=1}^{2} a_{ij} \mathrm{d}\xi_i \mathrm{d}\xi_j + h^2 \mathrm{d}\xi_3^2 , \tag{8.6}$$

where $a_{ij} \in C^2(\mathbf{R}^3)$ satisfies

$$\sup \Big\{ |a_{ij}(\xi) - \delta_{ij}| \ : \ \ell \le |\xi_\perp| \le 10\ell \Big\} \le c(K)\varepsilon . \tag{8.7}$$

Moreover, $a_{ij} = \delta_{ij}$ away from the set $\ell \le |\xi_\perp| \le 10\ell$, i.e.

$$\mathrm{d}s_\Omega^2 = \mathrm{d}\xi_1^2 + \mathrm{d}\xi_2^2 + h^2 \mathrm{d}\xi_3^2 \qquad \textit{on the domain} \qquad |\xi_\perp| \le \ell \quad \textit{or} \ \ |\xi_\perp| \ge 10\ell . \tag{8.8}$$

The functions Ω, a_{ij} and h also satisfy

$$\Omega \equiv a_{ij} \equiv h \equiv 1 \quad \textit{on the domain} \quad |\xi_3| \ge 3\ell , \tag{8.9}$$

$$\Omega = f(\xi_3)(1 + \varepsilon \mathcal{O}_2^\ell(|\xi_\perp|)), \quad \textit{and} \quad h = 1 + \varepsilon \mathcal{O}_2^\ell(|\xi_\perp|) \tag{8.10}$$

with

$$f(\xi_3) := \left(\frac{|\mathbf{B}(x(0,\xi_3))|}{b} \right)^{1/2} , \tag{8.11}$$

and

$$\Omega \equiv 1, \quad \partial_{\xi_\perp} h \equiv 0, \quad \text{on the domain} \quad |\xi_\perp| \geq 10\ell \, . \tag{8.12}$$

Globally, the following bounds hold

$$\|h - 1\|_\infty, \ \ \|\Omega - 1\|_\infty \leq c(K)\varepsilon \, , \tag{8.13}$$

$$\|\nabla^\gamma a_{ij}\|_\infty, \ \ \|\nabla^\gamma h\|_\infty, \ \ \|\nabla^\gamma \Omega\|_\infty \leq c(K)\varepsilon^\gamma \ell^{-\gamma} \, , \quad \gamma = 1, 2 \, . \tag{8.14}$$

Moreover, there exists an orthonormal basis $\{e_1, e_2, e_3\}$ *in the* $\mathrm{d}s^2_\Omega$ *metric such that* $e_3 = h^{-1}\partial_{\xi_3}$ *everywhere, and* $e_j = \partial_{\xi_j}$, $j = 1, 2$, *apart from the region* $\{\xi \, : \, \ell \leq |\xi_\perp| \leq 10\ell, |\xi_3| \leq 4\ell\}$.

Remark. The estimates in Lemma 8.2 actually depend only on the variational lengthscale of the direction of the magnetic field, L_n, and they are independent of the variational lengthscale of its strength, L_v. The proof given in Section A.4 uses a construction that relies only on the field line structure and on the logarithmic gradient of $|\mathbf{B}|$ along the field line. However,

$$\frac{\nabla_{\mathbf{n}} |\mathbf{B}|}{|\mathbf{B}|} = -\mathrm{div}\, \mathbf{n}$$

since $\mathbf{B} = |\mathbf{B}|\mathbf{n}$ is divergence-free, therefore derivatives of $\mathbf{n}$ alone control the errors.

We define the spin-up projection associated with a field line through a given point.

Definition 8.3. *Given* $z \in \mathcal{P}$, *the field line* $\varphi_z(\tau)$, *the associated coordinates* $\xi^z(x)$ *and the inverse function* $x^z(\xi)$ *as defined in Lemma 8.2. Then the* **spin-up projection** *associated with* $\varphi_z(\tau)$ *is given by a 2 by 2 matrix*

$$P_z^\uparrow(x) := \frac{1}{2}\Big[1 + \boldsymbol{\sigma} \cdot \mathbf{n}\Big(x^z(0, \xi_3^z(x)) \Big) \Big] \tag{8.15}$$

at any point $x \in \mathbf{R}^3$.

Note that $P_z^\uparrow$ is constant on the level sets of ξ_3^z.

8.2 Cylindrical partition of unity and grid of field lines

We start with a technical lemma.

Lemma 8.4. *Given $y \in \mathcal{P}$ and the associated coordinates $\{\xi_k^y\}$, $k = 1, 2, 3$, as constructed in Lemma 8.2, then for any sufficiently small $\varepsilon \le \varepsilon(K)$ and any $z \in \mathbf{R}^3$*

$$\frac{1}{2} \le \frac{|\xi_\perp^y(z)|}{|y - \pi(z)|} \le 2 \, , \tag{8.16}$$

$$\|P_y^\uparrow(z) - P_{\pi(z)}^\uparrow(z)\| \le cK\varepsilon\ell^{-1}|y - \pi(z)| \, , \tag{8.17}$$

where $\|\cdot\|$ denotes the standard norm of 2 by 2 matrices.

Proof. Denote $u = \pi(z)$ and set $q(\tau) := \xi^y(\varphi_u(\tau)) - \xi^y(\varphi_y(\tau)) \in \mathbf{R}^3$ and let $r(\tau) := q_\perp(\tau) = (q_1(\tau), q_2(\tau))$. We have $|q(0)| = |r(0)| = |u - y|$ and by $\dot\varphi(\tau) = \mathbf{n}(\varphi(\tau))$ and Lemma 8.2 we can estimate

$$\begin{aligned}
|\dot q(\tau)| &\le \|D_x\xi\|_\infty|\dot\varphi_u(\tau) - \dot\varphi_y(\tau)| + \|D_x^2\xi\|_\infty|\varphi_u(\tau) - \varphi_y(\tau)| \\
&\le (\|D_x\xi\|_\infty\|\nabla\mathbf{n}\|_\infty + \|D_x^2\xi\|_\infty)|\varphi_u(\tau) - \varphi_y(\tau)| \\
&\le (\|D_x\xi\|_\infty\|\nabla\mathbf{n}\|_\infty + \|D_x^2\xi\|_\infty)\|(D_x\xi)^{-1}\|_\infty|q(\tau)| \\
&\le c(K)\varepsilon\ell^{-1}|q(\tau)|
\end{aligned} \tag{8.18}$$

for $|\tau| \le 4\ell$ and $\dot q(\tau) \equiv 0$ for $|\tau| \ge 4\ell$. Therefore $\sup_\tau |q(\tau)| \le |u - y|e^{c(K)\varepsilon}$ by Gromwall's inequality and

$$\sup_\tau |\dot r(\tau)| \le \sup_\tau |\dot q(\tau)| \le cK\varepsilon\ell^{-1}e^{c(K)\varepsilon}|u - y| \, .$$

Combining this with $|r(0)| = |u - y|$ we obtain $\frac{1}{2}|u - y| \le \sup_\tau |r(\tau)| \le 2|u - y|$ if ε is sufficiently small. Note that for some τ

$$|\xi_\perp^y(z)| = \Big|\xi_\perp^y(\varphi_{\pi(z)}(\tau)) - \xi_\perp^y(\varphi_y(\tau))\Big| = |r(\tau)| \, ,$$

which concludes the proof of (8.16).

For the proof of (8.17) we again set $u = \pi(z)$ and by Definition 8.3 and Lemma 8.2 we estimate

$$\begin{aligned}
\|P_y^\uparrow(z) - P_u^\uparrow(z)\| &\le \|\nabla\mathbf{n}\|_\infty\Big|x^y(0, \xi_3^y(z)) - x^u(0, \xi_3^u(z))\Big| \\
&\le \|\nabla\mathbf{n}\|_\infty\Big(\Big|x^y(0, \xi_3^y(z)) - x^y(0, \xi_3^u(z))\Big| + \Big|x^y(0, \xi_3^u(z)) - x^u(0, \xi_3^u(z))\Big|\Big) \\
&\le \|\nabla\mathbf{n}\|_\infty(2\|D_u\xi^u\|_\infty + \|D_ux^u\|_\infty)|y - u| \\
&\le cK\varepsilon\ell^{-1}|y - u| \, ,
\end{aligned}$$

using $|x^y(\xi)-x^y(\xi')| \le 2|\xi-\xi'|$ that follows from (8.5) if ε is sufficiently small. This completes the proof of (8.17). $\square$

We construct a grid of field lines. Choose a square lattice $\mathcal{Y} := \{y_j \;:\; j \in \mathbf{Z}^2\}$ on $\mathcal{P}$ with spacing $b^{-1/2}$, i.e., $|y_j - y_k| = b^{-1/2}|j-k|$, $j,k \in \mathbf{Z}^2$. Applying Lemma 8.2 to each field line φ_{y_j}, we construct conformal factors Ω_j, orthonormal bases $\{e_1^{(j)}, e_2^{(j)}, e_3^{(j)}\}$ and coordinate functions $\xi^{(j)} = (\xi_1^{(j)}, \xi_2^{(j)}, \xi_3^{(j)})$. We now construct a set of Gaussian localization functions with a transversal lengthscale of order $b^{-1/2}$ that are essentially supported around the field lines φ_{y_j}. Let $\eta \le \frac14$ be a small positive number to be specified later and we define

$$v_j(x) = \exp\Big(-\frac{\eta b}{4}[\xi_\perp^{(j)}(x)]^2\Big) . \tag{8.19}$$

We set $P_j^{\uparrow}(x) := P_{y_j}^{\uparrow}(x)$ to be the 2 by 2 spin-up projection matrix associated with the field line through y_j (see Definition 8.3).

Lemma 8.5. *If ε is sufficiently small depending only on K, then for any $\gamma > 0$, $\kappa \ge 0$ we have*

$$\sum_{j\in\mathbf{Z}^2} (\eta b)^\kappa [\xi_\perp^{(j)}(x)]^{2\kappa} v_j(x)^\gamma \;\le\; c(\gamma,\kappa)\eta^{-1} , \tag{8.20}$$

$$\sum_{j\in\mathbf{Z}^2} v_j(x)^\gamma \;\ge\; c(\gamma)\eta^{-1} , \tag{8.21}$$

uniformly in $x \in \mathbf{R}^3$. Moreover, there is a universal constant C_0 and for any $0<\lambda<1$ there exists $0 < \eta(\lambda) \le \frac14$ such that for any $\eta \le \eta(\lambda)$

$$\sum_{j\in\mathbf{Z}^2} v_j^4(x)\Big[b\Big(\lambda - \eta^2 b[\xi_\perp^{(j)}(x)]^2\Big)P_j^{\uparrow}(x) + C_0\ell^{-2}\Big] \ge 0 . \tag{8.22}$$

Proof. Since $(\eta b\xi_\perp^2)^\kappa \exp(-\frac{\gamma\eta b}{8}\xi_\perp^2) \le c(\gamma,\kappa)$ uniformly in $\xi_\perp$, it is sufficient to estimate $\sum_j v_j^{\gamma/2}$ for the proof of (8.20). Using (8.16) we obtain

$$\sum_{j\in\mathbf{Z}^2} v_j^{\gamma/2}(x) \le \sum_j \exp\Big(-\frac{\gamma\eta b}{16}|y_j - \pi(x)|^2\Big) \le c(\eta\gamma)^{-1}$$

since y_j runs through a square grid with spacing $b^{-1/2}$. The proof of (8.21) is similar.

For the proof of (8.22) we define $k \in \mathbf{Z}^2$ to be an index such that $|y_k - \pi(x)| \le b^{-1/2}$. Then by (8.16) and Schwarz' inequality

$$|y_k - y_j|^2 \le 2b^{-1} + 2|\pi(x) - y_j|^2 \le 2b^{-1} + 4|\xi_\perp^j(x)|^2 .$$

Combining this estimate with (8.17) and using that $(P^{\uparrow})^2 = P^{\uparrow}$ we have

$$P_j^{\uparrow} \geq \tfrac{1}{2}P_k^{\uparrow} - 2\|P_j^{\uparrow} - P_k^{\uparrow}\|^2 \geq \tfrac{1}{2}P_k^{\uparrow} - \ell^{-2}|y_k - y_j|^2 \geq \tfrac{1}{2}P_k^{\uparrow} - 4\ell^{-2}|\xi_{\perp}^j|^2 - cb^{-1}\ell^{-2}$$

and

$$P_j^{\uparrow} \leq 2P_k^{\uparrow} + 2(P_j^{\uparrow} - P_k^{\uparrow})^2 \leq 2P_k^{\uparrow} + 4\ell^{-2}|\xi_{\perp}^j|^2 + cb^{-1}\ell^{-2}$$

if ε is sufficiently small. We omitted the x argument for brevity. Therefore we can use (8.20) and (8.21) to estimate

$$\begin{aligned}
&\sum_j v_j^4 b(\lambda - \eta^2 b|\xi_{\perp}^j|^2)P_j^{\uparrow} \\
\geq\ & \frac{b}{2}\sum_j v_j^4(\lambda - 4\eta^2 b|\xi_{\perp}^j|^2)P_k^{\uparrow} - c\sum_j v_j^4\Big(\lambda b|\xi_{\perp}^j|^2 + c\lambda + \eta^2 b^2|\xi_{\perp}^j|^4 + \eta^2 b|\xi_{\perp}^j|^2\Big)\ell^{-2} \\
\geq\ & \frac{b}{2}(c\eta^{-1}\lambda - c)P_k^{\uparrow} - c\eta^{-1}\ell^{-2} \\
\geq\ & -\sum_j v_j^4(x)c\ell^{-2}
\end{aligned}$$

if η is sufficiently small. We can choose C_0 to be the universal constant c in the last formula and the proof of (8.22) is completed. □

9 Dirac operator on $\mathbf{R}^3$ with a general metric

The following sections summarize basic information about the Dirac operator over a non-flat manifold. More details are found in [ES-III] (the sign of $\mathbf{A}$ is chosen to be the opposite in this paper). The presentation here is simplified because the spinor bundle is trivial and we can work in a global orthonormal basis.

Throughout this section we shall consider $\mathbf{R}^3$ with a general Riemannian metric $g = (\cdot,\cdot)$ and we shall consider the Dirac operator for this particular Riemannian manifold. The Dirac operator will be an unbounded self-adjoint operator in $L_g^2(\mathbf{R}^3) \otimes \mathbf{C}^2$ (the subscript g refers to the fact that the measure is the volume form of g).

Let $\{e_1, e_2, e_3\}$ be a global orthonormal basis of vectorfields and let $\{e^1, e^2, e^3\}$ be the dual basis. If X is a vectorfield on $\mathbf{R}^3$, we denote by

$$P_X := \frac{1}{2}\Big[1 + \sum_{j=1}^{3}(X, e_j)\sigma^j\Big] \tag{9.1}$$

the *spin projection* in direction X with respect to the basis $\{e_1, e_2, e_3\}$.

We also introduce a *covariant derivative* on $L^2_g(\mathbf{R}^3) \otimes \mathbf{C}^2$ by

$$\nabla_X := \partial_X + \tfrac{i}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\omega}(X) \ . \tag{9.2}$$

Here we define

$$\boldsymbol{\omega}(X) := \Big((\nabla_X e_3, e_2),\ (\nabla_X e_1, e_3),\ (\nabla_X e_2, e_1) \Big) \ , \tag{9.3}$$

where ∇_X refers to the Levi-Civita connection on vectorfields for the metric g on $\mathbf{R}^3$.

If α is a (real) 1-form we define the corresponding covariant derivative on $L^2_g(\mathbf{R}^3) \otimes \mathbf{C}^2$ (see Proposition 2.9 in [ES-III])

$$\nabla^\alpha_X := \nabla_X + i\alpha(X) \ . \tag{9.4}$$

The magnetic 2-form is $\beta := \mathrm{d}\alpha$. We define the *Dirac operator* by

$$\mathcal{D}^\alpha := \sum_{j=1}^{3} \sigma^j (-i\nabla^\alpha_{e_j}) \ . \tag{9.5}$$

It is a symmetric operator in $L^2_g(\mathbf{R}^3) \otimes \mathbf{C}^2$ (Theorem 3.2 in [ES-III]). Note that $\mathcal{D}^\alpha$ also depends on the metric g and the choice of $\{e_1, e_2, e_3\}$ but this fact will usually be suppressed in the notation.

For notational convenience we introduce the following *vector of covariant derivatives*

$$\boldsymbol{\Pi}^\alpha := (-i\nabla^\alpha_{e_1}, -i\nabla^\alpha_{e_2}, -i\nabla^\alpha_{e_3}) \ . \tag{9.6}$$

With this notation we may write $\mathcal{D}^\alpha = \boldsymbol{\sigma} \cdot \boldsymbol{\Pi}^\alpha$. Note that the components of $\boldsymbol{\Pi}^\alpha$ are not self-adjoint, however the components of the vector

$$\mathbf{D}^\alpha = (D^\alpha_1, D^\alpha_2, D^\alpha_3) := \boldsymbol{\Pi}^\alpha - \frac{i}{2}\Big(\mathrm{div}\ e_1, \mathrm{div}\ e_2, \mathrm{div}\ e_3\Big) \tag{9.7}$$

are self-adjoint operators.

For any one form $\lambda = \lambda_1 e^1 + \lambda_2 e^2 + \lambda_3 e^3$ we define $\sigma(\lambda) := \lambda_1\sigma^1 + \lambda_2\sigma^2 + \lambda_3\sigma^3$. The Lichnerowicz' formula (see, e.g., Theorem 3.4 in [ES-III]) states that

$$[\mathcal{D}^\alpha]^2 = [\boldsymbol{\Pi}^\alpha]^* \cdot \boldsymbol{\Pi}^\alpha + \frac{1}{4}R + \sigma(\star\beta) \ , \tag{9.8}$$

where R is the scalar curvature of g and $\star$ denotes the Hodge dual.

In terms of D_j^α operators, the Lichnerowicz' formula reads as

$$[\mathcal{D}^\alpha]^2 = [\mathbf{D}^\alpha]^2 + \frac{1}{4}R + \frac{1}{4}\sum_{j=1}^3 [\mathrm{div}\ e_j]^2 + \frac{1}{2}\sum_{j=1}^3 \partial_{e_j}(\mathrm{div}\ e_j) + \sigma(\star\beta) \ . \tag{9.9}$$

For the flat Euclidean metric with the standard orthonormal basis we have $\boldsymbol{\omega} \equiv 0$. In this case if a denotes the 1-form dual to the vector field $\mathbf{A} = (A_1, A_2, A_3)$ then $\mathbf{\Pi}^a = \mathbf{p_A}$ where $\mathbf{p}_A$ is the vector of operators $(-i\partial_1 + A_1, -i\partial_2 + A_2, -i\partial_3 + A_3)$. Therefore we obtain the usual Dirac operator $\mathcal{D} = \mathcal{D}^a$ defined in Section 1 with $\beta = \mathrm{d}a$ given by (8.1). Moreover, $P_X = P_z^\uparrow$ if $X(x) = \mathbf{n}(x^z(0, \xi^z(x)))$ from (8.15) and (9.1).

9.1 Gauge transformation

In the previous construction $\mathcal{D}^\alpha$ and $\mathbf{\Pi}^\alpha$ depend on α and also on $\{e_1, e_2, e_3\}$. Up to a unitary equivalent gauge transformation, however, $\mathcal{D}^\alpha$ and $\mathbf{\Pi}^\alpha$ depend only on the metric g and the magnetic 2-form β. Similarly, the spin projection P_X defined in (9.1) is gauge-invariant.

More precisely, given another 1-form α' with $\mathrm{d}\alpha' = \beta$ and another orthonormal basis $\{e_1', e_2', e_3'\}$ with the same orientation, we denote the corresponding operators by $\mathcal{D}'$ and $\mathbf{\Pi}'$ and let P_X' be the spin projection. There exist a real valued function $\phi(x)$ and a continuous function $R(x) \in SO(3)$ on $\mathbf{R}^3$ such that $\alpha' = \alpha + \mathrm{d}\phi$ and $\sum_k w_k e_k' = \sum_k (R\mathbf{w})_k e_k$ for any $\mathbf{w} \in \mathbf{R}^3$. Let $U_R(x) \in SU(2)$ denote the image of $R(x)$ under the isomorphism $SO(3) \to SU(2)/\{\pm 1\}$. The requirement that $U_R(x)$ be a continuous function of x determines U_R uniquely up to a global sign. In particular

$$U_R(\boldsymbol{\sigma} \cdot \mathbf{v})U_R^* = \boldsymbol{\sigma} \cdot (R\mathbf{v}) \tag{9.10}$$

for any $\mathbf{v} \in \mathbf{R}^3$, i.e. $R(\psi, \boldsymbol{\sigma}\psi) = (U_R\psi, \boldsymbol{\sigma}U_R\psi)$ for any $\psi \in \mathbf{C}^2$, where $(\psi, \boldsymbol{\sigma}\psi)$ denotes the vector $\Big((\psi, \sigma^1\psi), (\psi, \sigma^2\psi), (\psi, \sigma^3\psi)\Big) \in \mathbf{R}^3$.

We define the unitary operator of the form

$$[\mathcal{U}_{R,\phi}\psi](x) = e^{i\phi(x)}U_R(x)\psi(x) \ , \tag{9.11}$$

then

$$P_X' = \mathcal{U}_{R,\phi}^* P_X \mathcal{U}_{R,\phi} \tag{9.12}$$

and

$$\mathcal{D}' = \mathcal{U}_{R,\phi}^* \mathcal{D}^\alpha \mathcal{U}_{R,\phi} \ , \qquad \text{and} \quad \mathbf{w} \cdot \mathbf{\Pi}' = (R\mathbf{w}) \cdot \mathcal{U}_{R,\phi}^* \mathbf{\Pi}^\alpha \mathcal{U}_{R,\phi} \tag{9.13}$$

for any $\mathbf{w} \in \mathbf{R}^3$. In particular, the spectrum of $\mathcal{D}^\alpha$ and the functions

$$\mathrm{tr}\Big(\frac{1}{([\mathcal{D}^\alpha]^2 + c)^2}(x,x)\Big) \qquad \text{and} \qquad \mathrm{tr}\Big(\frac{1}{[\mathcal{D}^\alpha]^2 + c}\mathcal{D}^\alpha \varphi^2 \mathcal{D}^\alpha \frac{1}{[\mathcal{D}^\alpha]^2 + c}(x,x)\Big)$$

depend only on g and β, where φ is any function on $\mathbf{R}^3$ and $c > 0$ is a constant.

9.2 Change of the Dirac operator under a conformal change of the metric

Let Ω be a positive real function on $\mathbf{R}^3$ and let $g_\Omega := \Omega^2 g$ be a metric which is conformal to g. Consider the $(f_1, f_2, f_3) := (\Omega^{-1}e_1, \Omega^{-1}e_2, \Omega^{-1}e_3)$ orthonormal basis in g_Ω. Given a 1-form α we let $\nabla_X^{\alpha,\Omega}$ and $\mathcal{D}_\Omega^\alpha$ denote the corresponding covariant derivative and Dirac operator. With the notation

$$\mathbf{\Pi}_\Omega^\alpha := \Big(-i\nabla_{f_1}^{\alpha,\Omega}, -i\nabla_{f_2}^{\alpha,\Omega}, -i\nabla_{f_3}^{\alpha,\Omega}\Big) \tag{9.14}$$

we have $\mathcal{D}_\Omega^\alpha = \boldsymbol{\sigma}\cdot\mathbf{\Pi}_\Omega^\alpha$. Then from Section 4 of [ES-III]

$$\mathcal{D}_\Omega^\alpha = \Omega^{-2}\mathcal{D}^\alpha\Omega \tag{9.15}$$

and

$$\nabla_X^{\alpha,\Omega} = \nabla_X^\alpha + \tfrac{1}{4}\Omega^{-1}[\sigma(X^*), \sigma(\mathrm{d}\Omega)] \tag{9.16}$$

for any vector X, where X^* refers to the 1-form which is dual to the vector X relative to the metric g, and $\sigma(X^*), \sigma(\mathrm{d}\Omega)$ are computed in the $\{e_1, e_2, e_3\}$ basis. In particular,

$$\mathbf{\Pi}_\Omega^\alpha = \Omega^{-1}\mathbf{\Pi}^\alpha - \tfrac{i}{4}\Omega^{-2}\Big([\sigma^1, \sigma(\mathrm{d}\Omega)], [\sigma^2, \sigma(\mathrm{d}\Omega)], [\sigma^3, \sigma(\mathrm{d}\Omega)]\Big)\ . \tag{9.17}$$

9.3 Constant approximation of the magnetic field along a field line

The goal of this section is to express the Dirac operator with a non-homogeneous regular magnetic field as a sum of a constant field Dirac operator and some error terms in a neighborhood of a given field line. This can be done if the original Dirac operator is already written in an appropriate orthonormal basis and with a carefully selected vector potential. The basis and the vector potential are determined by the local magnetic field.

Given an extended (D, K)-regular field $\mathbf{B}$. Consider the corresponding 2-form β and a fixed field line. Let the coordinates (ξ_1, ξ_2, ξ_3), the new metric $g_\Omega = \Omega^2\mathrm{d}s^2$ with a conformal factor Ω and the orthonormal basis $\{e_1, e_2, e_3\}$ be as constructed in Lemma 8.2, associated with the given field line. Let α denote a vector potential, $\mathrm{d}\alpha = \beta$, to be chosen later. Let $\mathbf{\Pi}^\alpha$ be given by (9.6) and let $\mathcal{D}^\alpha := \boldsymbol{\sigma}\cdot\mathbf{\Pi}^\alpha$.

On the central line and in the regime $|\xi_\perp| \geq 10\ell$ the magnetic field β is constant in the $\mathrm{d}s_\Omega^2$ metric:

$$\beta(e_1, e_2) = \Omega^{-2}\beta(\Omega e_1, \Omega e_2) = \Omega^{-2}|\mathbf{B}| = b, \qquad \beta(e_j, e_3) = 0, \quad j = 1, 2\ .$$

This observation gives rise to the following definition.

Definition 9.1. *Given a field line, the associated coordinate system ξ and the conformal factor Ω as above such that magnetic field β is constant in the $\mathrm{d}s^2_\Omega$ metric with strength $b = \beta(e_1, e_2)$. Then the magnetic field β_c given by*

$$\beta_c := b\, \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2$$

is called the approximating constant magnetic field along the field line.

The magnetic field β_c is clearly constant in the $\mathrm{d}\xi^2 = \sum_{j=1}^3 \mathrm{d}\xi_j^2$ metric. A convenient gauge is defined as $\alpha_c := \frac{b}{2}[\xi_1 \mathrm{d}\xi_2 - \xi_2 \mathrm{d}\xi_1]$, then $\beta_c = \mathrm{d}\alpha_c$.

In particular $\beta = \beta_c$ along the central line and in the regime $|\xi_\perp| \geq 10\ell$. We compute the norm of the difference field $\delta\beta := \beta - \beta_c$ and the norm of its derivative in the $\mathrm{d}\xi^2$ metric. Using (8.5), (8.7), (8.8), (8.13) and (5.5), (5.6) we obtain

$$\delta\beta(\xi) = \varepsilon b \mathcal{O}_2^\ell(|\xi_\perp|) \tag{9.18}$$

and $\delta\beta(\xi) \equiv 0$ if $|\xi_\perp| \geq 10\ell$.

Next, we define an appropriate gauge α for the original magnetic field β, $\mathrm{d}\alpha = \beta$, such that $\alpha - \alpha_c$ be small. The following Lemma was given in [ES-I] (Proposition 2.3). Although it was stated in a slightly weaker form, the explicit formula (2.20) of [ES-I] gives the following stronger result with a straightforward computation:

Lemma 9.2 (A-formula). *Given any C^2 magnetic 2-form β on $\mathbf{R}^3$ with Euclidean cooordinates (ξ_1, ξ_2, ξ_3). For $k, m \in \mathbf{N}$ we define*

$$b_{k,m}(\xi_\perp) := \int_0^{\xi_1} u^k \sup_{z_2, z_3} \|\nabla^m \beta(u, z_2, z_3)\| \mathrm{d}u + \int_0^{\xi_2} u^k \sup_{z_1, z_3} \|\nabla^m \beta(z_1, u, z_3)\| \mathrm{d}u\ .$$

Then there exists a 1-form α generating β, $\mathrm{d}\alpha = \beta$, such that

$$\|\alpha(\xi)\| \;\leq\; c\Big[b_{0,0}(\xi_\perp) + b_{1,1}(\xi_\perp)\Big]\ , \tag{9.19}$$

$$\|\nabla\alpha(\xi)\| \;\leq\; c\Big[\sup\{\|\beta(u)\| \ :\ |u_\perp| \leq |\xi_\perp|\} + b_{0,1}(\xi_\perp) + b_{1,2}(\xi_\perp)\Big]\ . \tag{9.20}$$

We apply this lemma to the magnetic 2-form $\delta\beta$ and we denote by $\delta\alpha$ the generating 1-form. We define $\alpha := \alpha_c + \delta\alpha$, then α generates the original magnetic field β, $\mathrm{d}\alpha = \beta$ and it is close to the linear gauge α_c of the constant field β_c using (9.18) and Lemma 9.2:

$$(\alpha - \alpha_c)(\xi) = \varepsilon b \ell \mathcal{O}_1^\ell(|\xi_\perp|^2)\ . \tag{9.21}$$

The norm of the left hand side is computed with respect to the standard metric.

Definition 9.3. *With the notations above, the Dirac operator*

$$\widetilde{\mathcal{D}} := \sum_{k=1}^{3} \sigma^k [-i\partial_{\xi_k} + \alpha_c(\partial_{\xi_k})] \tag{9.22}$$

with a constant field β_c in the $\mathrm{d}\xi^2$ metric will be called the approximating constant field Dirac operator along the field line.

By the properties of the coordinate vectorfields ∂_{ξ_k} and the orthonormal basis $\{e_1, e_2, e_3\}$ in the g_Ω metric from Lemma 8.2 and by the definitions (9.2), (9.4), (9.6) we have, for sufficiently small ε,

$$\mathcal{D}^\alpha = \boldsymbol{\sigma} \cdot \mathbf{\Pi}^\alpha = \widetilde{\mathcal{D}} + \sum_{k=1}^{3} \sigma^k [\alpha(e_k) - \alpha_c(\partial_{\xi_k})] + \sum_{k=1}^{3} \mathcal{K}_k(-i\partial_{\xi_k}) + \mathcal{M}_0 \;, \tag{9.23}$$

where $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$ and $\mathcal{K}_k, \mathcal{M}_0$ are 2 by 2 matrix valued functions. We use the bounds (8.7), (8.10), (8.14) and the estimate (9.21) to obtain, for $k = 1, 2, 3$,

$$\Big| [\alpha(e_k) - \alpha_c(\partial_{\xi_k})](\xi) \Big| \;=\; \varepsilon b \ell \mathcal{O}_1^\ell(|\xi_\perp|^2) \;. \tag{9.24}$$

We obtain from (9.23) and (9.24) that

$$\mathcal{D}^\alpha = \widetilde{\mathcal{D}} + \sum_{k=1}^{3} \mathcal{K}_k(-i\partial_{\xi_k}) + \mathcal{M} \tag{9.25}$$

with matrix valued functions that satisfy

$$\mathcal{M} = b\ell\mathcal{O}_1^\ell(|\xi_\perp|) + \ell^{-1}\mathcal{O}_1^\ell(1) \;, \quad \mathcal{K}_{1,2} = \mathcal{O}_1^\ell(|\xi_\perp|^\gamma) \;, \quad \mathcal{K}_3 = \mathcal{O}_1^\ell(|\xi_\perp|) \tag{9.26}$$

for any $\gamma \geq 0$ if $\varepsilon \leq \varepsilon(K)$. These estimates follow from Lemma 8.2, especially from the fact that $e_k = \partial_{\xi_k}$, $k = 1, 2$, apart from the region $\ell \leq |\xi_\perp| \leq 10\ell$, where (8.6) holds, i.e. $\mathcal{K}_1, \mathcal{K}_2$ are supported in this region.

10 Positive energy regime: Proof of Proposition 6.2

We first notice that both sides of (6.5) scale as ℓ^{-2}, hence it is sufficient to prove the result for $\ell := 1$. We can apply the constructions of Section 8 for the magnetic field $\mathbf{B}$ to obtain conformal factors Ω_j, orthonormal bases $\{e_1^{(j)}, e_2^{(j)}, e_3^{(j)}\}$, coordinate functions $\xi^{(j)} = (\xi_1^{(j)}, \xi_2^{(j)}, \xi_3^{(j)})$, spin-up projections $P_j^\uparrow$ and Gaussian localization functions v_j concentrated along the field line passing through y_j. We recall that y_j was a lattice with spacing $b^{-1/2}$ on the supporting plane (see Section 8.2).

We first estimate

$$\mathcal{D}^2 = \Big(1 - \varepsilon^{-2}\tfrac{1}{2b}\Big)\mathcal{D}^2 + \varepsilon^{-2}\tfrac{1}{2b}[\mathbf{p}_{\mathbf{A}}^2 + \boldsymbol{\sigma}\cdot\mathbf{B}] \geq \tfrac{1}{2}\mathcal{D}^2 + \varepsilon^{-2}\tfrac{1}{2b}\mathbf{p}_{\mathbf{A}}^2 - \varepsilon^{-2}$$

since $b = |\mathbf{B}_\infty| \geq \varepsilon^{-2}$ and $\sup|\mathbf{B}| \leq 2b$ if ε is sufficiently small.

Using this estimate and (8.20)–(8.21) we have

$$\begin{aligned} \mathcal{D}^2 + \mu\varepsilon^{-5} - M\chi^2 V \;&\geq\; \tfrac{1}{2}\Big(\mathcal{D}^2 + \varepsilon^{-2}\tfrac{1}{b}\mathbf{p}_{\mathbf{A}}^2 + \mu\varepsilon^{-5} - 2M\chi^2 V\Big) \\ &\geq\; c\eta\sum_j \Big(\mathcal{D}v_j^4\mathcal{D} + \varepsilon^{-2}\tfrac{1}{b}\mathbf{p}_{\mathbf{A}}\cdot v_j^2\mathbf{p}_{\mathbf{A}} + \mu\varepsilon^{-5}v_j - cM\chi^2 V v_j^6\Big) \end{aligned} \tag{10.1}$$

if ε is sufficiently small (depending on μ). Notice from the explicit formula (8.19) that

$$\Big|[\mathbf{p}_{\mathbf{A}}, v_j]\Big|^2 = |\nabla v_j|^2 \leq c\eta b v_j\ . \tag{10.2}$$

Therefore by Schwarz' inequality $\mathbf{p}_{\mathbf{A}}\cdot v_j^2\mathbf{p}_{\mathbf{A}} \geq \frac{1}{2}v_j\mathbf{p}_{\mathbf{A}}^2 v_j - c\eta b v_j$, and using this estimate in (10.1), including the negative error term into $\mu\varepsilon^{-5}v_j$ and subtracting the pointwise inequality (8.22) we obtain for any $0<\lambda<1$, $\eta\leq\eta(\lambda)$ (see Lemma 8.5) that

$$\begin{aligned} &\mathcal{D}^2 + \mu\varepsilon^{-5} - M\chi^2 V \\ &\geq\; c\eta\sum_j \Big(\mathcal{D}v_j^4\mathcal{D} + \varepsilon^{-2}\tfrac{1}{2b}v_j\mathbf{p}_{\mathbf{A}}^2 v_j - v_j^4 b(\lambda - \eta^2 b[\xi_\perp^{(j)}]^2)P_j^\uparrow + \tfrac{\mu}{2}\varepsilon^{-5}v_j - cM\chi^2 V v_j^6\Big)\ . \end{aligned} \tag{10.3}$$

The error term $C_0 v_j^4$ in (8.22) has been absorbed into $\mu\varepsilon^{-5}v_j$ if ε is small enough depending on η,μ. The inequality (8.22) has been subtracted to prepare for a later step. Hence

$$\begin{aligned} &\Big|\mathrm{Tr}\Big(\mathcal{D}^2 + \mu\varepsilon^{-5} - M\chi^2 V\Big)_-\Big| \\ &\leq\; c\eta\sum_j \Big|\mathrm{Tr}\Big(\mathcal{D}v_j^4\mathcal{D} + \varepsilon^{-2}\tfrac{1}{2b}v_j\mathbf{p}_{\mathbf{A}}^2 v_j - v_j^4 b(\lambda - \eta^2 b[\xi_\perp^{(j)}]^2)P_j^\uparrow + \tfrac{\mu}{2}\varepsilon^{-5}v_j - cM\chi^2 V v_j^6\Big)_-\Big| \end{aligned} \tag{10.4}$$

by $|\mathrm{Tr}(\sum_j H_j)_-| \leq \sum_j |\mathrm{Tr}(H_j)_-|$ that follows from the variational principle. The following lemma is the cylindrically localized version of Proposition 6.2.

Proposition 10.1. *With the notations above and setting $W := cM\chi^2 V$ we have*

$$\begin{aligned} &\Big|\mathrm{Tr}\Big(\mathcal{D}v_j^4\mathcal{D} + \varepsilon^{-2}\tfrac{1}{2b}v_j\mathbf{p}_{\mathbf{A}}^2 v_j - v_j^4 b(2^{-7} - \eta^2 b[\xi_\perp^{(j)}]^2)P_j^\uparrow + \tfrac{\mu}{2}\varepsilon^{-5}v_j - W v_j^6\Big)_-\Big| \\ &\qquad\leq\; c\int\Big((v_j^2 W)^{5/2} + b(v_j^2 W)^{3/2}\Big) \end{aligned} \tag{10.5}$$

for each j if ε is small enough depending on K,μ,η.

Choosing $\lambda := 2^{-7}$ and $\eta := \eta(2^{-7})$ (see Lemma 8.5), Proposition 6.2 directly follows from this proposition, from (10.4) and (8.20). $\square$

Proof of Proposition 10.1. The proof contains three transition steps that are performed locally around each field line from the grid constructed in Section 8.2. First we replace $\mathcal{D}$ with a Dirac operator $\mathcal{D}_\Omega$ in a metric that is conformal to the Euclidean one. The conformal factor Ω is chosen such that the strength of the magnetic field becomes constant along a field line. Second we replace the volume form $\mathrm{d}x$ with the volume form $\mathrm{d}\xi = \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2 \wedge \mathrm{d}\xi_3$, where ξ is the coordinate system associated with the chosen field line. Then we perform a gauge transformation so that $\mathcal{D}_\Omega$ becomes close to the Dirac operator $\widetilde{\mathcal{D}} := \boldsymbol{\sigma} \cdot (-i\partial_\xi + \alpha_c(\partial_\xi))$ with a constant magnetic field $\beta_c = \mathrm{d}\alpha_c = b\ \mathrm{d}\xi_1 \wedge \xi_2$ in the linear gauge $\alpha_c = \frac{b}{2}[\xi_1 \mathrm{d}\xi_2 - \xi_2 \mathrm{d}\xi_1]$. Finally, the operator $\widetilde{\mathcal{D}}$ can be analyzed explicitly.

For each fixed j we consider the constructions in Section 9 with the metric $g_j := \Omega_j^2 \mathrm{d}x^2$. We apply Section 9.2 to the Euclidean metric with the standard basis vectors $\{\partial_1, \partial_2, \partial_3\}$. The vectors $\{\Omega_j^{-1}\partial_1, \Omega_j^{-1}\partial_2, \Omega_j^{-1}\partial_3\}$ form an orthonormal basis in g_j. Let $\mathcal{D}_j^a$ and $\mathbf{\Pi}_j^a$ denote the corresponding Dirac operator and the vector of derivative operators as defined in (9.15) and (9.14). We recall that a is the 1-form dual to the vector potential $\mathbf{A}$ in the standard metric. Using the estimates in Lemma 8.2 to the formula (9.17), we obtain

$$\mathbf{\Pi}_j^a = \Omega_j^{-1}\mathbf{p_A} + \varepsilon\mathcal{O}_2(1)\ , \qquad \text{and} \quad \mathcal{D}_j^a = \Omega_j^{-1}\mathcal{D} + \varepsilon\mathcal{O}_2(1)\ , \tag{10.6}$$

where the error terms are functions. By Schwarz' inequality we obtain the following pointwise bound

$$|\mathbf{p_A}\psi|^2 \geq \tfrac{1}{8}|\mathbf{\Pi}_j^a\psi|^2 - c(K)\varepsilon^2|\psi|^2 \quad \text{and} \quad |\mathcal{D}\psi|^2 \geq \tfrac{1}{8}|\mathcal{D}_j^a\psi|^2 - c(K)\varepsilon^2|\psi|^2 \tag{10.7}$$

since $\frac{1}{2} \leq \Omega_j \leq 2$ if ε is sufficiently small. Therefore

$$v_j\mathbf{p}_\mathbf{A}^2 v_j \geq \tfrac{1}{8}v_j[\mathbf{\Pi}_j^a]^* \cdot \mathbf{\Pi}_j^a v_j - c(K)\varepsilon^2 v_j^2 \tag{10.8}$$

and

$$\mathcal{D}v_j^4\mathcal{D} \geq \tfrac{1}{8}[\mathcal{D}_j^a]^* v_j^4 \mathcal{D}_j^a - c(K)\varepsilon^2 v_j^4\ , \tag{10.9}$$

where star denotes the adjoint in the standard L^2-space. By applying the inequalities (10.8) and (10.9) we have

$$\begin{aligned} &\Big|\mathrm{Tr}\Big(\mathcal{D}v_j^4\mathcal{D} + \varepsilon^{-2}\tfrac{1}{2b}v_j\mathbf{p}_\mathbf{A}^2 v_j - v_j^4 b(2^{-7} - \eta^2 b[\xi_\perp^{(j)}]^2)P_j^\uparrow + \tfrac{\mu}{2}\varepsilon^{-5}v_j - Wv_j^6\Big)_-\Big| \qquad (10.10)\\ &\leq \ \tfrac{1}{8}\Big|\mathrm{Tr}\Big([\mathcal{D}_j^a]^* v_j^4\mathcal{D}_j^a + \varepsilon^{-2}\tfrac{1}{2b}v_j[\mathbf{\Pi}_j^a]^* \cdot \mathbf{\Pi}_j^a v_j - v_j^4 b(\tfrac{1}{16} - 8\eta^2 b[\xi_\perp^{(j)}]^2)P_j^\uparrow + \tfrac{\mu}{4}\varepsilon^{-5}v_j - 8Wv_j^6\Big)_-\Big|\ . \end{aligned}$$

The error terms in (10.8) and (10.9) have been absorbed into the $\frac{\mu}{2}\varepsilon^{-5}v_j$ term using $v_j \le 1$ if ε is sufficiently small.

The right hand side of (10.10) is invariant under an $SU(2)\times U(1)$ gauge transformation $\mathcal{U}_{R,\phi}$ as defined in (9.11). We shall choose R to be the rotation from $\{\Omega_j^{-1}\partial_1, \Omega_j^{-1}\partial_2, \Omega_j^{-1}\partial_3\}$ to the basis $\{e_1, e_2, e_3\}$ constructed in Lemma 8.2 and ϕ to be such that $\alpha = a + \mathrm{d}\phi$, where α is constructed in Section 9.3. In particular $P^\uparrow$ becomes $\sigma^\uparrow := \frac{1}{2}[1+\sigma^3]$ according to (9.12) since $\mathbf{n} = e_3$ along the central field line. Therefore the right hand side of (10.10) continues as

$$(10.10) = \frac{1}{8}\Big|\mathrm{Tr}\Big([\mathcal{D}^\alpha]^* v^4 \mathcal{D}^\alpha + \varepsilon^{-2}\frac{1}{2b}v[\mathbf{\Pi}^\alpha]^*\cdot\mathbf{\Pi}^\alpha v - v^4 b(\tfrac{1}{16} - 8\eta^2 b\xi_\perp^2)\sigma^\uparrow + \tfrac{\mu}{4}\varepsilon^{-5}v - 8Wv^6\Big)_-\Big| , \tag{10.11}$$

where we also omitted the j index for brevity, i.e. $\mathcal{D}^\alpha = \mathcal{D}_j^\alpha$, $\mathbf{\Pi}^\alpha = \mathbf{\Pi}_j^\alpha$, $v = v_j$ and $\xi_\perp = \xi_\perp^{(j)}$ for the rest of this section.

Now we translate our problem from the standard $L^2(\mathrm{d}x)$ space to the $L^2_\xi := L^2(\mathrm{d}\xi)$ space. We change the measure from the volume form $\mathrm{d}x$ to $\mathrm{d}\xi = \mathrm{d}\xi_1\wedge\mathrm{d}\xi_2\wedge\mathrm{d}\xi_3$. Since these two volume forms are comparable by a factor of at most 4 by (8.13) of Lemma 8.2 if ε is sufficiently small, we can use Lemma A.8 from the Appendix to obtain

$$(10.11) \le \frac{1}{8}\Big|\mathrm{Tr}_{L^2_\xi}\Big([\mathcal{D}^\alpha]^* v^4 \mathcal{D}^\alpha + \varepsilon^{-2}\frac{1}{2b}v[\mathbf{\Pi}^\alpha]^*\cdot\mathbf{\Pi}^\alpha v - v^4 b(\tfrac{1}{4} - 8\eta^2 b\xi_\perp^2)\sigma^\uparrow + \tfrac{\mu}{4}\varepsilon^{-5}v - 32Wv^6\Big)_-\Big|$$

where the trace and the adjoints are computed in the $L^2(\mathrm{d}\xi)$ space.

We remark that already on the right hand side of (10.10) it could have been natural to transform the trace on $L^2(\mathrm{d}x)$ to the trace on $L^2(\Omega_j^3\mathrm{d}x)$, according to Lemma A.8.

We introduce the function

$$G = G(\xi) := 1 + \sqrt{b}\min\{|\xi_\perp|, 1\} , \tag{10.12}$$

and we notice that

$$\sup_\xi G(\xi)^p v(\xi)^q \le c(p,q)\eta^{-p/2} , \quad p\ge 0,\ q > 0 . \tag{10.13}$$

We recall the definition of D_b and the decomposition (9.25) from Section 9.3. Since $\ell = 1$, the estimates (9.26) are translated into

$$\mathcal{M} = \mathcal{O}(G^2), \quad \|\nabla\mathcal{M}\| = \sqrt{b}\mathcal{O}(G), \quad \mathcal{K}_{1,2} = \mathcal{O}_1(|\xi_\perp|^\gamma), \quad \mathcal{K}_3 = \mathcal{O}_1(|\xi_\perp|) . \tag{10.14}$$

for any $\gamma \ge 0$. Here $\mathcal{O}(G^k)$ denotes the following class of functions on $\mathbf{R}^3$

$$\mathcal{O}(G^k) := \Big\{F(\xi) \ : \ \sup_\xi \frac{|F(\xi)|}{G^k(\xi)} < \infty\Big\} .$$

Hence

$$\int v^4|\mathcal{D}^\alpha\psi|\mathrm{d}\xi \geq \tfrac{1}{2}\int |v^2\widetilde{\mathcal{D}}\psi|^2\mathrm{d}\xi - c\sum_{k=1}^3\int |v^2\mathcal{K}_k\partial_{\xi_k}\psi|^2\mathrm{d}\xi - \int v^4 O(G^4)|\psi|^2\mathrm{d}\xi\ . \tag{10.15}$$

In the second term on the right hand side we first commute v through the derivative. Notice that $[\partial_{\xi_k}, v] = \eta b v\mathcal{O}(|\xi_\perp|)$ for $k=1,2$ and $[\partial_{\xi_3}, v] = 0$. Then we use the estimates (10.14) and (10.13) to obtain

$$\begin{aligned}\sum_{k=1}^3\int |v^2\mathcal{K}_k\partial_{\xi_k}\psi|^2\mathrm{d}\xi &\leq 2\sum_{k=1}^3\int v^2\|\mathcal{K}_k\|^2|\partial_{\xi_k}v\psi|^2\mathrm{d}\xi + 2\sum_{k=1}^2\int v^2\|\mathcal{K}_k\|^2[\eta b O(|\xi_\perp|)]^2|v\psi|^2\mathrm{d}\xi \\ &\leq cb^{-1}\sum_{k=1}^3\int |\partial_{\xi_k}v\psi|^2\mathrm{d}\xi + c\int v^2|\psi|^2\mathrm{d}\xi \end{aligned}\tag{10.16}$$

if ε is sufficiently small. From the last term in (10.15) we obtain a similar error term as in (10.16) using (10.13), hence

$$\int v^4|\mathcal{D}^\alpha\psi|\mathrm{d}\xi \geq \tfrac{1}{2}\int |v^2\widetilde{\mathcal{D}}\psi|^2\mathrm{d}\xi - cb^{-1}\sum_{k=1}^3\int |\partial_{\xi_k}v\psi|^2\mathrm{d}\xi - c\int v^2|\psi|^2\mathrm{d}\xi\ . \tag{10.17}$$

We also define for $k = 1, 2, 3$

$$\Pi_{\eta,k} := -i\partial_{\xi_k} + (1+2\eta)\alpha_c(\partial_{\xi_k})$$

and

$$\mathcal{D}_\eta := \boldsymbol{\sigma}\cdot\mathbf{\Pi}_\eta = \sum_{k=1}^3 \sigma^k\Pi_{\eta,k}$$

which is a Dirac operator with constant field $(1+2\eta)b\ \mathrm{d}\xi_1\wedge\mathrm{d}\xi_2$ in the $\mathrm{d}\xi^2$ metric. Notice that

$$\mathcal{D}_\eta v^2 = v^2\widetilde{\mathcal{D}} + 2i\eta b v^2(\sigma^1\xi_1 + \sigma^2\xi_2)\sigma^\uparrow\ . \tag{10.18}$$

This identity, called the *magnetic localization formula*, was introduced in [ES-II].

Hence, using (10.18) to continue (10.17), we obtain

$$\int v^4|\mathcal{D}^\alpha\psi|\mathrm{d}\xi \ \geq\ \tfrac{1}{4}\int |\mathcal{D}_\eta v^2\psi|^2\mathrm{d}\xi - 8\eta^2b^2\int v^4\xi_\perp^2|\sigma^\uparrow\psi|^2\mathrm{d}\xi \tag{10.19}$$

$$
\begin{aligned}
&-cb^{-1}\int\Big(\sum_{k=1}^{3}|\partial_{\xi_k}v\psi|^2\Big)\mathrm{d}\xi - c\int v^2|\psi|^2\mathrm{d}\xi\\
\geq\ & \tfrac{1}{8}\int|\mathcal{D}_\eta v^2\psi|^2\mathrm{d}\xi+\tfrac{b}{4}\int|\sigma^{\uparrow}v^2\psi|^2\mathrm{d}\xi-8\eta^2b^2\int v^4\xi_\perp^2|\sigma^{\uparrow}\psi|^2\mathrm{d}\xi\\
&-cb^{-1}\sum_{k=1}^{3}\int|\partial_{\xi_k}v\psi|^2\mathrm{d}\xi-c\int v^2|\psi|^2\mathrm{d}\xi\ .
\end{aligned}
$$

In the last step we used that $\mathcal{D}_\eta^2\geq 2b(1+2\eta)\sigma^{\uparrow}\geq 2b\sigma^{\uparrow}$, i.e., that on the spin-up subspace $\{\psi\ :\ \sigma^{\uparrow}\psi=\psi\}$ the constant field Pauli operator is bounded from below by twice of the constant field.

We shall control the second negative error term on the right hand side of (10.19) by the term $\varepsilon^{-2}(2b)^{-1}v[\mathbf{\Pi}^\alpha]^*\cdot\mathbf{\Pi}^\alpha v$. Notice that the following inequality holds pointwise

$$
|\mathbf{\Pi}^\alpha\psi|^2\geq\frac{1}{2}\Big(\sum_{k=1}^{3}|\partial_{e_k}\psi|^2\Big)-4\|\alpha(\xi)\|^2|\psi|^2-4(\sup\|\boldsymbol{\omega}\|)^2|\psi|^2
$$

using (9.2), (9.4) and (9.6). We can estimate $\|\alpha(\xi)\|\leq cb^{1/2}G(\xi)$ from (9.24) and the explicit choice of α_c. We also estimate $\|\boldsymbol{\omega}\|\leq cK\varepsilon$ by Lemma 8.2 and the same lemma is used to estimate the transition from $\sum_j|\partial_{e_j}\psi|^2$ to $\sum_j|\partial_{\xi_j}\psi|^2$. Therefore

$$
|\mathbf{\Pi}^\alpha\psi|^2\geq\frac{1}{4}\Big(\sum_{k=1}^{3}|\partial_{\xi_k}\psi|^2\Big)-cbG^2|\psi|^2\ ,
$$

if ε is sufficiently small, hence

$$
\frac{1}{2\varepsilon^2b}\int|\mathbf{\Pi}^\alpha v\psi|^2\mathrm{d}\xi\geq\frac{1}{8\varepsilon^2b}\int\Big(\sum_{k=1}^{3}|\partial_{\xi_k}v\psi|^2\Big)\mathrm{d}\xi-c\eta^{-1}\varepsilon^{-2}\int v^2|\psi|^2\mathrm{d}\xi \tag{10.20}
$$

using (10.13).

Combining (10.19) and (10.20) we obtain

$$
\begin{aligned}
&[\mathcal{D}^\alpha]^*v^4\mathcal{D}^\alpha+\varepsilon^{-2}\tfrac{1}{2b}v[\mathbf{\Pi}^\alpha]^*\cdot\mathbf{\Pi}^\alpha v-v^4b(\tfrac{1}{4}-8\eta^2b\xi_\perp^2)\sigma^{\uparrow}+\nu\varepsilon^{-5}v^2-32Wv^6\\
&\qquad\geq\tfrac{1}{8}v^2\mathcal{D}_\eta^2v^2-32Wv^6
\end{aligned}
$$

if ε is sufficiently small depending on ν,η and K. Since $\mathrm{Tr}(X^*HX)_-\leq\|X^*X\|\mathrm{Tr}H_-$,

$$
\begin{aligned}
\Big|\mathrm{Tr}\Big(\tfrac{1}{8}v^2\mathcal{D}_\eta^2v^2-32Wv^6\Big)_-\Big| &\leq \tfrac{1}{8}\Big|\mathrm{Tr}\Big(\mathcal{D}_\eta^2-256Wv^2\Big)_-\Big| \qquad (10.21)\\
&\leq c\int\Big(b(1+2\eta)(256Wv^2)^{3/2}+(256Wv^2)^{5/2}\Big)\mathrm{d}\xi\ ,
\end{aligned}
$$

where in the last step we used the Lieb-Thirring inequality for a constant magnetic field [LSY-II]. This completes the proof of Proposition 10.1. □

Remark. The reader may have found it confusing that along the proof of the positive energy regime we used the Birman-Schwinger principle (4.3) back and forth several times. It occured first in (4.3), (4.6), then in (6.7), (6.10), and finally, implicitly, in (10.21), when we referred to the Lieb-Thirring inequality with a constant magnetic field whose proof also relies on the Birman-Schwinger principle. The frequent changes back to an expression on the sum of the negative eigenvalues were purely for the purpose of compact presentation of the intermediate results. It would have been possible to use only (4.3) and stay with the resolvent language all the time since all estimates done for the operators are equally valid for the resolvents. In this case, of course, we could not have referred directly to the result of [LSY-II] on the constant field case, rather to the details of that proof.

11 Zero mode regime: Proof of Proposition 7.2

First notice that the inequalities in Proposition 7.2 are scale invariant in powers of ℓ; both sides of (7.5) scale like ℓ and both sides of (7.6) scale like ℓ^{-1}. Therefore we can set $\ell = 1$ for the proof. The arguments for weak magnetic fields and for extended (D, K)-regular fields are different.

11.1 Weak magnetic field

For weak fields (7.5) and (7.6) will be estimated by a universal constant c if $\varepsilon \leq \varepsilon(K)$. We need the following lemma:

Lemma 11.1. *Let X, Y be self-adjoint operators such that $X \geq 0$, $X + Y \geq 0$ and $\|Y\| \leq M$ for some constant $M > 0$. Then*

$$\frac{1}{(X+Y+2M)^2} \leq \frac{4}{(X+M)^2} \leq \frac{4}{X^2+M^2} \tag{11.1}$$

Proof. Consider the resolvent expansion

$$\frac{1}{X+Y+2M} = \frac{1}{X+M} - \frac{1}{X+M}(Y+M)\frac{1}{X+Y+2M}$$

hence by Schwarz' inequality

$$\frac{1}{(X+Y+2M)^2} \quad \leq \quad \frac{2}{(X+M)^2} + 2\frac{1}{X+M}(Y+M)\frac{1}{(X+Y+2M)^2}(Y+M)\frac{1}{X+M}$$

$$\begin{aligned}
&\le \frac{2}{(X+M)^2} + 2(2M)^{-2}\frac{1}{X+M}(Y+M)(Y+M)\frac{1}{X+M}\\
&\le \frac{4}{(X+M)^2}
\end{aligned}$$

since $(Y+M)^2 \le 2\|Y\|^2 + 2M^2 \le 4M^2$. By positivity of X we have $(X+M)^2 \ge X^2 + M^2$ which completes the proof. $\square$

If $\|\mathbf{B}\|_\infty \le c\varepsilon^{-2}$ then using Lemma 11.1 we obtain

$$R[P]^2(u,u) = \Big[(-i\nabla + \mathbf{A})^2 + \boldsymbol{\sigma}\cdot\mathbf{B} + \varepsilon^{-5}\Big]^{-2}(u,u) \le 4\Big[(-i\nabla + \mathbf{A})^2 + \tfrac{1}{2}\varepsilon^{-5}\Big]^{-2}(u,u)$$

if ε is sufficiently small. By the diamagnetic inequality we can continue this estimate as

$$R[P]^2(u,u) \le 4\Big[-\Delta + \tfrac{1}{2}\varepsilon^{-5}\Big]^{-2}(u,u) \le c\varepsilon^5 \ .$$

This proves (7.5).

For the proof of (7.6) we define a smooth function $0 \le \chi \le 1$ such that $\chi(u) = 1$, $|\nabla\chi| \le c$, $|\nabla^2\chi| \le c$ and $\mathrm{supp}(\chi)\cap\mathrm{supp}(\varphi) = \emptyset$. For brevity we set $R := R[P]$ with $P = \varepsilon^{-5}$. Since the inequality (7.6) is gauge invariant, we can choose the Poincaré gauge $\widehat{\mathbf{A}}$ centered at z_0 to generate the magnetic field. In particular, $\|\widehat{\mathbf{A}}\|_\infty \le c\varepsilon^{-2}$ since by assumption $\mathbf{B}$ is supported on $\widetilde{D}$ and $\|\mathbf{B}\|_\infty \le c\varepsilon^{-2}$.

Let $\{X,Y\} := XY + YX$ denote the anticommutator. Notice that $[R,\chi] = R\{\mathcal{D},[\mathcal{D},\chi]\}R$ and

$$\{\mathcal{D},[\mathcal{D},\chi]\} = (-i)\{\boldsymbol{\sigma}\cdot(-i\nabla), \boldsymbol{\sigma}\cdot\nabla\chi\} + (-i)\{\boldsymbol{\sigma}\cdot\widehat{\mathbf{A}}, \boldsymbol{\sigma}\cdot\nabla\chi)\} \ .$$

We can compactly write

$$\{\mathcal{D},[\mathcal{D},\chi]\} = \sum_{j=1}^{3}(-i\nabla + \widehat{\mathbf{A}})_j\widehat{\mathcal{K}}_j + \widehat{\mathcal{M}} \ ,$$

where the $\widehat{\mathcal{K}}_j$ and $\widehat{\mathcal{M}}$ are 2 by 2 matrix valued functions and from the estimate on $\widehat{\mathbf{A}}$ and the derivatives of χ we easily obtain that $\sup_x \|\widehat{\mathcal{K}}_j(x)\|, \|\widehat{\mathcal{M}}(x)\| \le c\varepsilon^{-2}$.

Therefore commuting χ through first, estimating $\varphi^2 \le 1$, then using $R\mathcal{D}^2R \le R$ and applying a Schwarz' inequality we get

$$\begin{aligned}
\chi R\mathcal{D}\varphi^2\mathcal{D}R\chi &= R\{\mathcal{D},[\mathcal{D},\chi]\}^* R\mathcal{D}\varphi^2\mathcal{D}R\{\mathcal{D},[\mathcal{D},\chi]\}R\\
&\le R\{\mathcal{D},[\mathcal{D},\chi]\}^* R\{\mathcal{D},[\mathcal{D},\chi]\}R\\
&\le 4R\widehat{\mathcal{M}}^*R\widehat{\mathcal{M}}R + 4\sum_{j=1}^{3}R\widehat{\mathcal{K}}_j^*(-i\nabla + \widehat{\mathbf{A}})_jR(-i\nabla + \widehat{\mathbf{A}})_j\widehat{\mathcal{K}}_jR \ . \qquad (11.2)
\end{aligned}$$

In the first term we use $R \le P^{-1} = \varepsilon^5$ for the middle resolvent then we use the boundedness of $\widehat{\mathcal{M}}$ to arrive at the resolvent square, R^2, that was estimated above in the proof of (7.5).

In the second term we use that $\mathcal{D}^2 + P = (-i\nabla + \widehat{\mathbf{A}})^2 + \boldsymbol{\sigma} \cdot \mathbf{B} + P \ge (-i\nabla + \widehat{\mathbf{A}})^2 + \frac{P}{2}$ since $\|\mathbf{B}\| \le c\varepsilon^{-2} \le \frac{P}{2} = \frac{1}{2}\varepsilon^{-5}$ if ε is sufficiently small, therefore $(-i\nabla + \widehat{\mathbf{A}})_j R(-i\nabla + \widehat{\mathbf{A}})_j \le 1$. The estimate of the second term then can be completed by using $\|\widehat{\mathcal{K}}_j^* \widehat{\mathcal{K}}_j\| \le c\varepsilon^{-4}$ and referring to the estimate of the square of the resolvent in (7.5). This finishes the proof of Proposition 7.2 for the case of weak magnetic field (case (i)).

11.2 Strong magnetic field

Here we prove Proposition 7.2 for the case (ii). Throughout the proof we fix u and let $z = \pi(u)$ be its base point on the supporting plane $\mathcal{P}$. Consider the construction of Lemma 8.2, in particular the coordinate functions $\xi = (\xi_1, \xi_2, \xi_3)$ and the conformal factor Ω. We know that $\frac{1}{2} \le \Omega \le 2$ and $\|\nabla\Omega\|_\infty \le 1$ if ε is sufficiently small. Recall that we set $\ell = 1$, therefore the bounds on the right hand side of (7.5) and (7.6) saturate to $c(\varepsilon, K)b$ since $P = \varepsilon^{-5}$ and $|\mathbf{B}(u)|$ is comparable with $b := |\mathbf{B}_\infty| \ge 1$.

11.2.1 Transformation into good coordinates

Similarly to the positive energy regime in Section 10, we perform three transition steps. We first change $\mathcal{D}$ into $\mathcal{D}_\Omega^a := \Omega^{-2}\mathcal{D}\Omega$ and the underlying measure to $\mathrm{d}x$ to $\Omega^3\mathrm{d}x$, then we change the measure from $\Omega^3\mathrm{d}x$ to $\mathrm{d}\xi$ and finally we perform a gauge transformation. Recall that $\mathcal{D}_\Omega^a$ is self-adjoint on $L^2(\mathrm{d}s_\Omega^2)$ (see [ES-III]). We set $R_\Omega^a[P] := ([\mathcal{D}_\Omega^a]^2 + P)^{-1}$ and we assume that ε is sufficiently small so that $P \ge 2^9$.

Then Lemma A.9 from Appendix A.6 states that

$$\mathrm{tr}\ R[P]^2(u,u) \le 2^9\ \mathrm{tr}\ (R_\Omega^a[P])^2_{L^2(\Omega)}(u,u) \tag{11.3}$$

$$\mathrm{tr}\ \Big(R[P]\mathcal{D}\varphi^2\mathcal{D}R[P]\Big)(u,u)$$

$$\le 2^{12}\mathrm{tr}\ \Big(R_\Omega^a[P]\mathcal{D}_\Omega^a\varphi^2\mathcal{D}_\Omega^a R_\Omega^a[P]\Big)_{L^2(\Omega)}(u,u) + 2^{12}P\,\mathrm{tr}\ (R_\Omega^a[P])^2_{L^2(\Omega)}(u,u) \tag{11.4}$$

where the operator kernels on the right hand side are computed in the $L^2(\Omega) := L^2(\Omega^3\mathrm{d}x)\otimes\mathbf{C}^2$ space.

Gauge transformation of the form (9.11) leaves the diagonal elements of operator kernels invariant hence we can use the basis $\{e_1, e_2, e_3\}$ constructed in Lemma 8.2 and vector potential α constructed in Section 9.3 to express the right hand sides of (11.3), (11.4) using $\mathcal{D}_\Omega^\alpha$ instead of $\mathcal{D}_\Omega^a$ as in the proof of Proposition 10.1.

We recall that in this gauge the decomposition (9.25) holds, i.e.,

$$\mathcal{D}_{\Omega}^{\alpha} = \widetilde{\mathcal{D}} + \sum_{k=1}^{3} \mathcal{K}_k \partial_{\xi_k} + \mathcal{M} \tag{11.5}$$

with $\widetilde{\mathcal{D}}$ given in (9.22) and $\mathcal{K}_k$, $\mathcal{M}$ satisfy the estimates (10.14) with $\ell = 1$.

We apply Lemma A.7 from Appendix A.5 to compare operator kernels on the measure spaces with volume forms $\mathrm{d}\mu := \Omega^3 \mathrm{d}x$ and $\mathrm{d}\nu := \mathrm{d}\xi = \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2 \wedge \mathrm{d}\xi_3$. Since these two volume forms are comparable at every point by Lemma 8.2, we obtain from (A.33) that

$$\begin{aligned}
\mathrm{tr}\ (R_{\Omega}^{\alpha}[P])^2_{L^2(\Omega)}(u,u) &\leq c\ \mathrm{tr}\ (R_{\Omega}^{\alpha}[P]^* R_{\Omega}^{\alpha}[P])_{L^2(\mathrm{d}\xi)}(u,u) \\
\mathrm{tr}\ \Big(R_{\Omega}^{\alpha}[P] \mathcal{D}_{\Omega}^{\alpha} \varphi^2 \mathcal{D}_{\Omega}^{\alpha} R_{\Omega}^{\alpha}[P] \Big)_{L^2(\Omega)}(u,u) &\leq c\ \mathrm{tr}\ \Big(\Big[\varphi \mathcal{D}_{\Omega}^{\alpha} R_{\Omega}^{\alpha}[P] \Big]^* \varphi \mathcal{D}_{\Omega}^{\alpha} R_{\Omega}^{\alpha}[P] \Big)_{L^2(\mathrm{d}\xi)}(u,u) \ ,
\end{aligned}$$

where the adjoints and the operator kernels on the right hand sides are computed in the $L^2(\mathrm{d}\xi) \otimes \mathbf{C}^2$ space.

Therefore case (ii) of Proposition 7.2 has been reduced to proving that for $\varepsilon \leq \varepsilon(K)$

$$\mathrm{tr}\ \Big[R_{\Omega}^{\alpha}[P]^* R_{\Omega}^{\alpha}[P] \Big](u,u) \leq cb \ , \tag{11.6}$$

$$\mathrm{tr}\ \Big[(\varphi \mathcal{D}_{\Omega}^{\alpha} R_{\Omega}^{\alpha}[P])^* \varphi \mathcal{D}_{\Omega}^{\alpha} R_{\Omega}^{\alpha}[P] \Big](u,u) \leq cb \ , \tag{11.7}$$

where the adjoints and the operator kernels are computed on $L^2(\mathrm{d}\xi) \otimes \mathbf{C}^2$.

Remark: The operator $\mathcal{D}_{\Omega}^{\alpha}$ in general is not self-adjoint in $L^2(\mathrm{d}\xi) \otimes \mathbf{C}^2$, but $\widetilde{\mathcal{D}}$ is.

11.2.2 Proof of (11.6) and (11.7)

In this proof we will omit Ω and α from the notation of $\mathcal{D}_{\Omega}^{\alpha}$ and we will simply use $\mathcal{D}$ for this operator. This should not be confused with the notation $\mathcal{D}$ (see (1.2) with $h = 1$) used elsewhere in the paper.

We recall that $\widetilde{\mathcal{D}}$ denotes the Dirac operator with a constant field (see (9.22)). We also need the notations $\mathbf{\Pi} = (\Pi_1, \Pi_2, \Pi_3)$, $\widetilde{\mathbf{\Pi}} = (\widetilde{\Pi}_1, \widetilde{\Pi}_2, \widetilde{\Pi}_3)$ from $\mathcal{D} = \boldsymbol{\sigma} \cdot \mathbf{\Pi}$, $\widetilde{\mathcal{D}} = \boldsymbol{\sigma} \cdot \widetilde{\mathbf{\Pi}}$. We note that $\widetilde{\mathcal{D}}$ can be decomposed as

$$\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}_{\perp} + N \tag{11.8}$$

with

$$\widetilde{\mathcal{D}}_{\perp} := \sum_{j=1}^{2} \sigma^j \widetilde{\Pi}_j \qquad \text{and} \qquad N := \sigma^3 \widetilde{\Pi}_3 = \sigma^3(-i\partial_{\xi_3}) \ . \tag{11.9}$$

These operators are self-adjoint on $L^2(\mathrm{d}\xi^2)$ and

$$[\widetilde{\mathcal{D}}_\perp, \widetilde{\Pi}_3] = 0, \qquad \{\widetilde{\mathcal{D}}_\perp, N\} = 0, \qquad \text{and} \qquad \widetilde{\mathcal{D}}^2 = \widetilde{\mathcal{D}}_\perp^2 + N^2 = \widetilde{\mathcal{D}}_\perp^2 + \widetilde{\Pi}_3^2 \,. \tag{11.10}$$

We need two decompositions of the error term $\mathcal{E} := \mathcal{D} - \widetilde{\mathcal{D}}$ as in (9.25)

$$\mathcal{E} = \mathbf{\Pi} \cdot \mathcal{K} + \mathcal{M} \tag{11.11}$$

$$\mathcal{E} = \widetilde{\mathbf{\Pi}} \cdot \widetilde{\mathcal{K}} + \widetilde{\mathcal{M}} \,. \tag{11.12}$$

Here $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$ is a vector of 2 by 2 matrices and $\mathbf{\Pi} \cdot \mathcal{K} := \sum_{j=1}^3 \Pi_j \mathcal{K}_j$. The matrices $\mathcal{K}_j, \mathcal{M}$ satisfy (10.14) with $\ell = 1$ and the same estimates hold for $\widetilde{\mathcal{K}}_j, \widetilde{\mathcal{M}}$ as well. The following estimates are straightforward from (10.14) and (9.21) if $\varepsilon \leq \varepsilon(K)$

$$[\Pi_j, \mathcal{K}_k] = \mathcal{O}(1), \qquad [\Pi_j, \mathcal{M}] = b^{1/2}\mathcal{O}(G), \qquad j,k = 1,2,3 \,, \tag{11.13}$$

and the same holds for $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{M}}$. In particular the following relations also hold:

$$\mathcal{E} = \mathcal{K} \cdot \mathbf{\Pi} + \mathcal{M}_0 \,, \quad \mathcal{E} = \widetilde{\mathcal{K}} \cdot \widetilde{\mathbf{\Pi}} + \widetilde{\mathcal{M}}_0 \tag{11.14}$$

with $\mathcal{M}_0, \widetilde{\mathcal{M}}_0 = \mathcal{O}(G^2)$.

The following lemma collects various operator inequalities related to the diamagnetic inequality. The proof is postponed to Section A.7.

Lemma 11.2. *With the notations above we have the following operator inequalities in the space $L^2(\mathrm{d}\xi) \otimes \mathbf{C}^2$ if $\varepsilon \leq \varepsilon(K)$:*

$$\Pi_j^* \Big(\frac{1}{\mathcal{D}^2 + P} \Big)^* \Big(\frac{1}{\mathcal{D}^2 + P} \Big) \Pi_j \;\leq\; cb \,, \tag{11.15}$$

$$\Pi_j^* \Big(\frac{\mathcal{D}}{\mathcal{D}^2 + P} \Big)^* \Big(\frac{\mathcal{D}}{\mathcal{D}^2 + P} \Big) \Pi_j \;\leq\; cb \,, \qquad j = 1,2,3 \tag{11.16}$$

$$\Pi_k^* \Pi_j^* \Big(\frac{1}{\mathcal{D}^2 + P} \Big)^* \frac{1}{\mathcal{D}^2 + P} \Pi_j \Pi_k \;\leq\; cb^2, \qquad j,k = 1,2,3 \tag{11.17}$$

$$\mathcal{E}^* \Big(\frac{1}{\mathcal{D}^2 + P} \Big)^* \frac{1}{\mathcal{D}^2 + P} \mathcal{E} \;\leq\; \mathcal{O}(G^4) \tag{11.18}$$

$$\mathcal{E}^* \Big(\frac{\mathcal{D}}{\mathcal{D}^2 + P} \Big)^* \frac{\mathcal{D}}{\mathcal{D}^2 + P} \mathcal{E} \;\leq\; \mathcal{O}(G^4) \,. \tag{11.19}$$

For the constant field operator we have

$$\widetilde{\Pi}_j \frac{1}{\widetilde{\mathcal{D}}^2 + P} \widetilde{\Pi}_j \leq cb \,, \quad j = 1,2,3. \tag{11.20}$$

The next lemma estimates the diagonal elements of explicitly computable operators with a constant magnetic field. The proof is given in Section A.8. We set $H(\xi) := \min\{|\xi_\perp|, 1\}$ and we recall that $G = 1 + \sqrt{b}H$.

Lemma 11.3. *With the notations above and for any constants $P \geq 1$, $b \geq 1$ with $P \leq cb$ we have*

$$\operatorname{tr} \frac{1}{(\widetilde{\mathcal{D}}^2 + P)^2}(u,u) \leq cb \,. \tag{11.21}$$

For any $k = 1, 2 \ldots$, $m = 1, 2, \ldots$ and for any 2 by 2 matrix valued function $\mathcal{F}$ with $\|\mathcal{F}(x)\| = \mathcal{O}(G(x))$ we also have

$$\begin{aligned}
\operatorname{tr} \frac{1}{\widetilde{\mathcal{D}}^2 + P} G^{2m} \frac{1}{\widetilde{\mathcal{D}}^2 + P}(u,u) &\leq c(m)b \,, && (11.22)\\
\operatorname{tr} \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P}[\mathcal{F}^k]^* \frac{1}{\widetilde{\mathcal{D}}^2 + P}\mathcal{F}^k \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P}(u,u) &\leq c(k)b \,, && (11.23)\\
\operatorname{tr} \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P} H^{2m} \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P}(u,u) &\leq c(m)b^{1-m} \,, && (11.24)\\
\operatorname{tr} \frac{\widetilde{\Pi}_j}{\widetilde{\mathcal{D}}^2 + P} H^{2m} \frac{\widetilde{\Pi}_j}{\widetilde{\mathcal{D}}^2 + P}(u,u) &\leq c(m)b^{2-m} \,, \qquad j = 1,2,3 \,. && (11.25)
\end{aligned}$$

Let $\widetilde{\mathcal{W}}$ denote either the identity I, or $\widetilde{\mathcal{D}}$, or $\widetilde{\Pi}_3$, then

$$\operatorname{tr} \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P}[\mathcal{F}^k]^* \frac{\widetilde{\mathcal{W}}}{\widetilde{\mathcal{D}}^2 + P} G^{2m} \frac{\widetilde{\mathcal{W}}}{\widetilde{\mathcal{D}}^2 + P}\mathcal{F}^k \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P}(u,u) \leq c(k,m)b \tag{11.26}$$

(we recall that $[\widetilde{\Pi}_3, \widetilde{\mathcal{D}}] = 0$). Let $\widetilde{\mathcal{U}}$ denote either the identity I or $\widetilde{\mathcal{D}}$ or $\widetilde{\Pi}_j$, $j = 1, 2, 3$, and let $0 \leq \varphi \leq 1$ be a function with dist$(u, supp(\varphi)) \geq 1$, then

$$\begin{aligned}
\operatorname{tr} \frac{\widetilde{\mathcal{U}}}{\widetilde{\mathcal{D}}^2 + P}\varphi^2 \frac{\widetilde{\mathcal{U}}}{\widetilde{\mathcal{D}}^2 + P}(u,u) &\leq ce^{-c\sqrt{b}} \,, && (11.27)\\
\operatorname{tr} \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P}[\mathcal{F}^k]^* \frac{\widetilde{\mathcal{U}}}{\widetilde{\mathcal{D}}^2 + P} G^m \varphi^2 G^m \frac{\widetilde{\mathcal{U}}}{\widetilde{\mathcal{D}}^2 + P}\mathcal{F}^k \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P}(u,u) &\leq c(k,m)e^{-c\sqrt{b}} \,. && (11.28)
\end{aligned}$$

Armed with these lemmas, we complete the proof of (11.6) and (11.7). We start with (11.6). We introduce the notation $(\cdots)^*A$ for A^*A if A is a long expression. All adjoints are computed in the L^2_ξ space.

We use $\mathcal{D}^2 = \widetilde{\mathcal{D}}^2 + \mathcal{D}\mathcal{E} + \mathcal{E}\widetilde{\mathcal{D}}$ in the following resolvent expansion:

$$\Big(\cdots\Big)^* \frac{1}{\mathcal{D}^2 + P} \le 3\Big((A) + (B) + (C)\Big)$$

with

$$\begin{aligned}
(A) &:= \Big(\cdots\Big)^* \frac{1}{\widetilde{\mathcal{D}}^2 + P} \\
(B) &:= \Big(\cdots\Big)^* \frac{1}{\mathcal{D}^2 + P}\mathcal{D}\mathcal{E}\frac{1}{\widetilde{\mathcal{D}}^2 + P} \\
(C) &:= \Big(\cdots\Big)^* \frac{1}{\mathcal{D}^2 + P}\mathcal{E}\widetilde{\mathcal{D}}\frac{1}{\widetilde{\mathcal{D}}^2 + P} \ .
\end{aligned}$$

Term (A) is explicit from (11.21). In term (B) we first use (11.19) in the middle to arrive at (11.22) with $m = 2$.

In term (C) we use $\mathcal{E} = \widetilde{\mathbf{\Pi}} \cdot \widetilde{\mathcal{K}} + \widetilde{\mathcal{M}}$ and we expand the resolvent in the middle once more. The result is

$$(C) \le 4\Big((C1) + (C2) + (C3) + (C4)\Big)$$

with

$$\begin{aligned}
(C1) &:= \Big(\cdots\Big)^* \frac{1}{\widetilde{\mathcal{D}}^2 + P}\widetilde{\mathcal{M}}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P} \\
(C2) &:= \Big(\cdots\Big)^* \frac{1}{\mathcal{D}^2 + P}(\mathcal{D}\mathcal{E} + \mathcal{E}\widetilde{\mathcal{D}})\frac{1}{\widetilde{\mathcal{D}}^2 + P}\widetilde{\mathcal{M}}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P} \\
(C3) &:= \Big(\cdots\Big)^* \frac{1}{\widetilde{\mathcal{D}}^2 + P}\widetilde{\mathbf{\Pi}} \cdot \widetilde{\mathcal{K}}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P} \\
(C4) &:= \Big(\cdots\Big)^* \frac{1}{\mathcal{D}^2 + P}(\mathcal{D}\mathcal{E} + \mathcal{E}\widetilde{\mathcal{D}})\frac{1}{\widetilde{\mathcal{D}}^2 + P}\widetilde{\mathbf{\Pi}} \cdot \widetilde{\mathcal{K}}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P} \ .
\end{aligned}$$

Term (C1) is explicit from (11.23) after estimating one of the resolvents in the middle by $P^{-1} \le 1$. Term (C2) is split into two terms,

$$(C2) \le 2\Big((C21) + (C22)\Big) \ ,$$

with

$$\begin{aligned}
(C21) &:= \Big(\cdots\Big)^* \frac{1}{\mathcal{D}^2 + P}\mathcal{D}\mathcal{E}\frac{1}{\widetilde{\mathcal{D}}^2 + P}\widetilde{\mathcal{M}}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P} \\
(C22) &:= \Big(\cdots\Big)^* \frac{1}{\mathcal{D}^2 + P}\mathcal{E}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P}\widetilde{\mathcal{M}}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P}
\end{aligned}$$

In (C21) we use (11.19) and finally (11.26) with $\widetilde{\mathcal{W}} = I$, $m = k = 2$. In (C22) we use (11.18) first then (11.26) with $\widetilde{\mathcal{W}} = \widetilde{\mathcal{D}}$, $m = k = 2$. In the term (C3) we use (11.20) then (11.24) with $m = 1$ together with the estimates (10.14) used for $\widetilde{\mathcal{M}}$.

Finally, for the term (C4) we estimate

$$(C4) \le 2\Big((C41) + (C42)\Big)$$

$$(C41) := \Big(\cdots\Big)^* \frac{1}{\mathcal{D}^2 + P}\mathcal{D}\mathcal{E}\frac{1}{\widetilde{\mathcal{D}}^2 + P}\widetilde{\mathbf{\Pi}} \cdot \widetilde{\mathcal{K}}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P}$$

$$(C42) := \Big(\cdots\Big)^* \frac{1}{\mathcal{D}^2 + P}\mathcal{E}\widetilde{\mathcal{D}}\frac{1}{\widetilde{\mathcal{D}}^2 + P}\widetilde{\mathbf{\Pi}} \cdot \widetilde{\mathcal{K}}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P}$$

In (C41) we first use (11.19) to arrive at

$$(C41) \le c\Big(\cdots\Big)^* G^2 \frac{1}{\widetilde{\mathcal{D}}^2 + P}\widetilde{\mathbf{\Pi}} \cdot \widetilde{\mathcal{K}}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P} \ . \tag{11.29}$$

For the term (C42) we first observe the following inequality

Lemma 11.4. *With the notations above*

$$\Big(\cdots\Big)^* \frac{1}{\mathcal{D}^2 + P}\mathcal{E}\widetilde{\mathcal{D}} \le b\mathcal{O}(G^8) \ . \tag{11.30}$$

Proof of Lemma 11.4. We can write $\widetilde{\mathcal{D}} = \mathbf{\Pi} \cdot \widehat{\mathcal{N}} + \widehat{\mathcal{M}}$, with $\widehat{\mathcal{N}}, \widehat{\mathcal{M}} = O(G^2)$. Therefore

$$\mathcal{E}\widetilde{\mathcal{D}} = (\mathbf{\Pi} \cdot \mathcal{K} + \mathcal{M})(\mathbf{\Pi} \cdot \widehat{\mathcal{N}} + \widehat{\mathcal{M}}) = \sum_{j,k=1}^{3} \Pi_j \Pi_k b^{-1/2}\mathcal{O}(G^4) + \sum_{j=1}^{3} \Pi_j \mathcal{O}(G^4) + b^{1/2}\mathcal{O}(G^4)$$

after commuting $\mathbf{\Pi}$ through using (11.13) and (10.14). Therefore (11.30) follows from (11.17) and (11.15). □

Armed with Lemma 11.4, we see that

$$(C42) \le (C43) := cb\Big(\cdots\Big)^* G^4 \frac{1}{\widetilde{\mathcal{D}}^2 + P}\widetilde{\mathbf{\Pi}} \cdot \widetilde{\mathcal{K}}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P} \ . \tag{11.31}$$

Since $G^2 \le G^4 \le G^8$ and $b \ge 1$, it is sufficient to estimate (C43) that will complete the estimate of (C41) from (11.29) as well.

To estimate (C43), we first separate the term $\widetilde{\mathbf{\Pi}} \cdot \widetilde{\mathcal{K}}$ into terms containing $\widetilde{\Pi}_3 \mathcal{K}_3$ and $\widetilde{\Pi}_\perp \mathcal{K}_\perp := \widetilde{\Pi}_1 \mathcal{K}_1 + \widetilde{\Pi}_2 \mathcal{K}_2$ by Schwarz' inequality. We arrive at

$$(C43) \leq c\Big(\cdots\Big)^* G^4 \frac{1}{\widetilde{\mathcal{D}}^2 + P} \widetilde{\Pi}_3 (b^{1/2} \widetilde{\mathcal{K}}_3) \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P} + cb\Big(\cdots\Big)^* G^4 \frac{1}{\widetilde{\mathcal{D}}^2 + P} \widetilde{\Pi}_\perp \widetilde{\mathcal{K}}_\perp \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P} \, .$$

The first term is explicit from (11.26) with $\widetilde{\mathcal{W}} = \widetilde{\Pi}_3$, $m = 2$, $k = 1$, using that $b^{1/2} \widetilde{\mathcal{K}}_3 = \mathcal{O}(G)$. In the second term we estimate $G^8 \leq b^4$ in the middle, then estimate one of the resolvents by $P^{-1} \leq 1$ and first use (11.20) and finally (11.24) with $m = 5$, together with (10.14) for $\widetilde{\mathcal{K}}$. This completes the proof of (11.6).

Now we prove (11.7). We need the following lemma.

Lemma 11.5. *With the notations above, we have*

$$\operatorname{tr} \Big(\cdots\Big)^* \varphi \mathcal{E} \frac{1}{\widetilde{\mathcal{D}}^2 + P} (u, u) \;\; \leq \;\; c \, e^{-c\sqrt{b}} \, , \tag{11.32}$$

$$\operatorname{tr} \Big(\cdots\Big)^* \varphi \mathcal{E} \frac{1}{\widetilde{\mathcal{D}}^2 + P} \mathcal{M} \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P} (u, u) \;\; \leq \;\; c \, e^{-c\sqrt{b}} \, . \tag{11.33}$$

Proof. For the proof of both inequalities (11.32) we write $\mathcal{E} = \widetilde{\mathcal{K}} \cdot \widetilde{\mathbf{\Pi}} + \widetilde{\mathcal{M}}$, then we separate the terms by a Schwarz' inequality and we use (11.27) and (11.28), respectively, with appropriately chosen $\widetilde{\mathcal{U}}$. □

For the operator on the left hand side of (11.7) we use a resolvent expansion and a Schwarz' inequality to obtain

$$\Big(\cdots\Big)^* \varphi \frac{\mathcal{D}}{\mathcal{D}^2 + P} \leq 3\Big((D) + (E) + (F)\Big)$$

with

$$\begin{aligned} (D) \;&:=\; \Big(\cdots\Big)^* \varphi \mathcal{D} \frac{1}{\widetilde{\mathcal{D}}^2 + P} \\ (E) \;&:=\; \Big(\cdots\Big)^* \varphi \frac{\mathcal{D}}{\mathcal{D}^2 + P} \mathcal{D} \mathcal{E} \frac{1}{\widetilde{\mathcal{D}}^2 + P} \\ (F) \;&:=\; \Big(\cdots\Big)^* \varphi \frac{\mathcal{D}}{\mathcal{D}^2 + P} \mathcal{E} \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P} \, . \end{aligned}$$

The estimate of (D) is trivial by $\mathcal{D} = \widetilde{\mathcal{D}} + \mathcal{E}$ applying a Schwarz' inequality and using (11.27) and (11.32) for these two terms, respectively.

In term (E) we use

$$\frac{\mathcal{D}}{\mathcal{D}^2+P}\mathcal{D} = I - P\frac{1}{\mathcal{D}^2+P} \tag{11.34}$$

and separate it by a Schwarz' inequality:

$$(E) \le 2\Big(\cdots\Big)^*\varphi\mathcal{E}\frac{1}{\widetilde{\mathcal{D}}^2+P} + 2P^2\Big(\cdots\Big)^*\varphi\frac{1}{\mathcal{D}^2+P}\mathcal{E}\frac{1}{\widetilde{\mathcal{D}}^2+P}\,.$$

For the first term we can use (11.32), for the second one we use $\varphi \le 1$, (11.18) then (11.22) with $m=2$.

Finally for the term (F) we write

$$(F) \le 2\Big((F1)+(F2)\Big)$$

with

$$(F1) := \Big(\cdots\Big)^*\varphi\frac{\mathcal{D}}{\mathcal{D}^2+P}\mathbf{\Pi}\cdot\mathcal{K}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2+P}$$

$$(F2) := \Big(\cdots\Big)^*\varphi\frac{\mathcal{D}}{\mathcal{D}^2+P}\mathcal{M}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2+P}\,.$$

For (F1) we first estimate $\varphi \le 1$, then use (11.16) and (11.24) with $m=1$ together with (10.14).

For (F2) we need one more resolvent expansion:

$$(F2) \le 2\Big((F21)+(F22)\Big)$$

with

$$\begin{aligned}(F21) \;&:=\; \Big(\cdots\Big)^*\varphi\mathcal{D}\frac{1}{\widetilde{\mathcal{D}}^2+P}\mathcal{M}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2+P}\\ (F22) \;&:=\; \Big(\cdots\Big)^*\varphi\frac{\mathcal{D}}{\mathcal{D}^2+P}(\mathcal{D}\mathcal{E}+\mathcal{E}\widetilde{\mathcal{D}})\frac{1}{\widetilde{\mathcal{D}}^2+P}\mathcal{M}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2+P}\,.\end{aligned}$$

We split (F21) further by using $\mathcal{D} = \widetilde{\mathcal{D}} + \mathcal{E}$:

$$(F21) \le 2\Big(\cdots\Big)^*\varphi\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2+P}\mathcal{M}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2+P} + 2\Big(\cdots\Big)^*\varphi\mathcal{E}\frac{1}{\widetilde{\mathcal{D}}^2+P}\mathcal{M}\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2+P}$$

The first term is explicit by (11.23) after $\varphi \le 1$ and estimating $\mathcal{D}^2$ by the resolvent. The second term was estimated in (11.33).

Finally, to estimate (F22), we use again (11.34) and we split it as follows

$$
\begin{aligned}
(F22) \;\leq\; & 3\Big(\cdots\Big)^* \varphi \mathcal{E} \frac{1}{\widetilde{\mathcal{D}}^2+P} \mathcal{M} \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2+P} \\
& +3P^2\Big(\cdots\Big)^* \varphi \frac{1}{\mathcal{D}^2+P} \mathcal{E} \frac{1}{\widetilde{\mathcal{D}}^2+P} \mathcal{M} \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2+P} \\
& +3\Big(\cdots\Big)^* \varphi \frac{\mathcal{D}}{\mathcal{D}^2+P} \mathcal{E} \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2+P} \mathcal{M} \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2+P} \; .
\end{aligned}
$$

The first term is estimated in (11.33). For the second term we use $\varphi \leq 1$ and (11.18) first, then (11.26) with $\widetilde{\mathcal{W}} = 1$, $m = k = 2$. For the third term we use $\varphi \leq 1$ and (11.19) first, then (11.26) with $\widetilde{\mathcal{W}} = \widetilde{\mathcal{D}}$, $m = k = 2$.

This completes the proof of (11.7). $\square$

A Proof of the technical lemmas

A.1 Proof of Proposition 2.3 on the tempered lengthscale

Proof. We recall the definitions of $B_L(x), b_L(x)$ from (2.1), (2.2) and we notice that $B_L(x)$ is increasing in L, while $b_L(x)$ is decreasing. Since $\mathbf{B}$ and its derivatives are locally bounded and $\mathbf{B}$ is not constant everywhere, we easily obtain that the sets appearing on the right hand sides of (2.3) and (2.4) are non-empty and bounded. Therefore L_m, L_s and L_c are positive finite valued functions.

We notice that if $B_{L(x)}(x) > L^{-2}(x)$ then $L_m(x) < L(x) < L_c(x)$, i.e. $L_c(x) = L_v(x)$. We claim that $L_c(x) = L_v(x)$ implies that (2.8) and (2.9) hold even if $L(x)$ is replaced with $L_c(x)$ which is a stronger statement. The validity of this stronger form of (2.8) follows directly from (2.4) and $L_v(x) \le L_s(x)$. This also implies that

$$B_{L_c(x)}(x) - b_{L_c(x)}(x) \le 2L_c(x) \cdot \sup\{|\nabla \mathbf{B}(y)| \; : \; |x-y| \le L_c(x)\} \le 2b_{L_c(x)}(x) \; ,$$

therefore $b_{L_c(x)}(x) \ge \frac{1}{3}B_{L_c(x)}(x)$, in particular $\mathbf{B}(y) \ne 0$ and $\mathbf{n}(y)$ is well defined for all y with $|y-x| \le L_c(x)$. This proves (2.9) with $L(x)$ replaced with $L_c(x)$ from the definition (2.5) and from $L_c(x) \le L_n(x)$. We also proved that if $L_c(x) = L_v(x)$ then the suprema in (2.3), (2.4) and (2.5) are actually maxima by the continuity of $\mathbf{B}$ and its derivatives and by $\mathbf{B} \ne 0$.

Finally, we have to show that $L(x)$ is tempered. Notice that it is sufficient to show that

$$|x-y| \le L(x) \Longrightarrow \frac{1}{2} \le \frac{L(y)}{L(x)} \tag{A.1}$$

for any $x, y \in \mathbf{R}^3$ because the inequality $L(y)/L(x) \le 2$ easily follows from this. To see it, we assume that $L(y) > 2L(x)$. Then $|x-y| \le L(x)$ implies $|x-y| \le L(y)$, so using (A.1) with x, y interchanged we arrive at a contradiction.

Now we show that (A.1) holds. Let x, y be two points with $|x-y| \le L(x) = \frac{1}{2}L_c(x)$ and we have to show that $L(x) \le 2L(y) = L_c(y)$. This is obvious if $B_{L(x)}(y) \le L(x)^{-2}$, since then $L(x) \le L_m(y)$ and $L_m(y) \le L_c(y)$ by definition.

Thus we can assume that $B_{L(x)}(y) > L(x)^{-2}$. Since $|x-y| \le L(x)$, we know that

$$\Big\{ z \; : \; |y-z| \le L(x) \Big\} \subset \Big\{ z \; : \; |x-z| \le 2L(x) = L_c(x) \Big\} \; , \tag{A.2}$$

thus $B_{L(x)}(y) \le B_{L_c(x)}(x)$, hence $B_{L_c(x)}(x) > L(x)^{-2} > L_c(x)^{-2}$, i.e. $L_c(x) > L_m(x)$, so $L_c(x) = L_v(x)$.

We will now check that for $\gamma = 1, \dots 4$

$$L(x)^\gamma \sup \Big\{ \Big| \nabla^\gamma |\mathbf{B}(z)| \Big| \; : \; |y-z| \le L(x) \Big\} \le b_{L(x)}(y) \tag{A.3}$$

and

$$L(x)^{\gamma} \sup\Big\{\Big|\nabla^{\gamma}\mathbf{n}(z)\Big| \ : \ |y-z| \leq L(x)\Big\} \leq 1 \tag{A.4}$$

hold, which will imply $L(x) \leq L_v(y)$, hence $L(x) \leq L_c(y)$. But as we showed above, $L_c(x) = L_v(x)$ implies that (2.8) and (2.9) hold with $L(x)$ replaced by $L_c(x)$. From (A.2) and $L(x) < L_c(x)$ we therefore immediately conclude (A.3), (A.4). $\square$

A.2 Proof of the covering Lemma 5.4.

Introduce the notation $D_x^* := B\Big(x, 40\ell(x)\Big)$ and $D_i^* := D_{x_i}^*$. Let S be any compact subset of $\mathbf{R}^3$. First we show how to find a finite set of points within S so that the balls $\widehat{D}_i$ cover S and they enjoy the finite overlapping property. Let $\overline{D}_x := B(x, \ell(x)/20)$ and we cover S by the collection of balls $\overline{D}_x$, $x \in S$. By compactness, we can choose points $\{x_\alpha\} \subset S$, with a finite index set $\alpha \in A$, such that the balls $\{\overline{D}_\alpha\}_{\alpha\in A}$ cover S. Now we discard certain points from the collection $\{x_\alpha\}$ and relabel the rest by $\{x_i\}$.

Let x_1 be the point with the biggest value $\ell(x_1)$ among all values $\{\ell(x_\alpha) \ : \ \alpha \in A\}$. Then let x_2 be the point with the biggest value $\ell(x_2)$ among all values $\ell(x_\alpha)$ such that $x_\alpha \in \mathbf{R}^3 \setminus \overline{D}_1$. Then let x_3 be the point with biggest value $\ell(x_3)$ among all values $\ell(x_\alpha)$ such that $x_\alpha \in \mathbf{R}^3 \setminus (\overline{D}_1 \cup \overline{D}_2)$, etc. until all x_α's are covered by $\overline{D}_i$'s. This selects a subcollection of the points $\{x_\alpha\}$ and they are relabelled to $x_1, x_2, \ldots$.

We claim that the collection of $\widehat{D}_i$'s cover S. Consider any $y \in S$, then $y \in \overline{D}_\alpha$ for some α. But x_α is covered by some $\overline{D}_i$. We choose the smallest such index i. By the maximality of the radii in the selection procedure, we know that $\ell(x_\alpha) \leq \ell(x_i)$, so $|y - x_i| \leq |y - x_\alpha| + |x_\alpha - x_i| \leq (\ell(x_\alpha) + \ell(x_i))/20 \leq \ell(x_i)/10$, hence $y \in \widehat{D}_i$.

We claim that the union of the D_i^* balls have the finite covering property with a sufficiently big universal N. From construction, the balls $D_i^{\#} := B(x_i, \ell(x_i)/40)$ are disjoint. Fix a point $y \in \mathbf{R}^3$ and let I be the set of indices i such that $y \in D_i^*$, $i \in I$. Choosing $\varepsilon < 1/40$, we see that $y \in D_i^*$, i.e., $|x_i - y| \leq 40\ell(x_i)$ implies $1/2 \leq \ell(y)/\ell(x_i) \leq 2$ for all $i \in I$. Hence the balls $D_i^{\#}$, $i \in I$, all have radius at least $\ell(y)/80$ and they are within a ball of radius $81\ell(y)$ about y. From their disjointness it follows that their number is universally bounded, i.e. the number of D_i^*'s covering any y is bounded by a universal number N.

This completes the construction of the covering balls for any compact set S satisfying (i) with a universal covering property.

We denote by $P(S)$ the points $\{x_1, x_2, \ldots\}$ obtained in this procedure and note that $P(S) \subset S$. Let $\widetilde{H}(S) := \bigcup_{i\in P(S)} \widetilde{D}_i$ and $H^*(S) := \bigcup_{i\in P(S)} D_i^*$

Now we show how to choose points in the whole space. Fix an arbitrary point x, and let $A_k := \{y \ : \ 4^k\ell(x) \leq |y - x| \leq 4^{k+1}\ell(x)\}$, $k = 1, 2, \ldots$, be a sequence of annuli. Clearly

$\widetilde{D}_x \cup \bigcup_k A_k = \mathbf{R}^3$. For each annulus we construct the points $P(A_k)$ defined above and we let

$$P := P(\widetilde{D}_x) \cup \bigcup_{k=1}^{\infty} P(A_k).$$

This will be our final set of points $\{x_1, x_2, \ldots\}$ after relabelling. It is clear that the balls $\widehat{D}_i = B(x_i, \ell(x_i)/10)$, $x_i \in P$, cover the space.

Next we prove the finite covering property for the balls $\{\widetilde{D}_{x_i} \; : \; x_i \in P\}$, i.e. that the number of balls that cover any given point of $\mathbf{R}^3$ is universally bounded. We need a lemma whose proof is given later.

Lemma A.1. *Fix any point $x \in \mathbf{R}^3$.*
(i) Let $L_k := \sup\{\ell(u) \; : \; u \in A_k\}$, $k \geq 1$, then $L_k \leq 4^{k+1}\ell(x)$.
(ii) If $y \in A_k$, then $y \notin H^*(A_m)$ for any $|m-k| \geq 5$.

Recall that for each m every point in $\mathbf{R}^3$ is covered by at most N balls $\widetilde{D}$ with center $z \in P(A_m)$ and similarly for balls with center in $P(\widetilde{D}_x)$. Hence (ii) of Lemma A.1 shows that any y is covered by at most $12N$ balls with center from P. This completes the proof of the finite covering property of the balls $\{\widetilde{D}_{x_i} \; : \; x_i \in P\}$.

Finally we show property (ii) of the Definition 5.3. If $\widetilde{D}_i \cap \widetilde{D}_j \neq \emptyset$, then ℓ_i, ℓ_j are comparable by (2.7). Therefore D_i^* covers x_j, but any point is covered only by finitely many D_i^*'s, hence $\widetilde{D}_j$ can be intersected by finitely many $\widetilde{D}_i$'s.
Proof of Lemma A.1. (i) Suppose that there exists $u \in A_k$ with $\ell(u) > 4^{k+1}\ell(x)$. Then $|x-u| \leq 4^{k+1}\ell(x) < \ell(u)$, hence $\ell(u) \leq 2\ell(x)$ by (2.7) which is a contradiction.

(ii) Suppose that there is a point $z \in P(A_m)$ such that $y \in D_z^*$, i.e., $|y-z| \leq 40\ell(z)$. Using (2.7) this implies $\ell(z) \leq 2\ell(y)$ assuming $\varepsilon < 1/40$. Hence $|y-z| \leq 80\ell(y) \leq 80 \cdot 4^{k+1}\ell(x)$ by (i). But $z \in A_m$, so $|y-z| \geq (4^m - 4^{k+1})\ell(x)$ which is a contradiction if $m \geq k+5$. Suppose now that $m \leq k-5$. Then $|y-z| \geq (4^k - 4^{m+1})\ell(x)$ which contradicts to $|y-z| \leq 40\ell(z) \leq 40 \cdot 4^{m+1}\ell(x)$. $\square$

A.3 Proof of the localization Proposition 7.1

As a preparation for the proof we define a distance function on the collection $\{x_i\}_{i \in I}$ obtained in Lemma 5.4. Let $i \in I$ be a fixed index. We define the following compact sets successively

$$S_0(i) := D_i \, ,$$

$$S_{k+1}(i) := \bigcup_{j\,:\,D_j\cap S_k(i)\neq\emptyset} \widetilde{D}_j \ ,$$

and we denote

$$m_k := \mathrm{card}\{j \in I \ : \ D_j \cap S_k(i) \neq \emptyset\} \ .$$

Lemma A.2. *(i) The sets $S_0(i), S_1(i), \ldots$ are increasing.*

(ii) Let $u \in S_k(i)$, $v \notin int(S_{k+1}(i))$, then $|u-v| > 4\ell(u)$.

(iii) $\bigcup_k S_k(i) = \mathbf{R}^3$.

(iv) $m_k \leq N^{k+1}$ with the universal constant N from Lemma 5.4.

(v) For sufficiently small ε and for any nonnegative function G such that $G(x)$ and $G(y)$ are comparable whenever $|x-y| \leq \varepsilon^{-1}\ell(x)$, we have

$$\sup_{S_k(i)} G \leq 2^k \sup_{S_0(i)} G \ . \tag{A.5}$$

Proof. For simplicity, we omit i from the arguments since i is fixed.

(i) Since the balls $\{D_j\}$ cover $\mathbf{R}^3$ and $D_j \subset \widetilde{D}_j$, we see that $S_k \subset S_{k+1}$.

(ii) Let $u \in D_j$ for some j. From $|u-x_j| \leq \ell(x_j)$, it follows that $\ell(x_j)$, $\ell(u)$ are comparable. Since $\widetilde{D}_j \subset S_{k+1}$, we have $|v-x_j| \geq 10\ell(x_j)$, so $|v-u| \geq 9\ell(x_j) > 4\ell(u)$.

(iii) Suppose that $S := \bigcup_k S_k$ is not the whole $\mathbf{R}^3$ and select a point $z \in \partial S$. Then we can find a sequence of points $z_k \in \partial S_k$ converging to z such that $|z_k - z|$ monotonically decreasing (for example, we can choose the point $z_k \in S_k$ closest to z). Since $\ell(z) > 0$, we see that $|z - z_n| \leq \ell(z)$ for some n, hence $\ell(z_n)$ and $\ell(z)$ are comparable. We have

$$|z_{n+1} - z_n| \leq |z_{n+1} - z| + |z - z_n| \leq 2\ell(z) \leq 4\ell(z_n)$$

which contradicts (ii).

(iv) It is clear that $m_0 \leq N$ by Definition 5.3 (ii). By induction we show that $m_{k+1} \leq Nm_k$. This again follows from Definition 5.3 (ii), since each $\widetilde{D}_j$ in the definition of S_{k+1} may intersect at most N balls from the collection $\{D_j\}_{j\in I}$.

(v) Straightforward by induction on k and by the definition of $S_k(i)$. $\square$

This lemma gives an integer valued distance on the collection $\{x_j\}_{j\in I}$:

$$d_{ij} := \min\{k \ : \ x_j \in S_k(i)\} \ .$$

Clearly $d_{ii} = 0$, and $d_{ij} + d_{jk} \geq d_{ik}$, but the distance function is not symmetric. However, we have

Lemma A.3. *For sufficiently small ε the distance function satisfies*

$$d_{ji} \leq 7d_{ij} + 1 \; . \tag{A.6}$$

Proof. The proof goes by induction on the value of d_{ij}. If $d_{ij} = 0$, $x_j \in D_i$, then $x_i \in \widetilde{D}_j$, i.e., $d_{ji} \leq 1$. Suppose that (A.6) is proven for all (i, j) pairs with $d_{ij} \leq d$ and let now $d_{ij} = d + 1$. Then there exists m such that $x_m \in S_d(i)$, $d_{im} \leq d$ and $x_j \in \widetilde{D}_a$ for some index a with $|x_a - x_m| \leq \ell_a + 10\ell_m$. If ε is sufficiently small, the radii ℓ_a, ℓ_j and ℓ_m are comparable, and it easily follows that $d_{ja} \leq 3$ and $d_{am} \leq 3$. Therefore $d_{ji} \leq d_{ja} + d_{am} + d_{mi} \leq 6 + 7d + 1 < 7d_{ij} + 1$. □

For every $i \in I$, $k \in \mathbf{N}$ we define

$$u_k^{(i)} := \sum_{j \,:\, D_j \cap S_k(i) \neq \emptyset} \theta_j^2 \; . \tag{A.7}$$

Notice that $\text{supp}(u_k^{(i)}) \subset S_{k+1}(i)$, and $u_k^{(i)} \equiv 1$ on $S_k(i)$. Moreover

$$|\nabla u_k^{(i)}(x)| \leq cN\ell(x)^{-1}. \tag{A.8}$$

To see this, we notice that every x is covered by not more than N balls D_j and only these support those θ_j's which do not vanish at x. Moreover ℓ_j is comparable to $\ell(x)$ for all these j indices, hence $\|\nabla \theta_j\|_\infty \leq c\ell(x)^{-1}$.

In the rest of this section we set

$$R_f = R(f) := (T + f)^{-1}, \qquad R_i[f] := (T_i + f)^{-1}$$

for simplicity, in accordance with the notations (6.1), (6.2).

Proof of Proposition 7.1. We start with an auxiliary lemma.

Lemma A.4. *For any number $\mu \geq 0$ and real function χ on $\mathbf{R}^3$,*

$$\Big\| R_{P+\mu}^{1/2} (P + \mu) R_{P+\mu}^{1/2} \Big\| \;\leq\; 1 \, , \tag{A.9}$$

$$\Big\| R_{P+\mu}^{1/2} [T, \chi] R_{P+\mu}^{1/2} \Big\| \;\leq\; c_0 \Big\| P^{-1/2} |\nabla \chi| \; \Big\|_\infty . \tag{A.10}$$

Proof of Lemma A.4. The first inequality is trivial by inserting $P + \mu \leq T + P + \mu$. For the second inequality we use

$$[T, \chi] = A^*[A, \chi] + [A^*, \chi]A \; , \tag{A.11}$$

and it is sufficient to estimate one of these terms;

$$\left\| R_{P+\mu}^{1/2} A^*[A,\chi] R_{P+\mu}^{1/2} \right\| \le \left\| R_{P+\mu}^{1/2} A^* \right\| \, \left\| [A,\chi] R_{P+\mu}^{1/2} \right\| .$$

Using $\|M\| = \|MM^*\|^{1/2}$, we obtain that the first factor is bounded by 1 (again, using $A^*A = T \le T + P + \mu$). For the second factor we need pointwise commutator bounds

$$[A,\chi][A,\chi]^* \le c_0 |\nabla\chi|^2, \quad [A^*,\chi][A^*,\chi]^* \le c_0 |\nabla\chi|^2 \,, \tag{A.12}$$

that follows from (7.2).

Hence, we estimate the second factor as

$$\left\| [A,\chi] R_{P+\mu}^{1/2} \right\| \le c_0 \left\| R_{P+\mu}^{1/2} |\nabla\chi|^2 R_{P+\mu}^{1/2} \right\|^{1/2} \le c_0 \left\| R_{P+\mu}^{1/2} P R_{P+\mu}^{1/2} \right\|^{1/2} \left\| P^{-1} |\nabla\chi|^2 \right\|_\infty^{1/2} ,$$

and we use (A.9). □

The key lemma is the following (recall the definition of $\widetilde{\chi}_i$ from Section 5.2):

Lemma A.5. *For sufficiently small ε we have the following estimates for any $i, j \in I$*

$$\begin{aligned}
\widetilde{\chi}_i R_{P+\mu} \theta_j F^2 \theta_j R_{P+\mu} \widetilde{\chi}_i \;&\le\; c(4c_0\varepsilon)^{2(d_{ij}-1)_+} F_i^2 (P_i+\mu)^{-1} \widetilde{\chi}_i R_{P+\mu} \widetilde{\chi}_i \,, && \text{(A.13)}\\
\widetilde{\chi}_i R_{P+\mu} F^2 R_{P+\mu} \widetilde{\chi}_i \;&\le\; c F_i^2 (P_i+\mu)^{-1} \widetilde{\chi}_i R_{P+\mu} \widetilde{\chi}_i \,, && \text{(A.14)}\\
\widetilde{\chi}_i R_{P+\mu}^2 \widetilde{\chi}_i \;&\le\; (P_i+\mu)^{-2} \,, && \text{(A.15)}\\
\chi_i A R_{P+\mu}^2 A^* \chi_i \;&\le\; c(P_i+\mu)^{-1} \,. && \text{(A.16)}
\end{aligned}$$

Proof of Lemma A.5. For brevity, we denote $R := R_{P+\mu}$. First we show (A.13). We assume that ε_0 is small enough so that $F(x_i)$ and $F(x_j)$ are comparable as long as $\widetilde{D}_j \cap \widetilde{D}_i \neq \emptyset$ (see (7.1)).

We first consider the case $d_{ij} \le 1$. Then $F^2 \le cF_j^2 \le cF_i^2$ on the support of θ_j. We also use

$$\theta_j^2 \le c(P_j+\mu)^{-1}(P+\mu) \le c(P_i+\mu)^{-1}(P+\mu) \,. \tag{A.17}$$

Hence (A.13) follows from

$$\widetilde{\chi}_i R \theta_j F^2 \theta_j R \widetilde{\chi}_i \le c F_i^2 \widetilde{\chi}_i R \theta_j^2 R \widetilde{\chi}_i \le c F_i^2 (P_i+\mu)^{-1} \widetilde{\chi}_i R (P+\mu) R \widetilde{\chi}_i$$

and finally we use (A.9) to estimate $R(P+\mu)R \le R$.

To prove (A.13) for $d = d_{ij} \geq 2$, we recall the definition of the functions $u_k^{(i)}$ (A.7). For brevity, we omit the superscript i. We successively insert the functions $u_1, u_2, \ldots u_{d-1}$ where $d = d_{ij}$:

$$\begin{aligned}
\widetilde{\chi}_i R\theta_j &= \widetilde{\chi}_i u_1 R\theta_j = \widetilde{\chi}_i R[T, u_1]R\theta_j = \widetilde{\chi}_i R[T, u_1]u_2 R\theta_j = \widetilde{\chi}_i R[T, u_1]R[T, u_2]R\theta_j \\
&= \ldots = \widetilde{\chi}_i R[T, u_1]R[T, u_2]R \ldots R[T, u_{d-1}]R\theta_j \; .
\end{aligned}$$

We used that $u_1 \equiv 1$ on the support of $\widetilde{\chi}_i$, $u_{k+1} \equiv 1$ on the support of ∇u_k and $\mathrm{supp}(u_{d-1}) \cap \mathrm{supp}(\theta_j) = \emptyset$. Therefore we can first estimate $\theta_j F^2 \theta_j \leq cF_j \theta_j^2$, then use the succesive insertions to obtain

$$\begin{aligned}
&\widetilde{\chi}_i R\theta_j F^2 \theta_j R\widetilde{\chi}_i \qquad\qquad \text{(A.18)} \\
&\leq \; cF_j^2 \widetilde{\chi}_i R^{1/2} \left[\prod_{k=1}^{d-1} \Big(R^{1/2}[T, u_k]R^{1/2} \Big) \right] R^{1/2} \theta_j^2 R^{1/2} \left[\prod_{k=1}^{d-1} \Big(R^{1/2}[T, u_k]R^{1/2} \Big) \right]^* R^{1/2} \widetilde{\chi}_i \; .
\end{aligned}$$

First we use that

$$R^{1/2}\theta_j^2 R^{1/2} \leq c(P_j + \mu)^{-1} R^{1/2}(P + \mu)R^{1/2} \leq c(P_j + \mu)^{-1}$$

by (A.17) and (A.9). Then we use (A.10) to estimate the commutator norms and we use (A.8) to get

$$\Big\| P^{-1/2} |\nabla u_k| \Big\|_\infty \leq c\varepsilon^{5/2} \leq \varepsilon$$

for sufficiently small ε. We obtain

$$\widetilde{\chi}_i R\theta_j F^2 \theta_j R\widetilde{\chi}_i \leq c(c_0\varepsilon)^{2(d_{ij}-1)} F_j^2 (P_j + \mu)^{-1} \widetilde{\chi}_i R\widetilde{\chi}_i \qquad \text{(A.19)}$$

By (A.5), we see that $F_j^2(P_j + \mu)^{-1} \leq 16^d F_i^2 (P_i + \mu)^{-1}$ because part (v) of Lemma A.2 applies both to the function $G = F$ and $G = (P + \mu)^{1/2}$. This completes the proof of (A.13).

To prove (A.14) we insert a partition of unity

$$\widetilde{\chi}_i RF^2 R\widetilde{\chi}_i = \sum_{j \in I} \widetilde{\chi}_i R\theta_j F^2 \theta_j R\widetilde{\chi}_i \; .$$

We use (A.13), (iv) of Lemma A.2 and that

$$\sum_{j \in I} (4c_0\varepsilon)^{2(d_{ij}-1)_+} \leq 1 + N + \sum_{p=1}^{\infty} (4c_0\varepsilon)^p N^p \leq N + 2 \qquad \text{(A.20)}$$

if $\varepsilon \le \varepsilon_0$, where the universal constant N is from Lemma 5.4. This proves (A.14).

The proof of (A.15) is straight-forward by applying (A.14) with $F \equiv 1$,

$$\widetilde{\chi}_i R_{P+\mu}^2 \widetilde{\chi}_i \le c(P_i + \mu)^{-1} \widetilde{\chi}_i R_{P+\mu} \widetilde{\chi}_i, \tag{A.21}$$

and then using $R_{P+\mu} \le (P+\mu)^{-1}$ which is bounded by $c(P_i+\mu)^{-1}$ on the support of $\widetilde{\chi}_i$ since P and P_i are comparable on this set.

For the proof of (A.16) we insert $\widetilde{\chi}_i$ that is identically 1 on the support of χ_i and use (A.21)

$$\chi_i A R_{P+\mu}^2 A^* \chi_i = \chi_i A \widetilde{\chi}_i R_{P+\mu}^2 \widetilde{\chi}_i A^* \chi_i \le c(P_i+\mu)^{-1} \chi_i A \widetilde{\chi}_i R_{P+\mu} \widetilde{\chi}_i A^* \chi_i \ .$$

We can remove $\widetilde{\chi}_i$ and use $A R_{P+\mu} A^* \le 1$ to finish the proof. □

The next lemma is a strengthening of (A.13) in Lemma A.5. Notice that in (A.13) we lost a resolvent, and the right hand side is not locally trace class in the high momentum regime. The following lemma remedies this:

Lemma A.6. *For sufficiently small ε*

$$\theta_i^2 R_{P+\mu} \theta_j F^2 \theta_j R_{P+\mu} \theta_i^2 \le c(4c_0\varepsilon)^{2(d_{ij}-1)_+} F_i^2 \theta_i^2 \Big(R_i^2[P_i] + P_i^{-1} R_i[P_i] A_i^* \varphi_i^2 A_i R_i[P_i] \Big) \theta_i^2 \ . \tag{A.22}$$

Proof of Lemma A.6. For simplicity, we let $R := R_{P+\mu}$ and $R_i := R_i[P_i]$ in this proof. We start with the identity

$$\widehat{\chi}_i R = R_i \widehat{\chi}_i + R_i \Big(\widehat{\chi}_i (P_i - P - \mu) + A_i^* [A, \widehat{\chi}_i] + [A^*, \widehat{\chi}_i] A \Big) R \ , \tag{A.23}$$

since A and A_i coincide on the support of $\widehat{\chi}_i$ by (7.3). After a Schwarz' inequality

$$\begin{aligned} &\widehat{\chi}_i R \theta_j F^2 \theta_j R \widehat{\chi}_i \qquad\qquad\qquad (A.24) \\ &\le \; c\Big(R_i \widehat{\chi}_i \theta_j F^2 \theta_j \widehat{\chi}_i R_i + R_i \widehat{\chi}_i (P_i - P - \mu) R \theta_j F^2 \theta_j R (P_i - P - \mu) \widehat{\chi}_i R_i \\ &\quad + R_i A_i^* [A, \widehat{\chi}_i] R \theta_j F^2 \theta_j R [A, \widehat{\chi}_i]^* A_i R_i + R_i [A^*, \widehat{\chi}_i] A R \theta_j F^2 \theta_j R A^* [A^*, \widehat{\chi}_i]^* R_i \Big) \ . \end{aligned}$$

The first term is estimated as

$$R_i \widehat{\chi}_i \theta_j F^2 \theta_j \widehat{\chi}_i R_i \le c F_i^2 R_i^2 \mathbf{1}(d_{ij} \le 1) \ , \tag{A.25}$$

since $F \le cF_i$ on the support of $\theta_j \widehat{\chi}_i$.

Since $\chi_i \equiv 1$ on the support of $\widehat{\chi}_i$, we can freely insert χ_i in the last three terms of (A.24) replacing $R\theta_j F^2 \theta_j R$ with $\chi_i R\theta_j F^2\theta_j R\chi_i$ everywhere and apply (A.13) after multiplying it by χ_i from both sides:

$$\chi_i R\theta_j F^2 \theta_j R\chi_i \le c(4c_0\varepsilon)^{2(d_{ij}-1)_+} F_i^2 (P_i+\mu)^{-1}\chi_i R\chi_i \ .$$

In the second term of (A.24) we use $R \le (P+\mu)^{-1}$ and that $P+\mu$ and $P_i+\mu$ are comparable on the support of $\widehat{\chi}_i$. Hence

$$R_i\widehat{\chi}_i(P_i - P - \mu)R\theta_j F^2\theta_j R(P_i-P-\mu)\widehat{\chi}_i R_i \le c(4c_0\varepsilon)^{2(d_{ij}-1)_+} F_i^2 R_i^2 \ , \tag{A.26}$$

using $(P_i - P - \mu)^2 \le (P_i+\mu)^2$ on the support of $\widehat{\chi}_i$.

In the third term of (A.24) we again estimate $R \le (P+\mu)^{-1} \le cP_i^{-1}$ on the support of $\widehat{\chi}_i$, we use (A.12) and that $|\,[A,\widehat{\chi}_i][A,\widehat{\chi}_i]^*| = |\nabla\widehat{\chi}_i|^2 \le c\varepsilon^2 P_i\varphi_i^2$ to obtain

$$R_i A_i^*[A,\widehat{\chi}_i]R\theta_j F^2\theta_j R[A,\widehat{\chi}_i]^* A_i R_i \le c(4c_0\varepsilon)^{2d_{ij}} F_i^2 P_i^{-1} R_i A_i^* \varphi_i^2 A_i R_i \ . \tag{A.27}$$

Finally, the fourth term of (A.24) satisfies

$$\begin{aligned} R_i[A^*,\widehat{\chi}_i]AR\theta_j F^2\theta_j RA^*[A^*,\widehat{\chi}_i]^* R_i &\le c(4c_0\varepsilon)^{2(d_{ij}-1)_+} F_i^2 P_i^{-1} R_i[A^*,\widehat{\chi}_i]A\chi_i R\chi_i A^*[A^*,\widehat{\chi}_i]^* R_i \\ &\le c(4c_0\varepsilon)^{2d_{ij}} F_i^2 R_i^2 \ , \end{aligned} \tag{A.28}$$

since $\chi_i \equiv 1$ on the support of $\widehat{\chi}_i$, we can omit it, and we used $ARA^* \le 1$ and (A.12). Lemma A.6 follows from (A.24)–(A.28) using that $\theta_i\widehat{\chi}_i = \theta_i$. □

Finally, we complete the proof of Proposition 7.1. We use $R = R_{P+\mu}$ for brevity. We insert three partitions of unity and perform a weighted Schwarz' inequality

$$\begin{aligned} RF^2R &= \sum_{i,j,k\in I} \theta_i^2 R\theta_j F^2\theta_j R\theta_k^2 \qquad\qquad \text{(A.29)} \\ &\le \sum_{i,j,k\in I} \Big(\varepsilon^{(d_{kj}-1)_+ - (d_{ij}-1)_+}\theta_i^2 R\theta_j F^2\theta_j R\theta_i^2 + \varepsilon^{(d_{ij}-1)_+ - (d_{kj}-1)_+}\theta_k^2 R\theta_j F^2\theta_j R\theta_k^2 \Big) \ . \end{aligned}$$

Using (A.6) we see as in (A.20) that

$$\sum_{k\in I}\varepsilon^{(d_{kj}-1)_+} \le \sum_{k\in I}\varepsilon^{(d_{jk}-8)_+/7} \le N^9$$

if ε is small enough. Here N is from Lemma 5.4. Therefore (A.29) implies

$$RF^2R \le cN^9\sum_{i,j}\varepsilon^{-(d_{ij}-1)_+}\theta_i^2 R\theta_j F^2\theta_j R\theta_i^2 \ .$$

We use (A.22) and sum up the index j similarly to (A.20) with a possible smaller ε

$$\begin{aligned} RF^2R &\le cN^8\sum_{i,j}(16c_0^2\varepsilon)^{(d_{ij}-1)_+}F_i^2\theta_i^2\Big(R_i^2+P_i^{-1}R_iA_i^*\varphi_i^2A_iR_i\Big)\theta_i^2 \\ &\le cN^{10}\sum_i F_i^2\theta_i^2\Big(R_i^2+P_i^{-1}R_iA_i^*\varphi_i^2A_iR_i\Big)\theta_i^2\ . \qquad \square \end{aligned}$$

A.4 Proof of Lemma 8.2 on the magnetic coordinates

We give the construction of the new coordinates and conformal metric but we do not follow the explicit bounds along the proof as they easily follow by scaling.

We consider $z\in\mathcal{P}$ fixed in the proof and omit the notation z in the sub- and superscripts. We define the function

$$\kappa(\tau) := -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\log|\mathbf{B}(\varphi(\tau))| \tag{A.30}$$

where $\kappa(\tau)\in C^3(\mathbf{R})$. At each point $\varphi(\tau)$ we consider a small spherical cap of the sphere $\mathcal{S}(\tau)$, going through $\varphi(\tau)$, orthogonal to the field line φ and having curvature $|\kappa(\tau)|$. The different spheres should curve in a direction determined by the sign of $\kappa(\tau)$: positive curvature means a sphere with outward normal pointing in the direction of $\dot\varphi$. We consider now a small cylindrical tubular neighborhood, $\widetilde{\mathcal{N}} := \{x\ :\ \inf_\tau \operatorname{dist}(x,\varphi(\tau))\le 10\ell\}$, along the C^5-curves φ, in which the spherical caps define a C^3-foliation. Since $\mathbf{B}$ is extended (D,K)-regular, such neighborhood exists if $\varepsilon\le\varepsilon(K)$ is sufficiently small. The foliation naturally extends the function τ onto $\widetilde{\mathcal{N}}$ such that it is constant on the leaves and $\tau\in C^3(\mathbf{R}^3)$.

This foliation can be extended to the whole of $\mathbf{R}^3$ in the following way. Since $\mathbf{B}=\mathbf{B}_\infty$ outside of $D=B(z_0,\ell)$, for some $z_0\in\mathbf{R}^3$, in case of $|z-z_0|\ge 2\ell$ we have $\kappa(\tau)\equiv 0$ and φ is a straight line, so the foliating spherical caps are parallel flat discs and they can be trivially extended to a foliation of $\mathbf{R}^3$ with parallel planes.

Now we consider the case $|z-z_0|<2\ell$, where we only know $\kappa(\tau)\equiv 0$ for $|\tau|\ge 3\ell$. In this region the spherical caps are again parallel flat discs and they can be extended to parallel planes. That leaves a parallel slab unfoliated between the planes passing through $\varphi(-3\ell)$ and $\varphi(3\ell)$. The width of the slab is $(6\pm cK\varepsilon)\ell$.

We consider the smooth function $F:\mathbf{R}^3\mapsto\mathbf{R}$ of the form $F(x)=\lambda(\mathbf{n}_\infty\cdot x)$, where $\lambda:\mathbf{R}\to\mathbf{R}$ is smooth and is chosen such that $F(\varphi(\tau))=\tau$ for $|\tau|\ge 3\ell$, $\|\lambda'-1\|_\infty\le cK\varepsilon\ell$ and $\|\lambda^{(\gamma)}\|_\infty\le cK\varepsilon\ell^{-\gamma}$, $\gamma=2,3,4$. The level sets of F define a parallel foliation of $\mathbf{R}^3$ which coincides with the previous foliation outside of the slab.

Let χ be a smooth cutoff function supported on the ball $\widetilde{D}:=B(z_0,6\ell)$, $\|\nabla^\gamma\chi\|_\infty\le c\ell^{-\gamma}$, $\gamma=1,\dots,4$, and $\chi\equiv 1$ on $B(z_0,5\ell)$. Since $|z-z_0|\le 2\ell$, we note that $\widetilde{D}\subset\widetilde{\mathcal{N}}$, so τ is already defined on $\mathrm{supp}(\chi)$. We define the function

$$t := \chi\tau + (1-\chi)F\ .$$

An easy calculation shows that $t \in C^3(\mathbf{R}^3)$, $\|\nabla t - \mathbf{n}_\infty\| \le cK\varepsilon$ and the level sets of t define a regular foliation of $\mathbf{R}^3$. This is clearly an extension of the foliation given by τ on D and the leaves are planes on $\widetilde{D}^c$. Moreover, if we define a smaller tubular neighborhood $\mathcal{N}$ of the central field line as $\mathcal{N} := \{x\ :\ \inf_\tau \mathrm{dist}(x, \varphi(\tau)) \le 2\ell\}$, then we note that $\mathcal{N} \subset B(z_0, 5\ell) \cup \{|\tau| \ge 3\ell\}$, therefore $t \equiv \tau$ on $\mathcal{N}$. Let $N := \|\nabla t\|^{-1}\nabla t$ be the unit C^2-vectorfield orthogonal to the foliation. We remark that the integral curves of N typically do not coincide with the field lines except on the field line φ and in the region far away from z_0.

Armed with this foliation, we introduce new coordinates on $\mathbf{R}^3$. On the plane $\mathcal{P}$ we choose Euclidean orthonormal coordinates ξ_1, ξ_2 with origin at z and dual to the basis $\{p_1, p_2\}$, i.e. $x - z = \xi_1 p_1 + \xi_2 p_2$. Clearly $(\partial_{\xi_j}, \partial_{\xi_k}) = \delta_{jk}$ for $j, k = 1, 2$. For simplicity we set $\partial_j := \partial_{\xi_j}$ and $\nabla_j := \nabla_{\partial_j}$ in this proof.

We extend the coordinate system ξ_1, ξ_2 defined on the plane $\mathcal{P}$ by setting ξ_1, ξ_2 constant on the integral curves of N. It is easy to check that $\xi_1, \xi_2 \in C^3(\mathbf{R}^3)$. Together with t they define a regular set of coordinates on $\mathbf{R}^3$. The central line is given by $(0, 0, t)$ in these coordinates. Let $b(t) := |\mathbf{B}(0, 0, t)| \in C^3(\mathbf{R})$ be the strength of the magnetic field along the central line, note that $b(t)$ is comparable with b for all t by (5.1). We define $\xi_3 = \xi_3(t)$ to be the solution of $\frac{\mathrm{d}}{\mathrm{d}t}\xi_3 = [b(t)/b]^{1/2}$ with $\xi_3(0) = 0$, clearly $\xi_3 \in C^4(\mathbf{R})$. We reparametrize the coordinate t with $\xi_3 := \xi_3(t)$. In this way we defined a new coordinate system, $\{\xi_1, \xi_2, \xi_3\}$, with origin at z. We shall view the coordinates ξ_1, ξ_2, ξ_3 as C^3 functions of $x \in \mathbf{R}^3$ and whenever the dependence on z is relevant, we use the notation $\xi^z = (\xi_1^z, \xi_2^z, \xi_3^z)$. It is easy to check that the function $(\hat{z}, \xi) \mapsto \xi^z$ belongs to $C^3(\mathbf{R}^5)$ and the vectorfields $\partial_i = \partial_{\xi_i}$ are C^2.

We note that in the trivial case, $|z - z_0| \ge 2\ell$, we simply have $\xi^z(x) = R^t(x - z)$, where $R := [p_1|p_2|\mathbf{n}_\infty]$ is the 3 by 3 matrix with columns $p_1, p_2, \mathbf{n}_\infty$, and all statements of Lemma 8.2 are trivial with $\Omega \equiv h \equiv 1$.

From now on we shall assume that $|z - z_0| < 2\ell$. The relations (8.3)–(8.4) and (8.6), i.e., the fact that $\mathrm{d}s^2$ has no $\mathrm{d}\xi_j\,\mathrm{d}\xi_3$ $(j = 1, 2)$ components follow directly from the construction. From the regularity of the magnetic field (Definition 5.1) it easily follows that the Jacobian of the change of coordinates $x \mapsto \xi^z(x)$ is close to the matrix R^t and it varies regularly in z. This proves that the function $(\hat{z}, \xi) \mapsto x^z(\xi)$ is well defined and $C^3(\mathbf{R}^5)$, it also proves (8.5) and (8.7) by the inverse function theorem.

The metric is diagonal in the ξ coordinate system on the plane $\mathcal{P}$, i.e. for $t = 0$. The key point is to show that it remains diagonal within the tubular neighborhood $\mathcal{N}$. The diagonal metric elements will define the functions Ω and h.

We derive a differential equation for the metric components $g_{jk} := (\partial_j, \partial_k)_g \in C^2(\mathbf{R}^3)$ where $j, k = 1, 2$ within $\mathcal{N}$. We have $\partial_t g_{jk} = \partial_t(\partial_j, \partial_k)_g = (\nabla_t \partial_j, \partial_k)_g + (\partial_j, \nabla_t \partial_k)_g$. Using that $\partial_t, \partial_j, \partial_k$, are coordinate fields, i.e, have vanishing Lie derivatives we have $\nabla_t\partial_j = \nabla_j\partial_t$. Recall that $N = g_{tt}^{-1/2}\partial_t$ is the unit normal to the spherical foliation, where $g_{tt} := (\partial_t, \partial_t)_g$. Then $\nabla_j\partial_t = g_{tt}^{1/2}\nabla_j N + \partial_j(g_{tt}^{1/2})N$ and therefore we have

$$\partial_t g_{jk} = g_{tt}^{1/2}[(\nabla_j N, \partial_k)_g + (\partial_j, \nabla_k N)_g] = 2g_{tt}^{1/2}K_{jk}\ ,$$

where K_{jk} is the second fundamental form of the leaves of the foliation. For a sphere immersed in $\mathbf{R}^3$ we have $K_{jk} = \kappa g_{jk}$, where κ is the curvature. We recall the choice of κ from (A.30) and that $t \equiv \tau$ on $\mathcal{N}$. Thus

$$\partial_t g_{jk} = 2g_{tt}^{1/2}\kappa(t)g_{jk}. \tag{A.31}$$

This proves that since g_{12} is zero on the supporting plane $t = 0$, it is zero everywhere in $\mathcal{N}$. It also proves that $g_{11} = g_{22}$ everywhere in $\mathcal{N}$ since they satisfy the same equation and initial condition. The same relations trivially hold for the region $|\xi_\perp| \geq 10\ell$, where $g_{11} = g_{22} = g_{12} = 1$. Moreover, we define $\Omega := g_{11}^{-1/2} \in C^2(\mathbf{R}^3)$ and we obtain that within $\mathcal{N}$ as well as in the regime $|\xi_\perp| \geq 10\ell$ the conformal metric can be written as

$$\mathrm{d}s_\Omega^2 = \mathrm{d}\xi_1^2 + \mathrm{d}\xi_2^2 + \Omega^2 g_{tt}\mathrm{d}t^2 = \mathrm{d}\xi_1^2 + \mathrm{d}\xi_2^2 + \Omega^2 g_{tt} f(\xi_3)^{-2}\mathrm{d}\xi_3^2$$

using the definition of f and ξ_3. This proves (8.8) with $h := \Omega g_{tt}^{1/2} f(\xi_3)$. Since the new coordinates form an orthonormal system for $|\xi_3| \geq 3\ell$ and also for $|\xi_\perp| \geq 10\ell$ modulo a change of variables in the third direction, the identities (8.9) and (8.12), respectively, follow from the definitions.

Along the central line we have $g_{tt}(\varphi(t)) = 1$. Thus (A.31), (A.30) and $g_{11} \equiv 1$ for $|\tau| \geq 3\ell$ implies that $g_{11} = b/|\mathbf{B}|$, i.e., $\Omega = f(\xi_3)$ and $h \equiv 1$ along the central line. Then (8.5) implies (8.10). The global bounds (8.13)–(8.14) also follow from the smoothness of the contstruction, i.e. from (8.5). The details are left to the reader.

Finally, the orthonormal basis $\{e_1, e_2, e_3\}$ in $\mathrm{d}s_\Omega^2$ is defined by first constructing $e_1' := \partial_1$, $e_2' := \partial_2$, $e_3' := h^{-1}\partial_3$ which are automatically orthonormal apart from the region $\{\xi\ :\ \frac{3}{2}\ell \leq |\xi_\perp| \leq 9\ell,\ |\xi_3| \leq 3\ell\}$. On this region we apply a Gram-Schmidt orthonormalization procedure to obtain $\{e_1, e_2, e_3\}$ from $\{e_1', e_2', e_3'\}$. $\square$

A.5 Comparison of operators on equivalent L^2-spaces

Let $\mathrm{d}\mu$ and $\mathrm{d}\nu$ be two positive measures on $\mathbf{R}^d$, that are mutually and uniformly absolutely continuous, i.e. $\mathrm{d}\nu(x) = F(x)\mathrm{d}\mu(x)$ with a a positive bounded function F with bounded inverse F^{-1}. We let

$$C_F := \|F\|_\infty \|F^{-1}\|_\infty = \frac{\max F}{\min F}\ .$$

Consider the spaces $L^2(\mathrm{d}\mu)$ and $L^2(\mathrm{d}\nu) = L^2(F\mathrm{d}\mu)$ and let A be any operator defined on $L^2(\mathrm{d}\mu)$. Since these two spaces are the same as sets, we can consider A acting on $L^2(F\mathrm{d}\mu)$ as well. We denote this operator by A_F. Let $(\cdot,\cdot) := (\cdot,\cdot)_{L^2(\mathrm{d}\mu)}$ and and $(\cdot,\cdot)_F := (\cdot,\cdot)_{L^2(F\mathrm{d}\mu)}$. Similar convention is used for $\|\cdot\|$ and $\|\cdot\|_F$ and for the traces over these L^2-spaces: $\mathrm{Tr} := \mathrm{Tr}_{L^2(\mathrm{d}\mu)}$ and $\mathrm{Tr}_F := \mathrm{Tr}_{L^2(F\mathrm{d}\mu)}$

Lemma A.7. *Let A be a Hilbert-Schmidt operator on $L^2(\mathrm{d}\mu)$ with a kernel $A(x,y)$. Then A_F is also Hilbert-Schmidt on $L^2(\mathrm{d}\nu)$, and the kernels of these operators satisfy*

$$A(x,y) = A_F(x,y)F(y) \; . \tag{A.32}$$

Moreover, the diagonal kernels of A^*A and $A_F^*A_F$ are comparable:

$$C_F^{-1}\|F\|_\infty^{-1}(A^*A)(x,x) \le \Big(A_F^*A_F\Big)(x,x) \le C_F\|F^{-1}\|_\infty(A^*A)(x,x) \; . \tag{A.33}$$

Furthermore, if $0 < \alpha \le A^*A \le \beta$ for some constants α, β, then

$$C_F^{-1}\alpha \le A_F^*A_F \le C_F\beta \; . \tag{A.34}$$

If, in addition, A is of trace class, then so is A_F and their diagonal kernels satisfy

$$A(x,x) = A_F(x,x)F(x) \; . \tag{A.35}$$

Proof. Recalling the conventions at the end of Section 4, the identities (A.32) and (A.35) are obvious. For (A.33) we estimate

$$\begin{aligned}(A_F^*A_F)(x,x) &= \int A_F^*(x,y)A_F(y,x)F(y)\mathrm{d}\mu(y) \\ &= F^{-2}(x)\int |A(x,y)|^2F(y)\mathrm{d}\mu(y) \\ &\le C_F\|F^{-1}\|_\infty(A^*A)(x,x) \; .\end{aligned}$$

The lower bound is proven similarly.

For (A.34) we notice that

$$(\psi, A_F^*A_F\psi)_F = \|F^{1/2}A\psi\|^2 \le (\max F)\|A\psi\|^2 \le \beta(\max F)\|\psi\|^2$$

and

$$\|\psi\|^2 \le \|F^{-1/2}\psi\|_F^2 \le (\max F^{-1})\|\psi\|_F^2 \; ,$$

which proves the upper bound. The proof of the lower bound in (A.34) is similar. $\square$

Lemma A.8. *Let A_k be a finite collection of closed operators on $L^2(\mathrm{d}\mu)$, let W_1, W_2 be nonnegative functions on $\mathbf{R}^d$. Then*

$$\Big|\ \mathrm{Tr}_F\Big(\sum_k (A_k)_F^*(A_k)_F + W_1 - W_2\Big)_-\Big| \le \Big|\ \mathrm{Tr}\Big(\sum_k A_k^*A_k + W_1 - C_F W_2\Big)_-\ \Big| . \tag{A.36}$$

Proof of Lemma A.8. By the variational principle

$$\mathrm{Tr}_F\Big(\sum_k (A_k)_F^*(A_k)_F + W_1 - W_2\Big)_- = \inf\Big\{\mathrm{Tr}_F\Big(\sum_k (A_k)_F^*(A_k)_F + W_1 - W_2\Big)\gamma\ :\ 0\le\gamma\le 1\Big\}$$

where the infimum is over all finite rank density matrices γ on $L^2(F\mathrm{d}\mu)$. We can write $\gamma = \sum_n \lambda_n (f_n,\cdot)_F f_n$ with $0\le\lambda_n\le 1$ and $\{f_n\}$ being orthonormal in $L^2(F)$.

Define the operator $\widetilde{\gamma} := (\min F)\sum_n \lambda_n (f_n,\cdot) f_n$ on L^2. Since

$$(\phi,\widetilde{\gamma}\phi) = (\min F)\sum_n \lambda_n |(f_n, F^{-1}\phi)_F|^2 = (\min F)(F^{-1}\phi, \gamma F^{-1}\phi)_F \le \|\phi\|^2,$$

$\widetilde{\gamma}$ is a density matrix on L^2. Furthermore, for any $A = A_k$

$$\begin{aligned}
\mathrm{Tr}_F A_F^* A_F\gamma &= \sum_n \lambda_n \|A_F f_n\|_F^2 = \sum_n \lambda_n \|A f_n\|_F^2 \ge (\min F)\sum_n \lambda_n \|A f_n\|^2 \\
&= (\min F)\sum_n \lambda_n \mathrm{Tr}|A^*Af_n\rangle\langle f_n| = \mathrm{Tr} A^*A\widetilde{\gamma}\ .
\end{aligned}$$

The potential term is estimated as

$$\begin{aligned}
\mathrm{Tr}_F(W_1 - W_2)\gamma &= \sum_n \lambda_n (f_n, (W_1-W_2)f_n)_F \\
&\ge (\min F)\sum_n \lambda_n (f_n, (W_1 - C_F W_2) f_n) \\
&= \mathrm{Tr}(W_1 - C_F W_2)\widetilde{\gamma}\ .
\end{aligned}$$

Therefore

$$\mathrm{Tr}_F\Big(\sum_k (A_k)_F^*(A_k)_F + W_1 - W_2\Big)\gamma \ge \mathrm{Tr}\Big(\sum_k A_k^*A_k + W_1 - C_F W_2\Big)\widetilde{\gamma}\ ,$$

and (A.36) follows from the variational principle. $\square$

A.6 Comparison of Dirac operators under a conformal transformation

Let $\Omega : \mathbf{R}^3 \to \mathbf{R}_+$ be a C^1-function satisfying

$$\frac{1}{2} \leq \Omega(x) \leq 2 \tag{A.37}$$

and

$$\|\nabla\Omega\|_\infty \leq \ell^{-1} \tag{A.38}$$

with some constant $\ell > 0$. We define the metric $\mathrm{d}s_\Omega^2 := \Omega^2 \mathrm{d}s^2$ that is conformally equivalent to the Euclidean metric $\mathrm{d}s^2$. Let $\mathcal{D}$ be a Dirac operator in the $\mathrm{d}s^2$ metric, then a Dirac operator in the $\mathrm{d}s_\Omega^2$ metric is given by $\mathcal{D}_\Omega := \Omega^{-2}\mathcal{D}\Omega$. Notice that $\mathcal{D}_\Omega$ is self-adjoint on $L^2(\mathrm{d}s_\Omega^2) \otimes \mathbf{C}^2$ (see [ES-III]). The following lemma compares certain resolvent kernels of $\mathcal{D}$ and $\mathcal{D}_\Omega$ on $L^2(\mathrm{d}s^2) \otimes \mathbf{C}^2$ and on $L^2(\mathrm{d}s_\Omega^2) \otimes \mathbf{C}^2$, respectively.

Lemma A.9. *Let $P \geq 2^9\ell^{-2}$ be a number. Under the conditions (A.37), (A.38) we have*

$$\mathrm{tr}\left[\frac{1}{(\mathcal{D}^2 + P)^2}\right]_{L^2}(x,x) \leq 2^9\ \mathrm{tr}\left[\frac{1}{(\mathcal{D}_\Omega^2 + P)^2}\right]_{L^2_\Omega}(x,x) \qquad x \in \mathbf{R}^3\,. \tag{A.39}$$

The left hand side is the diagonal of an operator kernel on $L^2(\mathrm{d}s^2) \otimes \mathbf{C}^2$, the right hand side is the diagonal of an operator kernel on $L^2(\mathrm{d}s_\Omega^2) \otimes \mathbf{C}^2$. Moreover, if $0 \leq \varphi \leq 1$ is a bounded function then

$$\begin{aligned} &\mathrm{tr}\left[\frac{1}{\mathcal{D}^2 + P}\mathcal{D}\varphi^2\mathcal{D}\frac{1}{\mathcal{D}^2 + P}\right]_{L^2}(x,x) \\ &\qquad \leq\ 2^{12}\ \mathrm{tr}\left[\frac{1}{\mathcal{D}_\Omega^2 + P}\mathcal{D}_\Omega\varphi^2\mathcal{D}_\Omega\frac{1}{\mathcal{D}_\Omega^2 + P} + P\frac{1}{(\mathcal{D}_\Omega^2 + P)^2}\right]_{L^2_\Omega}(x,x) \end{aligned} \tag{A.40}$$

for any $x \in \mathbf{R}^3$.

Proof of Lemma A.9. Let $\mathcal{V} : L^2(\mathrm{d}s^2) \otimes \mathbf{C}^2 \to L^2(\mathrm{d}s_\Omega^2) \otimes \mathbf{C}^2$ be a unitary map given by $\mathcal{V}\psi := \Omega^{-3/2}\psi$. Notice that

$$\mathcal{D} = \mathcal{V}^*(\Omega^{1/2}\mathcal{D}_\Omega\Omega^{1/2})\mathcal{V}\,,$$

therefore the unitary operator $\Omega^{1/2}\mathcal{D}_\Omega\Omega^{1/2}$ on $L^2(\mathrm{d}s_\Omega^2) \otimes \mathbf{C}^2$ is unitarily equivalent to $\mathcal{D}$ on $L^2(\mathrm{d}s^2) \otimes \mathbf{C}^2$. In particular, for any real function f

$$f(\mathcal{D}) = \Omega^{3/2} f(\Omega^{1/2}\mathcal{D}_\Omega\Omega^{1/2})\Omega^{-3/2}\,. \tag{A.41}$$

From (A.41) we obtain

$$\frac{1}{(\mathcal{D}^2+P)^2} \le \frac{4}{(\mathcal{D}^2+P)^2+3P^2} = \Omega^{3/2}\frac{4}{([\Omega^{1/2}\mathcal{D}_\Omega\Omega^{1/2}]^2+P)^2+3P^2}\Omega^{-3/2} \ .$$

In particular,

$$\frac{1}{(\mathcal{D}^2+P)^2}(x,x) \le \left(\frac{4}{(\Omega^{1/2}\mathcal{D}_\Omega\Omega\mathcal{D}_\Omega\Omega^{1/2}+P)^2+3P^2}\right)_{L^2}(x,x) \ .$$

Here the right hand side is the $L^2(\mathrm{d}s^2)\otimes\mathbf{C}^2$ kernel of the corresponding bounded non self-adjoint operator. However, the same operator can be viewed on $L^2(\mathrm{d}s^2_\Omega)\otimes\mathbf{C}^2$ as well, where it is self-adjoint. Using (A.35) from Lemma A.7 we know that the two diagonal kernels differ by a factor $\Omega^3(x)$.

To conclude (A.39), it is therefore sufficient to show that

$$\frac{1}{(\Omega^{1/2}\mathcal{D}_\Omega\Omega\mathcal{D}_\Omega\Omega^{1/2}+P)^2+3P^2} \le \Omega^{-1/2}\frac{32}{(\mathcal{D}^2_\Omega+P)^2}\Omega^{-1/2} \tag{A.42}$$

as self-adjoint operators on $L^2(\mathrm{d}s^2_\Omega)\otimes\mathbf{C}^2$. Using

$$\Omega^{1/2}\mathcal{D}_\Omega\Omega\mathcal{D}_\Omega\Omega^{1/2} = \Omega^{3/2}\mathcal{D}^2_\Omega\Omega^{1/2}+\Omega^{1/2}[\mathcal{D}_\Omega,\Omega]\mathcal{D}_\Omega\Omega^{1/2}$$

and a Schwarz' inequality, we obtain

$$\begin{aligned}
&(\Omega^{1/2}\mathcal{D}_\Omega\Omega\mathcal{D}_\Omega\Omega^{1/2}+P)^2+3P^2 \qquad\qquad (A.43)\\
\ge\ & \frac{1}{2}(\Omega^{3/2}\mathcal{D}^2_\Omega\Omega^{1/2})^*(\Omega^{3/2}\mathcal{D}^2_\Omega\Omega^{1/2}) - 2\Big(\Omega^{1/2}[\mathcal{D}_\Omega,\Omega]\mathcal{D}_\Omega\Omega^{1/2}\Big)^*\Big(\Omega^{1/2}[\mathcal{D}_\Omega,\Omega]\mathcal{D}_\Omega\Omega^{1/2}\Big)+P^2\\
=\ & \frac{1}{2}\Big[\Omega^{1/2}\mathcal{D}^2_\Omega\Omega^3\mathcal{D}^2_\Omega\Omega^{1/2} - 4\Omega^{1/2}\mathcal{D}_\Omega[\mathcal{D}_\Omega,\Omega]^*\Omega[\mathcal{D}_\Omega,\Omega]\mathcal{D}_\Omega\Omega^{1/2}+2P^2\Big]\\
\ge\ & \frac{1}{2}\Big[\frac{1}{8}\Omega^{1/2}\mathcal{D}^4_\Omega\Omega^{1/2} - 8\Omega^{1/2}\mathcal{D}_\Omega[\mathcal{D}_\Omega,\Omega]^*[\mathcal{D}_\Omega,\Omega]\mathcal{D}_\Omega\Omega^{1/2}+2P^2\Big] \ .
\end{aligned}$$

Using that $[\mathcal{D}_\Omega,\Omega]=\Omega^{-2}[\mathcal{D},\Omega]\Omega$ and (A.38) we can estimate

$$\Big\|[\mathcal{D}_\Omega,\Omega]^*[\mathcal{D}_\Omega,\Omega]\Big\|_\infty \le 4\|\nabla\Omega\|^2_\infty \le 2^{-7}P \ , \tag{A.44}$$

so we can continue

$$(A.43) \ge \frac{1}{2}\Omega^{1/2}\left[\frac{1}{8}\mathcal{D}^4_\Omega - \frac{1}{16}P\mathcal{D}^2_\Omega + P^2\right]\Omega^{1/2} \ge \frac{1}{32}\Omega^{1/2}(\mathcal{D}^2_\Omega+P)^2\Omega^{1/2} \ .$$

This completes the proof of (A.42) and hence (A.39).

For the proof of (A.40) we can use the argument above to reduce the problem to estimating the diagonal element of the self-adjoint operator

$$T := R\Omega^{1/2}\mathcal{D}_\Omega\Omega^{1/2}\varphi^2\Omega^{1/2}\mathcal{D}_\Omega\Omega^{1/2}R \quad \text{with} \quad R := \frac{1}{(\Omega^{1/2}\mathcal{D}_\Omega\Omega^{1/2})^2 + P}$$

viewed on $L^2(\mathrm{d}s_\Omega^2) \otimes \mathbf{C}^2$, where $\mathcal{D}_\Omega$ is self-adjoint. The resolvent can be written as $R = \Omega^{-1/2}R_1\Omega^{-1/2}$ with

$$R_1 := \frac{1}{\mathcal{D}_\Omega\Omega D_\Omega + P\Omega^{-1}} = \Omega^{-1}\frac{1}{\mathcal{D}_\Omega^2 - \mathcal{D}_\Omega[\mathcal{D}_\Omega, \Omega]\Omega^{-1} + P\Omega^{-2}} ,$$

and we can expand

$$R_1 = \Omega^{-1}\frac{1}{\mathcal{D}_\Omega^2 + P} + R_1\Big[\mathcal{D}_\Omega[\mathcal{D}_\Omega, \Omega]\Omega^{-1} - P(\Omega^{-2} - 1)\Big]\frac{1}{\mathcal{D}_\Omega^2 + P} .$$

Therefore, using a Schwarz' inequality, (A.37) and $0 \le \varphi \le 1$, we can estimate

$$\begin{aligned} T &\le 2\Omega^{-1/2}R_1\mathcal{D}_\Omega\varphi^2\mathcal{D}_\Omega R_1\Omega^{-1/2} \\ &\le 4\Omega^{-1/2}\frac{1}{\mathcal{D}_\Omega^2 + P}\Omega^{-1}\mathcal{D}_\Omega\varphi^2\mathcal{D}_\Omega\Omega^{-1}\frac{1}{\mathcal{D}_\Omega^2 + P}\Omega^{-1/2} \\ &\quad +8\Big(\cdots\Big)^*\Big(\mathcal{D}_\Omega R_1\mathcal{D}_\Omega[\mathcal{D}_\Omega, \Omega]\Omega^{-1}\frac{1}{\mathcal{D}_\Omega^2 + P}\Omega^{-1/2}\Big) \\ &\quad +8\Big(\cdots\Big)^*\Big(\mathcal{D}_\Omega R_1 P(\Omega^{-2} - 1)\frac{1}{\mathcal{D}_\Omega^2 + P}\Omega^{-1/2}\Big) . \end{aligned} \tag{A.45}$$

Here we used the shorthand notation $(\cdots)^*A$ for the operator A^*A where A is a long expression.

In the first term on the right hand side of (A.45) we use $\mathcal{D}_\Omega\Omega^{-1} = \Omega^{-1}\mathcal{D}_\Omega + [\mathcal{D}_\Omega, \Omega^{-1}]$ and (A.44) to obtain

$$\frac{1}{\mathcal{D}_\Omega^2 + P}\Omega^{-1}\mathcal{D}_\Omega\varphi^2\mathcal{D}_\Omega\Omega^{-1}\frac{1}{\mathcal{D}_\Omega^2 + P} \le \frac{8}{\mathcal{D}_\Omega^2 + P}\mathcal{D}_\Omega\varphi^2\mathcal{D}_\Omega\frac{1}{\mathcal{D}_\Omega^2 + P} + 2^{-8}P\frac{1}{(\mathcal{D}_\Omega^2 + P)^2} ,$$

and both terms explicitly appear on the right hand side of (A.40). For the other two terms it is sufficient to show that

$$R_1\mathcal{D}_\Omega^2 R_1 \le 4P^{-1} , \tag{A.46}$$

$$\mathcal{D}_\Omega R_1\mathcal{D}_\Omega^2 R_1\mathcal{D}_\Omega \le 4 , \tag{A.47}$$

and then the last two terms in (A.45) can be estimated by the second term on the right hand side of (A.40) using (A.37), (A.38).

For the proof of (A.46) and (A.47) we first use $\mathcal{D}_\Omega^2 \le 2\mathcal{D}_\Omega \Omega \mathcal{D}_\Omega + 2P\Omega^{-1} = 2R_1^{-1}$ to cancel one of the resolvents. The proof of (A.46) is then completed by estimating the other R_1 by $2P^{-1}$. For the proof of (A.47) we notice that

$$\mathcal{D}_\Omega R_1 \mathcal{D}_\Omega = \mathcal{D}_\Omega \frac{1}{\mathcal{D}_\Omega \Omega \mathcal{D}_\Omega + P\Omega^{-1}} \mathcal{D}_\Omega \le \mathcal{D}_\Omega \frac{2}{\mathcal{D}_\Omega^2 + 2P\Omega^{-1}} \mathcal{D}_\Omega \le 2 \; .$$

This completes the proof of Lemma A.9. $\square$.

A.7 Proof of Lemma 11.2: apriori bound on the full resolvent

Using (A.34) from Lemma A.7 and since the volume forms $\mathrm{d}\nu$ and $\mathrm{d}\mu = \Omega^3 \mathrm{d}x$ are comparable at every point, it is sufficient to prove (11.15)–(11.19) in the space $L^2(\mathrm{d}\mu) = L^2(\Omega^3 \mathrm{d}x)$. In this space $\mathcal{D} = \mathcal{D}_\Omega^\alpha$ and the components of $\mathbf{D} = \mathbf{D}_\Omega^\alpha = \mathbf{\Pi}_\Omega^\alpha - \frac{i}{2}(\mathrm{div}_\Omega f_1, \mathrm{div}_\Omega f_2, \mathrm{div}_\Omega f_3)$ are self-adjoint. We recall that $\mathbf{\Pi}_\Omega^\alpha$ was given in (9.14) and $\mathbf{D}$ was defined in general in (9.7). Throughout the proof we will work in the space $L^2(\Omega^3 \mathrm{d}x)$, and we adapt the notation $\mathcal{D} = \mathcal{D}_\Omega^\alpha$, $\mathbf{D} = \mathbf{D}_\Omega^\alpha$ in this section. We also recall that $\Pi_j = D_j + i d_j$ with $d_j := \frac{1}{2}\mathrm{div}_\Omega \, e_j$.

Using Lichnerowicz' formula (9.9), $\sup_x \|\star\beta(x)\| = \sup_x \|\mathbf{B}(x)\| \le cb$ and that all geometric terms are bounded by (8.13) and (8.14), we can estimate

$$\mathcal{D}^2 \ge \mathbf{D}^2 - cb \; . \tag{A.48}$$

We recall that $\ell = 1$, $b \ge \varepsilon^{-2} \ge 1$ and $P = \varepsilon^{-5} \ge 1$.

For the proof of (11.15) we start with a Schwarz' inequality

$$\Pi_j^* \Big(\frac{1}{\mathcal{D}^2 + P} \Big)^2 \Pi_j \le 2 D_j \Big(\frac{1}{\mathcal{D}^2 + P} \Big)^2 D_j + 2 \sup |d_j|$$

and use that $|d_j| \le c$. We estimate one of the resolvents trivially and use (A.48)

$$D_j \Big(\frac{1}{\mathcal{D}^2 + P} \Big)^2 D_j \le D_j \frac{b}{\mathcal{D}^2 + Pb} D_j \le D_j \frac{b}{\mathbf{D}^2 - cb + Pb} D_j \le D_j \frac{b}{D_j^2 + Pb/2} D_j \le b$$

for sufficiently small ε. This completes the proof of (11.15).

The proof of (11.16) is identical just we estimate $(\mathcal{D}/(\mathcal{D}^2+P))^2$ by $(\mathcal{D}^2+P)^{-1}$.

For the proof of (11.17) we first compute $\Pi_j \Pi_k = D_j D_k + i(D_j d_k + D_k d_j) - (\partial_{e_k} d_k) + d_j d_k$. We use a Schwarz' inequality, the estimate (11.15) and the boundedness of d_j's together with their derivatives, we obtain

$$\Pi_k^* \Pi_j^* \Big(\frac{1}{\mathcal{D}^2 + P} \Big)^2 \Pi_j \Pi_k \le 2 D_k D_j \Big(\frac{1}{\mathcal{D}^2 + P} \Big)^2 D_j D_k + cb \; . \tag{A.49}$$

We apply Lemma 11.1 to estimate the resolvent square, using that $\mathcal{D}^2$ and $\mathbf{D}^2$ differ only by an operator bounded by $cb \leq Pb/2$ if ε is sufficiently small:

$$\Big(\frac{1}{\mathcal{D}^2+P}\Big)^2 \leq \frac{b^2}{(\mathcal{D}^2+Pb)^2} \leq \frac{4b^2}{(\mathbf{D}^2)^2+(Pb)^2/4} \; . \tag{A.50}$$

We expand $(\mathbf{D}^2)^2 = \sum_j D_j^4 + \sum_{j<k}(D_j^2D_k^2 + D_k^2D_j^2)$ and use the following commutator identity for A, B self-adjoint operators

$$A^2B^2+B^2A^2 = \Big[AB^2A + AB[A,B] + [B,A]BA + \tfrac{1}{2}\Big([A,[A,B]]B + B[[B,A],A]\Big)\Big] + \Big[A \leftrightarrow B\Big]$$

(the second square bracket contains the same expression as the first one with A and B interchanged). After several Schwarz's inequalities, we obtain

$$\begin{aligned} A^2B^2+B^2A^2 \;\; &\geq \;\; \tfrac{1}{2}\Big(AB^2A + BA^2B - A^2 - B^2\Big) - c[A,B][B,A] \\ &\quad -c[A,[A,B]][[B,A],A] - c[B,[B,A]][[A,B],B] \; . \end{aligned} \tag{A.51}$$

Using the formula given in Theorem 2.12 [ES-III] for the curvature of the covariant derivative $\nabla = \nabla^{\alpha,\Omega}$ we obtain

$$\begin{aligned} [D_j, D_k] \;\; &= \;\; -\nabla_{[e_j,e_k]} - \partial_{e_j} d_k + \partial_{e_k} d_j \\ &\quad -\frac{1}{4}\sum_{a,b=1}^{3} (e_a, \mathcal{R}(e_j,e_k)e_b)\sigma^a\sigma^b + i\beta(e_j,e_k) \; , \end{aligned}$$

where $\mathcal{R}$ is the Riemannian curvature, $j,k = 1,2,3$. In short, we can write

$$[D_j, D_k] = \sum_{a=1}^{3} U_{jk}^a D_a + W_{jk} \; ,$$

where U_{jk}^k, W_{jk} are 2 by 2 matrix valued functions with $\|U_{jk}^a\|_\infty \leq c$, $\|\nabla U_{jk}^a\|_\infty \leq c$ and $\|W_{jk}\|_\infty \leq cb$, $\|\nabla W_{jk}\|_\infty \leq cb$ using (8.14). These estimates guarantee bounds on the double commutators as well.

From these estimates and (A.51) it follows that

$$D_j^2D_k^2 + D_k^2D_j^2 \geq \tfrac{1}{2}\Big(D_jD_k^2D_j + D_kD_j^2D_k\Big) - cb\mathbf{D}^2 - cb^2 \; .$$

Therefore

$$(\mathbf{D}^2)^2 + (Pb)^2/4 \;\geq\; \tfrac{1}{2}\Big[\sum_j D_j^4 + \sum_{j<k}\Big(D_j D_k^2 D_j + D_k D_j^2 D_k\Big)\Big] + \sum_j \Big(\tfrac{1}{2}D_j^4 - cbD_j^2\Big) + (Pb)^2/4 - cb^2\ . \tag{A.52}$$

The second line is bigger than $(Pb)^2/8$ if ε is sufficiently small ($P=\varepsilon^{-5}$). Every term in the first line is nonnegative, so we can complete the estimate (A.49) using (A.50) and (A.52)

$$D_jD_k\frac{4b^2}{(\mathbf{D}^2)^2+(Pb)^2/4}D_kD_j \leq D_jD_k\frac{8b^2}{D_kD_j^2D_k+(Pb)^2/8}D_kD_j \leq 8b^2\ .$$

This completes the proof of (11.17).

The proof of (11.18) and (11.19) are straightforward from (11.11), (11.15), (11.16) and (10.14). Finally (11.20) is proven in the same way as (11.15) but now directly on the space $L^2(\mathrm{d}\xi)\otimes\mathbf{C}^2$. $\square$

A.8 Proof of Lemma 11.3: Estimates on the resolvent with a constant field

The proof of (11.21)–(11.28) may be done by straightforward explicit calculations since the magnetic Schrödinger operator with a constant field is exactly solvable. We show below how to obtain these estimates in a reasonably short way.

We work in the ξ coordinate system, and use that $\xi_\perp(u)=0$. Because of translation invariance in the third direction, we can assume $\xi(u)=0$, so in the ξ coordinates we need to estimate the operator kernels at $(0,0)$.

We recall the decompositions (11.8)–(11.10) and let

$$-\widetilde{\Delta} := \widetilde{\Pi}_1^2+\widetilde{\Pi}_2^2 = (-i\partial_1-\tfrac{b}{2}\xi_2)^2+(-i\partial_2+\tfrac{b}{2}\xi_1)^2$$

be the two dimensional magnetic Laplacian that commutes with $\widetilde{\Pi}_3$. For simplicity, we denoted $\partial_j := \partial_{\xi_j}$. By Lichnerowicz' formula (9.8), $\widetilde{\mathcal{D}}^2 = -\widetilde{\Delta}+\widetilde{\Pi}_3^2+\sigma^3 b$ and recall that $\widetilde{\Pi}_3 = -i\partial_3$.

The key idea is that the heat kernel of $\widetilde{\Delta}$ has a closed form (see, e.g., Chapter 15 in [S79])

$$e^{t\widetilde{\Delta}}(\xi_\perp,\zeta_\perp) = \frac{b}{4\pi\sinh(bt)}\exp\Big[-\frac{b\coth(bt)}{4}(\xi_\perp-\zeta_\perp)^2-\frac{ib}{2}(\xi_2\zeta_1-\xi_1\zeta_2)\Big]\ . \tag{A.53}$$

Then the resolvent can be expressed as

$$\frac{1}{\widetilde{\mathcal{D}}^2+P} = \int_0^\infty e^{-t(P+\sigma^3 b)} e^{t\widetilde{\Delta}} e^{t\partial_3^2}\,\mathrm{d}t\ . \tag{A.54}$$

We define the following norm on $\mathbf{R}^3$

$$|\!|\!|\xi|\!|\!| := (b\xi_\perp^2 + P\xi_3^2)^{1/2}\ .$$

Lemma A.10. *Let $P \le cb$, then the following bounds hold*

$$\Big\|\frac{1}{\widetilde{\mathcal{D}}^2+P}(\xi,\zeta)\Big\| \;\le\; cbP^{-1/2}\,\frac{e^{-c|\!|\!|\xi-\zeta|\!|\!|}}{|\!|\!|\xi-\zeta|\!|\!|}\ , \tag{A.55}$$

$$\Big\|\frac{\widetilde{\Pi}_3}{\widetilde{\mathcal{D}}^2+P}(\xi,\zeta)\Big\| \;\le\; cb\,\frac{e^{-c|\!|\!|\xi-\zeta|\!|\!|}}{|\!|\!|\xi-\zeta|\!|\!|^2}\ , \tag{A.56}$$

$$\Big\|\frac{\widetilde{\Pi}_j}{\widetilde{\mathcal{D}}^2+P}(\xi,\zeta)\Big\| \;\le\; cb^{3/2}P^{-1/2}\,\frac{e^{-c|\!|\!|\xi-\zeta|\!|\!|}}{|\!|\!|\xi-\zeta|\!|\!|^2}\ , \qquad j=1,2 \tag{A.57}$$

$$\Big\|\frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2+P}(\xi,\zeta)\Big\| \;\le\; c\,e^{-c|\!|\!|\xi-\zeta|\!|\!|}\Big[\frac{1}{|\xi-\zeta|^2}+b\Big]\ , \tag{A.58}$$

where $\|\cdot\|$ refer to the 2 by 2 matrix norm of the operator kernel as a function from $\mathbf{R}^3\times\mathbf{R}^3$ into the set of 2 by 2 matrices.

Remark. If $b \le cP$, then the same estimates (A.55)–(A.58) hold with b replaced by P everywhere, including in the definition of $|\!|\!|\cdot|\!|\!|$.

Proof. From (A.53) and (A.54) we estimate

$$\Big\|\frac{1}{\widetilde{\mathcal{D}}^2+P}(\xi,\zeta)\Big\| \le cb\int_0^\infty \frac{e^{-tP+tb}}{\sqrt{t}\sinh(bt)}\exp\Big(-\frac{1}{4t}\Big[bt\coth(bt)(\xi_\perp-\zeta_\perp)^2+(\xi_3-\zeta_3)^2\Big]\Big)\,\mathrm{d}t\ .$$

We split the integration into two regimes: $bt\le 1$ and $bt\ge 1$. In the first regime we use $bt\coth(bt)\ge 1$, $\sin(bt)\ge bt$. In the second regime we estimate $\coth(bt)\ge 1$ and $\sinh(bt)\ge \frac{1}{4}e^{bt}$. We obtain

$$\begin{aligned}\Big\|\frac{1}{\widetilde{\mathcal{D}}^2+P}(\xi,\zeta)\Big\| \;\le\; & c\Big[\int_0^{1/b}\frac{e^{-tP}}{t^{3/2}}\,\exp\Big(-\frac{(\xi-\zeta)^2}{4t}\Big)\mathrm{d}t \\ & +b\,\exp\Big(-\frac{b}{4}(\xi_\perp-\zeta_\perp)^2\Big)\int_{1/b}^\infty\frac{e^{-tP}}{t^{1/2}}\exp\Big(-\frac{(\xi_3-\zeta_3)^2}{4t}\Big)\mathrm{d}t\Big]\end{aligned}$$

$$\leq \; c(P^{1/2} + bP^{-1/2}) \, \frac{e^{-c\|\xi-\zeta\|}}{\|\xi - \zeta\|} \; ,$$

after extending both integrations over $(0, \infty)$ and using the resolvent kernels of the one and three dimensional free Laplacians. The proofs of (A.56)–(A.57) are similar and left to the reader.

For the proof of (A.58), explicit calculation and trivial estimates yield

$$\begin{aligned}
\Big\| \frac{\widetilde{\mathcal{D}}}{\widetilde{\mathcal{D}}^2 + P}(\xi,\zeta) \Big\| \; &\leq \; c \int_0^\infty \frac{e^{-t(P-b)} b^2 t (\coth(bt) - 1) |\xi_\perp - \zeta_\perp|}{t^{3/2} \sinh(bt)} \\
&\quad \times \exp\Big(- \frac{b \coth(bt)}{4} |\xi_\perp - \zeta_\perp|^2 - \frac{(\xi_3 - \zeta_3)^2}{4t} \Big) \, \mathrm{d}t \\
&\leq \; c \int_0^{1/b} \frac{e^{-t(P+b)} |\xi_\perp - \zeta_\perp|}{t^{5/2}} \, \exp\Big(- \frac{(\xi - \zeta)^2}{4t} \Big) \, \mathrm{d}t \\
&\quad + cb \, e^{-cb|\xi_\perp - \zeta_\perp|^2} \int_{1/b}^\infty \frac{e^{-tP} |\xi_\perp - \zeta_\perp|}{t^{3/2}} \, \exp\Big(- \frac{(\xi_3 - \zeta_3)^2}{4t} \Big) \, \mathrm{d}t \\
&\leq \; c \int_0^{1/b} \frac{e^{-t(P+b)}}{t^2} \, \exp\Big(- c \frac{(\xi - \zeta)^2}{t} \Big) \, \mathrm{d}t \\
&\quad + cb^{1/2} \, e^{-cb|\xi_\perp - \zeta_\perp|^2} \int_{1/b}^\infty \frac{e^{-tP}}{t^{3/2}} \, \exp\Big(- \frac{(\xi_3 - \zeta_3)^2}{4t} \Big) \, \mathrm{d}t \\
&\leq \; c \, e^{-\sqrt{b+P}|\xi - \zeta|} \int_0^\infty \frac{1}{t^2} \, \exp\Big(- c \frac{(\xi - \zeta)^2}{t} \Big) \, \mathrm{d}t \\
&\quad + cb \, e^{-cb|\xi_\perp - \zeta_\perp|^2 - c\sqrt{P}|\xi_3 - \zeta_3|} \\
&\leq \; c \, e^{-c\|\xi - \zeta\|} \Big[\frac{1}{|\xi - \zeta|^2} + b \Big] \, . \qquad \Box
\end{aligned}$$

With the estimates of Lemma A.10 at hand, the proof of Lemma 11.3 is straightforward. For example, the proof of (11.21) is as follows

$$\operatorname{tr} \frac{1}{(\widetilde{\mathcal{D}}^2 + P)^2}(0,0) = \int_{\mathbf{R}^3} \Big\| \frac{1}{\widetilde{\mathcal{D}}^2 + P}(0,\xi) \Big\|^2 \mathrm{d}\xi \leq cb^2 P^{-1} \int_{\mathbf{R}^3} \frac{e^{-c\|\xi\|}}{\|\xi\|^2} \mathrm{d}\xi \leq cbP^{-3/2}$$

after a change of variables. The other inequalities are proved similarly. $\Box$

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