INVARIANT PRE-FOLIATIONS FOR NON-RESONANT NON-UNIFORMLY HYPERBOLIC SYSTEMS

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Abstract. Let $\{x_i\}_{i\in\mathbb{N}}$ be a regular orbit of a C^2 dynamical system f. Let S be a subset of its Lyapunov exponents. Assume that all the Lyapunov exponents in S are negative and that the sums of Lyapunov exponents in S do not agree with any Lyapunov exponent in the complement of S. Denote by $E^S_{x_i}$ the linear spaces spanned by the spaces associated to the Lyapunov exponents in S.

We show that there are smooth manifolds $W^S_{x_i}$ such that $f(W^S_{x_i}) \subset W^S_{x_{i+1}}$ and $T_{x_i}W^S_{x_i} = E^S_{x_i}$.

We establish the same results for orbits satisfying dichotomies and whose rates of growth satisfy similar non-resonance conditions.

These systems of invariant manifolds are, in general, not a foliation.

1. Introduction and statement of results

When studying the behavior of an orbit of a dynamical system f it is natural to study also the behavior of its linearization and wonder whether there are non-linear analogues for the features found in the study of the linearization.

Very often we can classify the tangent vectors along an orbit into subspaces with different rates of exponential growth either in the future or in the past (we will discuss later several precise definitions of *rates of growth*). Since these subspaces and combinations of them are invariant, the question of the existence of invariant objects for the full system related to this linear spaces naturally arises.

The goal of this paper is to show that, under appropriate non-resonance conditions for the rates of growth, indeed one can find manifolds tangent to the spaces that grow at the indicated rates. As it will become clear, the non-resonance conditions are necessary for the existence of such invariant manifolds.

For the case of a fixed point of a local diffeomorphism of a Banach space near a fixed point, invariant manifolds associated to non-resonant subsets of the spectrum have been considered in [dlL97] (See also [ElB01], [CFdlL02]). Particular cases of non-resonant manifolds for uniformly hyperbolic systems were considered in [Pes73],[dlLW95], [JPdlL95].

As remarked in Section 8.3 of [dlL97] and in Section 2 of [CFdlL02], the results for fixed points, imply results for more general sets using the device

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of lifting, see [HP70]. Given a dynamical system, f we consider the action $\tilde{f}$ on the Banach space of C^0 vector fields defined by

$$[\tilde{f}v](x) = \exp_x^{-1} f(\exp_{f^{-1}(x)} v(f^{-1}(x))) \tag{1}$$

The zero vector field is, clearly, a fixed point of $\tilde{f}$. Moreover, the linearization of $\tilde{f}$ at the zero field is f_*, the push forward of f acting on C^0 vector fields. Hence, under assumptions on the spectrum of f_*, we can associate invariant manifolds for $\tilde{f}$. As it is well known from [Mat68], the spectral properties of f_* are related to the growth rates of the linearized system.

In this paper, we develop a theory of non-resonant manifolds for orbits that satisfy some rather weak notion of hyperbolicity. This notion is based on properties of each individual orbit and does not require the uniformity assumptions that are required in the lifting approach. Indeed, we introduce a very weak notion of rates of growth — see Definition 1.2 — which generalizes at the same time the notions of exponential dichotomies and the notions of Lyapunov exponents of non-uniform hyperbolic theory.

We note that the non-resonant conditions are automatically satisfied by the most contractive part of the stable spectrum. Hence, the results here generalize the classical results on strong stable manifolds. Nevertheless, in contrast with the strong stable manifolds, the non-resonant manifolds constructed here do not integrate to foliations — see [JPdlL95] —.

For the sake of simplicity, we will formulate the results only for dynamical systems with discrete time $\mathbb{Z}$. Analogous results are true for flows of time $\mathbb{R}$. The results for continuous times follow by taking time one maps of the flows. Of course, it is possible to give a direct proof of the results for flows following the arguments presented here for diffeomorphisms.

As a motivation for the study of the manifolds considered here, we will mention that they give one possible precise meaning to the idea of *slow manifolds* which is used in many heuristic calculations of asymptotic behavior in dynamics — see [Fra88, MP92] for a discussion of several possible meanings of *slow manifolds* —. From a mathematical point of view, we point that one can prove by following the arguments in [dlLMM86] that the solutions of cohomology equations are smooth on the manifolds considered here. The fact that the manifolds considered here do not lead to a foliation provides an obstruction to smooth equivalence of dynamical systems which is not related to periodic orbits and which is not captured by non-autonomous linearization — see [dlL92], [JPdlL95] —.

1.1. **Notions of rates of growth.**

There are two widely used methods to formalize the rates of growth of vectors along an orbit.

One possibility — considered in [SS74], [Fen74], [Fen77] — is to require that there is an uniform expansion or contraction, but that there is an spread on the rates.

That is, for $\mu_1 < \lambda_1 < \cdots < \mu_p < \lambda_p$, one characterizes the bundles E^i_x by:

$$v \in E^i_x \iff |Df^n(x)v| \le \begin{cases} C\exp(\lambda_i n)|v| & n \ge 0 \\ C\exp(\mu_i n)|v| & n \le 0. \end{cases} \tag{2}$$

We note that a consequence of (2) is that the angle between the different spaces E^i_x is bounded from below.

Remark 1.1. As it is well known (see e.g. [Mat68]) if the spectrum of f_* — the push-forward by f — acting on C^0 vector fields is contained in annuli of outer radius e^{λ_i} and inner radius e^{μ_i}, then, every orbit satisfies (2).

Under conditions (2), it is possible to develop a theory of invariant manifolds and foliations based on lifting to actions of bundles as in (1). See [HP70], [HPS77] for the origins of the theory and [CFdlL02] for results for non-resonant invariant manifolds. In this paper, we will, however, base our study on properties of individual orbits rather than on lifting them to bundles.

Another characterization of rates of growth is the existence of Lyapunov exponents considered in [Ose68].

$$v \in E^i_x \iff \lim_{n\to\pm\infty} \frac{1}{n}\ln(|Df^n(x)v|) = \lambda_i. \tag{3}$$

for some real numbers $\lambda_1, \ldots, \lambda_p$.

Notice that Definition (3) ignores polynomial terms in the rate and it can be quite non-uniform along the orbit. As it is well known, Oseledecs' theorem ensures that, given a measure ρ invariant under the system, one has (3) ρ-almost everywhere and, moreover, one can find sets of measure arbitrarily close to full measure where there is some uniformity in the deterioration of the hyperbolicity properties.

One can consider that condition (2) requires that there is an exponential rate which is uniform along the orbit but one has to allow a spread on the rate. On the other hand, in condition (3), there is an exponential rate but one only requires that the exponential rate happens in an averaged sense and that it does not need to be uniform along the orbit.

Neither of the characterizations of rates of growth (2) and (3) is more general than the other. Even if a vector is in one of the subbundles in (2), it may fail to have a Lyapunov exponent if the rates of growth keep on oscillating. Such points are easy to construct in hyperbolic systems (e.g. horseshoes) that have a symbolic dynamics. On the other hand, systems admitting Lyapunov exponents may have fluctuations that destroy the possibility of uniformity.

We introduce a new definition that encompasses both of the previous definitions (2) and (3). We allow the existence of an spread in the exponential rate as well as a deterioration of the constants along the orbit.

Definition 1.2. *Given* $\lambda = \{\lambda_1, \ldots, \lambda_p\}$ *and* $\mu = \{\mu_1, \ldots, \mu_p\}$ *such that* $\mu_1 \le \lambda_1 < \mu_2 \le \lambda_2 < \cdots < \mu_p \le \lambda_p$, $\ell > 1$, $\varepsilon > 0$, *we say that a* *point x has a $\lambda, \mu, \varepsilon, \ell$ regular orbit if we can find invariant decompositions $T_{f^m(x)}M = \oplus_{i=1}^p E^i_{f^m(x)}$ such that:*

(i) $v \in E^i_{f^m(x)}$ *implies for $n \geq 0$,*

$$\begin{aligned} &\ell^{-1} \exp(\mu_i n) \exp(-\varepsilon|m|)|v| \\ &\leq |Df^n(f^m(x))v| \\ &\leq \ell \exp(\lambda_i n) \exp(\varepsilon|m|)|v|. \end{aligned} \tag{4}$$

and for $n < 0$,

$$\begin{aligned} &\ell^{-1} \exp(\lambda_i n) \exp(-\varepsilon|m|)|v| \\ &\leq |Df^n(f^m(x))v| \\ &\leq \ell \exp(\mu_i n) \exp(\varepsilon|m|)|v|. \end{aligned} \tag{5}$$

(ii) angle $(E^i_{f^m(x)}, E^j_{f^m(x)}) \geq \ell^{-1} \exp(-\varepsilon|m|), \quad i \neq j.$

We denote $I_i = [\mu_i, \lambda_i]$. We also denote by $\Lambda_{\lambda,\mu,\varepsilon,\ell}$ the set of $\lambda, \mu, \varepsilon, \ell$ regular orbits.

We recall that if f is a C^1 system and ρ is an ergodic invariant probability measure, Oseledecs multiplicative ergodic theorem implies that, if the Lyapunov exponents are γ_i, $i = 1, \dots, p$, then, for $\varepsilon > 0$, the sets $\Lambda_{\gamma-\varepsilon,\gamma+\varepsilon,\varepsilon,\ell}$ can be made to have measure as close to full as desired by choosing ℓ big enough. That is, ρ-almost all orbits are regular but the constant ℓ cannot be chosen uniformly. The sets $\Lambda_{\gamma+\varepsilon,\gamma-\varepsilon,\varepsilon,\ell}$ are often called Pesin sets.

Condition (2) clearly implies Definition 1.2 taking $\varepsilon = 0$ and ℓ a suitable large number. A fortiori, it is shown in [Mat68] that the fact that the spectrum of the push-forward is contained in annuli of inner radii $\exp \mu_i$ and outer radii $\exp \lambda_i$ is equivalent to the fact that all orbits satisfy Definition 1.2 with $\varepsilon = 0$, ℓ chosen uniformly for all points.

1.2. **Non-resonance.**

Given two intervals $I_1, I_2 \subset \mathbb{R}$, we denote

$$I_1 + I_2 = \{t = t_1 + t_2 : \; t_1 \in I_1, t_2 \in I_2\}.$$

Of course, when the intervals consist of one number, the above operation corresponds to the sum of numbers.

Definition 1.3. *Let $\{I_i\}_{i=1}^p$ be a collection of intervals. Given $S \subset \{1, 2, \dots, p\}$, we say that a subset $\{I_i\}_{i \in S}$ is non-resonant if for $j \in \mathbb{N}$, $j \geq 2$, and any collection $i_1, \dots, i_j \in S$ of indices (perhaps repeated), we have:*

$$I_{i_1} + \cdots + I_{i_j} \cap \Big(\bigcup_{i \in S^c} I_i \Big) = \emptyset. \tag{6}$$

We say that a subset $\{I_i\}_{i \in S}$ is contractive when

$$\bigcup_{i \in S} I_i \subset \mathbb{R}^-. \tag{7}$$

We denote $I^S = \cup_{i\in S} I_i$ and $I^{S^c} = \bigcup_{i\in S^c} I_i$.

We denote by $E^S_x = \bigoplus_{i\in S} E^i_x$ and $E^{S^c}_x = \bigoplus_{i\in S^c} E^i_x$.

We clearly have $T_xM = E^S_x \oplus E^{S^c}_x$. We denote Π^S_x, $\Pi^{S^c}_x$ the projections associated to this decomposition. We call $d_S(x) = \dim E^S_x$ and $d_{S^c}(x) = \dim E^{S^c}_x$.

For collections of intervals I_i and subsets S satisfying (6) and (7), we denote by N_S the integer defined by

$$N_S = \left[\frac{\min\{t \in I^{S^c}\}}{\max\{t \in I^S\}}\right] \tag{8}$$

where $[t]$ denotes integer part of t.

It is clear that for collections satisfying (7), then (6) is automatically verified if $j > \max\{N_S, 0\}$.

Remark 1.4. Note that in part (i) of Definition 1.2 we have only assumed that in the invariant space E^i_x we have the rates of growth (4), (5).

In the literature, it is often assumed that the space E^i_x is precisely characterized by (4), (5).

One interesting example of spaces where assumption (i) of Definition applies but which is not characterized by (4), (5) is Cartesian product systems in which the two factors overlap.

For example, if we take the Cartesian product of a system by itself, $F(x, x) = (f(x), f(y))$ and E^i_x is a spectral decomposition for f, we see that $\tilde{E}^i_{x,y} = E^i_x \times \{0\}$ is an space admissible for our definition. On the other hand, the only spectral space is $(E^i_x \times \{0\}) \oplus (\{0\} \times E^i_y)$.

It is perfectly possible to have situations such as those mentioned here satisfy the non-resonance conditions in Definition 1.3. Note that, there we are only assuming that (6) happens for $j \geq 2$.

Remark 1.5. For collections which satisfy (7), property (6) amounts to a finite number of conditions. It is clear that if we enlarge slightly the intervals both conditions will remain valid.

Remark 1.6. When the intervals are points, it is instructive to compare condition (6) to the conditions in Sternberg linearization theorem. The conditions in Sternberg linearization theorem require that no interval contains sums of points in other intervals. Here we only require that the numbers of the set I^{S^c} cannot be obtained as sums of numbers in the set I^S.

Indeed, the proof presented here has some similarities with the proof of Sternberg theorem. We start by computing a polynomial approximation to the desired object and, then, we use an analytic argument to show that the very approximate polynomial solutions can be modified to become a true solution of the problem.

A Sternberg theorem along orbits under full non-resonance conditions can be found in [Yom88]. Similar results are crucial for [Yom87]. A related theory is the theory of the non-autonomous normal forms [GK98] develop under uniformity conditions on the bundle. It seems clear that one could develop a similar theory under assumptions on the behavior of individual orbits.

Remark 1.7. An important example of non-resonant set is when S includes the I_i contained in $(-\infty, l)$ where $l < 0$, $l \notin \bigcup_{i=1}^{p} I_i$. In such a case, the bundle E_x^S is the strongly stable bundle and our results give the usual strongly stable invariant manifolds.

Note that the strong stable manifold admits a characterization

$$W_s^{ss} = \{y | d(f^n(x), f^n(y)) \leq C_{y,x}\lambda^n\} \tag{9}$$

which makes it clear that $y \in W_x^{ss}$ is an equivalence relation. This makes it clear that the set of strong stable manifolds is a lamination.

For more general non-resonant manifolds, there is not a characterization in terms of rates of growth. In [JPdlL95] one can find examples of situations where the non-resonant manifolds of neighboring points have non-trivial intersections.

1.3. **Statement of main results.**

Definition 1.8. *We will say that a family of maps $\{w_n\}_{n\in\mathbb{Z}}$ is uniformly k differentiable if the maps are k differentiable and $\sup_{n\in\mathbb{Z}} \|D^k w_n\|_{C^0} < \infty$. We will also say that a family of parameterized manifolds $\{W_n\}_{n\in\mathbb{Z}}$ is uniformly k differentiable if there are parameterizations of W_n that are uniformly k differentiable.*

The main result of this paper is:

Theorem 1.9. *Let f be a C^r diffeomorphism, $r \in \mathbb{N} \cup \{\infty\}$, $r \geq 2$, of a compact manifold M. Let $x \in \Lambda_{\lambda,\mu,\varepsilon,\ell}$ and $\{I_i\}_{i\in S}$ be a non-resonant set. Assume that $r \geq N_S + 1$ and that $\varepsilon < \delta/2$ and δ is small enough such that if we consider the enlarged intervals $\tilde{I}_i = I_i + [-2\delta, 2\delta]$, the set $\{\tilde{I}_i\}_{i\in S}$ is still a non-resonant and contractive set.*

Then, there exist maps $w_n : B(0,1) \subset \mathbb{R}^d \longrightarrow M$, $n \in \mathbb{Z}$, in such a way that

(a) $w_n(0) = f^n(x)$.
(b) w_n *are uniformly* C^r.

If we denote by W_n the range of w_n we have:

(c) $f(W_n) \subset W_{n+1}$.
(d) $T_{f^n(x)}W_n = E^S_{f^n(x)}$.

Moreover, there exists K such that the manifolds W_n contain a disk of radius

$$K\ell^{-2}\sqrt{\tanh(\delta)}\exp(-(2\varepsilon+\delta)|n|)$$

and

(e) *If W_n, $\hat{W}_n$ are families of manifolds satisfying* (c), (d), *and both of them are uniformly C^m for some $m \leq r$ then $T^i_{f^n(x)}W_n = T^i_{f^n(x)}\hat{W}_n$, $i \leq m$.*

(f) *If* W_n, $\hat{W}_n$ *are sequences of manifolds satisfying* (c) *and* (d)*, and both of them are uniformly* C^k*,* $k > N_S$*, then,* $W_n \cap B_n = \hat{W}_n \cap B_n$ *for some balls* B_n *around* $f^n(x)$*.*

In particular, if the manifolds are uniformly C^{N_S+1}*, they have to agree with the manifolds* range w_n *and hence they are uniformly* C^r*.*

(g) *If* W_n *is a set of manifolds satisfying* (c), (d), $\sup_{n\in\mathbb{N}} \|W_n\|_{C^{N_s}} < \infty$ *and* W_0 *is a* C^k *manifold for some* $N_S < k \le r$*, then* $\sup_{n\in\mathbb{N}} \|W_{-n}\|_{C^k} < \infty$*.*

The meaning of the above result is that if the set S of rates of growth is non-resonant, we can find some collections of smooth manifolds that are non-linear analogues of the linear subspaces that are invariant for the manifold S.

The final conclusions of Theorem 1.9 are uniqueness conclusions that say that these systems of leaves are unique under some regularity properties.

Roughly, part (f) tells us that the manifolds are unique when they are regular beyond a critical value. In particular, when we know that they are more regular than this critical value, they are as regular as the map. Hence, part (f) is a bootstrap of regularity argument which starts working when the regularity is higher than a critical value related to rates of contraction. Similar bootstrap arguments appear in rigidity theory. See, for example, [dlL92].

Part (g) tells us that the invariant manifolds are unique provided that we have some low regularity in the past. Then, they are roughly as regular as the map. In Section 3, we will provide an example that illustrates the role of the critical regularity in the uniqueness properties.

It is important to remark that the proof will only use constructions in a neighborhood of an orbit.

Remark 1.10. We have formulated Theorem 1.9 only for maps on finite dimensional manifolds.

Nevertheless, there are versions, roughly along the same lines for maps on infinite dimensional Banach spaces which have smooth cutoff functions (e.g. Hilbert spaces).

However, we note that some of the arguments we present – notably the construction of the Lyapunov metric – require some serious modification (it is somewhat easier for Hilbert spaces).

We will not consider these situations in this paper.

Remark 1.11. Given the deterioration of the hyperbolicity properties (measured by ε) the sizes of these manifolds cannot be chosen uniformly and they could become very small. Nevertheless, we can bound the size by an amount that is bounded from below by ε. In particular, in the case that the exponential rate properties do not deteriorate, i.e. $\varepsilon = 0$ — as it is assumed in the results in [Fen74], [Fen77] — then, we obtain that the sizes are uniform.

Remark 1.12. Even in the case that the orbit is a fixed point, the non-resonance condition is necessary for the existence of an invariant manifold.

As we will see, the candidates for a jet satisfy functional equations which may fail to have solutions if the non-resonance conditions are violated. Hence, in those examples where there is no candidate for a jet, there cannot be a smooth invariant manifold.

We refer to Example 5.5 of [dlL97] for more details.

Remark 1.13. Note that, in contrast with many of the results in uniform normal hyperbolicity which are valid for $r \geq 1$, our results are only valid for $r \geq 2$.

It seems to us that the proof will work for $r = 1+\delta$ with mainly notational difficulties. Nevertheless, the proof with $r = 1$ does not work in general since the paper [Pug84] contains an example of a C^1 system satisfying Definition 1.2 for which there is no C^1 invariant manifold.

The paper [Pug84] contains the conjecture that in the case that the stable manifold is one-dimensional, indeed one could get stable manifolds even for $r = 1$.

Remark 1.14. If one has hyperbolicity properties for all the orbits, one could, perhaps hope that the results given here for individual orbits could be made coherent to integrate the distribution E^S_x to give a foliation. For the stable foliation, such procedure is carried out in [Fen74] [Fen77]. Nevertheless, in the generality considered here, the leaves produced here do not integrate to a foliation. Indeed, even in the very uniform case when $\varepsilon = 0$, in [JPdlL95] one can find examples — C^ω close to linear automorphisms of the torus — where these invariant manifolds cross in arbitrarily small neighborhoods.

In some particular cases — e.g. maps of the torus close to linear and when S corresponds to the intervals contained in $\mathbb{R}^-$ — in [dlLW95], there is a way of integrating the foliations based not on local properties but on global behavior. The leaves produced in [dlLW95] are not very smooth and, hence, are very different from the ones considered here. In particular, in the proof of [dlLW95] one has to take into account global properties of the manifold.

2. Proof of Theorem 1.9

The proof of Theorem 1.9 starts very similar to the proof of the invariant manifold in [Pes76] [Pes77] and subsequent papers. We start by defining a Lyapunov metric, which is singular with respect to the Euclidean metric but makes the hyperbolicity properties of the problem uniform. This allows us to choose a convenient system of coordinates in neighborhoods of each of the points in the orbit.

In a second step we see that, writing $W_n = \text{graph } V_n$, under the non-resonance conditions, it is possible to determine uniquely candidates V_n^0 for the jet of the functions V_n. That is, if the functions V_n were differentiable enough, we could take derivatives of the invariance equation and obtain functional equations satisfied by the sequence of jets. We will show that these functional equations admit unique solutions. We emphasize that at this point we only require the non-resonance condition (6) and not the contractive hypothesis (7).

In a third and final step, we use the computed candidates for jets to show that we can transform the equation satisfied by $U_n \equiv V_n - V_n^0$ into an equation that can be dealt with by the contraction mapping principle in some appropriate spaces by using the assumption (7).

The motivation of this scheme is that for functions that vanish at high order at the origin, by taking norms based on derivatives of order N_S the action of a contraction λ on the right is a contraction by λ^{N_S}. One of the consequences of this study will be that the U_n vanish to high order at the origin so that the V_n^0 are indeed the jets of the V_n.

2.1. **The Lyapunov metric and coordinates around the orbit.** The main goal of this section is to establish Lemma 2.1 which provides us with a system of coordinates around an orbit. The main idea — rather standard in the study of non-uniformly hyperbolic systems — is that one can define a Lyapunov metric around an orbit which makes the hyperbolicity uniform. Once we express all the properties in this metric, many of the methods of the theory of uniformly hyperbolic systems start to apply. In our case, we will reduce the problem to the study of systems with exponential dichotomies.

We will use different norms and scalar products. The subindexes E, L, R will stand for Euclidean, Lyapunov and Riemannian norms or scalar products respectively. We also recall that, given a set D, the modulus of continuity of $h_{|D}$ is

$$\omega(h,\eta) = \sup_{y,z\in D,\, |y-z|\le\eta} \|h(y)-h(z)\|.$$

Lemma 2.1. *Let M be a compact C^∞ n-dimensional manifold. Given a C^r map $f: M \to M$, $r \in \mathbb{N}\cup\{\infty\}$, $r \ge 2$, $x \in \Lambda_{\lambda,\mu,\varepsilon,\ell}$, $\delta > 2\varepsilon$, $\tau > 0$ and a fixed (Euclidean) orthogonal decomposition $\mathbb{R}^d = \bigoplus_{i=1}^p E^i$ such that $\dim E^i = \dim E^i_{f^k(x)}$, there exists a sequence of C^∞ maps*

$$\Phi_k : B(0,1) \subset \mathbb{R}^d \to M, \quad k \in \mathbb{Z}$$

such that

(i) $\Phi_k(0) = f^k(x)$.
(ii) $D\Phi_k(0)E^i = E^i_{f^k(x)}$.

If we denote by $f_k = \Phi_{k+1}^{-1} \circ f \circ \Phi_k$ we have:

(iii) $\exp(\mu_i - 2\delta)\|v\|_E \le \|Df_k(0)v\|_E \le \exp(\lambda_i + 2\delta)\|v\|_E$, $v \in E^i$.
(iv) range $(\Phi_k) \supset \{y : d_R(f^k(x), y) \le (2/\pi)\Gamma p^{-1/2}\ell^{-2}\sqrt{\tanh\delta}\exp(-(2\varepsilon + \delta)|k|)\}$, *where Γ is some constant.*
(v) $\sup_{k,\, |x|\le 1,\, 2\le j\le r} \|D^j f_k(x)\| \le \tau$.
(vi) *The modulus of continuity of $D^r f_k$ is bounded independently of k and can be made as small as we want.*

Proof. For a regular orbit we define the Lyapunov inner product in $E^i_{f^k(x)}$ by

$$\begin{aligned}\langle v,w\rangle_L = & \sum_{n\geq 0} e^{-2(\lambda_i+\delta)n}\langle Df^n(f^k(x))v, Df^n(f^k(x))w\rangle_R \\ & + \sum_{n<0} e^{-2(\mu_i-\delta)n}\langle Df^n(f^k(x))v, Df^n(f^k(x))w\rangle_R, \\ & v,w\in E^i_{f^k(x)}\end{aligned} \tag{10}$$

and in $T_{f^k(x)}M$

$$\langle v,w\rangle_L = \sum_{i=1}^p \langle \Pi^i_{f^k(x)}v, \Pi^i_{f^k(x)}w\rangle_L, \quad v,w\in T_{f^k(x)}M,$$

where $\Pi^i_{f^k(x)}$ are the projectors to $E^i_{f^k(x)}$. Note that the subspaces $E^i_{f^k(x)}$ are orthogonal with respect to the Lyapunov metric, and that, by Definition 1.2, denoting by φ_k the minimum angle between the subspaces $E^i_{f^k(x)}$, $\|\Pi^i_{f^k(x)}\| \leq \frac{1}{\sin\varphi_k} \leq \frac{\pi}{2\varphi_k} \leq \frac{\pi}{2}\ell\exp(\varepsilon|k|)$. By the previous definitions, if $v\in E^i_{f^k(x)}$,

$$\begin{aligned}\|v\|_L^2 &\leq \sum_{n\geq 0} e^{-2(\lambda_i+\delta)n}\|Df^n(f^k(x))v\|_R^2 + \sum_{n<0} e^{-2(\mu_i-\delta)n}\|Df^n(f^k(x))v\|_R^2 \\ &\leq \Big[\sum_{n\geq 0} e^{-2(\lambda_i+\delta)n}\ell^2 e^{2\lambda_i n}e^{2\varepsilon|k|} + \sum_{n<0} e^{-2(\mu_i-\delta)n}\ell^2 e^{2\mu_i n}e^{2\varepsilon|k|}\Big]\|v\|_R^2 \\ &= (\ell e^{\varepsilon|k|})^2/\tanh\delta\,\|v\|_R^2\end{aligned}$$

and if $v\in T_{f^k(x)}M$,

$$\|v\|_L^2 = \sum_{i=1}^p \|\Pi^i_{f^k(x)}v\|_L^2 \leq \Big(\sum_{i=1}^p \frac{\pi^2}{4}\ell^4 e^{4\varepsilon|k|}/\tanh\delta\Big)\|v\|_R^2$$

and hence

$$\|v\|_L \leq \Big(\sqrt{p}(\pi/2)\ell^2 e^{2\varepsilon|k|}/\sqrt{\tanh\delta}\Big)\|v\|_R. \tag{11}$$

Next, we define a family of linear maps C_k.

Given the orthogonal decomposition $\mathbb{R}^d = \bigoplus_{i=1}^p E^i$ we define $C_{k|E^i}: E^i \to E^i_{f^k(x)}$ as follows. We take an orthonormal basis $\{e^i_1,\ldots,e^i_{d_i}\}$ in each E^i with respect to the Euclidean scalar product and an orthonormal basis $\{u^i_1,\ldots,u^i_{d_i}\}$ in each $E^i_{f^k(x)}$ with respect to $\langle\cdot,\cdot\rangle_L$. We just define $C_{k|E^i}$ by the relations $C_k e^i_j = u^i_j$, $1\leq j\leq d_i$, $1\leq i\leq p$. With this definition C_k becomes an isometry from $(\mathbb{R}^d,\|\cdot\|_E)$ to $(T_{f^k(x)}M,\|\cdot\|_L)$. Indeed, $\|C_k(\sum\lambda_j e_j)\|_L^2 = \|\sum\lambda_j u_j\|_L^2 = \sum|\lambda_j|^2 = \|\sum\lambda_j e_j\|_E^2$. Moreover, from (11) we have

$$\|C_k^{-1}v\|_E = \|v\|_L \le \frac{\sqrt{p}(\pi/2)\ell^2 e^{2\varepsilon|k|}}{\sqrt{\tanh\delta}}\|v\|_R.$$

We also have

$$\begin{aligned}
\|C_k v\|_R &\le \sum_{i=1}^p \|\Pi^i_{f^k(x)} C_k v\|_R \le \sum_{i=1}^p \|\Pi^i_{f^k(x)} C_k v\|_L \\
&\le \sqrt{p}(\sum_{i=1}^p \|\Pi^i_{f^k(x)} C_k v\|_L^2)^{1/2} = \sqrt{p}\|C_k v\|_L = \sqrt{p}\|v\|_E. \qquad (12)
\end{aligned}$$

Let exp be the exponential map of the Riemannian metric in M. We define

$$\Phi_k(y) = \exp_{f^k(x)}(\gamma_k C_k y)$$

and we claim that if we choose γ_k in a suitable way, Φ_k satisfies all the conclusions of Lemma 2.1.

Indeed, conclusions (i) and (ii) are immediately satisfied. Since $D\Phi_k(0) = \gamma_k C_k$, $D\Phi_k(0)$ transforms the Euclidean metric into γ_k times the Lyapunov metric.

We have

$$\begin{aligned}
Df_k(0) &= D\Phi_{k+1}^{-1}(f^{k+1}(x))Df(f^k(x))D\Phi_k(0) \\
&= \frac{1}{\gamma_{k+1}} C_{k+1}^{-1} Df(f^k(x))\gamma_k C_k
\end{aligned}$$

and if $v \in E^i$

$$\|Df_k(0)v\|_E = \frac{\gamma_k}{\gamma_{k+1}}\|Df(f^k(x))C_k v\|_L.$$

Furthermore , if $v \in E^i$

$$\begin{aligned}
&\|Df(f^k(x))C_k v\|_L^2 \\
&= \sum_{n\geq 0} e^{-2(\lambda_i+\delta)n}\|Df^n(f^{k+1}(x))Df(f^k(x))C_k v\|_R^2 \\
&\quad + \sum_{n<0} e^{-2(\mu_i-\delta)n}\|Df^n(f^{k+1}(x))Df(f^k(x))C_k v\|_R^2 \\
&= \sum_{n\geq 0} e^{-2(\lambda_i+\delta)n}\|Df^{n+1}(f^k(x))C_k v\|_R^2 \\
&\quad + \sum_{n<0} e^{-2(\mu_i-\delta)n}\|Df^{n+1}(f^k(x))C_k v\|_R^2 \\
&= e^{2(\lambda_i+\delta)}\Big[\sum_{n\geq 0} e^{-2(\lambda_i+\delta)n}\|Df^n(f^k(x))C_k v\|_R^2 - \|C_k v\|_R^2\Big] \\
&\quad + e^{2(\mu_i-\delta)}\Big[\sum_{n<0} e^{-2(\mu_i-\delta)n}\|Df^n(f^k(x))C_k v\|_R^2 + \|C_k v\|_R^2\Big] \\
&\leq e^{2(\lambda_i+\delta)}\|C_k v\|_L^2.
\end{aligned}$$

Then

$$\|Df_k(0)v\|_E \leq \frac{\gamma_k}{\gamma_{k+1}} e^{(\lambda_i+\delta)}\|C_k v\|_L.$$

Recall that $\|C_k v\|_L = \|v\|_E$. This proves the upper bound claimed in (iii) in Lemma 2.1.

The proof of the lower bound is completely analogous. Following the previous computations we have

$$\|Df(f^k(x))C_k v\|_L \geq e^{\mu_i-\delta}\|C_k v\|_L$$

and hence

$$\exp(\mu_i-\delta)\frac{\gamma_k}{\gamma_{k+1}}\|v\|_E \leq \|Df_k(0)v\|_E \leq \exp(\lambda_i+\delta)\frac{\gamma_k}{\gamma_{k+1}}\|v\|_E.$$

We take $\gamma_k = \Gamma e^{-\delta|k|}$ with $\Gamma \leq 1$ to be determined later on. Then, if $k \geq 0$, $\frac{\gamma_k}{\gamma_{k+1}} = e^{\delta}$ and, if $k < 0$, $\frac{\gamma_k}{\gamma_{k+1}} = e^{-\delta}$.

To prove (iv) in Lemma 2.1, we observe that $\gamma_k C_k$ sends the unit ball of $\mathbb{R}^d$ to γ_k times the unit ball of $T_{f^k(x)}M$ with respect to the Lyapunov metric. From (11) the unit Lyapunov ball contains the Riemannian ball of radius $\gamma_k \frac{2}{\sqrt{p\pi}}\ell^{-2}e^{-2\varepsilon|k|}\sqrt{\tanh\delta}$ and finally $\exp_{f^k(x)}$ sends this ball to M.

For (v) we evaluate the derivative $D^r f_k = D^r(\Phi_{k+1}^{-1}\circ f\circ\Phi_k)$.

Recall the Faa di Bruno formulas

(13)

$$D^r f_k = \sum_{j=1}^{r} \sum_{\substack{l_1+\cdots+l_j=r \\ 1\leq l_1,\ldots,l_j\leq r}} c^{r,j}_{l_1,\ldots,l_j} D^j\Phi_{k+1}^{-1}\circ(f\circ\Phi_k)D^{l_1}(f\circ\Phi_k)\cdots D^{l_j}(f\circ\Phi_k)$$

and

$$D^l(f\circ\Phi_k)=\sum_{j=1}^{l}\sum_{\substack{m_1+\cdots+m_j=l\\ 1\le m_1,\ldots,m_j\le l}} c^{l,j}_{m_1,\ldots,m_j}D^jf\circ\Phi_k D^{m_1}\Phi_k\cdots D^{m_j}\Phi_k \tag{14}$$

where the c's are combinatorial coefficients which depend on the indices.

In our case, we were interested mainly in

$$\Phi_k=\exp_{f^k(x)}\circ(\gamma_k C_k)$$

and

$$\Phi_{k+1}^{-1}=\frac{1}{\gamma_{k+1}}C_{k+1}^{-1}\exp^{-1}_{f^{k+1}(x)}.$$

From now on K will stand for a constant which only depends on M and the properties of the exponential map.

We observe that

$$\|D^m\Phi_k\|\le K(\gamma_k\|C_k\|)^m.$$

Similarly,

$$\|D^j\Phi_{k+1}^{-1}\|\le K\frac{1}{\gamma_{k+1}}\|C_{k+1}^{-1}\|.$$

By (12) the norm of C_k as a linear map from $(\mathbb{R}^d,\|\cdot\|_E)$ to $(T_{f^{k+1}(x)}M,\|\cdot\|_R)$ is smaller than $\sqrt{p}$.

We deduce that

$$\begin{aligned}\|D^rf_k\| &\le K\ell^2\frac{\gamma_k^r}{\gamma_{k+1}}\frac{e^{2\varepsilon|k|}}{\sqrt{\tanh\delta}}\|f\|_{C^r}\\ &=\frac{K\ell^2\Gamma^{r-1}}{\sqrt{\tanh\delta}}e^{(2\varepsilon-(r-1)\delta)|k|}\|f\|_{C^r}.\end{aligned}$$

Therefore if we take Γ small enough we can adjust the smallness of the derivatives of f_k of order $r\ge 2$.

Finally we prove (vi) of Lemma 2.1. Since M is compact, all derivatives of f are uniformly continuous. Let B_z be the ball of center 0 and radius $\sqrt{p}\,\Gamma$ in T_zM (the radius $\sqrt{p}\,\Gamma$ is motivated by (12)). Since M is C^∞ and compact

$$\sup_{0\le m\le r+1}\sup_{z\in M}\sup_{x\in B_z}\|D^m\exp_z(x)\|<\infty.$$

For $m\le r$ and $\|y\|_E,\|z\|_E\le 1$ we have

$$\|D^m\Phi_k(y)\|_R=\|D^m\exp_{f^k(x)}(\gamma_kC_ky)(\gamma_kC_k)^{\otimes m}\|\le K\Gamma^m e^{-\delta|k|m},$$

and

$$\begin{aligned}\|D^m\Phi_k(y)-D^m\Phi_k(z)\| &\le \|D^{m+1}\exp_{f^{k+1}(x)}\|\,\|\gamma_kC_k\|^{m+1}\|y-z\|\\ &\le K\Gamma^{m+1}e^{-\delta|k|(m+1)}\|y-z\|.\end{aligned}$$

Then by (14) and the fact that $D^j f$, $0 \le j \le r$, are bounded and uniformly continuous, we get that

$$\|D^l(f \circ \Phi_k)\| \le K\Gamma^l e^{-\delta|k|l} \|f\|_{C^l}$$

and

$$\begin{aligned}\omega(D^l(f \circ \Phi_k), \eta) &= \sum_{j=1}^{l} K\Gamma^l e^{-\delta|k|l} \omega(D^j f, K\Gamma\eta) + K\Gamma^{l+1} e^{-\delta|k|(l+1)} \|f\|_{C^l}\eta \\ &= K\Gamma^l e^{-\delta|k|l} o(1) + K\Gamma^{l+1} e^{-\delta|k|(l+1)}\eta.\end{aligned}$$

Next we consider the modulus of continuity of $D^j \Phi_{k+1}^{-1} \circ (f \circ \Phi_k)$. For simplicity, we just bound the modulus of continuity using the mean value theorem assuming that $D^{j+1} f$ is uniformly bounded. We get

$$\omega(D^j \Phi_{k+1}^{-1} \circ (f \circ \Phi_k), \eta) \le K\ell^2 \frac{e^\delta}{\sqrt{\tanh \delta}} e^{2\varepsilon|k+1|} \|f\|_{C^1}\eta.$$

Finally by (13)

$$\begin{aligned}\omega(D^r f_k, \eta) \le & K\ell^2 \frac{e^\delta}{\sqrt{\tanh\delta}} e^{2\varepsilon|k+1|} \Gamma^r e^{-\delta|k|r} \|f\|_{C^1} \|f\|_{C^r}\eta \\ & + K \sum_{j=1}^{r} \sum_{l_j} \Gamma^{-1} e^{\delta|k+1|} \ell^2 \frac{e^{2\varepsilon|k+1|}}{\sqrt{\tanh\delta}} \Gamma^{r-l_j} e^{-\delta|k|(r-l_j)} \\ & \quad \left[\Gamma^{l_j} e^{-\delta|k|l_j} \omega(D^{l_j} f, K\Gamma\eta) + \Gamma^{l_j+1} e^{-\delta|k|(l_j+1)} \|f\|_{C^{l_j}}\eta\right] \\ \le & K\ell^2 \Gamma^{r-1} \frac{e^{(r+1)\delta}}{\sqrt{\tanh\delta}} \Big(e^{[2\varepsilon-(r-1)\delta]|k+1|} \omega(D^r f, K\Gamma\eta) + \Gamma e^{[2\varepsilon - r\delta]|k+1|} \|f\|_{C^r}\eta \Big).\end{aligned}$$

Then, if $\delta > 2\varepsilon/(r-1)$, $\omega(D^r f_k, \eta)$ is bounded independently of k, and, if $r \ge 2$, can be made small taking Γ small. □

Definition 2.2. *Let B_2 be the ball around the origin in $\mathbb{R}^d$ of radius 2. Given a sequence $\mathcal{F} = \{f_n\}_{n\in\mathbb{Z}}$, $f_n \in C^r(B_2, \mathbb{R}^d)$, satisfying $f_n(0) = 0$, we denote*

$$\begin{aligned} A_n &= Df_n(0), \\ \tilde{f}_n &= f_n - A_n, \\ \tilde{\mathcal{F}} &= \{\tilde{f}_n\}_{n\in\mathbb{Z}}, \\ \|\mathcal{F}\|_{C^r(B_2)} &= \sup_{n\in\mathbb{Z}} \|f_n\|_{C^r(B_2)}. \end{aligned}$$

Remark 2.3. Given a family of C^r maps, $\mathcal{F} = \{f_n\}_{n\in\mathbb{Z}}$ with $f_n(0) = 0$, we will consider the family $\mathcal{F}_\lambda = \{f_{n,\lambda}\}_{n\in\mathbb{Z}}$, with $\lambda > 0$ and

$$f_{n,\lambda}(x) = \lambda^{-1} f_n(\lambda x).$$

Notice that, if $r \ge 2$ and $\|\mathcal{F}\|_{C^r(B_2)} < \infty$, then $\|\tilde{\mathcal{F}}_\lambda\|_{C^r(B_2)}$ can be assumed to be arbitrarily small by taking λ small enough. Indeed, if $\mathcal{F} = \{f_n\}_{n\in\mathbb{Z}}$, it is clear that

$$\begin{aligned}|\tilde{f}_{n,\lambda}(x)| &= \lambda^{-1}|f_n(\lambda x) - Df_n(0)\lambda x| \\ &\le \lambda\|D^2 f_n \circ \lambda\|_{C^0(B_2)} \le \lambda\|\mathcal{F}\|_{C^r(B_2)},\end{aligned}$$

and that

$$\begin{aligned}|D\tilde{f}_{n,\lambda}(x)| &= |Df_n(\lambda x) - Df_n(0)| \\ &\le \lambda\|D^2 f_n \circ \lambda\|_{C^0(B_2)} \le \lambda\|\mathcal{F}\|_{C^r(B_2)}.\end{aligned}$$

For $2 \le i \le r$, one has

$$|D^i\tilde{f}_{n,\lambda}(x)| = \lambda^{i-1}|D^i f_n(\lambda x)| \le \lambda^{i-1}\|\mathcal{F}\|_{C^r(B_2)}.$$

Moreover if $\{\text{graph } V_n\}$ is a sequence of manifolds such that $V_n(0) = 0$ and $f_n(\text{graph } (V_n)) \subset \text{graph } (V_{n+1})$, then, defining $V_{n,\lambda}(x) = \lambda^{-1}V_n(\lambda x)$, we have that graph $V_{n,\lambda}$ is sequence of manifolds such that $f_{n,\lambda}(\text{graph } (V_{n,\lambda})) \subset \text{graph } (V_{n+1,\lambda})$.

Remark 2.4. If $r = 1 + \alpha$, $0 < \alpha < 1$, and Df_n are uniformly Hölder, that is, there exists $K > 0$ independent of n such that $|Df_n(x) - Df_n(y)| \le K|x-y|^\alpha$, then $\|\tilde{\mathcal{F}}\|_{C^{1+\alpha}(B_2)}$ can also be assumed to be small taking λ small. This can be easily checked since

$$|\tilde{f}_{n,\lambda}(x)| = \left|\int_0^1 (Df_n(t\lambda x) - Df_n(0))x\,dt\right| \le K\lambda^\alpha/(\alpha+1),$$

$$|D\tilde{f}_{n,\lambda}(x)| \le K\lambda^\alpha,$$

and

$$|D\tilde{f}_{n,\lambda}(x) - D\tilde{f}_{n,\lambda}(y)| = |Df_n(\lambda x) - Df_n(\lambda y)| \le K\lambda^\alpha|x-y|^\alpha.$$

However, if $r = 1$, it may not be possible to make $\|\tilde{\mathcal{F}}_\lambda\|_{C^1(B_2)}$ small. As an example, consider the family $\{f_n\}_{n\ge 1}$, where

$$f_n(x) = \frac{\sin(nx)}{n}.$$

It is clear that $\sup_n \|f_n\|_{C^1} = 1$. But, for all $\lambda > 0$,

$$\tilde{f}_{n,\lambda}(x) = \lambda^{-1}\left(\frac{\sin(\lambda n x)}{n} - \lambda x\right) \to -x, \qquad \text{when } n \to \infty,$$

which implies that $\|\tilde{\mathcal{F}}_\lambda\|_{C^1} \ge 1$, for all $\lambda > 0$.

As a consequence of Lemma 2.1, Theorem 1.9 will be immediately implied by the following theorem.

Theorem 2.5. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{Z}}$ be a family of C^r maps in $\mathbb{R}^q$, $r \in \mathbb{N}\cup\{\infty\}$, $r \ge 2$. Assume that there is a decomposition $\mathbb{R}^q = E^1 \oplus \cdots \oplus E^p$ invariant under A_n, that is, $A_n E^i = E^i$ and a set of real numbers $\tilde{\mu}_1 \le \tilde{\lambda}_1 < \tilde{\mu}_2 \le \cdots \le \tilde{\lambda}_{p-1} < \tilde{\mu}_p \le \tilde{\lambda}_p$, such that*

$$v \in E^i \iff e^{\tilde{\mu}_i}|v| \le |A_n v| \le e^{\tilde{\lambda}_i}|v| \qquad \textit{for all } n \in \mathbb{Z}. \tag{15}$$

Denote by $I_i = [\tilde\mu_i, \tilde\lambda_i]$. *Let* $S \subset \{1,\dots,p\}$ *be such that* $\{I_i : i \in S\}$ *satisfies* (6) *and* (7). *Denote* $E^S = \bigoplus_{i\in S} E^i$, $E^{S^c} = \bigoplus_{i\in S^c} E^i$. *Assume that* $\|\tilde{\mathcal F}\|_{C^{N_S+1}(B_2)}$ *is sufficiently small, the modulus of continuity of* $D^r\tilde f_n$ *is uniformly bounded with respect to* n *and* $r \ge N_S+1$. *Let* B_1 *be the unit ball in* E^S. *Then, there exist* C^r *maps* $V_n : B_1 \subset E^S \to E^{S^c}$ *in such a way that*

(a) $\sup_{n\in\mathbb Z} \|V_n\|_{C^r(B_1)} < \infty$.
(b) $V_n(0) = 0, \quad f_n\big(\mathrm{graph}(V_n)\big) \subset \mathrm{graph}(V_{n+1})$.
(c) $DV_n(0) = 0$.

Moreover, we have that

(d) *If* V_n, $\hat V_n$ *are families of maps satisfying* (b) *and*
$$\sup_{n\in\mathbb Z}\|V_n\|_{C^m(B_1)} < \infty, \qquad \sup_{n\in\mathbb Z}\|\hat V_n\|_{C^m(B_1)} < \infty$$
for some $m \le r$, *then* $D^iV_n(0) = D^i\hat V_n(0)$ *for all* $i \le m$.
(e) *If* V_n, $\hat V_n$ *satisfy* (b), $\sup_{n\in\mathbb Z}\|V_n\|_{C^{N_S+1}(B_1)} \le 1$, $\sup_{n\in\mathbb Z}\|\hat V_n\|_{C^{N_S+1}(B_1)} \le 1$, *then* $V_n = \hat V_n$. *In particular, if* $\hat V_n$ *satisfies* (b) *and* $\sup_{n\in\mathbb Z}\|\hat V_n\|_{C^{N_S+1}(B_1)} \le 1$, $\hat V_n$ *has to agree with the* V_n *produced by this theorem and hence* $\hat V$ *is* C^r *and* $\sup_{n\in\mathbb Z}\|\hat V_n\|_{C^r(B_1)} < \infty$.
(f) *If* V_n *satisfies* (b), $\sup_{n\in\mathbb Z}\|V_n\|_{C^{N_S}(B_1)} < \infty$ *and* $V_0 \in C^k(B_1)$ *for some* $N_S < k \le r$ *then* $\sup_{n\in\mathbb N}\|V_{-n}\|_{C^k(B_1)} < \infty$.

Remark 2.6. If we assume in (e) that $\sup_{n\in\mathbb Z}\|V_n\|_{C^{N_S+1}(B_1)}$, $\sup_{n\in\mathbb Z}\|\hat V_n\|_{C^{N_S+1}(B_1)}$ are just bounded, scaling the maps we get the suprema to be smaller than 1. This implies to work in a smaller domain and hence we obtain uniqueness in this smaller domain.

To apply Theorem 2.5 in proving Theorem 1.9, we take $\tilde\lambda_i = \lambda_i + 2\delta$, $\tilde\mu_i = \mu_i - 2\delta$, where δ is the small number we had to use in the estimates (iii) in Lemma 2.1. Recall that it is enough to take $\delta > 2\varepsilon$.

Note that if a set of intervals is non-resonant and contractive, then if we enlarge them by a sufficiently small quantity, then the enlarged intervals also satisfy the same properties. Hence, when applying Lemma 2.1 to the original situation, we have to pay attention to getting that δ is small enough that the non-resonance conditions and contractivity are still satisfied.

Then, taking $w_n(y) = \phi(y, V_n(y))$ all the conclusions follow.

The rest of the paper will be devoted to a proof of Theorem 2.5.

We denote by A_n^S and $A_n^{S^c}$ the restrictions of A_n to E^S and E^{S^c} respectively, and by Π^S and Π^{S^c} the projectors onto E^S and E^{S^c} respectively. Note that in this setting they are independent of n. Conclusion (b) of Theorem 2.5 can be expressed more explicitly as:

$$V_{n+1}\big(A_n^S x + \Pi^S \tilde f_n(x, V_n(x))\big) = A_n^{S^c} V_n(x) + \Pi^{S^c}\tilde f_n(x, V_n(x)). \tag{16}$$

2.2. **Obtaining a polynomial approximation.** To show that such a sequence of maps V_n exists and satisfies the uniqueness statements, we will start by determining their candidates to derivatives at zero.

That is, if a sequence $\{V_n\}_{n\in\mathbb{Z}}$ satisfies (16), we will show that the derivatives at the origin have to satisfy a functional equation. By studying this functional equation, we will show that, under the non-resonance conditions, there is one and only one bounded solution. Hence, if there is a sequence of differentiable maps, there is only one possibility for their jets at the origin. We will denote this only possibility by V_n^0. Incidentally, the analysis of this equation will show that if the non-resonance conditions are not met, it is possible that the equations for the jet do not have any solution. Hence, a fortiori, that there are no smooth invariant manifolds satisfying the conclusions of Theorem 1.9.

In later sections we will use V_n^0 to transform (16) into another equation which has better contraction properties.

Taking formally i derivatives of (16) at zero we obtain that, if the derivatives exist, they must satisfy

$$D^i V_{n+1}(0)(A_n^S)^{\otimes i} = A_n^{S^c} D^i V_n(0) + R_{n,i}, \tag{17}$$

where $R_{n,i}$ is an expression that involves only derivatives of V_n up to order $i-1$ and derivatives of f_n up to order i. We claim that the hierarchy of equations (17) can be solved recursively and that the solution is unique.

In order to solve (17), we recall that $D^i V_n(0)$ is a linear map from $(E^S)^{\otimes i}$ to E^{S^c}. We claim

Lemma 2.7. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{Z}}$ be a family of maps, $\|\mathcal{F}\|_{C^k(B_2)} < \infty$, such that $Df_n(0) = A_n$ satisfies the hypotheses of Theorem 2.5. Then, for all $1 \le i \le k$, the family of equations (17) have a unique solution in the Banach space*

$$\{\mathcal{T}^i = (T_n^i) :\ T_n^i \in L((E^S)^{\otimes i}, E^{S^c}),\ \|\mathcal{T}^i\| = \sup_n \|T_n^i\| < \infty\}.$$

Moreover the norm of the solution can be made as small as we want by taking $\|\tilde{\mathcal{F}}\|_{C^k(B_2)}$ small.

For $i = 1$ the only bounded solution is $DV_n(0) = 0$.

This result proves parts (c) and (d) of Theorem 2.5.

Proof of the lemma We will solve equations (17) in the components corresponding to the decomposition E^i. Notice that the decomposition

$$(E^S)^{\otimes i} = \bigoplus_{j_1,\dots,j_i\in S} E^{j_1} \otimes \cdots \otimes E^{j_i}$$

is preserved by the linear map

$$(A_n^S)^{\otimes i} = \bigoplus_{j_1,\dots,j_i\in S} A_n^{j_1} \otimes \cdots \otimes A_n^{j_i} : (E^S)^{\otimes i} \to (E^S)^{\otimes i},$$

where

$$A_n^{j_1} \otimes \cdots \otimes A_n^{j_i} : E^{j_1} \otimes \cdots \otimes E^{j_i} \to E^{j_1} \otimes \cdots \otimes E^{j_i}$$

and $\|A_n^{j_1} \otimes \cdots \otimes A_n^{j_i}\| = \|A_n^{j_1}\| \cdots \|A_n^{j_i}\|$.

If $T_n^i \in L((E^S)^{\otimes i}, E^{S^c})$, we can use the decomposition of $(E^S)^{\otimes i}$ and $E^{S^c} = \bigoplus_{i \in S^c} E^i$ to write

$$T_n^i = \bigoplus_{l \in S^c,\, j_1,\dots,j_i \in S} \tau^l_{n,j_1,\dots,j_i}$$

where

$$\tau^l_{n,j_1,\dots,j_i} : E^{j_1} \otimes \cdots \otimes E^{j_i} \to E^l$$

is linear. The same decomposition is made to the terms $R_{n,i}$ of (17), which also belong to $L((E^S)^{\otimes i}, E^{S^c})$, to obtain $R_{n,i} = \bigoplus_{l \in S^c,\, j_1,\dots,j_i \in S} R^l_{n,j_1,\dots,j_i}$.

Then, equations (17) are equivalent to the set of equations

$$\tau^l_{n+1,j_1,\dots,j_i} A_n^{j_1} \otimes \cdots \otimes A_n^{j_i} = A_n^l \tau^l_{n,j_1,\dots,j_i} + R^l_{n,j_1,\dots,j_i}, \tag{18}$$

with $n \in \mathbb{Z}, j_1, \dots, j_i \in S, l \in S^c$.

Equations (18) can be rewritten in either the form

$$\tau^l_{n,j_1,\dots,j_i} = (A_n^l)^{-1} \tau^l_{n+1,j_1,\dots,j_i} A_n^{j_1} \otimes \cdots \otimes A_n^{j_i} - (A_n^l)^{-1} R^l_{n,j_1,\dots,j_i} \tag{19}$$

or in the form

$$\tau^l_{n+1,j_1,\dots,j_i} = A_n^l \tau^l_{n,j_1,\dots,j_i} (A_n^{j_1} \otimes \cdots \otimes A_n^{j_i})^{-1} + R^l_{n,j_1,\dots,j_i} (A_n^{j_1} \otimes \cdots \otimes A_n^{j_i})^{-1}. \tag{20}$$

Both equations (19), (20) can be considered as fixed point equations and the right hand side of each of them defines continuous affine maps on the Banach space

$$\begin{aligned}\{\tau^l_{j_1,\dots,j_i} = (\tau^l_{n+1,j_1,\dots,j_i})_{n \in \mathbb{Z}} :\ & \tau^l_{n,j_1,\dots,j_i} \in L(E^{j_1} \otimes \cdots \otimes E^{j_i}, E^l), \\ & \|\tau^l_{j_1,\dots,j_i}\| = \sup_{n \in \mathbb{Z}} \|\tau^l_{n+1,j_1,\dots,j_i}\| < \infty\}.\end{aligned}$$

The map associated to (19) has a Lipschitz constant which is bounded from above by

$$\exp(\tilde\lambda_{j_1} + \cdots + \tilde\lambda_{j_i} - \tilde\mu_l) \tag{21}$$

while the map associated to (20) has a Lipschitz constant which is bounded from above by

$$\exp(\tilde\lambda_l - \tilde\mu_{j_1} - \cdots - \tilde\mu_{j_i}). \tag{22}$$

Since the intervals $\{I_i\}_{i \in S}$ are non-resonant, one of the two numbers (21), (22) is smaller than 1. Then, one of the two equations (19), (20) has a right hand side which defines a contraction. Hence, one of (19), (20) can be solved by the contraction mapping principle.

Since both equations (19), (20) are equivalent to (18), we have proved that the equations (18) have a unique solution.

All terms $R_{n,i}$ have a factor $D^j\tilde{f}(0,0)$, $2 \le j \le i$, which is small if $\|\tilde{\mathcal{F}}\|_{C^k(B_2)}$ is small. Therefore $R_{n,j_1,\dots,j_i}$ are also small and hence the non-homogeneous parts of (19), (20) are small. From that we deduce that the unique solutions of (18) are small.

Notice that the equations for the first derivative are homogeneous. It follows, then, that $DV_n(0) = 0$ is the unique bounded solution of equations (17) for $i = 1$. This establishes Lemma 2.7. □

2.3. **The high order part of the manifolds.** We now rewrite the equations of invariance in such a way that we can take advantage of the fact that we already know the low order terms of the expansion of the solution of the invariance equation.

We will write the sequence of maps $\mathcal{V} = \{V_n\}_{n\in\mathbb{Z}}$ in (16) in the form $\mathcal{V} = \mathcal{V}^0 + \mathcal{U}$ with $\mathcal{V}^0 = \{V_n^0\}_{n\in\mathbb{Z}}$, $\mathcal{U} = \{U_n\}_{n\in\mathbb{Z}}$ and

$$V_n = V_n^0 + U_n, \tag{23}$$

where V_n^0 is the unique polynomial of degree N_S whose derivatives at 0 satisfy equations (17) and $D^iU_n(0) = 0$, for $0 \le i \le N_S$. Note that $\|\mathcal{V}^0\|_{C^r}$ $r \in \mathbb{N}$ can be made as small as we need.

Next we reformulate our original problem in terms of the U_n.

As it turns out, this will be very convenient since we will have that the U_n are maps that vanish to high order. For such kind of maps, composing with contractions on the right, leads to a rather strong contraction factor.

If we substitute (23) in the equation of invariance (16) we obtain a fixed point equation for the operator σ acting on sequences of maps $\mathcal{U} = \{U_n\}_{n\in\mathbb{Z}}$, defined by

$$\begin{aligned} \sigma(\mathcal{U})_n(x) \;=\;& (A_n^{S^c})^{-1}\big[(V_{n+1}^0 + U_{n+1}) \circ \psi(\mathcal{U})_n(x) \\ & - \Pi^{S^c}\tilde{f}_n(x, V_n^0(x) + U_n(x))\big] - V_n^0(x), \end{aligned} \tag{24}$$

where

$$\psi(\mathcal{U})_n(x) = A_n^S x + \Pi^S \tilde{f}_n(x, V_n^0(x) + U_n(x)). \tag{25}$$

A simple algebraic manipulation shows that, at least formally — ignoring questions of whether the compositions can be defined —, it is equivalent to say that $\mathcal{U}$ is a fixed point of (24) than to say that $\mathcal{V}^0 + \mathcal{U}$ is a solution of (16). The plan will be to produce fixed points of (24) in a space of well behaved functions for which the algebraic manipulations involved in transforming (16) into (24) can be justified.

For the moment, we will consider only the case that $r \in \mathbb{N}$. Later we will see how this result can be extended to the cases $r = \omega, \infty$.

More precisely, we consider σ acting on the following spaces:

$$\begin{aligned} \mathcal{X}_k \;=\;& \{\mathcal{U} = \{U_n\}_{n\in\mathbb{Z}} :\ U_n \in C^k(B_1),\ D^iU_n(0) = 0,\ 0 \le i \le N_S, \\ & \|\mathcal{U}\|_{\mathcal{X}_k} \equiv \sup_{n\in\mathbb{Z}} \max_{N_S < i \le k} \|D^iU_n\|_{C^0} < \infty\}, \qquad \text{for } k > N_S, \end{aligned} \tag{26}$$

and

$$\begin{aligned}\mathcal{X}^0 &= \{\mathcal{U} = \{U_n\}_{n\in\mathbb{Z}} :\ U_n \in C^{N_S}(B_1),\ D^i U_n(0) = 0,\ 0 \le i \le N_S, \\ & \qquad \|\mathcal{U}\|_{\mathcal{X}^0} \equiv \sup_{n\in\mathbb{Z}} \sup_{x\in B_1\setminus\{0\}} |x|^{-1} |D^{N_S} U_n(x)| < \infty\}. \end{aligned} \tag{27}$$

Note that, by the mean value theorem, if $k > N_S$, $\mathcal{X}_k \subset \mathcal{X}^0$. Moreover, if $\mathcal{U} \in \mathcal{X}_k$, $\|\mathcal{U}\|_{\mathcal{X}^0} \le \|\mathcal{U}\|_{\mathcal{X}_k}$.

Let $\mathcal{B}_{N_S+1} = \{\mathcal{U} \in \mathcal{X}_{N_S+1} : \|\mathcal{U}\|_{\mathcal{X}_{N_S+1}} \le 1\}$ and $\mathcal{B}(\rho_{N_S+2}, \dots, \rho_k)$ be the set of families $\mathcal{U} = \{U_n\}_{n\in\mathbb{Z}} \in \mathcal{X}_k$ such that

$\sup_{n\in\mathbb{Z}} \|D^{N_S+1} U_n\|_{C^0} \le 1.$
$\sup_{n\in\mathbb{Z}} \|D^j U_n\|_{C^0} \le \rho_j, \qquad N_S + 1 < j \le k.$

Clearly $\mathcal{B}(\rho_{N_S+2}, \dots, \rho_k) \subset \mathcal{B}_{N_S+1}$. Note that if $\mathcal{U} \in \mathcal{X}_k$, we can reconstruct U_n from $D^{N_S+1} U_n$ and $\|D^i U_n(x)\|/|x|^{N_S+1-i} \le (1/(N_S+1-i)!)\|U_n\|_{C^{N_S+1}}$, for $0 \le i \le N_S + 1$. Therefore we get that if $\|D^{N_S+1}(\sigma\mathcal{U})_n\|_{C^0} \le 1$, we also have that $\|D^i(\sigma\mathcal{U})_n\|_{C^0} \le 1$, for $0 \le i \le N_S$.

Lemma 2.8. *Let σ be defined as in (24). Under appropriate smallness conditions in $\|\tilde{\mathcal{F}}\|_{C^{N_S+1}(B_2)}$ there exist numbers $\rho_{N_S+2}, \dots, \rho_k > 0$ such that*

(i) $\sigma(\mathcal{B}_{N_S+1}) \subset \mathcal{B}_{N_S+1}$.
(ii) *σ restricted to $\mathcal{B}_{N_S+1}$ is a contraction in the $\mathcal{X}^0$ norm.*

Moreover, under the same smallness conditions on $\|\tilde{\mathcal{F}}\|_{C^{N_S+1}(B_2)}$ needed for (i)*,* (ii)

(iii) $\sigma(\mathcal{B}(\rho_{N_S+2}, \dots, \rho_k)) \subset \mathcal{B}(\rho_{N_S+2}, \dots, \rho_k)$*, for $N_S + 2 \le k \le r$.*

For future reference, it will be important to note that the smallness conditions assumed in Lemma 2.8 are only smallness assumptions $\|\tilde{\mathcal{F}}\|_{C^{N_S+1}(B_2)}$ and do not change as we increase r.

Before proving the lemma we first establish some bounds for ψ. Given $\mathcal{U}, \hat{\mathcal{U}} \in \mathcal{B}_{N_S+1}$ we have

$$\|D\psi(\mathcal{U})_n\| \le \|A_n^S\| + K\|\tilde{\mathcal{F}}\|_{C^1}, \tag{28}$$

$$-\hat{\mathcal{U}}\|_{C^0}, |x|^{-1}\|\psi(\mathcal{U})_n(x)\| \le \|A_n^S\| + K\|\tilde{\mathcal{F}}\|_{C^1}, \tag{29}$$

$$|x|^{-1}\|(\psi(\mathcal{U})_n - \psi(\hat{\mathcal{U}})_n)(x)\| \le K\|\tilde{\mathcal{F}}\|_{C^1}\|\mathcal{U} - \hat{\mathcal{U}}\|_{\mathcal{X}^0}, \tag{30}$$

$$|x|^{-1}\|D(\psi(\mathcal{U})_n - \psi(\hat{\mathcal{U}})_n)(x)\| \le K\|\tilde{\mathcal{F}}\|_{C^2}\|\mathcal{U} - \hat{\mathcal{U}}\|_{\mathcal{X}^0}, \tag{31}$$

where K is a constant independent of n. Formula (28) is straightforward from the definition of ψ, in (25), while (29)–(31) follow from the fact that $\tilde{f}_n(0) = 0$, $D\tilde{f}_n(0) = 0$, $V_n^0(0) = 0$ and the derivatives of V_n^0 and U_n are uniformly bounded.

Proof of Lemma 2.8. It is clear that, by the choice of the polynomials V_n^0,

$$D^i\sigma(\mathcal{U})_n(0) = 0, \qquad 0 \le i \le N_S.$$

We observe that, for $k \geq 2$, by the Faa-di Bruno formula,

(32)
$$D^k(\sigma\mathcal{U})_n = (A_n^{S^c})^{-1} D^k U_{n+1} \circ \psi(\mathcal{U})_n D\psi(\mathcal{U})_n^{\otimes k} + B_n(\mathcal{U}) D^k U_n + R_{n,k}(\mathcal{U}),$$

where

$$\begin{aligned} B_n(\mathcal{U}) &= (A_n^{S^c})^{-1} D(V_{n+1}^0 + U_{n+1}) \circ \psi(\mathcal{U})_n \Pi^S D_2 \tilde{f}_n \circ \eta(\mathcal{U})_n \\ &\quad -(A_n^{S^c})^{-1} \Pi^{S^c} D_2 \tilde{f}_n \circ \eta(\mathcal{U})_n \end{aligned} \tag{33}$$

is linear,

$$\eta(\mathcal{U})_n = (\mathrm{Id}\,, V_n^0 + U_n) \tag{34}$$

and, since $k \geq N_S + 1$,

(35)
$$\begin{aligned} R_{n,k}(\mathcal{U}) &= \\ &(A_n^{S^c})^{-1} \Big(\sum_{l=2}^{N_S} \sum_{*} c_{j_1,\ldots,j_l}^{k,l} D^l V_{n+1}^0 \circ \psi(\mathcal{U})_n D^{j_1}\psi(\mathcal{U})_n \otimes \cdots \otimes D^{j_l}\psi(\mathcal{U})_n \\ &+ \sum_{l=2}^{k-1} \sum_{*} c_{j_1,\ldots,j_l}^{k,l} D^l U_{n+1} \circ \psi(\mathcal{U})_n D^{j_1}\psi(\mathcal{U})_n \otimes \cdots \otimes D^{j_l}\psi(\mathcal{U})_n \\ &+ D(V_{n+1}^0 + U_{n+1}) \circ \psi(\mathcal{U})_n \times \\ &\qquad \times \Pi^S \sum_{l=2}^{k} \sum_{*} c_{j_1,\ldots,j_l}^{k,l} D^l \tilde{f}_n \circ \eta(\mathcal{U})_n D^{j_1}\eta(\mathcal{U})_n \otimes \cdots \otimes D^{j_l}\eta(\mathcal{U})_n \\ &- \Pi^{S^c} \sum_{l=2}^{k} \sum_{*} c_{j_1,\ldots,j_l}^{k,l} D^l \tilde{f}_n \circ \eta(\mathcal{U})_n D^{j_1}\eta(\mathcal{U})_n \otimes \cdots \otimes D^{j_l}\eta(\mathcal{U})_n \Big), \end{aligned}$$

where $\sum_*$ stands for the sum over the indices j_i such that $1 \leq j_1, \ldots, j_l \leq k$ and $j_1 + \cdots + j_l = k$ and the coefficients $c_{j_1,\ldots,j_l}^{k,l}$ are combinatorial numbers.

Notice that a first derivative of $\tilde{f}_n$ appears as a factor in each term in $B_n(\mathcal{U})$. Also, $R_{n,k}(\mathcal{U})$ consists on a finite sum of terms. Some of them have explicitly a derivative of $\tilde{f}_n$ as a factor.

The other terms have a factor of the form

$$D^{j_1}\psi(\mathcal{U})_n \otimes \cdots \otimes D^{j_l}\psi(\mathcal{U})_n \tag{36}$$

where

$$j_1 + \cdots + j_l = k \quad \text{and} \quad l \leq k-1. \tag{37}$$

Because of (37), there is some i such that $j_i \geq 2$ and therefore, among the factors in (36), there is a factor of the form

$$D^{j_i}\psi(\mathcal{U})_n = D^{j_i}(\tilde{f}_n \circ \eta_n(\mathcal{U})).$$

Now we consider the case $k = N_S + 1$. The previous factors can be made arbitrarily small by assuming that $\|\tilde{\mathcal{F}}\|_{C^{N_S+1}}$ is sufficiently small. As a consequence, for any given $\nu > 0$, if $\|\tilde{\mathcal{F}}\|_{C^{N_S+1}}$ is small enough, we have that

$$\|B_n(\mathcal{U})\|,\ \|R_{n,N_S+1}(\mathcal{U})\|_{C^0} < \nu.$$

Since $\sup_n \|(A_n^{S^c})^{-1}\|\,\|A_n^S\|^{N_S+1} < 1$, there exists $\nu > 0$ such that

$$\gamma \equiv \|(A_n^{S^c})^{-1}\|\,(\|A_n^S\| + \nu)^{N_S+1} + 2\nu < 1$$

and hence

$$\|D^{N_S+1}(\sigma\mathcal{U})_n\|_{C^0} \le \sup_n \big(\|(A_n^{S^c})^{-1}\|\,(\|A_n^S\| + \nu)^{N_S+1} + \nu\big)\|D^{N_S+1}\mathcal{U}\|_{C^0} + \nu \le \gamma,$$

for all $\mathcal{U}$ in $\mathcal{B}_{N_S+1}$. This proves (i).

If $N_S + 1 < k \le r$, from (32) we have that

$$\|D^k(\sigma\mathcal{U})_n\| \le \gamma \sup_n \|D^k U_n\| + Q(\rho_{N_S+2}, \dots, \rho_{k-1}),$$

where Q is a polynomial. Therefore, since $\gamma < 1$, there exists $\rho_k > 0$ such that $\rho_k = \gamma\rho_k + Q(\rho_{N_S+2}, \dots, \rho_{k-1})$ and $\sigma(\mathcal{B}(\rho_{N_S+2}, \dots, \rho_k)) \subset \mathcal{B}(\rho_{N_S+2}, \dots, \rho_k)$.

To prove (ii) we take derivatives, and we obtain

$$
\begin{aligned}
& |x|^{-1}\|D^{N_S}(\sigma(\mathcal{U})_n - \sigma(\hat{\mathcal{U}})_n)(x)\| \\
\le\ & |x|^{-1}\|(A_n^{S^c})^{-1} D^{N_S}(U_{n+1} - \hat{U}_{n+1}) \circ \psi(\mathcal{U})_n(x) D\psi(\mathcal{U})_n(x)^{\otimes N_S}\| && (38)\\
+\ & |x|^{-1}\|(A_n^{S^c})^{-1}(D^{N_S}\hat{U}_{n+1} \circ \psi(\mathcal{U})_n(x) && (39)\\
& \qquad - D^{N_S}\hat{U}_{n+1} \circ \psi(\hat{\mathcal{U}})_n)(x) D\psi(\mathcal{U})_n(x)^{\otimes N_S}\| \\
+\ & |x|^{-1}\|(A_n^{S^c})^{-1} D^{N_S}\hat{U}_{n+1} \circ \psi(\hat{\mathcal{U}})_n(x) \times && (40)\\
& \qquad \times (D\psi(\mathcal{U})_n(x)^{\otimes N_S} - D\psi(\hat{\mathcal{U}})_n(x)^{\otimes N_S})\| \\
& +|x|^{-1}\|B_n(\mathcal{U})(x) D^{N_S}(U_n - \hat{U}_n)(x)\| && (41)\\
& +|x|^{-1}\|(B_n(\mathcal{U}) - B_n(\hat{\mathcal{U}}))(x) D^{N_S}\hat{U}_n(x)\| && (42)\\
& +|x|^{-1}\|(R_{n,N_S}(\mathcal{U}) - R_{n,N_S}(\hat{\mathcal{U}}))(x)\|. && (43)
\end{aligned}
$$

Taking into account inequalities (28) and (29), we bound (38) as follows,

$$
\begin{aligned}
& |x|^{-1}\|(A_n^{S^c})^{-1} D^{N_S}(U_{n+1} - \hat{U}_{n+1}) \circ \psi(\mathcal{U})_n(x) D\psi(\mathcal{U})_n(x)^{\otimes N_S}\| \\
& \quad \le |x|^{-1}|\psi(\mathcal{U})_n(x)|\,\|(A_n^{S^c})^{-1}\|\,\|D\psi(\mathcal{U})_n\|^{N_S}\|U_{n+1} - \hat{U}_{n+1}\|_{\mathcal{X}^0} \\
& \quad \le \|(A_n^{S^c})^{-1}\|\big(\|A_n^S\| + K\|\tilde{\mathcal{F}}\|_{C^1}\big)^{N_S+1}\|U_{n+1} - \hat{U}_{n+1}\|_{\mathcal{X}^0}.
\end{aligned}
$$

To bound (39), we use inequality (30) and the fact that $\mathcal{U}, \hat{\mathcal{U}} \in \mathcal{B}_{N_S+1}$. In this way,

$$
\begin{aligned}
& |x|^{-1}\|(A_n^{S^c})^{-1}(D^{N_S}\hat{U}_{n+1} \circ \psi(\mathcal{U})_n(x) - D^{N_S}\hat{U}_{n+1} \circ \psi(\hat{\mathcal{U}})_n(x)) D\psi(\mathcal{U})_n(x)^{\otimes N_S}\| \\
& \quad \le \|(A_n^{S^c})^{-1}\|\,\|D\psi(\mathcal{U})_n\|^{N_S}\|\hat{\mathcal{U}}\|_{C^{N_S+1}}|x|^{-1}\|(\psi(\mathcal{U})_n - \psi(\hat{\mathcal{U}})_n)(x)\| \\
& \quad \le K\|(A_n^{S^c})^{-1}\|\big(\|A_n^S\| + K\|\tilde{\mathcal{F}}\|_{C^1}\big)^{N_S}\|\hat{\mathcal{U}}\|_{C^{N_S+1}}\|\tilde{\mathcal{F}}\|_{C^1}\|\mathcal{U} - \hat{\mathcal{U}}\|_{\mathcal{X}^0}
\end{aligned}
$$

Term (40) can be bounded in the following way, using inequalities (31) and (28),

$$\begin{aligned}
&|x|^{-1}\|(A_n^{S^c})^{-1}D^{N_S}\hat{U}_{n+1}\circ\psi(\hat{\mathcal{U}})_n(x)(D\psi(\mathcal{U})_n(x)^{\otimes N_S}-D\psi(\hat{\mathcal{U}})_n(x)^{\otimes N_S})\|\\
&\quad\leq \|(A_n^{S^c})^{-1}\|\big(\|A_n^S\|+K\|\tilde{\mathcal{F}}\|_{C^1}\big)\|\hat{U}_{n+1}\|_{C^{N_S}}\\
&\qquad\quad |x|^{-1}\|D\psi(\mathcal{U})_n(x)^{\otimes N_S}-D\psi(\hat{\mathcal{U}})_n(x)^{\otimes N_S}\|\\
&\quad\leq N_S K\|(A_n^{S^c})^{-1}\|\big(\|A_n^S\|+K\|\tilde{\mathcal{F}}\|_{C^1}\big)^{N_S}\|\hat{U}_{n+1}\|_{C^{N_S}}\|\tilde{\mathcal{F}}\|_{C^2}\|\mathcal{U}-\hat{\mathcal{U}}\|_{\mathcal{X}^0},
\end{aligned}$$

where we have bounded

$$\begin{aligned}
&|x|^{-1}\|D\psi(\mathcal{U})_n(x)^{\otimes N_S}-D\psi(\hat{\mathcal{U}})_n(x)^{\otimes N_S}\|\\
&\quad\leq \sum_{j=1}^{N_S}|x|^{-1}\|D\psi(\mathcal{U})_n(x)-D\psi(\hat{\mathcal{U}})_n(x)\|\,\|D\psi(\mathcal{U})_n\|^{N_S-j}\|D\psi(\hat{\mathcal{U}})_n\|^{j-1}\\
&\quad\leq N_S K\big(\|A_n^S\|+K\|\tilde{\mathcal{F}}\|_{C^1}\big)^{N_S-1}\|\tilde{\mathcal{F}}\|_{C^2}\|\mathcal{U}-\hat{\mathcal{U}}\|_{\mathcal{X}^0}.
\end{aligned}$$

Term (41) can be easily bounded taking into account that $\|B_n(\mathcal{U})\|\leq\tilde{K}\|\tilde{\mathcal{F}}\|_{C^1}$, for some $\tilde{K}>0$.

To obtain a bound for (42) we proceed in the following way:

(44)

$$\begin{aligned}
\|B_n(\mathcal{U})-B_n(\hat{\mathcal{U}})\|\leq&\|(A_n^{S^c})^{-1}\|\,\|D(V_{n+1}^0+U_{n+1})\circ\psi(\mathcal{U})_n\Pi^S D_2\tilde{f}_n\circ\eta(\mathcal{U})_n\\
&\qquad-D(V_{n+1}^0+\hat{U}_{n+1})\circ\psi(\hat{\mathcal{U}})_n\Pi^S D_2\tilde{f}_n\circ\eta(\hat{\mathcal{U}})_n\|\\
&+\|\Pi^{S^c}D_2\tilde{f}_n\circ\eta(\mathcal{U})_n-\Pi^{S^c}D_2\tilde{f}_n\circ\eta(\hat{\mathcal{U}})_n\|.
\end{aligned}$$

Since $\|\eta(\mathcal{U})_n(x)-\eta(\hat{\mathcal{U}})_n(x)\|=\|U_n(x)-\hat{U}_n(x)\|$, the last term of (44) can be bounded immediately by

$$\|\tilde{\mathcal{F}}\|_{C^2}\|\mathcal{U}-\hat{\mathcal{U}}\|_{\mathcal{X}^0}.$$

The first term in (44) can be split into the three following terms that can be easily bounded

$$\begin{gathered}
\|D(U_{n+1}-\hat{U}_{n+1})\circ\psi(\mathcal{U})_n\Pi^S D_2\tilde{f}_n\circ\eta(\mathcal{U})_n\|,\\
\|\big(D(V_{n+1}^0+\hat{U}_{n+1})\circ\psi(\mathcal{U})_n-D(V_{n+1}^0+\hat{U}_{n+1})\circ\psi(\hat{\mathcal{U}})_n\big)\Pi^S D_2\tilde{f}_n\circ\eta(\mathcal{U})_n\|,\\
\|D(V_{n+1}^0+\hat{U}_{n+1})\circ\psi(\hat{\mathcal{U}})_n\big(\Pi^S D_2\tilde{f}_n\circ\eta(\hat{\mathcal{U}})_n-\Pi^S D_2\tilde{f}_n\circ\eta(\mathcal{U})_n\big)\|.
\end{gathered}$$

Finally, to bound (43), we recall that $R_{n,N_S}(\mathcal{U})$ is a finite sum of terms having expressions of the following forms:

1. Terms of the form

$$D^l V_{n+1}^0\circ\psi(\mathcal{U})_n D^{j_1}\psi(\mathcal{U})_n\otimes\cdots\otimes D^{j_l}\psi(\mathcal{U})_n,\qquad 2\leq l\leq N_S,\ j_1+\cdots+j_l=k.$$

The norm of their difference is easily bounded by

$$\begin{aligned}
&|x|^{-1}\|\big(D^l V_{n+1}^0\circ\psi(\mathcal{U})_n-D^l V_{n+1}^0\circ\psi(\hat{\mathcal{U}})_n\big)D^{j_1}\psi(\mathcal{U})_n\otimes\cdots\otimes D^{j_l}\psi(\mathcal{U})_n\|\\
&+|x|^{-1}\|D^l V_{n+1}^0\circ\psi(\hat{\mathcal{U}})_n\big(D^{j_1}\psi(\mathcal{U})_n\otimes\cdots\otimes D^{j_l}\psi(\mathcal{U})_n\\
&\qquad-D^{j_1}\psi(\hat{\mathcal{U}})_n\otimes\cdots\otimes D^{j_l}\psi(\hat{\mathcal{U}})_n\big)\|.
\end{aligned}$$

Both differences can be bounded by some constant times $\|\tilde{\mathcal{F}}\|_{C^2}\|\mathcal{U} - \hat{\mathcal{U}}\|_{\mathcal{X}^0}$.

2. Terms of the form

$$D^l U_{n+1} \circ \psi(\mathcal{U})_n D^{j_1}\psi(\mathcal{U})_n \otimes \cdots \otimes D^{j_l}\psi(\mathcal{U})_n, \quad 2 \le l \le N_S - 1,\ j_1 + \cdots + j_l = k.$$

Since $\mathcal{U} \in \mathcal{B}_{N_S+1}$, their difference can be bounded as in the previous case.

3. Terms of the form

$$D^l \tilde{f}_n \circ \eta(\mathcal{U})_n D^{j_1}\eta(\mathcal{U})_n \otimes \cdots \otimes D^{j_l}\eta(\mathcal{U})_n, \qquad 1 \le l \le N_S,\ j_1 + \cdots + j_l = k,$$

which can be treated as before, since $\tilde{f}_n$ is C^{N_S+1}.

4. Terms of the form

$$D(V^0_{n+1} + U_{n+1}) \circ \psi(\mathcal{U})_n \Pi^S D^l \tilde{f}_n \circ \eta(\mathcal{U})_n D^{j_1}\eta(\mathcal{U})_n \otimes \cdots \otimes D^{j_l}\eta(\mathcal{U})_n,$$

where $2 \le l \le N_S,\ j_1 + \cdots + j_l = k$, that can be bounded like the preceding ones.

It follows, then, that the Lipschitz constant of σ is less than

$$\|(A^{S^c}_n)^{-1}\|\big(\|A^S_n\| + \nu\big)^{N_S+1} + \nu,$$

where ν is as small as we need taking $\|\tilde{\mathcal{F}}\|_{C^{N_s+1}}$ small enough. Then, σ is a contraction. $\square$

By (i) and (ii) it is clear that there is a fixed point of σ in $\mathcal{X}^0$ — hence, a solution of (16) —. Moreover, because of (i), this fixed point of σ also belongs to the $\mathcal{X}^0$-closure of $\mathcal{B}(\rho_{N_S+2}, \dots, \rho_k)$, $N_S + 1 \le k \le r$.

By Lemma 2.8 and Proposition A2 in [LI73], we get the existence of a $C^{r-1+\mathrm{lip}}$ solution of the invariance equation. Furthermore, given any $\mathcal{U} \in \mathcal{B}(\rho_{N_S+2}, \dots, \rho_k)$, the sequence $\sigma^j(\mathcal{U})$ tends in the C^{r-1} norm to the fixed point of σ.

Now, to check that this solution is in fact C^r, we note first that

$$\|R_{n,r}(\mathcal{U}) - R_{n,r}(\hat{\mathcal{U}})\|_{C^0} \to 0,$$

when $\|\mathcal{U} - \hat{\mathcal{U}}\|_{C^{r-1}} \to 0$ and $\mathcal{U}, \hat{\mathcal{U}} \in \mathcal{B}(\rho_{N_S+2}, \dots, \rho_k)$.

This fact is trivial for all the terms involving derivatives up to order $r-1$ of $\tilde{\mathcal{F}}$, $\mathcal{U}$ and $\hat{\mathcal{U}}$, since these derivatives are in fact Lipschitz. The only terms involving r derivatives are

$$D(V^0_{n+1} + U_{n+1}) \circ \psi(\mathcal{U})_n \Pi^S D^r \tilde{f}_n \circ \eta(\mathcal{U})_n D\eta(\mathcal{U})_n^{\otimes r}$$

and

$$\Pi^{S^c} D^r \tilde{f}_n \circ \eta(\mathcal{U})_n D\eta(\mathcal{U})_n^{\otimes r}$$

which are continuous in $\mathcal{U}$ since $D^r\tilde{f}_n$ and η are continuous with modulus of continuity independent of n.

Next we consider the sequence of maps $\mathcal{U}^l = \{U^l_n\}_{n\in\mathbb{Z}}$ defined by

$$U^0_n = 0, \ \ n \in \mathbb{Z}, \qquad \mathcal{U}^{l+1} = \sigma(\mathcal{U}^l).$$

The preceding arguments show that the sequence $\mathcal{U}^l \in \mathcal{B}(\rho_{N_S+2}, \ldots, \rho_k)$ and converges in the C^{r-1} norm to a function $\mathcal{U}^\infty = \{U_n^\infty\}_{n\in\mathbb{Z}}$. The sequence of r-derivatives satisfies the recurrence relation

$$D^r U_n^{l+1} = (A_n^{S^c})^{-1} D^r U_{n+1}^l \circ \psi(\mathcal{U}^l)_n D\psi(\mathcal{U}^l)_n^{\otimes r} + B_n(\mathcal{U}^l) D^r U_n^l + R_{n,r}(\mathcal{U}^l).$$

That is, denoting $D^r U_n^l$ by T_n^l, $\mathcal{T}^l = \{T_n^l\}_{n\in\mathbb{Z}}$, we have that

$$\mathcal{T}^{l+1} = \mathcal{A}(\mathcal{U}^l)\mathcal{T}^l + \mathcal{R}(\mathcal{U}^l), \tag{45}$$

with

$$\begin{aligned} (\mathcal{A}(\mathcal{U}^l)\mathcal{T}^l)_n &= (A_n^{S^c})^{-1} T_{n+1}^l \circ \psi(\mathcal{U}^l)_n D\psi(\mathcal{U}^l)_n^{\otimes r} + B_n(\mathcal{U}^l) T_n^l, \\ \mathcal{R}(\mathcal{U}^l)_n &= R_{n,r}(\mathcal{U}^l). \end{aligned}$$

We have that $\mathcal{A}(\mathcal{U}^l)$ is a linear map from

$$\begin{aligned} \Xi = \{\mathcal{T} = \{T_n\}_{n\in\mathbb{Z}} :\ & T_n \in C^0(B(0,1), L^r(\mathbb{R}^d;\mathbb{R}^d), \\ & \omega(T_n,\eta) \text{ uniformly bounded in } n\} \end{aligned}$$

to itself. Notice that the terms of the sequence $\mathcal{T}^l$ belong to Ξ. Moreover both $\mathcal{A}(\mathcal{U})$ and $\mathcal{R}(\mathcal{U})$ are continuous in the C^0 norm when $\mathcal{U}$ is C^{r-1} with modulus of continuity of the $r-1$ derivative bounded.

We claim that the sequence $\mathcal{T}^l$ converges in the C^0 norm to a continuous map. Indeed, this limit will be the only bounded solution, $\mathcal{T}^\infty$, of the equation

$$\mathcal{T} = \mathcal{A}(\mathcal{U}^\infty)\mathcal{T} + \mathcal{R}(\mathcal{U}^\infty). \tag{46}$$

This equation has a unique solution since,

$$\|\mathcal{A}(\mathcal{U}^\infty)\| \le \sup_{n\in\mathbb{Z}} \|(A_n^{S^c})^{-1}\| \big(\|A_n^S\| + \nu\big)^r + \nu < \gamma < 1,$$

and, hence, the right hand side of (46) is a contraction. We can also assume that $\|\mathcal{A}(\mathcal{U}^l)\| \le \gamma$. To prove the claim we check that $\mathcal{T}^l \to \mathcal{T}^\infty$, when $l \to \infty$, in the C^0 norm.

$$\begin{aligned} \|\mathcal{T}^l - \mathcal{T}^\infty\| &\le \|\mathcal{A}(\mathcal{U}^{l-1})\mathcal{T}^{l-1} - \mathcal{A}(\mathcal{U}^\infty)\mathcal{T}^\infty\| + \|\mathcal{R}(\mathcal{U}^{l-1}) - \mathcal{R}(\mathcal{U}^\infty)\| \\ &\le \gamma\|\mathcal{T}^{l-1} - \mathcal{T}^\infty\| + d_l, \end{aligned}$$

where

$$d_l = \|(\mathcal{A}(\mathcal{U}^{l-1}) - \mathcal{A}(\mathcal{U}^\infty))\mathcal{T}^\infty\| + \|\mathcal{R}(\mathcal{U}^{l-1}) - \mathcal{R}(\mathcal{U}^\infty)\|.$$

Notice that, by the continuity of $\mathcal{A}$ and $\mathcal{R}$, $d_l \to 0$, when $l \to \infty$. Then it is clear that

$$\|\mathcal{T}^l - \mathcal{T}^\infty\| \le \gamma^{l-1}\|\mathcal{T}^1 - \mathcal{T}^\infty\| + \sum_{j=0}^{l-1} \gamma^j d_{l-j},$$

which tends to 0 when $l \to \infty$ since $\gamma < 1$ and $d_l \to 0$.

This proves that $D^r U_n^l$ tends in the C^0 norm to T_n^∞. To check that this map is the r derivative of $\mathcal{U}^\infty$, we simply note that for all n

$$D^{r-1}U_n^l(y) - D^{r-1}U_n^l(x) = \int_0^1 D^r U_n^l(x+s(y-x))(y-x)\,ds.$$

Since the integrand in the right hand side converges uniformly to the continuous map T_n^∞, we have

$$\begin{aligned} D^{r-1}U_n^\infty(y) - D^{r-1}U_n^\infty(x) &= T_n^\infty(x)(y-x) \\ &\quad + \int_0^1 [T_n^\infty(x+s(y-x)) - T_n^\infty(x)](y-x)\,ds \end{aligned}$$

and hence $D^r U_n^\infty(x) = T_n^\infty(x)$.

To prove the case when $r = \infty$, we note that when $f_n \in C^\infty$, we can find a sequence of positive numbers $\rho_{N_S+2}, \dots, \rho_k$ such that $\sigma(\mathcal{B}(\rho_{N_S+2}, \dots, \rho_k)) \subset \mathcal{B}(\rho_{N_S+2}, \dots, \rho_k)$, for all $k \geq N_S + 2$. According to the preceding arguments the fixed point of σ is $C^{r-1+\mathrm{lip}}$ for all k, and hence C^∞.

The case $r = \omega$ is much easier. It just suffices to observe that the previous arguments work exactly the same in a complex ball. We consider the Banach space of functions, analytic in the open ball, continuous on the closed ball, vanishing to order N_S at the origin topologized with the supremum of the $N_S + 1$ derivative.

We have established the existence claim in the theorem, (a) and (b). To prove the uniqueness claim (e) consider $\mathcal{V} = \{V_n\}_{n\in\mathbb{Z}}$, $\hat{\mathcal{V}} = \{\hat{V}_n\}_{n\in\mathbb{Z}}$. From (b), $\mathcal{V} = \sigma(\mathcal{V})$ and $\hat{\mathcal{V}} = \sigma(\hat{\mathcal{V}})$. Then

$$\|\mathcal{V} - \hat{\mathcal{V}}\|_{\mathcal{X}^0} = \|\sigma(\mathcal{V}) - \sigma(\hat{\mathcal{V}})\|_{\mathcal{X}^0} \leq \operatorname{Lip}\sigma_{|\mathcal{B}_{N_S+1}}\|\mathcal{V} - \hat{\mathcal{V}}\|_{\mathcal{X}^0}$$

and $\operatorname{Lip}\sigma_{|\mathcal{B}_{N_S+1}} < 1$ in the $\mathcal{X}^0$ norm. This shows that $\mathcal{V} = \hat{\mathcal{V}}$.

To prove claim (f), we observe that (16) shows that if $\hat{V}_n$ is C^k, $k \leq r$, for some n, then it is C^k for all n. To prove the claim, we just have to obtain uniform estimates for the derivatives, assuming they exist. We shall consider the case $k = N_S + 1$. The other cases follow by induction.

If we take $N_S + 1$ derivatives of (16), we obtain in a similar way as for (32)

$$\begin{aligned} D^{N_S+1}\hat{V}_n(x) &= (A_n^{S^c})^{-1} D^{N_S+1}\hat{V}_{n+1}\big(A_n^S x + \Pi^S \tilde{f}_n(x, \hat{V}_n(x))\big)(A_n^S)^{\otimes(N_S+1)} \\ &\quad + Q_{n,N_S+1}(\hat{\mathcal{V}}), \end{aligned} \tag{47}$$

where $Q_{n,N_S+1}(\hat{\mathcal{V}})$ contains terms with derivatives up to order N_S+1 of $\hat{V}_n$ and $\hat{V}_{n+1}$, but all of them multiplied by factors which involve derivatives of $\tilde{f}_n$ up to order $N_S + 1$.

From that, we conclude that

$$\begin{aligned} \|D^{N_S+1}\hat{V}_n\|_{C^0} &\leq \|(A_n^{S^c})^{-1}\|\,\|A_n^S\|^{N_S+1}\|D^{N_S+1}\hat{V}_{n+1}\|_{C^0} \\ &\quad + \nu(\|\hat{V}_n\|_{C^{N_S+1}} + \|\hat{V}_{n+1}\|_{C^{N_S+1}}), \end{aligned} \tag{48}$$

where ν can be made as small as we need by assuming that $\|\tilde{\mathcal{F}}\|_{C^r}$ is sufficiently small.

We observe that

$$\begin{aligned} \|\hat{V}_n\|_{C^{N_S+1}} &\le \sup(\|\hat{V}_n\|_{C^{N_S}}, \|D^{N_S+1}\hat{V}_n\|_{C^0}) \\ &\le \|\hat{\mathcal{V}}\|_{C^{N_S}} + \|D^{N_S+1}\hat{V}_n\|_{C^0}. \end{aligned} \tag{49}$$

Substituting (49) into (48) and using that $\|(A_n^{S^c})^{-1}\|\,\|A_n^S\|^{N_S+1} < 1$ we obtain that

$$\|D^{N_S+1}\hat{V}_n\|_{C^0} \le \gamma \|D^{N_S+1}\hat{V}_{n+1}\|_{C^0} + D,$$

where $\gamma < 1$ and D is some constant independent of n. From that, claim (f) follows easily. □

Notice that the lengths of the manifolds and the uniform bounds on their derivatives depend only on the properties of the orbit and the C^r properties of the map.

For the case that the orbits come from a splitting of the Mather spectrum, we can choose these constants uniformly for all the orbits in the manifold.

By Ascoli-Arzelá theorem, the mapping $x \mapsto V_x$ maps the compact set on M into a $C^{r-1+\mathrm{lip}}$ compact set of submanifolds. Moreover, the uniqueness statements we have proved show that in these topologies the mapping has a closed graph.

Hence, it should be continuous. We point out that the methods of [Pes73] allow to prove that the mapping $x \mapsto V_x$ is continuous when the manifolds are given in the C^r topology. To obtain a similar improvement with our methods, it would have been necessary to show the existence of a uniform modulus of continuity for the r derivatives in Theorem 2.5.

3. An example

The following example illustrates some of the subtle phenomena involved in the slow manifolds showing that uniqueness may hold or not depending very much on the details of the conditions. In particular, it shows that some of the limitations in Theorem 2.5 do belong.

The example is presented as a family of maps, as in the setting of Theorem 2.5, but, as it will be shown at the end of this section, this family of maps can be lifted to a smooth map from a four dimensional compact manifold to itself. The construction of such a lift is explicit, and follows the one in [Pug84].

Example 3.1. *Consider the sequence of maps $f_n : \mathbb{R}^2 \to \mathbb{R}^2$, $n \in \mathbb{Z}$, defined by:*

$$\begin{aligned} f_n(x_1, x_2) &= (\frac{1}{3}x_1, \frac{1}{20}x_2), \qquad n \ne 0, \\ f_0(x_1, x_2) &= (\frac{1}{3}x_1, \frac{1}{20}x_2 + \varphi(x_1, x_2)), \end{aligned} \tag{50}$$

where φ is a C^∞ real valued function with compact support – which we will think of as very small – not including $(0,0)$ *and*

$$\left| \sup_{(x_1,x_2)\in\mathbb{R}^2} \frac{\partial\varphi}{\partial x_2}(x_1,x_2)\right| < \frac{1}{20}. \tag{51}$$

Clearly, we have

$$Df_n(0,0) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{20} \end{pmatrix}, \qquad n \in \mathbb{Z}.$$

Moreover, condition (51) ensures that each f_n, $n \in \mathbb{Z}$, is a bijective map. Indeed, this assertion is trivial for $n \neq 0$. For $n = 0$, we remark that, given $(z_1, z_2) \in \mathbb{R}^2$, the equation

$$f_0(x_1, x_2) = (z_1, z_2)$$

is equivalent to

$$x_1 = 3z_1, \qquad x_2 = 20z_2 - 20\varphi(3z_1, x_2).$$

The second condition has a unique solution, since its right hand side defines a contraction on $\mathbb{R}$. Hence, because $\det Df_n(x_1, x_2) \neq 0$, for all $(x_1, x_2) \in \mathbb{R}^2$, the functions f_n, $n \in \mathbb{Z}$, are global diffeomorphisms.

The sequence of maps (50) verifies Definition 1.2 if we take as E^i_n the coordinate axes and we set $\lambda_1 = \mu_1 = \log(1/3)$, $\lambda_2 = \mu_2 = \log(1/20)$, $\ell = 1$, $\varepsilon = 0$. We take E^S_n to consist just of the first coordinate axis. In such case, $N_S = [\log 20/\log 3] = 2$.

The set of manifolds graph (V_n) satisfies the condition

$$f_n(\text{graph } (V_n)) = \text{graph } (V_{n+1}), \qquad \text{for all } n \in \mathbb{Z}$$

if and and only if the functions V_n satisfy

$$V_{n+1}(x_1) = \frac{1}{20}V_n(3x_1) + \delta_{n,0}\varphi(3x_1, V_n(3x_1)) \tag{52}$$

where δ is the δ of Kronecker.

Furthermore, it is easy to verify by induction in n that a sequence of functions V_n satisfying (52) satisfies also the initial condition

$$V_0 = \Psi,$$

where $\Psi : \mathbb{R} \to \mathbb{R}$ is a C^∞ function with compact support such that $\Psi(0) = 0$, if and only if it is of the form

$$V_n(x_1) = \begin{cases} \left(\frac{1}{20}\right)^n (\Psi + 20\varphi \circ (\mathrm{Id}, \Psi))(3^n x_1), & n \geq 1, \\ \left(\frac{1}{20}\right)^n \Psi(3^n x_1), & n \leq 0. \end{cases} \tag{53}$$

We see that

$$\frac{d^j}{dx_1^j}V_n(x_1) = \begin{cases} \left(\frac{3^j}{20}\right)^n \frac{d^j}{dx_1^j}(\Psi + 20\varphi \circ (\mathrm{Id}, \Psi))(3^n x_1), & n \geq 1, \\ \left(\frac{3^j}{20}\right)^n \frac{d^j}{dx_1^j}(\Psi)(3^n x_1), & n \leq 0. \end{cases} \tag{54}$$

We make the following observations.

(i) Since $V_n'(0) = \left(\frac{3}{20}\right)^n \Psi'(0)$ we see that the derivative is unbounded unless $\Psi'(0) = 0$. In such a case, $V_n'(0) = 0$, for all $n \in \mathbb{Z}$.

This phenomenon of boundedness of first derivatives implying tangency is an illustration of part (d) of Theorem 1.9.

(ii) Suppose that Ψ has support not containing 0.

If $j \le N_S$ – equivalently, $3^j/20 < 1$ –, we have that $|V_n^{(j)}(x_1)|$ is bounded uniformly on n in a ball around the origin.

This follows by observing that for $n > 0$, we have uniform boundedness in (54) because of the factor $\left(\frac{3^j}{20}\right)^n$. For $n < 0$, we have boundedness because for n sufficiently negative, the support of V_n is outside of the unit ball.

This illustrates that we cannot expect uniqueness by only assuming boundedness of derivatives of order less than N_S.

(iii) Let $j > N_S$. Choose $\varphi \equiv 0$ and Ψ such that $\Psi(0) = 0$, $\Psi'(0) = 0$, $\Psi^{(j)}(0) \neq 0$. Then, it is easy to see that $|V_n|_{C^j}$ is bounded for n negative but not for n positive.

(iv) If $j > N_S$, the only possibility of having uniform bounds for $|V_n^{(j)}(x_1)|$ for $n > 0$ is that

$$\Psi = -20\varphi \circ (\mathrm{Id}, \Psi). \tag{55}$$

We claim that this equation has a unique continuous solution which is C^∞ and has compact support. Indeed, for any $x \in \mathbb{R}$, condition (51) implies that the right hand side of the equation

$$y = -20\varphi(x, y)$$

is a contraction, and, hence, has a unique solution $\Psi(x)$. Since φ has compact support, $\Psi(x)$ also has compact support. The standard implicit function theorem ensures that Ψ is C^∞.

This determines uniquely the functions V_n for all values of n.

Note that for these functions $|V_n^{(j)}(x_1)|$ is uniformly bounded for $n < 0$.

This illustrates the fact that we have uniqueness under the assumption of uniformly boundedness of the derivatives of order bigger than N_S.

Now we show how to embed this family in a smooth map. In the following, we will denote by $\mathbb{S}^2$ the two sphere.

Proposition 3.2. *Consider the family of maps $\{f_n\}_{n\in\mathbb{Z}}$ of Example 3.1. Then, there exist a two dimensional compact smooth manifold, M^2, a smooth map $F : \mathbb{S}^2 \times M^2 \to \mathbb{S}^2 \times M^2$, a point $z_0 \in \mathbb{S}^2 \times M^2$ with orbit $\{z_n = F^n(z_0)\}$, and smooth two dimensional submanifolds $N_n \subset \mathbb{S}^2 \times M^2$ such that*

i) $z_n \in N_n$, $F(N_n) \subset N_{n+1}$,
ii) *there exists a diffeomorphism σ such that $F_{|N_n} = \sigma^{-1} \circ f_n \circ \sigma$, $n \in \mathbb{Z}$.*

Proof. The construction of the map F is performed in two steps. The first one consists in lifting the discrete family $\{f_n\}$ to a smooth family of maps $\{\tilde{g}_\lambda\}_{\lambda\in[0,1]}$, with $\tilde{g}_\lambda : \mathbb{S}^2 \to \mathbb{S}^2$. In the second one, with the aid of an auxiliary map on a compact smooth manifold, exhibiting a chosen hyperbolic dynamics — a Smale horseshoe, for instance –, the map F and the orbit are explicitly given.

We introduce some notations that will be used later. We fix $\mathbb{S}^2$ to be $\{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 = 1\}$, $\mathbb{S}^2_+ = \mathbb{S}^2 \cap \{z > 0\}$, $\mathbb{S}^2_- = \mathbb{S}^2 \cap \{z < 0\}$ and $E = \mathbb{S}^2 \cap \{z = 0\}$. Let $\pi : \mathbb{S}^2_- \to \mathbb{R}^2$ be the map

$$\pi(x, y, z) = \left(-\frac{x}{z}, -\frac{y}{z}\right). \tag{56}$$

Notice that π is the projection of $\mathbb{S}^2$ from the center of the sphere onto the plane $\{z = -1\}$. We remark that π is a diffeomorphism. We denote π^{-1} by σ.

It is clear that

$$\sigma(x_1, x_2) = \left(\frac{x_1}{\sqrt{1 + x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{1 + x_1^2 + x_2^2}}, \frac{-1}{\sqrt{1 + x_1^2 + x_2^2}}\right). \tag{57}$$

We also consider the antipodal map $\mu : \mathbb{S}^2 \to \mathbb{S}^2$, that is, $\mu(p) = -p$. In this way, σ is a chart of $\mathbb{S}^2$ covering $\mathbb{S}^2_-$ and $\mu \circ \sigma$ is a chart covering $\mathbb{S}^2_+$.

We define the one parameter family of maps $g_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g_\lambda(x_1, x_2) = (ax_1, bx_2 + \lambda\varphi(x_1, x_2)), \tag{58}$$

where $a = 1/3$ and $b = 1/20$. We have that $g_1 = f_0$ and $g_0 = f_n$, $n \in \mathbb{Z}\backslash\{0\}$. Now we define a lift of g_λ to $\mathbb{S}^2_-$ by

$$\tilde{g}^-_\lambda = \sigma \circ g_\lambda \circ \pi, \tag{59}$$

to $\mathbb{S}^2_+$ by

$$\tilde{g}^+_\lambda = \mu \circ \tilde{g}^-_\lambda \circ \mu^{-1}, \tag{60}$$

and finally

$$\tilde{g}_\lambda(p) = \begin{cases} \tilde{g}^+_\lambda(p), & p \in \mathbb{S}^2_+, \\ \tilde{g}^-_\lambda(p), & p \in \mathbb{S}^2_-. \end{cases}$$

From (56) and (57) we have that, for $(x, y, z) \in \mathbb{S}^2$, $z \neq 0$,

$$\tilde{g}_\lambda(x, y, z) = \left(\frac{ax}{\rho(x, y, z)}, \frac{by - \lambda z\varphi(-x/z, -y/z)}{\rho(x, y, z)}, \frac{z}{\rho(x, y, z)}\right),$$

where

$$\rho(x, y, z) = \sqrt{z^2 + a^2x^2 + (by - \lambda z\varphi(-x/z, -y/z))^2}.$$

We remark that ρ never vanishes. Since φ has compact support, $\tilde{g}_\lambda$ extends to a C^∞ map defined in the whole sphere. Indeed, if we take $R = \inf_{r\in\mathbb{R}}\{r : \ \mathrm{supp}(\varphi) \subset D_r(0)\}$, we have that $\varphi(u, v) = 0$, for $|(u, v)| > R$.

Then, for any $(x, y, z) \in \mathbb{S}^2$ such that $|z| < 1/\sqrt{1+R^2}$, we have $\rho(x, y, z) = \sqrt{z^2 + a^2x^2 + b^2y^2}$ and

$$\tilde{g}_\lambda(x, y, z) = \left(\frac{ax}{\rho(x, y, z)}, \frac{by}{\rho(x, y, z)}, \frac{z}{\rho(x, y, z)} \right),$$

which extends to a unique C^∞ function in a neighborhood of E and, hence, to $\mathbb{S}^2$. Notice that, for each $\lambda \in [0, 1]$, $\tilde{g}_\lambda : \mathbb{S}^2 \to \mathbb{S}^2$ is a diffeomorphism, which preserves $\mathbb{S}^2_-$ and $\mathbb{S}^2_+$, and that, restricted to $\mathbb{S}^2_-$, is conjugated to g_λ through π and σ. Moreover, the dependence on the parameter λ is smooth.

Now, we consider any compact two dimensional smooth manifold, M^2, and a diffeomorphism $h : M^2 \to M^2$ with an invariant hyperbolic subset Σ such that $h_{|\Sigma}$ is conjugated to the Bernoulli shift with two symbols. We can, for example, take h to be a Smale horseshoe. We can also assume that the Lyapunov exponents of this hyperbolic set are bigger than $-\log 3$, in order to avoid resonances. Let $q_0 \in \Sigma$ be the point corresponding to the sequence $(\cdots 11011 \cdots)$. Clearly, there exists a C^∞ function with compact support, $\eta : M^2 \to [0, 1]$, such that $\eta \equiv 1$ in a neighborhood of q_0 and vanishes outside a compact set which does not include $h^n(q_0)$, $n \neq 0$, that is, $\eta(h^n(q_0)) = \delta_{n,0}$, $n \in \mathbb{Z}$. We define $F : \mathbb{S}^2 \times M^2 \to \mathbb{S}^2 \times M^2$ by

$$F(p, q) = (\tilde{g}_{\eta(q)}(p), h(q)),$$

and we check that F satisfies the properties listed in Proposition 3.2. We consider the orbit of the point $z_0 = (\sigma(0, 0), q_0)$. It is clear that $z_n = F^n(z_0) = (\sigma(0, 0), q_n)$, where $q_n = h^n(q_0)$. By definition, the submanifolds $N_n = \mathbb{S}^2 \times \{q_n\} \equiv \mathbb{S}^2$ verify that $F(N_n) = N_{n+1}$ and $F_{|N_n} = \tilde{g}_{\eta(q_n)}$. Finally, $\tilde{g}_{\eta(q_n)}$ is conjugated by σ to f_n, which establishes the claim. □

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