Semiclassical Asymptotics for the Maxwell - Dirac System

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Abstract

We study the coupled system of Maxwell and Dirac equations from a semiclassical point of view. A rigorous nonlinear WKB-analysis, locally in time, for solutions of (critical) order $O(\sqrt{\varepsilon})$ is performed, where the small semiclassical parameter ε denotes the microscopic/macroscopic scale ratio.

Key words: Dirac equation, Maxwell equations, nonlinear geometrical optics, WKB-asymptotics

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1. Introduction and Scaling

The *Maxwell-Dirac system* (MD) is fundamental in the relativistic description of spin 1/2 particles. It represents the time-evolution of fast (relativistic) electrons and positrons within external and *self-consistent* generated electromagnetic fields:

$$\left\{\begin{aligned} &i\hbar\partial_s\Psi = \sum_{k=1}^{3}\alpha^k(-i\hbar c\partial_k - e(A_k + A_k^{ext}))\Psi + e(V+V^{ext})\Psi + mc^2\beta\Psi,\\ &\left(\frac{1}{c^2}\partial_s^2-\Delta\right)V = \frac{1}{\epsilon_0}\rho,\quad \left(\frac{1}{c^2}\partial_s^2-\Delta\right)A = \frac{1}{c\epsilon_0}j,\end{aligned}\right. \tag{1.1.1}$$

where the *particle-* and *current-densities* are defined by:

$$\rho := e|\Psi|^2,\quad j_k := e\langle\Psi,\alpha^k\Psi\rangle,\qquad k=1,2,3. \tag{1.1.2}$$

Here, $\Psi = \Psi(s,y)\in\mathbb{C}^4$ is the 4-vector of the *spinor field*, normalized s.t.

$$\int_{\mathbb{R}^3}|\Psi(s,y)|^2dy = 1, \tag{1.1.3}$$

with s, $y\equiv(y_1,y_2,y_3)$, denoting the time - resp. spatial coordinates in *Minkowski space*. Further, $V^{(ext)} = V^{(ext)}(s,y)\in\mathbb{C}$ is the *self-consistent* resp. *external electric potential* and $A_k^{(ext)} = A_k^{(ext)}(t,x)\in\mathbb{C}$, the corresponding *magnetic potential*, with $A=(A_1,A_2,A_3)$. In the following, the usual scalar-product for vectors $X,Y\in\mathbb{C}^4$

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will be denoted by $\langle X, Y\rangle$ and we shall also write $|X|^2 := \langle X, X\rangle$. The so-called *Dirac matrices* β, α^k, $k = 1, 2, 3$, are explicitly given by:

$$\beta := \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \quad \alpha^k := \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \tag{1.1.4}$$

with $\mathbb{I}_2$, the 2×2 identity matrix and σ^k the 2×2 *Pauli matrices*, *i.e.*

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.1.5}$$

Hence, α^k, β are *hermitian* and moreover one easily checks that the following identities hold for $k, l = 1, 2, 3$:

$$\begin{cases} \alpha^k\alpha^l + \alpha^l\alpha^k = 2\delta_{kl}, \\ \alpha^k\beta + \beta\alpha^k = 0. \end{cases} \tag{1.1.6}$$

Finally, the appearing physical constants are the normalized Planck constant $\hbar$, the speed of light c, the permittivity of the vacuum ϵ_0, the particle mass m and the charge e.

Additionally to (1.1.1), we impose the *Lorentz gauge condition*

$$\frac{1}{c}\partial_s V(s, y) + \operatorname{div} A(s, y) = 0, \tag{1.1.7}$$

for the initial potentials $V(0, y)$ and $A(0, y)$. The gauge is henceforth conserved during the time-evolution. It ensures that the corresponding *electromagnetic fields* E, B are uniquely determined by

$$E(s, y) := -\frac{1}{c}\partial_s A(s, y) - \nabla V(s, y), \quad B(s, y) := \operatorname{curl} A(s, y). \tag{1.1.8}$$

The MD equations are the underlying field equations of relativistic *quantum electrodynamics*, *cf.* [Sc], where one considers the system within the formalism of *second quantization*. Nevertheless, in order to obtain a deeper understanding for the interaction of matter and radiation, there is a growing interest in the MD system also for classical fields, since one can expect at least qualitative results, *cf.* [EsSe].

From the mathematical point of view, the strongly nonlinear MD system poses a hard problem in the study of PDE's arising from quantum physics. Well posedness and existence of solutions on all of $\mathbb{R}^3_y$ but only locally in time, has been proved almost forty years ago in [Gro]. On the other hand, only partial results (*i.e.* for small initial data) have been obtained in the quest of global-in-time solutions, [Ch], [G], [FST], let alone the study of other qualitative features of this system.

In this paper, we shall analyze the MD system in a *semiclassical* regime. To do so, we first rewrite the equations such that there remains only one (positive) *dimensionless* parameter

$$\delta = \frac{\hbar c\varepsilon_0}{e^2}, \tag{1.1.9}$$

which is obtained by replacing $y \to y/\bar{y}$, $s \to s/\bar{s}$ and $\Psi(s, y) \to \bar{y}^{-3/2}\Psi(s/\bar{s}, y/\bar{y})$, in order to maintain the normalization condition (1.1.3), with

$$c\bar{s} = \bar{y} \ \text{ and } \ \bar{y} = \frac{e^2}{mc^2\varepsilon_0}. \tag{1.1.10}$$

Here, we also replaced both, the external and the self-consistent potentials, by $A^{(ext)}(s, y) \to \lambda A^{(ext)}(s/\bar{s}, y/\bar{y})$ and $V^{(ext)}(s, y) \to \kappa V^{(ext)}(s/\bar{s}, y/\bar{y})$, with $\kappa = c\lambda$

and $\lambda = mc/e$. We assume for the following that A^{ext}, V^{ext} are of $O(1)$ in these units. In summary, we obtain the MD system in dimensionless form:

$$\begin{cases} i\delta\partial_s \Psi = \sum_{k=1}^{3} \alpha^k(-i\delta\partial_k - (A_k + A_k^{ext}))\Psi + (V + V^{ext})\Psi + \beta\Psi, \\ \Box V = \rho, \quad \Box A = j, \end{cases} \tag{1.1.11}$$

where from now on $\Box := \partial_s^2 - \Delta$. In (1.1.11), s, y represent the *microscopic* time and length scales. Note that in general, δ can not be considered as a *small* parameter, for example $\delta \approx 13$, in the case of electrons. Hence, semiclassical asymptotics in δ, *i.e.* on (1.1.11) directly, only make sense for highly charged and consequently heavy particles.

Therefore, we need to rescale the system (1.1.11) such that the time-evolution can be considered semiclassical, independent of the precise physical properties of the particles. We can suppose that the given external electromagnetic potentials are slowly varying w.r.t to the microscopic scales, *i.e.* $V^{ext} = V^{ext}(s\varepsilon/\delta, y\varepsilon/\delta)$ and likewise $A^{ext} = A^{ext}(s\varepsilon/\delta, y\varepsilon/\delta)$, where from now on ε denotes the small *semiclassical parameter*. Here, the δ is included in the scaling in order to eliminate it from the resulting equation. Hence, observing the evolution on macroscopic scales we are lead to:

$$y = \frac{\delta}{\varepsilon} x, \quad s = \frac{\delta}{\varepsilon} t. \tag{1.1.12}$$

Moreover, we want that the coefficients of all nonlinearities to be $O(1)$, *i.e.* they should *not* carry a positive power of ε. It turns out that there exists solutions ψ^ε, which obey this requirement. If we set

$$\frac{\delta}{\varepsilon}\Psi(s, y) = \psi^\varepsilon(t, x). \tag{1.1.13}$$

then, the normalization condition for (1.1.3) gives

$$\int_{\mathbb{R}^3} |\psi^\varepsilon(t, x)|^2 dx = \frac{\varepsilon}{\delta} \int_{\mathbb{R}^3} |\Psi(s, y)|^2 dx = \frac{\varepsilon}{\delta}. \tag{1.1.14}$$

This implies that we need to look for solutions ψ^ε s.t.

$$\psi^\varepsilon \sim O(\sqrt{\varepsilon}), \tag{1.1.15}$$

assuming, as mentioned above, that $\delta \sim O(1)$. We therefore end up with the following *semiclassical scaled* MD system:

$$i\varepsilon\partial_t\psi^\varepsilon = \sum_{k=1}^{3} \alpha^k(-i\varepsilon\partial_k - (A_k^\varepsilon + A^{ext}))\psi^\varepsilon + \beta\psi^\varepsilon + (V^\varepsilon + V^{ext})\psi^\varepsilon, \tag{1.1.16}$$

$$\Box V^\varepsilon = |\psi^\varepsilon|^2, \tag{1.1.17}$$

$$\Box A_k^\varepsilon = \langle \psi^\varepsilon, \alpha^k \psi^\varepsilon \rangle, \qquad k = 1, 2, 3, \tag{1.1.18}$$

subject to *Cauchy initial data*:

$$\begin{cases} \psi^\varepsilon\big|_{t=0} = \psi_I^\varepsilon(x) \sim O(\sqrt{\varepsilon}), \\ V^\varepsilon\big|_{t=0} = V_I^\varepsilon(x), \quad \partial_t V^\varepsilon\big|_{t=0} = \tilde{V}_I^\varepsilon(x), \\ A^\varepsilon\big|_{t=0} = A_I^\varepsilon(x), \quad \partial_t A^\varepsilon\big|_{t=0} = \tilde{A}_I^\varepsilon(x). \end{cases} \tag{1.1.19}$$

For this nonlinear system, we want to find an asymptotic description of $\psi^\varepsilon \sim O(\sqrt{\varepsilon})$ as $\varepsilon \to 0$, *i.e.* a semiclassical description. Note that, *equivalently*, one could consider asymptotic solutions of the form

$$\Phi^\varepsilon(t,x) := \frac{1}{\sqrt{\varepsilon}}\,\psi^\varepsilon(t,x) \sim O(1), \tag{1.1.20}$$

which do not vanish in the limit $\varepsilon \to 0$ and which again satisfy the semiclassical scaled DM system, modified by the fact that the right hand sides of (1.1.17), (1.1.18) are multiplied by an additional factor ε. This illustrates the fact that we are dealing with a *small coupling limit*. We further stress that in our scaling the mass is $O(1)$ and *fixed* as $\varepsilon \to 0$, which is different from the otherwise similar scaling used in [KS], where a classical mechanics analogue of the DM system has been studied.

The (rigorous) analysis of semiclassical asymptotics has a long tradition in quantum mechanics, the most common technique being the so called *WKB-method*. Quite recently, a semiclassical approach to the *linear* Dirac equation was taken in [BK] and also, using *Wigner measures*, in [GMMP], [Sp]. For a broader introduction on linear techniques and results, we refer to [MaFe], [Ro], [SMM] and the references given therein. Nonlinear extensions of the WKB-method can be found for example in [Ge], [Gr], where *scalar-valued* semilinear Schrödinger equations are analyzed. We remark that the case of the nonlinear Dirac 4-system introduces significant new difficulties in the WKB-analysis, some of them are already present in the linear setting.

Mathematically, our approach is inspired by techniques developed in [DoRa], which sometimes go under the name *weakly nonlinear geometrical optics*. Due to the appearance of the small parameter ε in front of each derivative in (1.1.16) we are in the regime of so-called *dispersive* weakly nonlinear geometrical optics, which differs in several aspects from the non-dispersive one. We remark that the latter case is much better studied in the so far existing literature and we refer to [JMR2], for a recent review.

To be more precise, we shall seek a local-in-time solution of (1.1.16), which asymptotically takes the following form:

$$\begin{cases} \psi^\varepsilon(t,x) = \sqrt{\varepsilon}\, u^\varepsilon(t,x,\phi(t,x)/\varepsilon), \\ u^\varepsilon(t,x,\theta) \sim \displaystyle\sum_{j=0}^{\infty} \varepsilon^{j/2} u_j(t,x,\theta). \end{cases} \tag{1.1.21}$$

Here, all the $u_j(t,x,\theta) \in \mathbb{C}^4$ being 2π-periodic w.r.t. $\theta \in \mathbb{R}$. Due to the factor $\varepsilon^{1/2}$, we call them *small semiclassical approximate solutions*, or *small WKB-solutions*.

As expected, there are two modes of the phase function $\phi_\pm$, which satisfy the (free) relativistic Hamilton-Jacobi equation, corresponding to positive resp. negative energies and of course, convergence of the expansion (1.1.21), can only hold on a time interval, which corresponds to the existence of smooth solutions $\phi_\pm$. In weakly nonlinear geometrical optics, the homogeneity of the nonlinearity determines the required order of smallness of the asymptotic solution and, as we shall see, the factor $\sqrt{\varepsilon}$ precisely fits with the cubic nonlinearities in (1.1.16). We will show that for this particular scale we obtain, on the one hand, *independent* propagation of the electronic resp. positronic phase function $\phi_\pm$ and, the other hand, *nonlinear interaction* of the corresponding principal amplitudes $u_{0,\pm}$.

In other words, we study solutions on the threshold of *adiabatic decoupling*, a phenomena which is already well known in the linear case, *cf.* [PST]. In particular, the importance of the $O(\sqrt{\varepsilon})$-scale for the (linear) Dirac equation is stressed in [FK], where one can also find a detailed description of the energy-transfer between electrons and positrons in terms of *two-scale* Wigner measures. These results, together with ours suggest that if one wants to obtain semiclassical $O(1)$-approximation in the strongly coupled regime, one needs to take into account simultaneously scales of order ε and $O(\sqrt{\varepsilon})$. These asymptotic solutions are then appropriate for heavily charged particles. We finally remark, that in the very recent paper [Je], coupled Gauge-fields are studied from a similar point of view as in our work.

This paper is organized as follows: We collect some preliminaries in section 2, then we shall determine the critical exponent and the corresponding eikonal equation of the approximate WKB-type solution in section 3. The corresponding ε-oscillations introduced by the nonlinearity are determined in section 4 and the nonlinear transport of the approximation along the rays of geometrical optics is obtained in section 5. Finally, in section 6 we shall prove that there exists a (local-in-time) solution of the MD system which stays close to the approximation and we also collect some further qualitative results.

2. Preliminaries

In the following, we will assume that no external electromagnetic fields are present:

$$V^{ext}(t,x) \equiv 0, \quad A^{ext}(t,x) \equiv 0. \tag{2.2.1}$$

Moreover, we assume that at $t = 0$ we have:

$$V^\varepsilon_I(x) = \tilde{V}^\varepsilon_I(x) \equiv 0, \quad A^\varepsilon_I(x) = \tilde{A}^\varepsilon_I(x) \equiv 0. \tag{2.2.2}$$

Neither of these assumptions changes the following analysis significantly. They are only imposed for the sake of simplicity. Note that the DM system is time-reversible, but w.r.o.g. we shall consider positive times only in the sequel.

Using the fundamental solution of the wave equation in dimension $d = 3$ and for times $t > 0$, we find the following expression for V, called *the retarded potential*:

$$\begin{aligned} V^\varepsilon[\psi^\varepsilon](t,x) = & \ \frac{1}{4\pi}\int_{|x-y|\le t} \frac{|\psi^\varepsilon(t-|x-y|,y)|^2}{|x-y|}\,dy \\ =: & \ \mathcal{G}_r(t,x) * |\psi^\varepsilon(t,x)|^2, \end{aligned} \tag{2.2.3}$$

where $*$ denotes the convolution w.r.t. (t,x) and

$$\mathcal{G}_r(t,x) := \frac{\Theta(t)}{4\pi|x|}\,\delta(t-|x|). \tag{2.2.4}$$

Likewise, A can be written as:

$$A^\varepsilon[\psi^\varepsilon](t,x) = \mathcal{G}_r(t,x) * \langle \psi^\varepsilon(t,x), \alpha^k \psi^\varepsilon(t,x)\rangle. \tag{2.2.5}$$

Using this representations, we shall rewrite (1.1.16)-(1.1.18) in the form of a *semilinear Dirac equation*:

$$\begin{cases} i\varepsilon\partial_t\psi^\varepsilon - \mathcal{D}^\varepsilon_A(t,x,\varepsilon D_x)\psi^\varepsilon = 0, & x\in\mathbb{R}^3,\ t>0, \\ \qquad\qquad \psi^\varepsilon\big|_{t=0} = \psi^\varepsilon_I(x), \end{cases} \tag{2.2.6}$$

where $\mathcal{D}^\varepsilon_A$ is a *matrix-valued* differential operator ($D_x := -i\nabla$). The corresponding ε*-dependent* symbol is given by

$$\mathcal{D}^\varepsilon_A(t,x,\xi) = \alpha\cdot(\xi - A^\varepsilon[\psi^\varepsilon](t,x)) + \beta + V^\varepsilon[\psi^\varepsilon](t,x), \tag{2.2.7}$$

where $x, \xi, \in \mathbb{R}^3, t \in \mathbb{R}$. Here and in the following we use the notation

$$\alpha \cdot \xi := \sum_{k=1}^{3} \alpha^k \xi_k. \tag{2.2.8}$$

Note that in the nonlinear equation (2.2.6), the potentials $A^\varepsilon[\psi^\varepsilon]$, $V^\varepsilon[\psi^\varepsilon]$ depend *non-locally* on ψ^ε, as indicated by the bracket-notation.

Multiplying (2.2.6) with ψ^ε and taking imaginary parts, we obtain the usual conservation law for $\|\psi^\varepsilon(t,x)\|_2$, hence the *conservation of charge*:

$$\int_{\mathbb{R}^3} \langle \psi^\varepsilon(t,x), \psi^\varepsilon(t,x)\rangle \, dx = \|\psi^\varepsilon(t,x)\|_2^2 = \text{const}. \tag{2.2.9}$$

The *free Dirac operator* will be denoted by

$$\mathcal{D}(\varepsilon D_x)\psi^\varepsilon := -i\varepsilon(\alpha \cdot \nabla)\psi^\varepsilon + \beta\psi^\varepsilon, \tag{2.2.10}$$

with symbol

$$\mathcal{D}(\xi) = \alpha \cdot \xi + \beta. \tag{2.2.11}$$

This 4×4 matrix has two different eigenvalues $h_\pm(\xi)$ of multiplicity 2 each:

$$h_\pm(\xi) := \pm\lambda(\xi), \quad \xi \in \mathbb{R}^3, \tag{2.2.12}$$

where

$$\lambda(\xi) := \sqrt{|\xi|^2 + 1}, \quad \xi \in \mathbb{R}^3. \tag{2.2.13}$$

As expected, the eigenvalues $h_\pm(\xi)$ are nothing but the *free Hamiltonian* for a *relativistic* particle. The positive resp. negative sign in (2.2.13) corresponds to *electrons* resp. *positrons*. By straightforward calculations we obtain:

Lemma 2.1. *The* spectral projectors $\Pi_\pm(\xi) : \mathbb{C}^4 \to \mathbb{C}^4$*, associated to* $h_\pm(\xi)$ *are given by*

$$\Pi_\pm(\xi) := \frac{1}{2}\left(\mathrm{id}_4 \pm \frac{1}{\lambda(\xi)}\, \mathcal{D}(\xi)\right), \qquad \Pi_\pm \Pi_\mp \equiv 0. \tag{2.2.14}$$

The matrix-valued symbol $\mathcal{D}(\xi)$ can therefore be decomposed into its positive and negative energy part in the following way:

$$\mathcal{D}(\xi) = h_+(\xi)\Pi_+(\xi) + h_-(\xi)\Pi_-(\xi). \tag{2.2.15}$$

Note that

$$h_\pm(-\xi) = h_\pm(\xi), \quad \text{whereas } \Pi_\pm(-\xi) = \Pi_\mp(\xi). \tag{2.2.16}$$

For later purpose, we also define:

Definition 2.2. The *partial inverse* $\Lambda_\pm(\xi) : \mathbb{C}^4 \to \mathbb{C}^4$, associated to $\Pi_\pm(\xi)$ is given by

$$\Lambda_\pm(\xi)\Pi_\pm(\xi) = 0, \qquad \Lambda_\pm(\xi)\mathcal{D}(\xi)X = (\mathrm{id}_4 - \Pi_\pm(\xi))X, \quad \forall X \in \mathbb{C}^4. \tag{2.2.17}$$

Finally, we recall the definition of *asymptotic equivalence*:

Definition 2.3. Let $\mathcal{O} \subseteq \mathbb{R}^n$, $n \geq 1$, be an open set, $a_j(y) \in C^\infty(\mathbb{R}^n; \mathbb{C}^n)$ and $a^\varepsilon \in C^\infty(]0, \varepsilon_0[\times\mathbb{R}^n; \mathbb{C}^n)$. Then we say that a^ε is asymptotically equivalent to the formal sum $\sum_{j=0}^\infty \epsilon^j a_j$ and write

$$a^\varepsilon(y) \sim \sum_{j=0}^{\infty} \varepsilon^j a_j(y), \tag{2.2.18}$$

if for every $m > 0$, every multiindex σ and every compact subset $\mathcal{K} \subset \mathcal{O}$, there exists a $C_{m,\sigma} > 0$, such that

$$\sup_{\mathcal{K}} \left| \partial_y^\sigma \left(a^\varepsilon(y) - \sum_{j=0}^{m} \varepsilon^j a_j(y) \right) \right| \leq C_{m,\sigma} \varepsilon^m. \tag{2.2.19}$$

3. Generalized WKB-Ansatz and the Eikonal Equation

At first we will show that the desired $O(\sqrt{\varepsilon})$-asymptotics for the spinor field fits into the framework of weakly nonlinear (dispersive) geometrical optics, as introduced in [DoRa], for nonlinear hyperbolic systems.

We plug the following *generalized WKB - Ansatz* into equation (2.2.6):

$$\begin{cases} \psi^\varepsilon(t,x) = \varepsilon^p \, u^\varepsilon(t,x,\phi(t,x)/\varepsilon), \\ u^\varepsilon(t,x,\theta) \sim \displaystyle\sum_{j=0}^{\infty} \varepsilon^{jp} u_j(t,x,\theta), \end{cases} \tag{3.3.1}$$

where the functions $u_j(t,x,\theta) \in \mathbb{C}^4$ are assumed to be sufficiently smooth and 2π-periodic w.r.t. $\theta \in \mathbb{R}$. This gives

(3.3.2)

$$\begin{aligned} 0 &= i\varepsilon^{p+1}\partial_t u^\varepsilon - \varepsilon^p \, \mathcal{D}_A^\varepsilon(t,x,\varepsilon D_x) u^\varepsilon \\ &= i\varepsilon^{p+1}(\partial_t u^\varepsilon + (\alpha \cdot \nabla) u^\varepsilon) - \varepsilon^p \beta u^\varepsilon + i\varepsilon^p \, (\partial_t \phi + (\alpha \cdot \nabla\phi))\partial_\theta u^\varepsilon + \varepsilon^{3p} \mathcal{N}^\varepsilon[u^\varepsilon], \end{aligned}$$

with a *nonlinearity*, $\mathcal{N}^\varepsilon : \mathbb{C}^4 \to \mathbb{C}^4$, defined by

$$\mathcal{N}^\varepsilon[u^\varepsilon] := ((\alpha \cdot A^\varepsilon[u^\varepsilon]) - V^\varepsilon[u^\varepsilon]) \, u^\varepsilon. \tag{3.3.3}$$

The strategy is now to expand the right hand side of (3.3.2) as

$$\varepsilon^p \sum_{j=0}^{\infty} \varepsilon^{jp} R_j(t,x) \tag{3.3.4}$$

and choose the coefficients u_j of (3.3.1) in such a way, that $R_j(t,x) \equiv 0$, $\forall j \in \mathbb{N}$.

It is important to note that the first term on the right hand side of (3.3.2) is of order ε^{p+1}, whereas the second and the third are $\sim O(\varepsilon^p)$. Since $A^\varepsilon[u^\varepsilon]$, $V^\varepsilon[u^\varepsilon]$ are of order ε^{2p}, by equation (2.2.3), (2.2.5), the function $\mathcal{N}^\varepsilon[u^\varepsilon]$ is of order ε^{3p}. This nonlinear term is supposed to be *small*, more precisely, it should not enter into the equation for R_0, describing terms of order $O(\varepsilon^p)$, but rather into expressions of $O(\varepsilon^{p+1})$. Thus we are led to the following *normalization condition*:

$$3p = p + 1, \tag{3.3.5}$$

implying $p = 1/2$. With this normalization we have $u^\varepsilon \sim O(\varepsilon^{1/2})$ (just as required by the scaling presented in the introduction), whereas the nonlinear term satisfies: $\mathcal{N}^\varepsilon[u^\varepsilon] \sim O(\varepsilon^{3/2})$.

Remark 3.1. The choice $p = 1/2$ gives the critical exponent in the sense that for amplitudes $O(\varepsilon^{1/2})$ one can prove simultaneously existence of the approximate smooth solution for times $t = O(1)$, *i.e.* on a time-scale *independent* of ε, and nontrivial nonlinear behavior in the principal term of the approximation, *cf.* [DoRa], [JMR2].

Setting $R_0(t,x)=0$, yields

$$i(\partial_t\phi+(\alpha\cdot\nabla\phi))\partial_\theta u_0-\beta u_0=0. \tag{3.3.6}$$

Since $u_j\in C^\infty(\mathbb{R}^4\times\mathbb{S}^1;\mathbb{C}^4)$ it can be *Fourier-expanded* w.r.t. θ

$$u_0(t,x,\theta)=\sum_{m\in\mathbb{Z}}u_{0,m}(t,x)e^{im\theta}. \tag{3.3.7}$$

By this procedure, we find the following equation for the coefficients $u_{0,m}$:

$$L(m\,d\phi(t,x))u_{0,m}:=(m\,\partial_t\phi+\mathcal{D}(m\nabla\phi))\,u_{0,m}=0, \tag{3.3.8}$$

where $\mathcal{D}(\nabla(m\phi))$ is the 4×4 symbol matrix of the free Dirac operator evaluated at $\xi=\nabla(m\phi(t,x))$. In order to have a nontrivial solution $u_{0,m}\neq 0$ we impose the condition, that there exists an open set $\Omega\subseteq\mathbb{R}^{1+3}$, having a nontrivial intersection with $\{t=0\}$, s.t.

$$\det L(m\,d\phi(t,x))=0,\quad \forall(t,x)\in\Omega\subseteq\mathbb{R}^{1+3}. \tag{3.3.9}$$

Using equations (2.2.12), (2.2.15), this is equivalent to

$$(m\,\partial_t\phi)^2-|m\nabla\phi|^2=1,\quad \forall(t,x)\in\Omega\subseteq\mathbb{R}^{1+3}. \tag{3.3.10}$$

Thus, for $m=\pm1$, the phase function ϕ satisfies (in Ω) the *eikonal equation* for the *Klein-Gordon operator*, *i.e.*

$$(\partial_t\phi)^2-|\nabla\phi|^2=1,\quad \forall(t,x)\in\Omega\subseteq\mathbb{R}^{1+3}. \tag{3.3.11}$$

Indeed, it is easy to see that the choices $m=\pm1$ are the only possibilities, since equation (3.3.11) gives

$$\begin{aligned}(m\partial_t\phi)^2-|m\nabla\phi|^2-1&=(m^2-1)((\partial_t\phi)^2-|\nabla\phi|^2)\\&=m^2-1,\end{aligned} \tag{3.3.12}$$

which is different from zero for all $m\neq\pm1$. Hence, in the Fourier-series (3.3.7), there appear only two nontrivial *harmonics*, which are associated to the eikonal equation (3.3.11): namely $\exp(i\phi(t,x)/\varepsilon)$, for $m=1$ and $\exp(-i\phi(t,x)/\varepsilon)$, for $m=-1$.
For $m=\pm1$ the equation (3.3.11) is fulfilled by two possible ϕ's, obtained from

$$\partial_t\phi_\pm(t,x)=h_\pm(\nabla\phi_\pm(t,x))\equiv\pm\sqrt{|\nabla\phi_\pm(t,x)|^2+1}=0. \tag{3.3.13}$$

This is the *Hamilton-Jacobi equation* for free relativistic particles. The following lemma guarantees existence and uniqueness of smooth solutions, where from now on, we shall denote by $D^2f(x)$, the Hessian of a given function $f:\mathbb{R}^3\to\mathbb{R}$.

Lemma 3.2. *Given $\phi_I\in C^\infty(\mathbb{R}^3;\mathbb{R})$, s.t. $\|D^2\phi_I(x)\|\leq C$, there exist $T_\pm>0$ and uniquely determined functions $\phi_\pm\in C^\infty(\Omega_\pm;\mathbb{R})$, where $\Omega_\pm:=[0,T_\pm)\times\mathbb{R}^3$, s.t.*

$$\begin{cases}\partial_t\phi_\pm(t,x)=h_\pm(\nabla\phi_\pm(t,x)),\quad \forall(t,x)\in\Omega_\pm,\\ \qquad \phi\big|_{t=0}=\phi_I(x).\end{cases} \tag{3.3.14}$$

Proof. We only proof the assertion for ϕ_+, since the other case is completely analogous. The initial value problem is *non-characteristic* everywhere, since

$$\partial_t\phi_+(0,x)=\sqrt{|\nabla\phi_+(0,x)|^2+1}\;\neq 0,\ \forall x\in\mathbb{R}^3. \tag{3.3.15}$$

Thus, $\partial_t \phi_+(0,x)$ can be obtained from the initial data $\phi_+(0,x) = \phi_I(x) \in C_b^\infty(\mathbb{R}^3)$ at each point $x \in \mathbb{R}^3$. Standard PDE theory then guarantees the existence of a unique smooth solution $\phi_+ \in C^\infty(\Omega_+;\mathbb{R})$, as long as

$$1 - t\,\|D^2 h_+(\nabla\phi_I)\|\,\|D^2\phi_I\| \neq 0. \tag{3.3.16}$$

Since $D^2H(\xi)$ is uniformly bounded, this condition holds by assumption and the assertion is proved. □

Remark 3.3. The assumption in lemma 3.2 can be relaxed to $\phi_I \in C^\infty(\mathbb{R}^3;\mathbb{R})$. In this case, however, one can not guarantee the existence a smooth solution ϕ in a space-time slab, but only in some open set $\mathcal{O} \subset \mathbb{R}^{1+3}$. In the following, this would lead to some technical difficulties, which we want to avoid, though, the whole procedure can be generalized to that case.

By equation (3.3.13), we have

$$-\phi_\pm(t,x) = \phi_\mp(t,x), \tag{3.3.17}$$

assuming that it holds initially at $\{t=0\} \subset \Omega$. In the following, we therefore consider only the solution to (3.3.13) with positive sign in front of the square root and write for it $\phi(t,x) \equiv \phi_+(t,x)$. Also, we henceforth denote by $\Omega := [0,T_+) \times \mathbb{R}^3$ the slab, in which existence of a smooth function ϕ is guaranteed. This we can do w.r.o.g. as will become clear in a moment:

Since for $\phi \equiv \phi_+$ it holds that $\partial_t\phi - h_+(\nabla\phi) = 0$, equation (3.3.8) implies the following *polarization conditions*, locally for all $(t,x) \in \Omega$:

$$(\Pi_-(\nabla\phi)u_{0,+1})(t,x) = 0 \;\Leftrightarrow\; (\Pi_+(\nabla\phi)u_{0,+1})(t,x) = u_{0,+1}(t,x). \tag{3.3.18}$$

Likewise, we get

$$(\Pi_+(\nabla\phi)u_{0,-1})(t,x) = 0 \;\Leftrightarrow\; (\Pi_-(\nabla\phi)u_{0,-1})(t,x) = u_{0,-1}(t,x). \tag{3.3.19}$$

One easily checks, using (2.2.16) and (3.3.17), that the conditions obtained with the choice $\phi = \phi_-$, are *equivalent* to (3.3.18), (3.3.19). Thus, equation (3.3.8) indeed carries two degrees of freedom for the phase, given by $\pm\phi$ (or equivalently ϕ_+, ϕ_-). The amplitudes are then rigidly linked, by (3.3.18), (3.3.19).

In summary, we find that the *principal term* $u_0(t,x,\theta)$ in our asymptotic description is given by

$$u_0(t,x,\phi(t,x)/\varepsilon) := u_{0,+1}(t,x)e^{i\phi(t,x)/\varepsilon} + u_{0,-1}(t,x)e^{-i\phi(t,x)/\varepsilon}, \tag{3.3.20}$$

where the amplitudes are polarized according to (3.3.18), (3.3.19). From now on we shall use the simplified notation $u_{0,\pm1} = u_{0,\pm}$.

4. Oscillations of the Nonlinearity

Let us determine the response of the wave equations (1.1.17), (1.1.18) to r.h.s. source terms induced by functions of the form (3.3.20):
To this end, we calculate:

$$|u_0(t,x,\phi(t,x)/\varepsilon)|^2 = |u_{0,+}(t,x)|^2 + |u_{0,-}(t,x)|^2. \tag{4.4.1}$$

The terms, which mix the electronic and positronic components cancel, since $\Pi_\pm$ is hermitian and $\Pi_\pm\Pi_\mp \equiv 0$. Hence, we get from (2.2.3) (at least formally), that the scalar potential V generated by the principal term u_0, is simply given by

$$V[u_0] = \mathcal{G}_r(t,x) * \left(|u_{0,+}(t,x)|^2 + |u_{0,-}(t,x)|^2\right). \tag{4.4.2}$$

In order to calculate the magnetic potential corresponding to u_0, we first note that, by definition, we have the following identity

$$\Pi_\pm(\xi)\,(\alpha\cdot\xi+\beta)=h_\pm(\xi)\;\Pi_\pm(\xi). \tag{4.4.3}$$

Differentiating w.r.t. ξ_k and multiplying (from the right) with $\Pi_\pm(\xi)$ gives

$$\begin{aligned}\Pi_\pm(\xi)\alpha^k\Pi_\pm(\xi)&=\Pi_\pm(\xi)(\partial_{\xi_k}h_\pm(\xi))\Pi_\pm(\xi)\\&=\pm\frac{\xi_k}{\sqrt{|\xi|^2+1}}\;\Pi_\pm(\xi),\end{aligned} \tag{4.4.4}$$

since $\Pi_\pm^2(\xi)=\Pi_\pm(\xi)$. The expression

$$\omega_\pm(\xi):=\nabla_\xi h_\pm(\xi)=\pm\frac{\xi}{\lambda(\xi)}, \tag{4.4.5}$$

is called the electronic resp. positronic *group velocity*, $\omega_\pm\in C^\infty(\mathbb{R}^3;\mathbb{R}^3)$. Using this definition, we obtain for $k=1,2,3$:

$$\begin{aligned}&\langle u_0(t,x,\phi(t,x)/\varepsilon),\alpha^k u_0(t,x,\phi(t,x)/\varepsilon)\rangle\\&=\ \omega_{+,k}(\nabla\phi(t,x))|u_{0,+}(t,x)|^2\ +\omega_{-,k}(\nabla\phi(t,x))|u_{0,-}(t,x)|^2\\&\ +\langle u_{0,+}(t,x),\alpha^k u_{0,-}(t,x)\rangle e^{-i2\phi(t,x)/\varepsilon}\\&\ +\langle u_{0,-}(t,x),\alpha^k u_{0,+}(t,x)\rangle e^{i2\phi(t,x)/\varepsilon}.\end{aligned} \tag{4.4.6}$$

The oscillating terms are usually called the *Zitterbewegung* of the Dirac-current, *cf.* [Sc], p. 195. The fact that the current-density corresponding to u_0 carries ε-oscillations is in sharp contrast to the WKB-approach for Schrödinger-type problems, see *e.g.* [Ge], [Gr].

The Zitterbewgung may cause severe problems since *a-priori* one cannot exclude the possibility of *resonant interactions* between the principal term u_0 and the magnetic potential $A^\varepsilon[u_0]$ obtained from (1.1.18) with r.h.s. given by (4.4.6). If this happens to be the case, our one-phase ansatz (3.3.1) breaks down and instead one would need to establish a so-called *resonant asymptotic expansion* in the spirit of [JMR1]. (We remark that so far, only the case of resonances in one spatial dimension can be treated rigorously, *cf.* [JMR2].)

We will show that these problems do not appear in our situation. To this end, we need to describe precisely what kind of ε-oscillations are present in $A^\varepsilon[u_0]$.

First we note that, by the superposition principle, every term appearing on the r.h.s. of (4.4.6) generates its own potential field. The nonoscillating terms of (4.4.6) lead to a standard hyperbolic problem, hence (2.2.5) gives

$$A_0[u_0](t,x):=\mathcal{G}_r(t,x)*\big(\omega_+(\nabla\phi)|u_{0,+}|^2+\omega_{-,k}(\nabla\phi)|u_{0,-}|^2\big)\,(t,x). \tag{4.4.7}$$

In order to treat the Zitterbewegung, let us define $Z:=(Z_1,Z_2,Z_3)$ by

$$\begin{aligned}Z_k(t,x,\phi(t,x)/\varepsilon):=\ &\langle u_{0,+}(t,x),\alpha^k u_{0,-}(t,x)\rangle e^{-i2\phi(t,x)/\varepsilon}\\&+\langle u_{0,-}(t,x),\alpha^k u_{0,+}(t,x)\rangle e^{i2\phi(t,x)/\varepsilon},\quad k=1,2,3.\end{aligned} \tag{4.4.8}$$

Using this definition, we can now prove the following lemma:

Lemma 4.1. *Let $\Omega\subseteq\mathbb{R}^{1+3}$ be the slab in which existence of a smooth phase $\phi\in C^\infty(\Omega;\mathbb{R})$, satisfying (3.3.14), is guaranteed. Then, given $Z\in C^\infty(\Omega\times\mathbb{S}^1;\mathbb{C}^3)$, as in (4.4.8), there exists a uniquely determined smooth $A^\varepsilon\in C^\infty(\Omega\times\mathbb{S}^1;\mathbb{C}^3)$, with*

$$A^\varepsilon(t,x,\theta)\sim\sum_{l=1}^\infty\varepsilon^l A_l(t,x,\theta), \tag{4.4.9}$$

s.t. $A^\varepsilon(t,x,\phi(t,x)/\varepsilon)$ *satisfies:*

$$\begin{cases} \Box A^\varepsilon(t,x) - Z(t,x) \sim 0, & \text{in } C^\infty(\Omega;\mathbb{C}^3), \\ A^\varepsilon\big|_{t=0} = \partial_t A^\varepsilon\big|_{t=0} = 0. \end{cases} \tag{4.4.10}$$

More precisely, $A_i(t,x,\phi(t,x)/\varepsilon)$ *can be written in the following form:*

$$A_l(t,x,\phi(t,x)/\varepsilon) = A_l^+(t,x)e^{i2\phi(t,x)/\varepsilon} + A_l^-(t,x)e^{-i2\phi(t,x)/\varepsilon}, \tag{4.4.11}$$

with principal amplitudes $A_1^\pm \in C^\infty(\Omega;\mathbb{C}^3)$*, given by*

$$A_{1,k}^\pm(t,x) = \pm\, \langle u_0(t,x), \alpha^k u_0(t,x)\rangle. \tag{4.4.12}$$

Proof. The proof can be done separately for each spatial component of $A^\varepsilon(t,x,\theta)$ and for both types of oscillations, corresponding to $\pm 2\phi$. Hence, we are lead to the following type of problem:

$$\begin{cases} \Box a^\varepsilon(t,x) = b(t,x)e^{\pm i2\phi(t,x)/\varepsilon}, & (t,x)\in\Omega, \\ a^\varepsilon\big|_{t=0} = \partial_t a^\varepsilon\big|_{t=0} = 0, \end{cases} \tag{4.4.13}$$

for some given $b \in C^\infty(\Omega;\mathbb{C})$. Let us define a new variable $f^\varepsilon(t,x)\in\mathbb{C}^5$ by

$$f^\varepsilon(t,x) := (\partial_1 a^\varepsilon, \partial_2 a^\varepsilon, \partial_3 a^\varepsilon, \partial_t a^\varepsilon, a^\varepsilon)^\top(t,x).$$

Further, denoting by

$$\hat b(t,x) := (0,0,0,b(t,x),0)^\top,$$

we can rewrite (4.4.13) in the form of a symmetric hyperbolic system

$$(\partial_t + (\lambda\cdot\nabla) + \kappa)\, f^\varepsilon(t,x) = \hat b(t,x)e^{\pm i2\phi(t,x)/\varepsilon}, \tag{4.4.14}$$

with λ^k, κ, denoting realvalued (symmetric) 5×5 matrices. In our case, these matrices are simply given by (see *e.g.* [R], p. 21, for more details):

$$\lambda^k := -\left(\delta_{mk}\delta_{n4} + \delta_{m4}\delta_{nk}\right)_{m,n}, \quad \kappa := -\left(\delta_{5m}\delta_{4n}\right)_{m,n}, \quad m,n = 1,\dots,5, \tag{4.4.15}$$

where δ_{ab} denotes the Kronecker symbol and $k = 1,2,3$.

It is now possible to use the existing results on linear geometrical optics, provided the phase $\pm 2\phi$ is *not* characteristic for the system (4.4.14), *i.e.*

$$\det\left(\pm 2\partial_t\phi(t,x) \pm 2\lambda\cdot\nabla\phi(t,x) + \kappa\right) \neq 0, \quad \forall (t,x)\in\Omega. \tag{4.4.16}$$

Computing this determinant, we obtain the condition

$$\pm 32\,(\partial_t\phi)^3((\partial_t\phi)^2 - |\nabla\phi|^2) \neq 0, \quad \forall (t,x)\in\Omega. \tag{4.4.17}$$

Since, by assumption, ϕ solves the Klein-Gordon eikonal equation (3.3.11) the second factor on the l.h.s. of (4.4.17) is equal to one and thus, different from zero in all of Ω. On the other hand we get from (3.3.11): $(\partial_t\phi)^3 = (|\nabla\phi|^2+1)^{3/2} \neq 0$, $\forall (t,x)\in\Omega$. Hence, condition (4.4.16) is fulfilled and the assertion follows from theorem 4.4 in [Ra]. In particular we get:

$$\begin{aligned} A_{1,k}^\pm(t,x) &= \frac{\mp 1}{-(\partial_t\phi)^2 + |\nabla\phi|^2}\, \langle u_0(t,x), \alpha^k u_0(t,x)\rangle \\ &= \pm\, \langle u_0(t,x), \alpha^k u_0(t,x)\rangle, \end{aligned} \tag{4.4.18}$$

which concludes the proof. ∎

Lemma 4.1 shows that the Zitterbewegung in (4.4.6) generates a magnetic potential which is small, *i.e.* at least of order $O(\varepsilon)$. Moreover the ε-oscillations, appearing in A^ε, are exactly the *same* as in the (4.4.6) and hence we can consistently proceed with our one-phase expansion for ψ^ε.

Remark 4.2. Although the MD system is hyperbolic, lemma 4.1 can be considered as an analogue of so-called *elliptic* high frequency asymptotics, the main feature of which is the fact that asymptotic solutions can be obtained by *local* (in t, x) algebraic relations. In other words, the Maxwell system can be considered *transparent* w.r.t to the oscillations generated by the Dirac equation.

The result of lemma 4.1 implies that the nonlinearity $\mathcal{N}^\varepsilon[u_0]$, defined in (3.3.3), admits an asymptotic expansion of the form

$$\mathcal{N}^\varepsilon[u_0](t,x,\theta) \sim \mathcal{N}_0[u_0](t,x,\theta) + \sum_{l=1}^{\infty} \varepsilon^l \mathcal{N}_l(t,x,\theta), \tag{4.4.19}$$

where, using the expressions (4.4.2), (4.4.7), we have:

$$\mathcal{N}_0(t,x,\theta) = \ ((\alpha \cdot A_0[u_0](t,x)) - V[u_0](t,x))\ u_0(t,x,\theta) \tag{4.4.20}$$

and for all $l \geq 1$:

$$\mathcal{N}_l(t,x,\theta) := \ (\alpha \cdot A_l(t,x,\theta))\, u_0(t,x,\theta), \tag{4.4.21}$$

with A_l given by (4.4.11).

Note that the expression (4.4.19) represents two kinds of ε-oscillations: Those described by phase-functions $\pm\phi$ are present in all terms $\mathcal{N}_l$ with $l \geq 0$, whereas ε-oscillations with phases $\pm 3\phi$ appear in $\mathcal{N}_l$ with $l > 0$. Also, note that u_0 enters in a *nonlocal* way only in the lowest order term (4.4.20).

5. Nonlinear Transport along Rays

We need to find an evolution equation, which determines u_0 from the initial data. To this end, let us define an operator $\mathbb{P}$, which projects on the set of harmonics corresponding to solutions of the eikonal equation (3.3.11):

Definition 5.1. Given some $v \in C^\infty(\mathbb{R}^4 \times \mathbb{S}^1; \mathbb{C}^4)$, which can be represented by

$$v(t,x,\theta) = \sum_{m\in\mathbb{Z}} v_m(t,x) e^{im\theta}, \tag{5.5.1}$$

we define the action of $\mathbb{P}$ on v, by

$$(\mathbb{P}v)(t,x,\theta) := (\Pi_+(\nabla\phi)v_{+1})(t,x)e^{i\theta} + (\Pi_-(\nabla\phi)v_{-1})(t,x)e^{-i\theta}. \tag{5.5.2}$$

In words: $\mathbb{P}$ picks modes corresponding to $m = \pm 1$ and multiplies them with the matrices $\Pi_\pm(\nabla\phi)$. Note that, at least in Ω, it holds true that

$$(\mathbb{P}u_0)(t,x,\theta) = u_0(t,x,\theta), \tag{5.5.3}$$

in view of (3.3.8) and (3.3.18), (3.3.19).

From (3.3.2), we have that the evolution of u_0 is determined by terms of order $O(\varepsilon^{p+1}) = O(\varepsilon^{3/2})$. Setting the corresponding coefficient in (3.3.4) equal to zero, *i.e.* $R_2(t,x) = 0$, yields

$$i(\partial_t\phi + (\alpha\cdot\nabla\phi))\partial_\theta u_2 - \beta u_2 + i(\partial_t u_0 + (\alpha\cdot\nabla)u_0) + \mathcal{N}_0[u_0] = 0, \tag{5.5.4}$$

with $\mathcal{N}_0[u_0]$ as in (4.4.20). Equation (5.5.4) implies

$$i(\partial_t + (\alpha \cdot \nabla))u_0 + \mathcal{N}_0[u_0] \in \operatorname{ran}\Big(i(\partial_t\phi + (\alpha \cdot \nabla\phi))\partial_\theta - \beta\Big). \tag{5.5.5}$$

Applying $\mathbb{P}$ to (5.5.4), eliminates the term including u_2, since $\mathbb{P}$ projects on the kernel of $i(\partial_t\phi + (\alpha \cdot \nabla\phi))\partial_\theta - \beta$ and we obtain

$$i\mathbb{P}\partial_t u_0 + i\mathbb{P}(\alpha \cdot \nabla)u_0 + \mathbb{P}\mathcal{N}_0[u_0] = 0. \tag{5.5.6}$$

Using the fact that $\mathbb{P}u_0 = u_0$, by (5.5.3), this gives

$$\mathbb{P}\partial_t(\mathbb{P}u_0) + \mathbb{P}(\alpha \cdot \nabla)(\mathbb{P}u_0) = i\mathbb{P}\mathcal{N}_0[\mathbb{P}u_0]. \tag{5.5.7}$$

This equation is similar to the one appearing in [DoRa], however, in contrast to the quoted work, our nonlinearity constitutes only the first term of an asymptotic expansion of the full $\mathcal{N}^\varepsilon[u_0]$.

We proceed by stating a useful identity:

$$\alpha^k \Pi_\pm(\xi) = \Pi_\mp(\xi)\alpha^k + \omega_{\pm,k}(\xi)\mathbb{I}_4, \tag{5.5.8}$$

obtained from straightforward calculations. After more lengthy but straightforward calculations, in which we apply the relations (4.4.3), (4.4.4) and (5.5.8), we can express the l.h.s. of (5.5.7) in the form of a transport operator:

$$\begin{aligned}\Pi_\pm\partial_t(\Pi_\pm u_0) + \Pi_\pm\alpha \cdot \nabla(\Pi_\pm u_0) = \\ \partial_t u_{0,\pm} + (\omega_\pm(\nabla\phi)\cdot\nabla)u_{0,\pm} + \frac{1}{2}\operatorname{div}(\omega_\pm(\nabla\phi))u_{0,\pm}.\end{aligned} \tag{5.5.9}$$

On the other hand, computing the action of the projector $\mathbb{P}$ on the nonlinear term $\mathcal{N}_0[u_0]$, we get

$$\begin{aligned}\mathbb{P}\mathcal{N}_0[\mathbb{P}u_0] = {} & e^{i\phi/\varepsilon}\left((A_0[u_0]\cdot\omega_+(\nabla\phi)) - V[u_0]\right) u_{0,+} \\ & + e^{-i\phi/\varepsilon}\left((A_0[u_0]\cdot\omega_-(\nabla\phi)) - V[u_0]\right) u_{0,-}\,.\end{aligned} \tag{5.5.10}$$

Here we have again used (4.4.4). Thus, we finally conclude, that the time-evolution of the *principal amplitudes* $u_{0,\pm}$ is governed by the following semilinear first-order system:

$$\begin{cases}(\partial_t + (\omega_+(\nabla\phi)\cdot\nabla))\, u_{0,+}(t,x) = \Gamma_+[u_0](t,x)\, u_{0,+}(t,x), \\ (\partial_t + (\omega_-(\nabla\phi)\cdot\nabla))\, u_{0,-}(t,x) = \Gamma_-[u_0](t,x)\, u_{0,-}(t,x),\end{cases} \tag{5.5.11}$$

where

$$\Gamma_\pm[u_0](t,x) := iA_0[u_0]\cdot\omega_\pm(\nabla\phi) - iV[u_0] - \frac{1}{2}\operatorname{div}(\omega_\pm(\nabla\phi)). \tag{5.5.12}$$

By construction, the polarization of $u_{0,\pm}$ is conserved during the evolution. The system (5.5.11) determines $(\mathbb{P}u_0)(t,x,\theta)$, from its initial data $(\mathbb{P}u_0)(0,x,\theta)$ and since $(\mathbb{P}u_0) = u_0$, we have completely constructed u_0.

Multiplying (5.5.11) by $\overline{u}_{0,+}$ resp. $\overline{u}_{0,-}$ and integrating by parts, we obtain the important property of charge-conservation:

$$\int_{\mathbb{R}^3} |u_{0,+}(t,x)|^2 + |u_{0,-}(t,x)|^2 \, dx = \|u_0(t,x)\|_2^2 = \text{const}. \tag{5.5.13}$$

Given u_0, determined by (5.5.11), it remains to construct the higher order terms $u_j(t,x,\theta)$, $j \geq 1$ of our approximate solution. This can be done by a similar construction as given in [DoRa]:

We expand the cubic nonlinearity $\mathcal{N}^\varepsilon[u^\varepsilon]$ in powers of ε:

$$\mathcal{N}^\varepsilon[u_0 + \varepsilon u_1 + \cdots] \sim \mathcal{N}^\varepsilon[u_0] + \varepsilon\mathcal{M}^\varepsilon[u_0,u_1] + \cdots, \tag{5.5.14}$$

where, using the definitions (4.4.2), (4.4.8), we easily compute:

$$\mathcal{M}^\varepsilon[u_0,u_1] = 2\,(\mathcal{G}_r * \langle u_0,u_1\rangle)u_0 + V[u_0]u_1 + \alpha\cdot(\mathcal{G}_r * Z)\,u_1 + \left(\sum_{k=1}^{3}\alpha^k\left(\mathcal{G}_r * \langle u_0,\alpha^k u_1\rangle + \mathcal{G}_r * \langle u_1,\alpha^k u_0\rangle\right)\right)u_0. \tag{5.5.15}$$

We need to apply lemma 4.1 to all terms appearing on the r.h.s of (5.5.14), which results in a similar expansion as given in (4.4.19). Hence, after rearranging terms in powers of ε, we can write

$$\mathcal{N}^\varepsilon[u_0+\varepsilon u_1+\cdots] \sim \mathcal{N}_0[u_0] + \varepsilon\mathcal{N}_1 + \varepsilon\mathcal{M}_0[u_0,u_1] + \cdots . \tag{5.5.16}$$

Consequently, for $j\geq 1$, the $O(\varepsilon^{j/2+1/2})$-coefficient is given by

$$R_j(t,x) = i(\partial_t\phi + (\alpha\cdot\nabla\phi))\partial_\theta u_j - \beta u_j + i(\partial_t + (\alpha\cdot\nabla))u_{j-2} + \mathcal{M}_0[u_0,u_{j-2}] + S(u_0,\dots,u_{n<j-2}), \tag{5.5.17}$$

where, as usual, we impose: $u_n(t,x,\theta)=0$, for all $n<0$. The source term S, only depends on lower order coefficients $u_0,\dots,u_{n<j-2}$. It is obtained by applying lemma 4.1 to higher order terms in the expansion (5.5.14), leading to contributions $\mathcal{N}_l$ and $\mathcal{M}_l$ with: $l+1=j/2$.

We can now decompose

$$u_j(t,x,\theta) = (\mathbb{P}u_j)(t,x,\theta) + (\mathrm{id}_4-\mathbb{P})u_j(t,x,\theta). \tag{5.5.18}$$

Note that in contrast to u_0, where, in view of (5.5.3), it holds

$$(\mathrm{id}_4-\mathbb{P})u_0(t,x,\theta) = 0, \tag{5.5.19}$$

we can not expect all higher order coefficients u_j to be polarized too. Hence, we need to determine separately $\mathbb{P}u_j$ and $(\mathrm{id}_4-\mathbb{P})u_j$. To this end, we introduce the following definition:

Definition 5.2. Again, let $v(t,x,\theta)$ be given as in definition 5.1, then we define a partial inverse $\mathbb{Q}$, associated to $\mathbb{P}$, by

$$(\mathbb{Q}v)(t,x,\theta) := (\Lambda_+(\nabla\phi)v_{+1})(t,x)e^{i\theta} + (\Lambda_-(\nabla\phi)v_{-1})(t,x)e^{-i\theta}, \tag{5.5.20}$$

where $\Lambda_\pm$ is the partial inverse to $\Pi_\pm$, defined by (2.2.17).

Assume now that we already know u_n, for $n<j$, then $(\mathrm{id}_4-\mathbb{P})u_j$ is determined by setting $(\mathbb{Q}R_j)(t,x,\theta)=0$. This gives

$$(\mathrm{id}_4-\mathbb{P})u_j = -\,\mathbb{Q}\Big(i(\partial_t + (\alpha\cdot\nabla))u_{j-2} + \mathcal{M}_0[u_0,u_{j-2}] + S(u_0,\dots,u_{n<j-2})\Big). \tag{5.5.21}$$

On the other hand, setting $(\mathbb{P}R_{j+2})(t,x,\theta)=0$, we obtain an evolution equation for $\mathbb{P}u_j$:

$$i\mathbb{P}\partial_t(\mathbb{P}u_j) + i\mathbb{P}(\alpha\cdot\nabla\mathbb{P}u_j) = -\mathbb{P}\mathcal{M}_0[u_0,u_j] + r(u_0,\dots,u_j), \tag{5.5.22}$$

where

$$r(u_0,\dots,u_j) := -\mathbb{P}S(u_0,\dots,u_{n<j}) - \mathbb{P}\left(i\partial_t + i(\alpha\cdot\nabla) - \beta\right)(\mathrm{id}_4-\mathbb{P})u_j. \tag{5.5.23}$$

Here, the first term on the r.h.s is already known by the inductive hypothesis and the second one is given by equation (5.5.21). Hence, by induction, one can construct all higher order coefficients $u_j(t,x,\theta)$, $j\geq 1$ in this way.

Note, that the left hand side of (5.5.22) is essentially a transport operator, which can be expressed as shown above. Thus (5.5.22) constitutes a *linear* first order system, which determines the so-called *propagating part* $\mathbb{P}u_j$ from its initial data.

Remark 5.3. The above construction can be generalized to the case, where, additionally *given* external potentials V^{ext}, A^{ext}, are included, or, equivalently, nonzero Cauchy initial data for the Maxwell equations (1.1.17), (1.1.18) are assumed. One checks that, instead of (3.3.13), the following Hamilton-Jacobi equation, corresponding to $m = 1$, holds:

$$\partial_t \phi_\pm \pm \sqrt{|\nabla \phi_\pm - A^{ext}(t,x)|^2 + 1} + V^{ext}(t,x) = 0. \tag{5.5.24}$$

Since no other harmonics with $m \neq 1$ exist, one again ends up with two phases $\phi_\pm(t,x)$, corresponding to the electronic resp. positronic degrees of freedom. In this case however, $-\phi_+(t,x) \neq \phi_-(t,x)$, in contrast to (3.3.17). Also, one obtains an additional matrix-valued *spin-transport term*, appearing on the left hand side of (5.5.11) and which can be found in [BK], [FK], [PST] *e.g.*.

We are now in the position to formulate our first theorem (in which we do not aim to impose the weakest possible assumptions). In the following, $C^\infty_{(0)}$ denotes the space of smooth function, compactly supported in $x \in \mathbb{R}^3$.

Theorem 5.4. *Assume that the initial data $\psi^\varepsilon_I(x)$ admits an asymptotic expansion of the form:*

$$\begin{cases} \psi^\varepsilon_I(x) = \sqrt{\varepsilon}\, u^\varepsilon(x, \phi_I(x)/\varepsilon), \\ u^\varepsilon(x,\theta) \sim \displaystyle\sum_{j=0}^\infty \varepsilon^{j/2} \chi_j(x,\theta), \end{cases} \tag{5.5.25}$$

where $\phi_I \in C^\infty(\mathbb{R}^3;\mathbb{R})$ satisfies $\|D^2\phi_I\| \leq C$. Further, let $\chi_j \in C^\infty_{(0)}(\mathbb{R}^3 \times \mathbb{S}^1; \mathbb{C}^4)$ be s.t.

$$(\mathbb{P}\chi_j)(x,\theta) = \chi_j(x,\theta), \quad \forall j \in \mathbb{N}. \tag{5.5.26}$$

Then, there exists a $0 < T^* \leq T$, a corresponding domain $\Omega^* := [0,T^*) \cap \Omega$ and a uniquely determined $u^\varepsilon \in C^\infty_{(0)}(\Omega^* \times \mathbb{S}^1; \mathbb{C}^4)$, with

$$u^\varepsilon(t,x,\theta) \sim \sqrt{\varepsilon} \sum_{j=0}^\infty \varepsilon^{j/2} u_j(t,x,\theta), \tag{5.5.27}$$

s.t. $u^\varepsilon(t,x,\phi(t,x)/\varepsilon)$ satisfies:

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon - \mathcal{D}^\varepsilon_A(t,x,\varepsilon D) u^\varepsilon \sim 0, \quad \forall (t,x) \in \Omega^*, \\ \qquad\qquad u^\varepsilon\big|_{t=0} = \psi^\varepsilon_I(x). \end{cases} \tag{5.5.28}$$

More precisely we have:
The principal term u_0 is given by (3.3.20), satisfies $(\mathbb{P}u_0)(t,x,\theta) = u_0(t,x,\theta)$ and solves (5.5.7) with initial data $(\mathbb{P}u_0)(0,x,\theta) = \chi_0(x,\theta)$.
For all $j \geq 13$, the infinite sequence of equations (5.5.21), (5.5.22), uniquely determines $u_j(t,x,\theta)$, with initial data $(\mathbb{P}u_j)(0,x,\theta) = \chi_j(x,\theta)$.

Proof. The existence of a smooth phase $\phi \in C^\infty(\Omega;\mathbb{R})$, on the slab $\Omega \subseteq \mathbb{R}^{1+3}$, is already guaranteed by lemma 3.2.
Next, consider the case $j = 0$: Since $\omega_\pm(\nabla\phi) \in \mathbb{R}^3$, defined by (4.4.5), satisfies for all multiindices σ, ν:

$$\sup_{(t,x)\in\Omega} |\, \partial_t^\sigma \partial_x^\nu\, \omega_{\pm,k}(\nabla\phi(t,x))| < \infty, \quad k = 1,2,3, \tag{5.5.29}$$

we find that the l.h.s. of (5.5.11) constitutes a linear symmetric hyperbolic system. From L^2-conservation property (5.5.13) the usual commutator estimates lead to

H^s-regularity, *i.e.* $u_0 \in C^1(\mathbb{R}^3; H^s)$ for all $s \geq 0$. Now, it is a standard results for the linear wave equations in $d = 3$ spatial dimensions, that source terms in $H^s(\mathbb{R}^3)$ generate solutions in $H^s(\mathbb{R}^3)$, *cf.* [Ho], chapter XXIII. This fact and Schauder's lemma imply that the maps

$$u_0(t, \cdot) \mapsto \Gamma_\pm[u_0(t, \cdot)] \tag{5.5.30}$$

are locally Lipschitz from $H^s((0,t) \times \mathbb{R}^3 \times \mathbb{S}^1)$ to itself, for all $s > 2$, uniformly for $0 \leq t < T$. By a standard Picard iteration we therefore obtain a local-in-time existence and uniqueness result in $H^s(\Omega^* \times \mathbb{S}^1)$, for every $s > 2$ and a Sobolev imbedding gives $u_0 \in C^1(\Omega^* \times \mathbb{S}^1; \mathbb{C}^4)$. The proof of the asserted regularity for the t-derivatives follows by using the differential equation to express them in terms of x-derivatives and the finite speed of propagation for solution of (5.5.11) implies that u_0 is compactly supported in $\mathbb{R}^3_x$ since χ_0 is.
Finally, for $j > 0$, we have that the amplitudes $u_{j,\pm}$ are determined by the linear symmetric hyperbolic system (5.5.22), (5.5.21) and the assertion is proved. $\square$

Once again, we stress the fact that we analyze the MD system in a *weakly coupled regime*. Indeed, the above result implies:

Corollary 5.5. *Let $u^\varepsilon(t, x, \phi(t, x)/\varepsilon)$ be as in theorem 5.4, then*

$$\begin{cases} V^\varepsilon[u^\varepsilon](t,x) \sim \varepsilon V[u_0](t,x) + O(\varepsilon^2), \\ A^\varepsilon[u^\varepsilon](t,x) \sim \varepsilon A_0[u_0](t,x) + O(\varepsilon^2), \end{cases} \tag{5.5.31}$$

where $V[u_0]$, $A_0[u_0]$ are nonoscillating and explicitly given by (4.4.2), (4.4.7).

6. Stability and further results

In theorem 5.4 we obtained a function u^ε, which solves the MD equation up to a residual $R^\varepsilon \sim 0$, compactly supported in $[0, T^*] \times \mathbb{R}^3$. We want to compare u^ε to a true solution ψ^ε and prove that $u^\varepsilon - \psi^\varepsilon \sim 0$ on $\Omega^* = [0, T^*] \times \mathbb{R}^3$.

Theorem 6.1. *Under the assumptions of theorem 5.4, there is an $\varepsilon^* \in (0, 1)$, s.t. for $\varepsilon < \varepsilon^*$, there exists a unique smooth $\psi^\varepsilon \in C^\infty_{(0)}(\Omega^*; \mathbb{C}^4)$, satisfying*

$$\begin{cases} i\varepsilon\partial_t \psi^\varepsilon - \mathcal{D}^\varepsilon_A(t, x, \varepsilon D)\psi^\varepsilon = 0, \quad \forall (t,x) \in \Omega^*, \\ \qquad \psi^\varepsilon\big|_{t=0} = \psi^\varepsilon_I(x), \end{cases} \tag{6.6.1}$$

which is asymptotically equivalent to u^ε, i.e.

$$\psi^\varepsilon(t,x) \sim u^\varepsilon(t, x, \phi(t,x)/\varepsilon) \quad \text{in } C^\infty_{(0)}(\Omega^*; \mathbb{C}^4). \tag{6.6.2}$$

Proof. Defining $v^\varepsilon := u^\varepsilon - \psi^\varepsilon$, we obtain for the following IVP:

$$\begin{cases} i(\varepsilon\partial_t + \varepsilon(\alpha \cdot \nabla))v^\varepsilon - \beta v^\varepsilon + \mathcal{N}^\varepsilon[u^\varepsilon + v^\varepsilon] - \mathcal{N}^\varepsilon[u^\varepsilon] = -R^\varepsilon, \quad \text{in } \Omega^*, \\ \qquad v^\varepsilon\big|_{t=0} = 0, \end{cases} \tag{6.6.3}$$

The nonlinearity can be handled analogous to the proof of lemma 6.2, in [DoRa], since for smooth sources the wave equation has smooth solutions, which moreover travel with finite speed. Having this in mind, the rest of the proof is a simple modification of the one of theorem 6.1 in [DoRa]. $\square$

As far as particle creation is concerned, the local-in-time solution $\psi^\varepsilon \sim O(\sqrt{\varepsilon})$ shows the following qualitative behavior:

Corollary 6.2. *Let ψ_I^ε be as in theorem 5.4. If $(\Pi_-\psi_I^\varepsilon)(x) = 0$, then for $0 \leq t < T^*$ it holds: $(\Pi_-\psi^\varepsilon)(t,x) \sim O(\varepsilon^{3/2})$, i.e. no positrons are generated, up to $O(\varepsilon^{3/2})$ and the analogous statement for electrons is valid, too.*

Proof. The assertion holds true, since a careful examination of the asymptotic expansion shows that both, u_0 *and* u_1, satisfy: $(\mathbb{P}u_j)(t,x,\theta) = u_j(t,x,\theta)$, in Ω^*. □

For completeness we shall also consider the *matrix-valued Wigner transform* corresponding to ψ^ε, *i.e.*

$$w^\varepsilon[\psi^\varepsilon](t,x,\xi) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \psi^\varepsilon\left(t, x + \frac{\varepsilon}{2}y\right) \otimes \overline{\psi^\varepsilon}\left(t, x - \frac{\varepsilon}{2}y\right) e^{i\xi\cdot y} dy, \tag{6.6.4}$$

where $\otimes$ denotes the tensor product of vectors. The 4×4-matrix $w^\varepsilon[\psi^\varepsilon]$ is a *phase-space description* of the quantum state ψ^ε.

Corollary 6.3. *Let $\psi^\varepsilon \sim O(\sqrt{\varepsilon})$ be the unique smooth local-time-solution of the MD system, as guaranteed by theorem (6.1) and let $w^\varepsilon[\psi^\varepsilon] \sim O(\varepsilon)$ be its Wigner transform. Then, up to extraction of subsequences, we have*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} w^\varepsilon[\psi^\varepsilon] = \mu \quad \textit{in } \mathcal{S}'([0,T^*)\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi) \textit{ weak-}\star, \tag{6.6.5}$$

where the matrix-valued Wigner measure μ is given by: $\mu = \mu_+ + \mu_-$, with

$$\mu_\pm(t,x,\xi) = u_{0,\pm}(t,x) \otimes \overline{u}_{0,\pm}(t,x)\,\delta(\xi \mp \nabla\phi(t,x)). \tag{6.6.6}$$

Proof. Since ϕ has no stationary points within Ω^*, a nonstationary phase argument implies that all Wigner matrix elements, which mix the electronic and positronic components are of order $O(\varepsilon^\infty)$. The assertion then follows from the well known results on Wigner measures, *cf.* [GMMP]. □

We finally remark on the case of the Dirac-Maxwell system where the Dirac particles have vanishing mass. Instead of (3.3.13) we obtain

$$\partial_t\phi = \pm|\nabla\phi|, \tag{6.6.7}$$

which is equivalent to the eikonal equation of the wave equation. It follows that in this case lemma 4.1 can not hold, since the phases $\pm\phi$ are characteristic for the wave equation. More precisely, they are indeed *everywhere* characteristic, *i.e.* in all of Ω, which again allows for an asymptotic description of the A^ε, similar to (4.4.9), (4.4.11), *cf.* [La] or [Ra], chapter 5. In this case, the ε-oscillations are also given by $\exp(\pm 2i\phi/\varepsilon)$, but the corresponding amplitudes $A_l^\pm$ are of course different. The main difference, however, is the fact that in this case the summation index runs from $l = 0$ to infinity, *i.e.* ε-oscillations are present already in the lowest order term. This leads to a more complicated structure of the transport equations for the amplitudes u_j, but apart from that all results remain valid.

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