HYDRODYNAMIC LIMIT OF ASYMMETRIC EXCLUSION PROCESSES UNDER DIFFUSIVE SCALING IN $d \geq 3$.

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Dedicated to József Fritz on his sixtieth birthday.

Abstract. We consider the asymmetric exclusion process. We start from a profile which is constant along the drift direction and prove that the density profile, under a diffusive rescaling of time, converges to the solution of a parabolic equation.

1. Introduction

Consider the asymmetric exclusion process evolving on the lattice $\mathbb{Z}^d$. This dynamics can be informally described as follows : fix a translation invariant transition probability $p(x,y) = p(0,y-x) = p(y-x)$. Each particle, independently from the others, waits a mean one exponential time, at the end of which being at x it chooses the site $x+y$ with probability $p(y)$. If the chosen site is unoccupied, the particle jumps, otherwise it stays where it is. In both cases, after its attempt, the particle waits a new mean one exponential time.

The configurations of the state space $\{0,1\}^{\mathbb{Z}^d}$ are denoted by the Greek letter η so that, for x in $\mathbb{Z}^d$, $\eta(x)$ is equal to 1 or 0, whether site x is occupied or not. For each density $0 \leq \alpha \leq 1$, the Bernoulli product measure with parameter α, denoted by ν_α, is invariant.

The macroscopic evolution of the process under Euler rescaling is described [14] by the first order quasilinear hyperbolic equation

$$\partial_t \rho \; + \; q \cdot \nabla F(\rho) \; = \; 0 \; , \tag{1.1}$$

where $F(a) = a(1-a)$ and $q \in \mathbb{R}^d$ is the mean drift of each particle : $q = \sum_z z p(z)$. Assume that the system starts from a product measure with slowly varying density $\rho_0(\varepsilon u)$. Under Euler scaling (times of order $t\varepsilon^{-1}$) the density has still a slowly varying profile $\lambda_\varepsilon(t, \varepsilon u)$ which converges weakly (in fact pointwisely at every continuity point, [7]) to the entropy solution of equation (1.1) with initial data ρ_0.

In the context of asymmetric interacting particle systems the Navier–Stokes equations takes the form

$$\partial_t \rho^\varepsilon \; + \; q \cdot \nabla F(\rho^\varepsilon) \; = \; \varepsilon \sum_{i,j} \partial_{u_i} \Big(a_{i,j}(\rho^\varepsilon) \partial_{u_j} \rho^\varepsilon \Big) \; , \tag{1.2}$$

where a is a diffusion coefficient. Three different interpretations have been proposed for the Navier–Stokes corrections :

(a) The incompressible limit ([3], [4]) : Consider a small perturbation of a constant profile α_0 : $\rho_0^\varepsilon = \alpha_0 + \varepsilon\varphi$. Assuming that this form persists at latter times ($\rho^\varepsilon(t,u) = \alpha_0 + \varepsilon\varphi(t,u)$) we obtain from (1.2) the following equation for $\varphi_\varepsilon = \varphi(t\varepsilon^{-1}, u)$

$$\partial_t \varphi_\varepsilon \;+\; \varepsilon^{-1} F'(\alpha_0) q \cdot \varphi_\varepsilon \;+\; (1/2) F''(\alpha_0) q \cdot \nabla \varphi_\varepsilon^2 \;=\; a_{i,j}(\alpha_0) \sum_{i,j} \partial^2_{u_i,u_j} \varphi_\varepsilon \;+\; O(\varepsilon)\;.$$

A Galilean transformation $m_\varepsilon(t,u) = \varphi_\varepsilon(t, u + \varepsilon^{-1} t F'(\alpha_0) q)$ permits to remove the diverging term of the last differential equation and to get a limit equation for $m = \lim_{\varepsilon\to 0} m_\varepsilon$

$$\partial_t m \;+\; (1/2) F''(\alpha_0) q \cdot \nabla m^2 \;=\; a_{i,j}(\alpha_0) \sum_{i,j} \partial^2_{u_i,u_j} m\;.$$

(b) First order correction to the hydrodynamic equation ([2], [8]) : Fix a smooth profile $\rho_0 \colon \mathbb{R}^d \to \mathbb{R}_+$ and consider a process starting from a product measure with slowly varying density $\rho_0(\varepsilon u)$. We have seen that under Euler scaling the density is still a slowly varying profile $\lambda_\varepsilon(t, \varepsilon u)$ which converges weakly to the entropy solution of equation (1.1) with initial data ρ_0. This second interpretation asserts that the solution of equation (1.2) with initial profile ρ_0 approximates λ_ε up to the order ε :

$$\varepsilon^{-1}\big(\lambda_\varepsilon - \rho^\varepsilon\big) \;\to\; 0$$

in a weak sense as $\varepsilon \downarrow 0$.

(c) Long time behaviour ([8], [1]) : The third interpretation consists in analyzing the behaviour of the solution of equation (1.2) in time scales of order $t\varepsilon^{-1}$. Let $b_\varepsilon(t,u) = \rho(t\varepsilon^{-1}, u)$. From (1.2) we obtain the following equation for b_ε :

$$\partial_t b_\varepsilon \;+\; \varepsilon^{-1} q \cdot \nabla F(b_\varepsilon) \;=\; \sum_{i,j} \partial_{u_i} \Big(a_{i,j}(b_\varepsilon) \partial_{u_j} b_\varepsilon \Big)\;.$$

To eliminate the diverging term $\varepsilon^{-1} q \cdot \nabla F(b_\varepsilon)$, assume that the initial data (and therefore the solution at any fixed time) is constant along the drift direction : $q \cdot \nabla \rho_0 = 0$. In this case we get the parabolic equation

$$\partial_t b_\varepsilon \;=\; \sum_{i,j} \partial_{u_i} \Big(a_{i,j}(b_\varepsilon) \partial_{u_j} b_\varepsilon \Big)$$

which describes the evolution of the system in the hyperplane orthogonal to the drift.

Notice that while the first and the third interpretation concern the behaviour of the system under diffusive rescaling, the second one is a statement on the process under Euler rescaling. Interpretation (a) and (b) have been proved [4], [8] for asymmetric simple exclusion processes in dimensions $d \ge 3$ and a double variational formula for the diffusion coefficient was deduced. As one would expect, the diffusion coefficients of the two interpretations are the same and may be expressed by a Green–Kubo formula [13]. It was also proved (Corollary 6.2, [9]) that the diffusion coefficient is strictly bounded below in the matrix sense by the diffusion coefficient that governs the evolution of the symmetric process and that it depends smoothly on the density [12].

In contrast with interpretations (a) and (b), the third one is meaningful in dimension $d \ge 2$. It has been proved in [1] for asymmetric zero range processes. The purpose of this paper is to give a rigorous proof of the third interpretation in the case of asymmetric exclusion processes in dimension $d \ge 3$. The proof in this context is much more demanding because the process is nongradient. In particular, we obtain a non-trivial diffusion coefficient.

2. Notation and Results

Fix a finite range probability measure $p(\cdot)$ on $\mathbb{Z}^d$. The exclusion process evolving on the discrete torus $\mathbb{T}_N^d = \{0, \dots, N-1\}^d$ associated to $p(\cdot)$ is the Markov process on the state space $\mathcal{X}_N = \{0,1\}^{\mathbb{T}_N^d}$ whose generator L_N acts on a local function f as

$$(L_N f)(\eta) \;=\; \sum_{x,y\in\mathbb{T}_N^d} p(y)\eta(x)\{1-\eta(x+y)\}[f(\sigma^{x,x+y}\eta) - f(\eta)]\;, \tag{2.1}$$

where $\sigma^{x,x+y}\eta$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$, $\eta(x+y)$:

$$(\sigma^{x,x+y}\eta)(z) \;=\; \begin{cases} \eta(z) & \text{if } z \neq x,\, x+y \;, \\ \eta(x) & \text{if } z = x+y \;, \\ \eta(x+y) & \text{if } z = x \;. \end{cases}$$

Fix α in $(0,1)$ and denote by ν_α^N the Bernoulli product measure on $\mathcal{X}_N$ with density α. Let L_N^* be the adjoint of L_N in $L^2(\nu_\alpha^N)$. This operator is obtained by replacing $p(y)$ by $p^*(y) = p(-y)$ in (2.1).

Denote by $\mathbb{T}^d$ the d-dimensional torus. Fix a continuous function $\rho_0 : \mathbb{T}^d \to [0,1]$ and denote by $\nu_{\rho_0(\cdot)}^N$ the product measure on $\{0,1\}^{\mathbb{T}_N^d}$ associated to ρ_0. This is the Bernoulli product measure on $\{0,1\}^{\mathbb{T}_N^d}$ with marginals given by

$$\nu_{\rho_0(\cdot)}^N\{\eta(x) = 1\} \;=\; \rho_0(x/N)$$

for x in $\mathbb{T}_N^d$.

For $N \geq 1$ and a configuration η, denote by $\pi^N(\eta)$ the empirical measure associated to η. This is the measure on $\mathbb{T}^d$ obtained by assigning mass N^{-d} to each particle of η:

$$\pi^N(\eta) \;=\; N^{-d}\sum_{x\in\mathbb{T}_N^d} \eta(x)\delta_{x/N}\;,$$

where δ_u stands for the Dirac measure on u. It has been proved in [14] that if particles are initially distributed according to $\nu_{\rho_0(\cdot)}^N$ for some profile $\rho_0 : \mathbb{T}^d \to [0,1]$, then $\pi^N(\eta_{tN})$ converges in probability to $\rho(t,u)du$, where ρ is the entropy solution of the Burgers equation

$$\partial_t \rho \;+\; q\cdot\nabla F(\rho) \;=\; 0\;, \tag{2.2}$$

where $F(a) = a(1-a)$ and $q \in \mathbb{R}^d$ is the mean drift of each particle: $q = \sum_z z p(z)$.

In this article, we investigate the diffusive behavior of the empirical measure π^N, that is, its evolution in times of order N^2.

As time increases, the solution of Burgers equation (1.2) converges to a stationary profile which is constant along the drift direction:

$$\lim_{t\to\infty} \rho(t,u) \;=\; \rho_\infty(u) \;=\; \int_0^1 \rho_0(u + rq)\, dr\;,$$

provided ρ_0 stands for the initial data. The limit should be understood pointwisely. In particular, in a time scale of order N^2, the profile of the empirical measure should immediately become constant along the drift direction.

We shall therefore assume that the initial state is a product measure $\nu^N_{\rho_0(\cdot)}$ associated to a profile ρ_0 constant along the drift direction:

$$q\cdot\nabla\rho_0(u) \;=\; 0 \tag{2.3}$$

for all u in $\mathbb{T}^d$. Assume furthermore that the profile is bounded away from 0 and 1:

$$\delta_0 \;\le\; \rho_0(u) \;\le\; 1-\delta_0 \tag{2.4}$$

for some $\delta_0>0$.

Theorem 2.1. *Assume that the initial state is distributed according to $\nu^N_{\rho_0(\cdot)}$, where the profile ρ_0 satisfies (2.3), (2.4). There exists a smooth matrix-valued function $a(\alpha)=\{a_{i,j}(\alpha),\,1\le i,j\le d\}$ with the following property. For each $t\ge 0$, $\pi^N(\eta_{tN^2})$ converges in probability to $\rho(t,u)du$, where the density ρ is the solution of the parabolic equation*

$$\begin{cases} \partial_t\rho \;=\; \sum_{i,j}\partial_{u_i}(a_{i,j}(\rho)\partial_{u_j}\rho)\;, \\ \rho(0,\cdot) \;=\; \rho_0(\cdot)\;. \end{cases} \tag{2.5}$$

In this theorem, $a_{i,j}(\alpha)=D_{i,j}(\alpha)+(1/2)(1-2\alpha)\sigma_{i,j}$, where $D_{i,j}(\alpha)$ is the matrix given by (5.9) and $\sigma_{i,j}$ the covariance matrix of the transition probability $p(\cdot)$:

$$\sigma_{i,j} \;=\; \sum_{y\in\mathbb{Z}^d} p(y)\,y_i\,y_j\;.$$

Notice that by the maximum principle, $\delta_0\le\rho(t,u)\le 1-\delta_0$ for all (t,u). Moreover, the solution of the hydrodynamic equation is constant along the drift direction,

$$\sum_{i=1}^d q_i\,(\partial_{u_i}\rho)(t,u) \;=\; 0$$

because so is the initial data.

This theorem is an elementary consequence of the following estimate on the relative entropy of the state of the process with respect to a local Gibbs state. For two measures μ, ν on $\{0,1\}^{\mathbb{T}^d_N}$, denote by $H_N(\mu|\nu)$ the relative entropy of μ with respect to ν:

$$H_N(\mu\,|\,\nu) \;=\; \sup_f\Big\{\int f\,d\mu \;-\; \log\int e^f d\nu\Big\}\;,$$

where the supremum is carried over all bounded, continuous functions, which in our finite setting coincide with all functions. For $t\ge 0$, denote by S^N_t the semigroup associated to the Markov process with generator (2.1) speeded up by N^2.

Theorem 2.2. *Under the assumptions of Theorem 2.1 on the initial profile ρ_0, let $\{\mu_N,\,N\ge 1\}$ be a sequence of probability measures on $\{0,1\}^{\mathbb{T}^d_N}$ whose entropy with respect to $\nu^N_{\rho_0(\cdot)}$ is of order $o(N^d)$:*

$$H_N(\mu_N\,|\,\nu^N_{\rho_0(\cdot)}) \;=\; o(N^d)\;.$$

Then, for every $t\ge 0$, the relative entropy of the state of the process at time tN^2 with respect to $\nu^N_{\rho(t,\cdot)}$ is also of order $o(N^d)$:

$$H_N(\mu_N S^N_t\,|\,\nu^N_{\rho(t,\cdot)}) \;=\; o(N^d)\;,$$

provided $\rho(t,u)$ is the solution of (2.5).

In view of this result, we can weaken the assumptions of Theorem 2.1 and assume only that the initial state has relative entropy of order $o(N^d)$ with respect to $\nu^N_{\rho_0(\cdot)}$.

3. Relative entropy estimates

We introduce in this section some auxiliary measures which will play a central role in the proof of Theorem 2.2. The statements presented here appeared essentially in the same form in [4] and [8]. We include their proof in sake of completeness.

Fix a profile ρ_0 constant along the drift direction and bounded away from 0 and 1 as in (2.4). Denote by $\rho(t,u)$ the smooth solution of the parabolic equation (2.5). Fix $0<\alpha<1$. For $N\ge 1$, denote by f_t^N the density of $\mu^N S_t^N$ with respect to ν_α^N. An elementary computation shows that f_t^N is the solution of

$$\partial_t f_t^N \;=\; N^2 L_N^* f_t^N \;,$$

where L_N^* is the adjoint of the generator L_N in $L^2(\nu_\alpha^N)$.

Denote by $\mathfrak{F}$ the space of functions $\mathfrak{f}\colon [0,1]\times\{0,1\}^{\mathbb{Z}^d}\to\mathbb{R}$ such that

1. There exists a finite set Λ such that for each β in $[0,1]$ the support of $\mathfrak{f}(\beta,\cdot)$ is contained in Λ.
2. For each configuration η, $\mathfrak{f}(\cdot,\eta)$ is a smooth function.
3. For each density β, the cylinder functions $\mathfrak{f}(\beta,\cdot)$, $\mathfrak{f}_1(\beta,\cdot)$ have zero mean with respect to ν_β. Here, $\mathfrak{f}_1(\beta,\cdot)$ stands for the derivative of $\mathfrak{f}(\beta,\eta)$ with respect to the first coordinate.

Let $\lambda:\mathbb{R}_+\times\mathbb{T}^d\to\mathbb{R}$ be defined by

$$\lambda(t,u) \;=\; \log\frac{\rho(t,u)(1-\alpha)}{\alpha[1-\rho(t,u)]}\;\cdot$$

$\lambda(t,u)$ is well defined because the solution $\rho(t,u)$ of the hydrodynamic equation (2.5) is bounded away from 0 and 1. Denote by $\psi_t^N(\eta)$ the density of $\nu_{\rho(t,\cdot)}^N$ with respect to ν_α:

$$\psi_t^N(\eta) \;=\; \frac{1}{Z_t}\exp\Big\{\sum_{x\in\mathbb{T}_N^d}\lambda(t,x/N)\eta(x)\Big\}\;,$$

where Z_t is a renormalizing constant.

For functions $\mathfrak{f}_i$ in $\mathfrak{F}$, $1\le i\le d$, a time $t\ge 0$ and integers $\ell\ll M\ll N$, define the density $\psi_{t,\mathfrak{f}}^N(\eta)=\psi_{t,\mathfrak{f}}^{N,M,\ell}(\eta)$ with respect to the reference measure ν_α^N by

$$\begin{aligned}\psi_{t,\mathfrak{f}}^N(\eta) \;=\; &\frac{1}{Z_t^{\mathfrak{f}}}\exp\Big\{\sum_{x\in\mathbb{T}_N^d}\lambda(t,x/N)\eta(x)\Big\}\;\times\\ &\times\;\exp\Big\{-\;N^{-1}\sum_{i=1}^d\sum_{x\in\mathbb{T}_N^d}\partial_{u_i}\lambda(t,x/N)\frac{1}{|\Lambda_{\ell'}|}\sum_{y\in\Lambda_{\ell'}}\mathfrak{f}_i(\eta^M(x),\tau_{x+y}\eta)\Big\}\;,\end{aligned}$$

where $Z_t^{\mathfrak{f}}$ is a renormalizing constant, $\Lambda_K=\{-K,\dots,K\}^d$ is a cube of length $2K+1$ centered at the origin, $\eta^K(x)$ is the mean density of particles in $x+\Lambda_K$:

$$\eta^K(x) \;=\; \frac{1}{|\Lambda_K|}\sum_{y\in x+\Lambda_K}\eta(y) \tag{3.1}$$

and $\ell'=\ell-A$ for a finite constant A chosen for the support of $\mathfrak{f}_i(\beta,\tau_y\eta)$ to be contained in Λ_ℓ for all $1\le i\le d$, $|y|\le\ell-A$. Throughout this article, A stands for a finite integer related to the support of the transition probability $p(\cdot)$ or to the support of some local function.

In the following, we will need to take M as a function of N and ℓ as an independent integer which increases to ∞ after N. In fact we will require M to be such that

$$\lim_{N\to\infty} \frac{|\Lambda_M|}{N} \;=\; 0\;, \quad \lim_{N\to\infty} \frac{N}{M|\Lambda_M|} \;=\; 0\;. \tag{3.2}$$

We present three elementary results which illustrate some properties of the density $\psi^N_{t,\mathfrak f}(\eta)$. Denote by $s_{\mathfrak f}$ the smallest integer m with the property that the common support of the local functions $\mathfrak f_i(\beta,\cdot)$, $1\le i\le d$, $0\le\beta\le 1$, is contained in Λ_m.

Lemma 3.1. *Assume that* $s_{\mathfrak f}\le \ell\le M$ *and that* $\lim_{N\to\infty}|\Lambda_M|/N = 0$. *Fix a density* f *with respect to the reference measure* ν^N_α. *There exists a finite constant* C, *depending only on* $\mathfrak f$ *and* $\rho(t,u)$, *such that*

$$\big|\, H_N(f\,|\,\nu^N_{\rho(t,\cdot)}) \;-\; H_N(f\,|\,\psi^N_{t,\mathfrak f})\,\big| \;\le\; CN^{d-1}$$

for all $N\ge 1$.

In the statement of this result and frequently in this article, if measures μ, ν have density f, g with respect to the reference measure ν^N_α, to keep notation simple, we denote by $H_N(f\,|\,g)$ the entropy of $f\,d\nu^N_\alpha$ with respect to $g\,d\nu^N_\alpha$ and by $E_f[\cdot]$ the expectation with respect to $f\,d\nu^N_\alpha$.

Proof. Fix a density f. By the explicit formula for the entropy, the difference $H_N(f\,|\,\nu^N_{\rho(t,\cdot)}) - H_N(f\,|\,\psi^N_{t,\mathfrak f})$ is equal to

$$\int f\,\log\frac{\psi^N_{t,\mathfrak f}}{\psi^N_t}\,d\nu^N_\alpha \;=\; O(N^{d-1}) \;-\; \log\frac{Z^{\mathfrak f}_t}{Z_t}\;\cdot$$

In particular, we just need to show that the second term on the right hand side is absolutely bounded by CN^{d-1}. By definition of the renormalizing constant $Z^{\mathfrak f}_t$, Z_t, the logarithm is equal to

$$\log E_{\nu^N_{\rho(t,\cdot)}}\Big[\exp\Big\{-\;N^{-1}\sum_{i=1}^d\sum_{x\in\mathbb T^d_N}\partial_{u_i}\lambda(t,x/N)(A_\ell\mathfrak f_i)(\eta^M(x),\tau_x\eta)\Big\}\Big]\;, \tag{3.3}$$

where, for a function $\mathfrak f$ in $\mathfrak F$ and a positive integer ℓ,

$$(A_\ell\mathfrak f)(\beta,\eta) \;=\; \frac{1}{|\Lambda_{\ell'}|}\sum_{y\in\Lambda_{\ell'}}\mathfrak f(\beta,\tau_y\eta)\;.$$

By Jensen inequality, (3.3) is bounded below by $-CN^{d-1}$. On the other hand, since $\ell\le M$, $A_\ell\mathfrak f_i(\eta^M(0),\eta)$ depends on the configuration η only through $\eta(z)$ for z in Λ_M. In particular, since $\nu^N_{\rho(t,\cdot)}$ is a product measure, by Hölder inequality, (3.3) is bounded above by

$$\frac{1}{d|\Lambda_M|}\sum_{i=1}^d\sum_{x\in\mathbb T^d_N}\log E_{\nu^N_{\rho(t,\cdot)}}\Big[\exp\Big\{-d|\Lambda_M|N^{-1}(\partial_{u_i}\lambda)(t,x/N)(A_\ell\mathfrak f_i)(\eta^M(x),\tau_x\eta)\Big\}\Big]\;.$$

Since by assumption $|\Lambda_M|N^{-1}$ vanishes as $N\uparrow\infty$, we may expand the exponential up to the second order to show that this expression is less than or equal to CN^{d-1}. This concludes the proof of the lemma. □

Taking $f=\psi^N_{t,\mathfrak f}$ in Lemma 3.1, we obtain a bound on the entropy of $\psi^N_{t,\mathfrak f}$ with respect to $\nu^N_{\rho(t,\cdot)}$.

Corollary 3.2. *Under the assumptions of Lemma 3.1, there exists a finite constant C, depending only on $\mathfrak{f}$ and $\rho(t,u)$, such that*

$$H_N(\psi_{t,\mathfrak{f}}^N \,|\, \nu_{\rho(t,\cdot)}^N) \;\le\; C N^{d-1}$$

for all $N \ge 1$.

Corollary 3.3. *Fix a smooth function $H : \mathbb{T}^d \to R$ and a function $\mathfrak{g}$ in $\mathfrak{F}$. There exists a finite constant C_0, depending only on $\mathfrak{f}$, $\mathfrak{g}$, H and $\rho(t,u)$, such that*

$$\Big| E_{\psi_{t,\mathfrak{f}}^N}\Big[N^{-d}\sum_{x\in\mathbb{T}_N^d} H(x/N)(A_\ell\mathfrak{g})(\eta^M(x),\tau_x\eta)\Big] \;-\; E_{\nu_{\rho(t,\cdot)}^N}\Big[N^{-d}\sum_{x\in\mathbb{T}_N^d} H(x/N)(A_\ell\mathfrak{g})(\eta^M(x),\tau_x\eta)\Big]\Big| \;\le\; C_0\sqrt{|\Lambda_M|/N}\;.$$

Proof. By the entropy inequality

$$E_{\psi_{t,\mathfrak{f}}^N}\Big[N^{-d}\sum_{x\in\mathbb{T}_N^d} H(x/N)(A_\ell\mathfrak{g})(\eta^M(x),\tau_x\eta)\Big]$$

is less than or equal to

$$\frac{H_N(\psi_{t,\mathfrak{f}}^N \,|\, \nu_{\rho(t,\cdot)}^N)}{\gamma N^{d-1}} + \frac{1}{\gamma N^{d-1}}\log E_{\nu_{\rho(t,\cdot)}^N}\Big[\exp\Big\{\gamma N^{-1}\sum_{x\in\mathbb{T}_N^d} H(x/N)(A_\ell\mathfrak{g})(\eta^M(x),\tau_x\eta)\Big\}\Big]$$

for every $\gamma>0$. By Corollary 3.2, the first term is bounded by $C\gamma^{-1}$. On the other hand, repeating the argument presented in the proof of Lemma 3.1, we show that the second term is less than or equal to

$$E_{\nu_{\rho(t,\cdot)}^N}\Big[N^{-d}\sum_{x\in\mathbb{T}_N^d} H(x/N)(A_\ell\mathfrak{g})(\eta^M(x),\tau_x\eta)\Big] + \frac{C\gamma|\Lambda_M|}{N} E_{\nu_{\rho(t,\cdot)}^N}\Big[N^{-d}\sum_{x\in\mathbb{T}_N^d} H(x/N)^2(A_\ell\mathfrak{g})(\eta^M(x),\tau_x\eta)^2\Big]$$

provided that $\gamma|\Lambda_M|N^{-1}$ vanishes as $N\uparrow\infty$. In this formula, C is a finite constant which depends on $\mathfrak{g}$ and H. In particular, the difference appearing inside the absolute value in the statement of the corollary is less than or equal to

$$\frac{C}{\gamma} \;+\; \frac{C\gamma|\Lambda_M|}{N}\;.$$

Taking $\gamma=\sqrt{N/|\Lambda_M|}$, we show that this expression is bounded by $C\sqrt{|\Lambda_M|/N}$. Replacing H by $-H$, we we conclude the proof of the corollary. □

4. Proof of Theorem 2.2

We prove in this section Theorem 2.2. In view of Lemma 3.1, Theorem 2.2 is a consequence of the following result.

Proposition 4.1. *Fix a measure μ^N such that $H_N(\mu^N \mid \nu^N_{\rho_0(\cdot)}) = o(N^d)$. Assume that the profile ρ_0 satisfies (2.3), (2.4). There exist sequences $\{\mathfrak{f}_{i,n}, n\ge 1\}$, $1\le i\le d$, of functions in $\mathfrak{F}$ such that*

$$\lim_{n\to\infty}\limsup_{\ell\to\infty}\limsup_{N\to\infty} N^{-d} H_N(\mu^N S^N_t \mid \psi^N_{t,\mathfrak{f}_n}) \;=\; 0$$

for every $t\ge 0$. In this formula, $\mathfrak{f}_n = (\mathfrak{f}_{1,n},\dots,\mathfrak{f}_{d,n})$.

The proof of Proposition 4.1 is divided in several steps. To keep notation simple, denote by $H^{\mathfrak{f}}_N(t)$ the relative entropy of $\mu^N S^N_t$ with respect to $\psi^N_{t,\mathfrak{f}}\, d\nu^N_\alpha$:

$$H^{\mathfrak{f}}_N(t) \;=\; H_N(\mu^N S^N_t \mid \psi^N_{t,\mathfrak{f}})\ .$$

In view of Lemma 3.1 and of Gronwall inequality, it is enough to show that for every $t\ge 0$,

$$H^{\mathfrak{f}}_N(t) \;\le\; o(N^d,\mathfrak{f}) \;+\; \gamma^{-1}\int_0^t ds\, H_N(\mu^N S^N_s \mid \nu^N_{\rho(s,\cdot)}) \tag{4.1}$$

for some $\gamma>0$. Here, $o(N^d,\mathfrak{f})$ stands for a finite constant such that

$$\lim_{n\to\infty}\limsup_{\ell\to\infty}\limsup_{N\to\infty} N^{-d} o(N^d,\mathfrak{f}_n) \;=\; 0\ .$$

The sequence $\{\mathfrak{f}_{i,n},\, n\ge 1\}$ is given by Theorem 5.1. To keep notation simple, we perform all computations for a single function $\mathfrak{f} = (\mathfrak{f}_1,\dots,\mathfrak{f}_d)$ and then replace it by the sequence $\mathfrak{f}_n$.

Recall that M depends on N through the relations (3.2) and that ℓ is an integer independent of N which increases to infinity after N. To prove (4.1), we start computing the time derivative of the entropy $H^{\mathfrak{f}}_N(t)$. On the one hand, a celebrated estimate of [15] gives that

$$\frac{d}{dt} H^{\mathfrak{f}}_N(t) \;\le\; \int f^N_t \Big\{ \frac{N^2 L^*_N \psi^N_{t,\mathfrak{f}}}{\psi^N_{t,\mathfrak{f}}} - \partial_t \log(\psi^N_{t,\mathfrak{f}}) \Big\}\, d\nu^N_\alpha\ . \tag{4.2}$$

On the other hand, a straightforward computation, presented in section 6, shows that the expression inside braces in the previous integral is equal to

$$\begin{aligned}
& N\sum_{i=1}^d \sum_{x\in\mathbb{T}^d_N} (\partial_{u_i}\lambda)(t,x/N)\Big\{\tau_x W^*_i - L^*_N(A_\ell \mathfrak{f}_i)(\eta^M(x),\tau_x\eta)\Big\} \\
&\quad + (1/2)\sum_{i,j=1}^d \sum_{x\in\mathbb{T}^d_N} (\partial^2_{u_i,u_j}\lambda)(t,x/N)\, \tau_x G_{i,j}(\eta) \\
&\quad + (1/2)\sum_{i,j=1}^d \sum_{x\in\mathbb{T}^d_N} (\partial_{u_i}\lambda)(t,x/N)(\partial_{u_j}\lambda)(t,x/N)\, \tau_x H_{i,j}(\eta) \\
&\quad - \sum_{x\in\mathbb{T}^d_N} (\partial_t\lambda)(t,x/N)\eta(x) \;+\; E_{\psi^N_{t,\mathfrak{f}}}\Big[\sum_{x\in\mathbb{T}^d_N} (\partial_t\lambda)(t,x/N)\eta(x)\Big] \;+\; o(N^d)\ .
\end{aligned} \tag{4.3}$$

In this formula, $o(N^d)$ is a term of order $N^d\ell M^{-1} \ll N^d$, $E_{\psi^N_{t,\mathfrak{f}}}$ stands for the expectation with respect to $\psi^N_{t,\mathfrak{f}}\, d\nu^N_\alpha$, W^*_i is the current in the i-th direction for the adjoint process and $G_{i,j}(\eta)$, $H_{i,j}(\eta)$ are local functions given by:

$$W_i^* \;=\; \sum_{y\in\mathbb{Z}^d} p^*(y)\, y_i\, \eta(0)[1-\eta(y)]\;, \quad G_{i,j}(\eta) \;=\; \sum_{y\in\mathbb{Z}^d} p^*(y)\, y_i\, y_j\, \eta(0)[1-\eta(y)]\;,$$

$$H_{i,j}(\eta) \;=\; \sum_{y\in\mathbb{Z}^d} p^*(y)\eta(0)[1-\eta(y)]\big\{y_i - \nabla_{0,y}\Gamma_{\mathfrak{f}_i(\eta^M(x),\cdot)}\big\} \;\times$$

$$\times\big\{y_j - \nabla_{0,y}\Gamma_{\mathfrak{f}_j(\eta^M(x),\cdot)}\big\}\;.$$

Here and below, $\nabla_{x,y}$ is the operator defined by

$$(\nabla_{x,y} f)(\eta) \;=\; f(\sigma^{x,y}\eta) - f(\eta)$$

and, for a local function h, Γ_h is the formal sum

$$\Gamma_h \;=\; \sum_{x\in\mathbb{Z}^d} \tau_x h\;.$$

Since h is a local function, even if the sum of translations is not defined, the gradient $\nabla_{0,y}\Gamma_h$ makes sense because only a finite number of terms do not vanish.

We consider separately the sums in (4.3). The goal is to replace each one by a simpler expression and a remainder denoted by $o(N^d)$. The remainder $o(N^d)$ stands for an expression which may depend on time and on the configuration but such that

$$\lim_{\ell\to\infty}\limsup_{N\to\infty} N^{-d}\Big|\int_0^t ds\, o(N^d) f_s^N\, d\nu_\alpha^N\Big| \;=\; 0$$

for every $t>0$. If the remainder vanishes only after taking the limit in $\mathfrak{f}_n$, we denote it by $o(\mathfrak{f}_n, N^d)$ and we require

$$\lim_{n\to\infty}\limsup_{\ell\to\infty}\limsup_{N\to\infty} N^{-d}\Big|\int_0^t ds\, o(\mathfrak{f}_n, N^d) f_s^N\, d\nu_\alpha^N\Big| \;=\; 0$$

for every $t>0$.

We start with the last term of (4.3). By Corollary 3.3, we may replace the expectation with respect to $\psi_{t,\mathfrak{f}}^N\, d\nu_\alpha^N$ with an expectation with respect to $\nu_{\rho(t,\cdot)}^N$ paying a price of order $N^d\sqrt{|\Lambda_M|/N}$. After this modification, the last line of (4.3), becomes

$$-\sum_{x\in\mathbb{T}_N^d} (\partial_t\lambda)(t,x/N)\,\{\eta(x)-\rho(t,x/N)\} \;+\; o(N^d)\;.$$

Since $\partial_t\lambda$ is a smooth function, we may further replace $\eta(x)$ by $\eta^\ell(x)$ paying a price absolutely bounded by $C\ell^2 N^{d-2}$ for some finite constant C.

To estimate the order N^{d+1} term of (4.3), we first take advantage of the assumption that the solution $\rho(t,u)$ is constant along the drift direction.

By paying a price of order $O(\ell^2 N^{d-1})$, we may replace the current W_i^* by an average $|\Lambda_\ell|^{-1}\sum_{y\in\Lambda_\ell}\tau_y W_i^*$. Here again one should keep in mind, that the average is in fact carried over a cube of length slightly smaller than $2\ell+1$ to ensure that all local functions $\tau_y W_i^*$ have support contained in Λ_ℓ.

Recall that $q=(q_1,\dots,q_d)$ denotes the drift of particles. The average of the current W_i^* can be written as

$$\frac{1}{|\Lambda_{\ell'}|}\sum_{z\in\Lambda'_\ell}\tau_z W_i^* \;=\; q_i^*\Big\{\eta^{\ell'}(0) - 2\eta^\ell(0)\eta^{\ell'}(0) + \eta^\ell(0)^2\Big\} \;+\; \frac{1}{|\Lambda_{\ell'}|}\sum_{z\in\Lambda'_\ell} w_i^*(\eta^\ell(0),\tau_z\eta)\;,$$

where $q_i^* = -q_i$ and

$$w_i^*(\alpha,\eta) \;=\; -\sum_{y\in\mathbb{Z}^d} p^*(y)\, y_i\, [\eta(0)-\alpha]\, [\eta(y)-\alpha] \;-\; \alpha \sum_{y\in\mathbb{Z}^d} p^*(y)\, y_i\, [\eta(y)-\eta(0)]\; .$$

The first term of the current gives no contribution since for any function J,

$$N\sum_{i=1}^d \sum_{x\in\mathbb{T}_N^d} (\partial_{u_i}\lambda)(t,x/N) q_i J(\eta^{\ell'}(0),\eta^\ell(x)) \;=\; 0$$

because $\sum_{1\le i\le d} q_i(\partial_{u_i}\lambda)(t,u) = \{\rho(t,u)[1-\rho(t,u)]\}^{-1}\sum_{1\le i\le d} q_i(\partial_{u_i}\rho)(t,u)$ vanishes for all (t,u). The first term of (4.3) becomes therefore

$$N\sum_{i=1}^d \sum_{x\in\mathbb{T}_N^d} (\partial_{u_i}\lambda)(t,x/N)\tau_x\Big\{(A_\ell w_i^*)(\eta^\ell(0),\eta) - L_N^*(A_\ell\mathfrak{f}_i)(\eta^M(0),\eta)\Big\}\; .$$

To ensure that the function which appears in $A_\ell w_i^*$ has mean zero with respect to the all canonical measures on the cube Λ_ℓ, we further replace $A_\ell w_i^*$ by $A_\ell^0 w_i^*$, where

$$(A_\ell^0 w_i^*)(\alpha,\eta) \;=\; (A_\ell w_i^*)(\alpha,\eta) \;+\; q_i\frac{\alpha(1-\alpha)}{|\Lambda_\ell|-1}\cdot$$

This replacement is pemitted because $\sum_i q_i\partial_{u_i}\rho = 0$.

Following the nongradient method, we add and subtract $\sum_{1\le j\le d} D_{i,j}(\eta^{\varepsilon N}(0))$ $[\eta^{\varepsilon N}(e_j)-\eta^{\varepsilon N}(0)]$. Since the diffusion coefficient is smooth, this expression is equal to $\sum_{1\le j\le d}\{d_{i,j}(\eta^{\varepsilon N}(e_j))-d_{i,j}(\eta^{\varepsilon N}(0))\}+(\varepsilon N)^{-2}$, where $d_{i,j}$ stands for the integral of $D_{i,j}$. In particular, after a summation by parts, the first line of (4.3), may be rewritten as

$$\begin{aligned} &N\sum_{i=1}^d \sum_{x\in\mathbb{T}_N^d} (\partial_{u_i}\lambda)(t,x/N)\tau_x V_i^{\varepsilon N,M,\ell}(\eta) \\ &\quad + \sum_{i,j=1}^d \sum_{x\in\mathbb{T}_N^d} (\partial^2_{u_i,u_j}\lambda)(t,x/N) d_{i,j}(\eta^{\varepsilon N}(x)) \;+\; O(N^{d-1})\; , \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} &V_i^{K,M,\ell}(\eta) \;=\; \\ &\quad (A_\ell^0 w_i^*)(\eta^\ell(0),\eta) \;+\; \sum_{j=1}^d D_{i,j}(\eta^K(0))[\eta^K(e_j)-\eta^K(0)] \;-\; L_N^*(A_\ell\mathfrak{f}_i)(\eta^M(0),\eta)\; . \end{aligned}$$

It is not difficult to see that there exists a finite constant $C(\alpha)$ such that $H(\mu^N\,|\,\nu_\alpha^N) \le C(\alpha)N^d$ for every probability measure μ^N on $\{0,1\}^{\mathbb{T}_N^d}$. In particular, by the usual two blocks estimate, since $d_{i,j}$ is Lipschitz continuous, for every $T>0$,

$$\begin{aligned} &\lim_{\ell\to\infty}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty} \\ &\quad \int_0^T dt\int \nu_\alpha^N(d\eta)\, f_t^N(\eta)\, N^{-d}\sum_{x\in\mathbb{T}_N^d}\big|\, d_{i,j}(\eta^{\varepsilon N}(x)) - d_{i,j}(\eta^\ell(x))\,\big| \;=\; 0\; . \end{aligned}$$

We may therefore replace in the second line of (4.4) the average of particles over a small macroscopic cube by the average over a large microscopic cube, i.e., replace $\eta^{\varepsilon N}(x)$ by $\eta^\ell(x)$.

On the other hand, the usual nongradient techniques, based on integration by parts formula, allows the replacement in (4.4) of $D_{i,j}(\eta^{\varepsilon N}(0))[\eta^{\varepsilon N}(e_j)-\eta^{\varepsilon N}(0)]$ by $D_{i,j}(\eta^\ell(0))[\eta^{\ell'}(e_j)-\eta^{\ell'}(0)]$. Here $\ell'=\ell-1$ for the previous function to depend only on the sites in Λ_ℓ. To keep notation simple, we will denote this expression by $D_{i,j}(\eta^\ell(0))[\eta^\ell(e_j)-\eta^\ell(0)]$. We refer to Chap. 7 of [6] for a proof of this replacement.

In subsection 6.2 we prove that we may replace $L_N^*(A_\ell\mathfrak{f}_i)(\eta^M(0),\eta)$ by $L_{\Lambda_\ell}^*(A_\ell\mathfrak{f}_i)$ $(\eta^\ell(0),\eta)$. Here $L_{\Lambda_\ell}^*$ stands for the restriction of the generator L_N^* to the cube Λ_ℓ. This means that we suppress all jumps from Λ_ℓ to Λ_ℓ^c and all jumps from Λ_ℓ^c to Λ_ℓ. In particular, this generator leaves $\eta^\ell(0)$ invariant and it is acting in fact only on the second coordinate. This replacement is one of the main technical point of the article. It is here that the special form of $\psi_{t,\mathfrak{f}}^N$ plays an important role, that we need the spatial averages and the particular size of M and ℓ presented in (3.2).

Up to this point, we transformed the first line of (4.3) in

$$N\sum_{i=1}^d\sum_{x\in\mathbb{T}_N^d}(\partial_{u_i}\lambda)(t,x/N)\tau_x V_i^\ell(\eta) \tag{4.5}$$
$$+\sum_{i,j=1}^d\sum_{x\in\mathbb{T}_N^d}(\partial^2_{u_i,u_j}\lambda)(t,x/N)d_{i,j}(\eta^\ell(x)) \;+\; o(N^d)\;,$$

where

$$V_i^\ell(\eta) \;=\; (A_\ell^0 w_i^*)(\eta^\ell(0),\eta) \;+\; \sum_{j=1}^d D_{i,j}(\eta^\ell(0))[\eta^\ell(e_j)-\eta^\ell(0)] \;-\; L_{\Lambda_\ell}^*(A_\ell\mathfrak{f}_i)(\eta^\ell(0),\eta)\;.$$

By the nongradient method, the first line can be shown to be of order $o(\mathfrak{f},N^d)$. Details are given in subsection 6.4.

It remains to consider the second and third line of (4.3). By the one block estimate the second line of (4.3) is equal to

$$(1/2)\sum_{i,j=1}^d\sum_{x\in\mathbb{T}_N^d}(\partial^2_{u_i,u_j}\lambda)(t,x/N)\,\sigma_{i,j}\tau_x F(\eta^\ell(0)) \;+\; o(N^d)\;,$$

where $\sigma_{i,j}$ is the symmetric matrix defined just after (2.5) and $F(a)=a(1-a)$. For $1\le i,j\le d$, let

$$J_{i,j}(\beta) \;=\; 2\beta(1-\beta)\Big\{D_{i,j}(\beta)-\beta\sigma_{i,j}\Big\}\;. \tag{4.6}$$

We prove in subsection 6.3 that the third line of (4.3) is equal to

$$(1/2)\sum_{i,j=1}^d\sum_{x\in\mathbb{T}_N^d}(\partial_{u_i}\lambda)(t,x/N)(\partial_{u_j}\lambda)(t,x/N)\sigma_{i,j}F(\eta^\ell(x))$$
$$+\;(1/2)\sum_{i,j=1}^d\sum_{x\in\mathbb{T}_N^d}(\partial_{u_i}\lambda)(t,x/N)(\partial_{u_j}\lambda)(t,x/N)J_{i,j}(\eta^\ell(x)) \;+\; o(\mathfrak{f}_n,N^d)\;.$$

In conclusion, we proved that (4.3) is equal to

$$\sum_{m=1}^{2}\sum_{i,j=1}^{d}\sum_{x\in\mathbb{T}^d_N} G^m_{i,j}(t,x/N)H^m_{i,j}(\eta^\ell(x)) \qquad (4.7)$$
$$-\sum_{x\in\mathbb{T}^d_N}(\partial_t\lambda)(t,x/N)\,\{\eta^\ell(x)-\rho(t,x/N)\}\;+\;o(\mathfrak{f}_n,N^d)\;.$$

where

$$G^1_{i,j}(t,u) = (\partial^2_{u_i,u_j}\lambda)(t,u)\;,\quad H^1_{i,j}(\beta)\;=\;d_{i,j}(\beta)\;+\;(1/2)\sigma_{i,j}F(\beta)\;,$$
$$G^2_{i,j}(t,u) = (\partial_{u_i}\lambda)(t,u)(\partial_{u_j}\lambda)(t,u)\;,\quad H^2_{i,j}(\beta)\;=\;(1/2)\{J_{i,j}(\beta)\;+\;\sigma_{i,j}F(\beta)\}\;.$$

An integration by parts shows that

$$\sum_{m=1}^{2}\sum_{i,j=1}^{d}\int_{\mathbb{T}^d} du\, G^m_{i,j}(t,u)H^m_{i,j}(\rho(t,u))\;=\;0\;.$$

In particular, in formula (4.7), we may replace the terms $H^m_{i,j}(\eta^\ell(x))$ by $H^m_{i,j}(\eta^\ell(x))-H^m_{i,j}(\rho(t,x/N))$ paying a price of order $o(N^d)$. A further elementary computation gives that

$$\sum_{m=1}^{2}\sum_{i,j=1}^{d} G^m_{i,j}(t,u)(H^m_{i,j})'(\rho(t,u))\;=\;(\partial_t\lambda)(t,u)$$

for every t and u, where $(H^m_{i,j})'$ stands for the derivative of $H^m_{i,j}$. Therefore, (4.7) becomes

$$\sum_{m=1}^{2}\sum_{i,j=1}^{d}\sum_{x\in\mathbb{T}^d_N} G^m_{i,j}(t,x/N)\,B^m_{i,j}(\eta^\ell(x),\rho(t,x/N))\;+\;o(\mathfrak{f}_n,N^d)\;,$$

where

$$B^m_{i,j}(a,b)\;=\;H^m_{i,j}(a)\;-\;H^m_{i,j}(b)\;-\;(H^m_{i,j})'(b)\,[a-b]\;.$$

At this point we may repeat the standard arguments of the relative entropy method do conclude. We refer to Chap. 6 of [6] for details.

5. Hilbert space of variances

We prove in this section the existence of functions $\mathfrak{f}_1,\dots\mathfrak{f}_d$ in $\mathfrak{F}$ which approximate the current in the Hilbert space of variances. We rely on recent results based on general duality presented in [10], [12].

For $0\le\alpha\le 1$, denote by $\mathcal{G}_\alpha$ the space of cylinder functions g such that $E_{\nu_\alpha}[g]=\partial_\alpha E_{\nu_\alpha}[g]=0$:

$$\tilde g(\alpha)\;=\;E_{\nu_\alpha}[g]\;=\;0\quad\text{and}\quad\tilde g'(\alpha)\;=\;\frac{d}{d\beta}E_{\nu_\beta}[g]\Big|_{\beta=\alpha}\;=\;0\;.$$

For each function g in $\mathcal{G}_\alpha$ we define $\|g\|_\alpha$ by

$$\|g\|^2_\alpha\;=\;|g|^2_\alpha\;+\;\|g\|^2_{-1,\alpha}\;, \qquad (5.1)$$

where

$$|g|^2_\alpha \;=\; \sup_{a\in\mathbb{R}^d} \Big\{ 2\sum_{i=1}^d a_i \sum_{x\in\mathbb{Z}^d} x_i < g\,;\, \eta(x) >_\alpha - \frac{\chi(\alpha)}{2} a\cdot \sigma a \Big\}\;,$$

$$\|g\|^2_{-1,\alpha} \;=\; \sup_{h\in\mathcal{G}_\alpha} \Big\{ 2 \ll g, h \gg_\alpha - \ll h, (-L^s) h \gg_\alpha \Big\}\;.$$

In this formula, $\chi(\alpha) = \alpha(1-\alpha)$, $a\cdot b$ stands for the inner product in $\mathbb{R}^d$ and $\ll \cdot,\cdot \gg_\alpha$ for the inner product in $\mathcal{G}_\alpha$ given by

$$\ll g, h\gg_\alpha = \sum_{x\in\mathbb{Z}^d} < g\,;\, \tau_x h >_\alpha\;,$$

where $< f_1\,;\, f_2 >_\alpha$ denotes the covariance of f_1, f_2 with respect to ν_α. Notice that in the sums which appear in the formulas above, all but a finite number of terms vanish because ν_α is a product measure. Theorem 5.1 is the main result of this section.

Theorem 5.1. *There exist a smooth matrix-valued function $D(\alpha) = \{D_{i,j}(\alpha),\, 1\le i,j\le d\}$ and a sequence of functions $\{\mathfrak{f}_{i,n},\, n\ge 1\}$ in $\mathfrak{F}$, $1\le i\le d$, such that*

$$\lim_{n\to\infty} \sup_{\alpha\in[0,1]} \Big\| w^*_i(\alpha,\eta) + \sum_{j=1}^d D_{i,j}(\alpha)[\eta(e_j)-\eta(0)] - L^*\mathfrak{f}_{i,n}(\alpha,\eta)\Big\|_\alpha \;=\; 0$$

for $1\le i\le d$. Moreover, for any vector v in $\mathbb{R}^d$,

$$\begin{aligned}\lim_{n\to\infty} \sum_{x\in\mathbb{Z}^d} &< \sum_{j=1}^d v_j \mathfrak{f}_{j,n}(\alpha,\eta), (-L^s)\tau_x \sum_{j=1}^d v_j \mathfrak{f}_{j,n}(\alpha,\eta) >_\alpha \\ &= \chi(\alpha)\, v\cdot\{D(\alpha)-\alpha\sigma\} v \end{aligned} \tag{5.2}$$

uniformly in α.

This result is a slight generalization of Corollary 10.1 and Lemma 10.4 in [8], proved in [9] using results presented in [13]. We have the advantage here to obtain uniformity up to the boundary. In sake of completeness, we present a simpler proof relying on the generalized duality developed in [10], [12].

To keep notatiom simple, we prove Theorem 5.1 for the current w_i obtained from w_i^* by replacing $p^*(\cdot)$ by $p(\cdot)$ and for the generator L in place of L^*.

Duality. For each $n\ge 0$, denote by $\mathcal{E}_n$ the subsets of $\mathbb{Z}^d$ with n points and let $\mathcal{E} = \cup_{n\ge 0}\mathcal{E}_n$ be the class of finite subsets of $\mathbb{Z}^d$. For each A in $\mathcal{E}$, let Ψ_A be the local function

$$\Psi_A \;=\; \prod_{x\in A} \frac{\eta(x)-\alpha}{\sqrt{\chi(\alpha)}}\;.$$

By convention, $\Psi_\phi = 1$. It is easy to check that $\{\Psi_A,\, A\in\mathcal{E}\}$ is an orthonormal basis of $L^2(\nu_\alpha)$. For each $n\ge 0$, denote by $\mathcal{D}_n$ the subspace of $L^2(\nu_\alpha)$ generated by $\{\Psi_A,\, A\in\mathcal{E}_n\}$, so that $L^2(\nu_\alpha) = \oplus_{n\ge 0}\mathcal{D}_n$. Functions in $\mathcal{D}_n$ are said to have degree n.

Consider a local function f. Since $\{\Psi_A : A\in\mathcal{E}\}$ is a basis of $L^2(\nu_\alpha)$, we may write

$$f = \sum_{n\ge 0}\sum_{A\in\mathcal{E}_n} \mathfrak{f}(\alpha, A)\Psi_A\;.$$

Note that the coefficients $\mathfrak{f}(\alpha,A)$ depend not only on f but also on the density α. Since f is a local function, $\mathfrak{f}\colon \mathcal{E}\to\mathbb{R}$ is a function of finite support.

Fix a local function f and denote by $\mathfrak{f}(\alpha, A)$ its Fourier coefficients. f has zero mean with respect to ν_α if and only if $\mathfrak{f}(\alpha,\phi)=0$. It belongs to $\mathcal{G}_\alpha$ if and only if $\mathfrak{f}(\alpha,\phi)=0$ and the degree one part is such that

$$\sum_{z\in\mathbb{Z}^d} \mathfrak{f}(\alpha,\{z\}) \;=\; 0\;.$$

In this case, we may rewrite the degree one piece as

$$\sqrt{\chi(\alpha)}\sum_{z\in\mathbb{Z}^d} \mathfrak{f}(\alpha,\{z\})[\eta(z)-\eta(0)]\;.$$

In conclusion, all functions f in $\mathcal{G}_\alpha$ may be written as

$$\sqrt{\chi(\alpha)}\sum_{z\in\mathbb{Z}^d} \mathfrak{f}(\alpha,\{z\})[\eta(z)-\eta(0)] \;+\; \sum_{n\ge 2}\sum_{A\in\mathcal{E}_n} \mathfrak{f}(\alpha,A)\Psi_A\;.$$

For $n\ge 0$, denote by π_n the projection on $\mathcal{D}_n$ so that $f=\sum_{n\ge 1}\pi_n f$ for f in $\mathcal{G}_\alpha$. In the formula above, the first term corresponds to $\pi_1 f$, the piece of f which has degree one, and the second term corresponds to $(I-\pi_1)f$, the piece of degree greater or equal to 2.

It is clear that a local function of type $h-\tau_x h$ belongs to the kernel of the inner product $\ll\cdot,\cdot\gg_\alpha$ defined above. This is the case of $\eta(z)-\eta(0)$ so that $\|f\|_{-1,\alpha}=\|(I-\pi_1)f\|_{-1,\alpha}$. In contrast, any function h of degree greater or equal to 2 is such that

$$\sum_{x\in\mathbb{Z}^d} x_i < h\,;\,\eta(x) >_\alpha =\; 0$$

for all i so that $|h|_\alpha=0$. Therefore, $|f|_\alpha=|\pi_1 f|_\alpha$ and

$$|\!|\!| f|\!|\!|_\alpha^2 \;=\; |\pi_1 f|_\alpha^2 \;+\; \|(I-\pi_1)f\|_{-1,\alpha}^2$$

for every local function f in $\mathcal{G}_\alpha$.

The generator on the Fourier coefficient. Let $\mathcal{E}_*$ be the class of all finite subsets of $\mathbb{Z}^d_*=\mathbb{Z}^d\backslash\{0\}$ and let $\mathcal{E}_{*,n}$ be the class of all subsets of $\mathbb{Z}^d_*$ with n points. For a local function f in $\mathcal{G}_\alpha$, define $\mathfrak{T}f:[0,1]\times\mathcal{E}_*\to\mathbb{R}$ by

$$(\mathfrak{T}f)(\alpha,A) \;=\; \sum_{z\in\mathbb{Z}^d} \mathfrak{f}(\alpha,[A\cup\{0\}]+z)\;,$$

where $\mathfrak{f}(\alpha,B)$ stands for the Fourier coefficients of f. In this context, a function $f(\alpha,\eta)$ belongs to $\mathcal{G}_\alpha$ if and only if $\mathfrak{f}(\alpha,\phi)=(\mathfrak{T}f)(\alpha,\phi)=0$. It has been in proved in [12] that for every zero-mean local functions f, g

$$\ll f,g\gg_\alpha = <(\mathfrak{T}f),(\mathfrak{T}g)> = \sum_{n\ge 0}\frac{1}{n+1}\sum_{A\in\mathcal{E}_{*,n}} (\mathfrak{T}f)(\alpha,A)\,(\mathfrak{T}g)(\alpha,A)\;. \tag{5.3}$$

For functions in $\mathcal{G}_\alpha$, this sum starts from 1 because $(\mathfrak{T}f)(\alpha,\phi)=(\mathfrak{T}g)(\alpha,\phi)=0$.

Observe that not every function $\mathfrak{f}:[0,1]\times\mathcal{E}_*\to\mathbb{R}$ is the image by $\mathfrak{T}$ of some local function f since

$$(\mathfrak{T}f)(\alpha,A) \;=\; (\mathfrak{T}f)(\alpha,S_zA) \tag{5.4}$$

for all z in A. Here, S_zA is the set defined by

$$S_zA \;=\; \begin{cases} A-z & \text{if } z\notin A,\\ (A-z)_{0,-z} & \text{if } z\in A\;. \end{cases}$$

Let $\mathfrak{f}_* : [0,1]\times\mathcal{E}_* \to \mathbb{R}$ be a finitely supported function satisfying (5.4). Define $\mathfrak{f} : [0,1]\times\mathcal{E} \to \mathbb{R}$ by

$$\mathfrak{f}(\alpha, B) \;=\; \begin{cases} |B|^{-1}\mathfrak{f}_*(\alpha, B\setminus\{0\}) & \text{if } B\ni 0\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{5.5}$$

An elementary computations shows that $\mathfrak{T} f(\alpha,\eta) = \mathfrak{f}_*$, if $f(\alpha,\eta)$ is the local function whose Fourier coefficients are $\mathfrak{f}(\alpha, A)$. Notice that $f(\alpha,\eta)$ belongs to $\mathcal{G}_\alpha$ if $\mathfrak{f}_*(\alpha,\phi) = 0$.

For any local function f, $\mathfrak{T}(Lf) = \mathfrak{L}_\alpha \mathfrak{T} f$, provided

$$\mathfrak{L}_\alpha = \mathfrak{L}_s + (1-2\alpha)\mathfrak{L}_d + \sqrt{\chi(\alpha)}\{\mathfrak{L}_+ + \mathfrak{L}_-\}$$

and, for $A\in\mathcal{E}_*$, $\mathfrak{v} : \mathcal{E}_*\to\mathbb{R}$ a finitely supported function,

$$\begin{aligned}
(\mathfrak{L}_s\mathfrak{v})\,(B) \;&=\; (1/2)\sum_{x,y\in\mathbb{Z}^d_*} s(y-x)[\mathfrak{v}(B_{x,y}) - \mathfrak{v}(B)] \;+\; \sum_{y\notin B} s(y)[\mathfrak{v}(S_y B) - \mathfrak{v}(B)]\;, \\
(\mathfrak{L}_d\mathfrak{v})(A) \;&=\; \sum_{\substack{x\in A, y\notin A\\ x,y\neq 0}} a(y-x)\{\mathfrak{v}(A_{x,y}) - \mathfrak{v}(A)) + \sum_{\substack{y\notin A\\ y\neq 0}} a(y)\{\mathfrak{v}(S_y A) - \mathfrak{v}(A)\}\;, \\
(\mathfrak{L}_+\mathfrak{v})(A) \;&=\; 2\sum_{x\in A, y\in A} a(y-x)\,\mathfrak{v}(A\backslash\{y\}) \\
&\quad + 2\sum_{x\in A} a(x)\{\mathfrak{v}(A\backslash\{x\}) - \mathfrak{v}(S_x[A\backslash\{x\}])\}\;, \\
(\mathfrak{L}_-\mathfrak{v})(A) \;&=\; 2\sum_{\substack{x\notin A, y\notin A\\ x,y\neq 0}} a(y-x)\,\mathfrak{v}(A\cup\{y\})\;.
\end{aligned}$$

In this formula, $A_{x,y}$ is the set defined by

$$A_{x,y} \;=\; \begin{cases} (A\backslash\{x\})\cup\{y\} & \text{if } x\in A,\ y\notin A, \\ (A\backslash\{y\})\cup\{x\} & \text{if } y\in A,\ x\notin A, \\ A & \text{otherwise}\;;. \end{cases}$$

Hilbert spaces. For two local functions f, g, let

$$\ll f, g \gg_{\alpha,1} = \ll f, (-L^s) g \gg_\alpha$$

and let $H_1(\alpha)$ be the Hilbert space generated by local functions f and the inner product $\ll\cdot,\cdot\gg_{\alpha,1}$. Denote by $\ll\cdot,\cdot\gg_1$ the scalar product on $\mathcal{E}_*$ defined by

$$\ll \mathfrak{f}, \mathfrak{g} \gg_1 \;=\; \sum_{n\geq 0}\frac{1}{n+1}\sum_{A\in\mathcal{E}_{*,n}} \mathfrak{f}(\alpha, A)(-\mathfrak{L}_s\mathfrak{g})(\alpha, A)$$

and by $\mathfrak{H}_1$ the Hilbert space generated by the finite supported functions endowed with the previous scalar product. From the previous definitions, for every local function f, g,

$$\ll f, g \gg_{1,\alpha} = \ll \mathfrak{T} f, \mathfrak{T} g \gg_1$$

To introduce the dual Hilbert spaces of H_1, $\mathfrak{H}_1$, for a local function f, consider the semi-norm $\|\cdot\|_{-1}$ given by

$$\|f\|^2_{-1,\alpha} \;=\; \sup_g \Big\{ 2 \ll f, g\gg_\alpha - \ll g, g \gg_{1,\alpha} \Big\}\;,$$

where the supremum is carried over all local functions g. Denote by H_{-1} the Hilbert space generated by the local functions and the semi-norm $\|\cdot\|_{-1}$. In the same way, for a finitely supported function $\mathfrak{f}:\mathcal{E}_*\to\mathbb{R}$, let

$$\|\mathfrak{f}\|_{-1}^2 \;=\; \sup_{\mathfrak{g}}\Big\{2<\mathfrak{f},\mathfrak{g}> - <\mathfrak{g},\mathfrak{g}>_1\Big\}\;,$$

where the supremum is carried over all finitely supported functions $\mathfrak{g}:\mathcal{E}_*\to\mathbb{R}$ and $<\cdot,\cdot>$ is the inner product on $L^2(\mathcal{E}^*)$ defined in (5.3). Denote by $\mathfrak{H}_{-1}$ the Hilbert space induced by the finitely supported functions $\mathfrak{f}:\mathcal{E}_*\to\mathbb{R}$ and the semi-norm $\|\cdot\|_{-1}$. By the identities for the L^2 and the H_1 norms, we obtain that

$$\|f(\alpha,\eta)\|_{-1,\alpha}^2 \;=\; \|(\mathfrak{T}f)(\alpha,\cdot)\|_{-1}^2 \tag{5.6}$$

The currents. Recall the definition of the current $w_i(\alpha,\eta)$ given in section 4. w_i is obtained from w_i^* by replacing $p^*(\cdot)$ by $p(\cdot)$ and can be expressed as

$$w_i \;=\; -\alpha(1-\alpha)\sum_{y\in\mathbb{Z}^d} p(y)\,y_i\,\Psi_{0,y} \;-\; \alpha\sum_{y\in\mathbb{Z}^d} p(y)\,y_i\,\{\eta(y)-\eta(0)\}\;.$$

Denote the first piece, which has degree 2, by $\alpha(1-\alpha)w_i^0$. On the other hand, since $\eta(e_k)-\eta(0)=\eta(e_k+x)-\eta(x)$ for the norm $|\cdot|_\alpha$ for any x, the piece which has degree one is equal to $\alpha\sum_{y\in\mathbb{Z}^d}\sum_{1\le j\le d}p(y)\,y_i\,y_j\{\eta(e_j)-\eta(0)\}$ so that

$$w_i \;=\; \alpha(1-\alpha)w_i^0 \;-\; \alpha\sum_{j=1}^d \sigma_{i,j}\,[\eta(e_j)-\eta(0)]\;.$$

Let $\mathfrak{w}_i=\mathfrak{T}w_i^0$. An straightforward computation gives that

$$\mathfrak{w}_i(\alpha,\{z\}) \;=\; -2\,z_i\,a(z)$$

for $z\neq 0$ and $\mathfrak{w}_i(\alpha,A)=0$ otherwise. Notice that $\mathfrak{w}_i$ does not depend on α. We have now all elements to prove the main result of this section.

Proof of Theorem 5.1. Fix $1\le i\le d$. By Theorem 4.1 in [12], $\mathfrak{w}_i$ belongs to $\mathfrak{H}_{-1}$ because $\mathfrak{w}_i(\alpha,\phi)=0$ and we are in $d\ge 3$.

It has been proved in Lemma 4.3 of [12] that for each $\lambda>0$ there exists a solution $\mathfrak{f}_{i,\lambda}(\alpha,A)$ of the resolvent equation

$$\lambda\mathfrak{f}_{i,\lambda} \;-\; \mathfrak{L}_\alpha\mathfrak{f}_{i,\lambda} \;=\; \mathfrak{w}_i$$

satisfying (5.4) and such that $\mathfrak{f}_{i,\lambda}(\alpha,\phi)=0$.

By Theorem 4.4 in [12], for any $k\ge 1$, there exists a finite constant C_k independent of α and λ such that

$$\lambda\sum_{n\ge 0}(1+n)^k <\pi_n\mathfrak{f}_{i,\lambda},\pi_n\mathfrak{f}_{i,\lambda}>_n \;+\; \sum_{n\ge 0}(1+n)^k <\pi_n\mathfrak{f}_{i,\lambda},(-\mathfrak{L}_s)\pi_n\mathfrak{f}_{i,\lambda}>_n\;\le\; C_k \tag{5.7}$$

for every $\lambda>0$ and α in $[0,1]$. In this formula, π_n stands for the projection on $\mathcal{E}_{*,n}$: $(\pi_n\mathfrak{f})(\alpha,A)=\mathfrak{f}(\alpha,A)\mathbf{1}\{A\in\mathcal{E}_{*,n}\}$, and $<\cdot,\cdot>_n$ for the inner product in $\mathcal{E}_{*,n}$ with respect to the counting measure:

$$<\mathfrak{f},\mathfrak{g}>_n = \sum_{A\in\mathcal{E}_{*,n}}\mathfrak{f}(\alpha,A)\mathfrak{g}(\alpha,A)\;.$$

The estimate is uniform in α because the current $\mathfrak{w}_i$ does not depend on α.

By section 6 of [12], for each z in $\mathbb{Z}^d$, $\mathfrak{f}_{i,\lambda}(\cdot, \{z\})$ is a smooth function in $[0,1]$ and there exists a subsequence $\lambda_k \downarrow 0$ such that $\mathfrak{f}_{i,\lambda_k}(\alpha, \{z\})$ converges uniformly, as well as its derivatives, to some smooth function $\mathfrak{f}_i(\alpha, \{z\})$.

By the proof of Lemma 2.8 of [11], taking a further subsequence, we may assume that $-(\mathfrak{L}_\alpha \mathfrak{f}_{i,\lambda_k})(\alpha, \cdot)$ converges weakly to $\mathfrak{w}_i$ in $\mathfrak{H}_{-1}$ for a countable dense subset of densities α in $[0,1]$. By Lemma 5.2 below, $-(\mathfrak{L}_\alpha \mathfrak{f}_{i,\lambda_k})(\alpha, \cdot)$ converges weakly to $\mathfrak{w}_i$ in $\mathfrak{H}_{-1}$ for all α in $[0,1]$.

Our goal is to replace the sequence $\mathfrak{f}_{i,\lambda_k}$ by a sequence $\mathfrak{h}_{i,n}$ of finite supported functions with all the above properties of $\mathfrak{f}_{i,\lambda_k}$ and for which $-(\mathfrak{L}_\alpha \mathfrak{h}_{i,n})(\alpha, \cdot)$ converges strongly to $\mathfrak{w}_i$ in $\mathfrak{H}_{-1}$, uniformly in α.

For each α fixed, we may take convex combinations of the functions $\mathfrak{f}_{i,\lambda_k}$ to obtain a new sequence $\mathfrak{g}_{i,n}$ such that $-\mathfrak{L}_\alpha \mathfrak{g}_{i,n}$ converges strongly to $\mathfrak{w}_i$ in $\mathfrak{H}_{-1}$. Lemma 5.2 below shows that the procedure can be made uniform in α. Indeed, fix $\varepsilon > 0$ and a finite set $\{\alpha_j, 1 \le j \le m\}$ in $[0,1]$. The standard procedure to derive a strong converging sequence from a weak, bounded converging sequence shows that there exist $M \ge 1$ and a probability $(\theta_1, \dots, \theta_M)$ in $\{1, \dots, M\}$, such that

$$\max_{1 \le j \le m} \|\mathfrak{L}_{\alpha_j} \mathfrak{g}(\alpha_j, \cdot) + \mathfrak{w}_i\|_{-1} \ \le\ \varepsilon\ ,$$

where

$$\mathfrak{g}(\alpha_j, \cdot) \ =\ \sum_{l=1}^{M} \theta_l\, \mathfrak{f}_{i,\lambda_l}(\alpha_j, \cdot)\ .$$

Notice that we are taking the same convex combination for all densities α_j. If m is equal to δ^{-1}, given by Lemma 5.2 below, and $\alpha_j = j\delta$, by Lemma 5.2,

$$\sup_{\alpha \in [0,1]} \|\mathfrak{L}_\alpha \mathfrak{g}(\alpha, \cdot) + \mathfrak{w}_i\|_{-1} \ \le\ 2\varepsilon\ ,$$

where $\mathfrak{g}(\alpha, \cdot)$ is obtained from $\mathfrak{f}_{i,\lambda}(\alpha, \cdot)$ through the same convex combination. We have thus constructed a convex combination which guarantees the strong convergence in $\mathfrak{H}_{-1}$ for all values of α. That is, there exists a sequence $\mathfrak{g}_{i,n}(\alpha, \cdot)$ such that

- For each $n \ge 1$, and each z in $\mathbb{Z}^d$, $\mathfrak{g}_{i,n}(\cdot, \{z\})$ is a smooth function of α which converges uniformly, as well as all its derivatives, to some smooth function $\mathfrak{f}_i(\alpha, \{z\})$.
- Each $\mathfrak{g}_{i,n}$ satisfies (5.4) and $\mathfrak{g}_{i,n}(\alpha, \phi) = 0$ because the functions f_{i,λ_k} have this property.
- The sequence converges uniformly to $\mathfrak{w}_i$ in $\mathfrak{H}_{-1}$:

$$\lim_{n \to \infty} \sup_{\alpha \in [0,1]} \|\mathfrak{L}_\alpha \mathfrak{g}_{i,n}(\alpha, \cdot) + \mathfrak{w}_i\|_{-1} \ =\ 0\ .$$

It remains to replace the functions $\mathfrak{g}_{i,n}$ by finite supported functions. Fix two integer m, ℓ and let $\mathfrak{h}_{i,n}(\alpha, A) = \mathfrak{g}_{i,n}(\alpha, A)\mathbf{1}\{|A| \le m, A < \Lambda_\ell\}$. The integers m, ℓ, which depend on n and increase to infinity with n, will be chosen later. Here, $A < \Lambda_\ell$ if there exists z in A such that $S_z A \subset \Lambda_\ell$. In this way, $\mathfrak{h}_{i,n}$ satisfies (5.4).

The sequence $\mathfrak{h}_{i,n}$ just defined has the first two properties of the sequence $\mathfrak{g}_{i,n}$ enumerated above because m and ℓ increase to infinity as $n \uparrow \infty$. To prove the third one, recall from the computations performed after (4.12) in [9] that

$$
\begin{aligned}
&\|\mathfrak{L}_\alpha \mathfrak{g}_{i,n}(\alpha,\cdot) - \mathfrak{L}_\alpha \mathfrak{h}_{i,n}(\alpha,\cdot)\|_{-1}^2 \\
&\quad \le \; C_0 \sum_{k=1}^{m+1} (1+k)\, \|\pi_k \mathfrak{g}_{i,n}(\alpha,\cdot) - \pi_k \mathfrak{h}_{i,n}(\alpha,\cdot)\|_{0,k}^2 \\
&\qquad + \sum_{k\ge m} (1+k)^2 < \pi_k \mathfrak{g}_{i,n}(\alpha,\cdot), (\mathfrak{L}_s \pi_k \mathfrak{g}_{i,n})(\alpha,\cdot) >_k
\end{aligned}
$$

for some finite constant C_0 independent of α. Here, $\|\cdot\|_{0,k}$ stands for the norm associated to the scalar product $<\cdot,\cdot>_k$ defined above. By (5.7), the second term on right hand side can be made uniformly small in α by choosing m large enough because each function $\mathfrak{g}_{i,n}$ is obtained as convex combinations of the solution of the resolvent equation. For a fixed finite set $\alpha_1,\dots,\alpha_r$, we may turn the first term as small as one wishes for $\{\alpha_i\, 1\le i\le r\}$ by taking ℓ large enough. By Lemma 5.2 below, we may turn the estimate uniform in α because the functions $\mathfrak{g}_{i,n}$ are convex combinations of the solution of the resolvent equation.

For each fixed n, the functions $\mathfrak{h}_{i,n}(\alpha,\cdot)$ has a uniform support. Since $\mathfrak{h}_{i,n}$ satisfies (5.4) and $\mathfrak{h}_{i,n}(\alpha,\phi)=0$, the local functions $f_{i,n}(\alpha,\eta)$ obtained from $\mathfrak{h}_{i,n}$ through (5.5) are in $\mathfrak{F}$.

We claim that the sequence $-\chi(\alpha) f_{i,n}(\alpha,\eta)$ has the properties required in the statement of the theorem. In view of the decomposition of the current w_i, by (5.1),

$$
\begin{aligned}
&\Big|\Big|\Big| w_i(\alpha,\eta) + \sum_{j=1}^d D_{i,j}(\alpha)[\eta(e_j)-\eta(0)] + \chi(\alpha) L f_{i,n}(\alpha,\eta)\Big|\Big|\Big|_\alpha^2 \qquad (5.8)\\
&\quad = \Big\| \chi(\alpha) w_i^0 + \chi(\alpha)(I-\pi_1) L f_{i,n}(\alpha,\eta)\Big\|_{-1,\alpha}^2 \\
&\qquad + \Big| \sum_{j=1}^d \{D_{i,j}(\alpha) - \alpha\sigma_{i,j}\}[\eta(e_j)-\eta(0)] + \chi(\alpha)\pi_1 L f_{i,n}(\alpha,\eta)\Big|_\alpha^2 .
\end{aligned}
$$

Since functions of degree 1 are in the kernel of the scalar product $\ll\cdot,\cdot\gg_\alpha$, we may replace $(I-\pi_1)Lf_{i,n}$ by $Lf_{i,n}$ on the first term on the right hand side. On the other hand, by definition of $\mathfrak{T}$, by identity (5.6) and since $\mathfrak{T} w_i^0 = \mathfrak{w}_i$, the first term on the right hand side of (5.8) is equal to

$$
\chi(\alpha)^2 \,\big\| \mathfrak{w}_i + \mathfrak{L}_\alpha \mathfrak{h}_{i,n}(\alpha,\cdot)\big\|_{-1}^2 .
$$

This expression vanishes, as $n\uparrow\infty$, uniformly in α, by construction of the sequence $\mathfrak{h}_{i,n}$.

On the other hand, an elementary computation, presented just after (5.4) in [12], shows that

$$
\pi_1 L f_{i,n}(\alpha,\eta) \;=\; \sum_{z\in\mathbb{Z}^d} a(z)\mathfrak{h}_{i,n}(\alpha,z)[\eta(z)-\eta(0)] .
$$

Since $\eta(z)-\eta(0) = \sum_{1\le j\le d} z_j[\eta(e_j)-\eta(0)]$ for the norm $|\cdot|_\alpha$, the second expression on the right hand side of (5.8) is equal to

$$
\Big| \sum_{j=1}^d \{D_{i,j}(\alpha) - \alpha\sigma_{i,j} + h_{i,j,n}(\alpha)\}[\eta(e_j)-\eta(0)]\Big|_\alpha^2 ,
$$

where

$$h_{i,j,n}(\alpha) \;=\; \chi(\alpha) \sum_{z\in\mathbb{Z}^d} a(z)\, z_j\, \mathfrak{h}_{i,n}(\alpha,\{z\})\ .$$

By construction, $\mathfrak{h}_{i,n}(\alpha,\{z\})$ converges to $\mathfrak{f}_i(\alpha,\{z\})$ uniformly, as $n\uparrow\infty$. In particular, if we define $D_{i,j}(\alpha)$ as

$$D_{i,j}(\alpha) \;=\; \alpha\sigma_{i,j} \;-\; \chi(\alpha) \sum_{z\in\mathbb{Z}^d} a(z)\, z_j\, \mathfrak{f}_i(\alpha,\{z\})\ , \tag{5.9}$$

it not difficult to show from the variational formula for the norm $|\cdot|_\alpha$ that the second term on the right hand side of (5.8) also vanishes as $n\uparrow\infty$, uniformly in α. This proves the first statement of the theorem.

Notice that $D_{i,j}(\cdot)$ inherits the smoothness of $\mathfrak{f}_i(\cdot,\{z\})$.

It remains to check identity (5.2). By definition of the scalar product $\ll\cdot,\cdot\gg_\alpha$ and by identity (5.3), for any vector v in $\mathbb{R}^d$, the left hand side of (5.2) with the sequence $-\chi(\alpha) f_{i,n}(\alpha,\eta)$ in place of $\mathfrak{f}_{i,n}(\alpha,\eta)$, is equal to

$$-\chi(\alpha)^2 < \sum_{j=1}^d v_j \mathfrak{h}_{j,n}(\alpha,\cdot), \sum_{j=1}^d v_j \mathfrak{L}_\alpha \mathfrak{h}_{j,n}(\alpha,\cdot) > \ .$$

Since $-\mathfrak{L}_\alpha\mathfrak{h}_{j,n}$ converges to $\mathfrak{w}_j$ in $\mathfrak{H}_{-1}$, uniformly in α, and since $\mathfrak{h}_{j,n}$ is (n,α)-uniformly bounded in $\mathfrak{H}_1$, the limit of the previous expression is equal to the limit of

$$\chi(\alpha)^2 \sum_{j,k=1}^d v_j v_k < \mathfrak{h}_{j,n}(\alpha,\cdot), \mathfrak{w}_k > = \; -\chi(\alpha)^2 \sum_{j,k=1}^d v_j v_k \sum_{z\in\mathbb{Z}^d_*} \mathfrak{h}_{j,n}(\alpha,\{z\}) z_k a(z)\ .$$

The last identity follows from the explicit formula for $\mathfrak{w}_k$. Notice that a factor $1/2$ appeared because in the definition of the inner product $<\cdot,\cdot>$, there is the term $(n+1)^{-1}$. By construction, $\mathfrak{h}_{j,n}(\alpha,\{z\})$ converges uniformly in α to $\mathfrak{f}_j(\alpha,\{z\})$. In particular, the previous sum converges uniformly to

$$-\chi(\alpha)^2 \sum_{j,k=1}^d v_j v_k \sum_{z\in\mathbb{Z}^d_*} \mathfrak{f}_j(\alpha,\{z\}) z_k a(z)$$

By definition (5.9) of the diffusion coefficient $D_{i,j}(\alpha)$, the previous expression is equal to

$$\chi(\alpha)\, v\cdot \{D(\alpha) - \alpha\sigma\} v\ .$$

This concludes the proof of the theorem. □

We conclude this section with a technical lemma needed above.

Lemma 5.2. *For each $\varepsilon>0$ and $k\ge 1$, there exists $\delta>0$ such that*

$$\sup_{|\alpha-\beta|\le\delta} \|\mathfrak{L}_\alpha \mathfrak{f}_{i,\lambda}(\alpha,\cdot) - \mathfrak{L}_\beta \mathfrak{f}_{i,\lambda}(\beta,\cdot)\|_{-1} \;\le\; \varepsilon$$

$$\sup_{|\alpha-\beta|\le\delta} \lambda \sum_{n\ge 1} (1+n)^k \sum_{A\in\mathcal{E}_{*,n}} \{\mathfrak{f}_{i,\lambda}(\alpha,A) - \mathfrak{f}_{i,\lambda}(\beta,A)\}^2 \;\le\; \varepsilon$$

for all $0<\lambda<1$.

The proof of this lemma is implicit in the proof of the regularity of $f_\lambda(\cdot,A)$ presented in Theorem 5.1 and Lemma 5.2 of [10]. We just need to write the equation for $\mathfrak{f}_{i,\lambda}(\alpha,\cdot) - \mathfrak{f}_{i,\lambda}(\beta,\cdot)$ as a resolvent equation with a right hand side which is a function in $\mathfrak{H}_{-1}$ times $O(\alpha-\beta)$. Details are left to the reader.

6. Technical bounds

We present in this section some technical lemmas and some computations omitted in section 3.

6.1. Computation of $N^2 L_N^* \psi_{t,\mathfrak{f}}^N / \psi_{t,\mathfrak{f}}^N$. Since L_N^* is the generator of the exclusion process associated to the transition probability $p^*(y) = p(-y)$,

$$\frac{N^2 L_N^* \psi_{t,\mathfrak{f}}^N(\eta)}{\psi_{t,\mathfrak{f}}^N(\eta)} \;=\; N^2 \sum_{x,y\in\mathbb{T}_N^d} \eta(x)[1-\eta(x+y)]p^*(y)\Big\{ \frac{\psi_{t,\mathfrak{f}}^N(\sigma^{x,x+y}\eta)}{\psi_{t,\mathfrak{f}}^N(\eta)} - 1\Big\}\;.$$

For each fixed bond (x,y), $\psi_{t,\mathfrak{f}}^N(\sigma^{x,x+y}\eta)/\psi_{t,\mathfrak{f}}^N(\eta)$ is an expression of order N^{-1} because $\mathfrak{f}_i(\cdot,\eta)$ is a smooth function for each fixed configuration η. We may therefore expand the exponential up to the second order. The order one term is exactly $N^2 L_N^* \log \psi_{t,\mathfrak{f}}^N$ and is responsible for the first two lines of (4.3) plus a remainder of order N^{d-1}. The second order term is equal to

$$(1/2) \sum_{x,y\in\mathbb{T}_N^d} \eta(x)[1-\eta(x+y)]p^*(y)\Big\{ N\{\lambda(t,x+y/N) - \lambda(t,x/N)\} \\ - \sum_{i=1}^d \sum_{z\in\mathbb{T}_N^d} (\partial_{u_i}\lambda)(t,z/N)\nabla_{x,x+y}(A_\ell \mathfrak{f}_i)(\eta^M(z),\tau_z\eta)\Big\}^2\;.$$

Since $\ell + s_{\mathfrak{f}} + A \le M$, the gradient $\nabla_{x,x+y}$ acts either on the first coordinate or on the second but never on both. $\mathfrak{f}_i(\cdot,\eta)$ being a smooth function, the contribution of the gradient $\nabla_{x,x+y}$ applied on the first coordinate is at most of order M^{-d}. Since there are $O(M^{d-1})$ boundary sites z for which $\nabla_{x,x+y}\eta^M(z)$ does not vanish, the total contribution of the gradient $\nabla_{x,x+y}$ acting on the first coordinate of $A_\ell\mathfrak{f}_i$ is of order M^{-1}.

We consider now the set of sites z for which the gradient $\nabla_{x,x+y}$ acts on the second coordinate of $A_\ell\mathfrak{f}_i$. In this case, z should be at a distance smaller than $\ell + A$ from x and we may replace $(\partial_{u_i}\lambda)(t,z/N)$ by $(\partial_{u_i}\lambda)(t,x/N)$ paying a price of order $\ell^{d+1}N^{-1}$. At this point, for a fixed i, after a change of variables $z' = z - x$, we may rewrite the sum appearing inside braces in the previous formula as

$$(\partial_{u_i}\lambda)(t,x/N)\tau_x\nabla_{0,y}\sum_{z\in\Lambda_{\ell+A}} \frac{1}{|\Lambda_\ell|}\sum_{w\in\Lambda_\ell} \mathfrak{f}_i(\eta^M(z),\tau_{z+w}\eta)\;.$$

Since the summation over z takes place on $\Lambda_{\ell+A}$, we may replace $\eta^M(z)$ by $\eta^M(0)$ paying a price of order ℓ/M. In this case the previous sum becomes

$$(\partial_{u_i}\lambda)(t,x/N)\tau_x\nabla_{0,y}\sum_{z\in\mathbb{Z}^d}\mathfrak{f}_i(\eta^M(0),\tau_z\eta) \;=\; (\partial_{u_i}\lambda)(t,x/N)\tau_x\nabla_{0,y}\Gamma_{\mathfrak{f}_i(\eta^M(0),\cdot)}$$

because the contribution of each fixed w is the same after replacing $\eta^M(z)$ by $\eta^M(0)$.

To obtain the third line of (4.3) and the correct order of the remainder, it remains to expand $N\{\lambda(t,x+y/N) - \lambda(t,x/N)\}$ and to develop the square.

6.2. Replacement of $L^*(A_\ell\mathfrak{f}_i)(\eta^M(0),\eta)$ by $L_{\Lambda_\ell}^*(A_\ell\mathfrak{f}_i)(\eta^\ell(0),\eta)$. Observe initially that the generator acts either on the first coordinate or on the second but never on both because we assumed that $s_{\mathfrak{f}} + \ell \le M$. Hence, we have to show that the action of the generator on the first coordinate is negligible. This is the content of the next result.

Lemma 6.1. *Fix a function* $\mathfrak{f}$ *in* $\mathfrak{F}$*, a smooth function* $G : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}$ *and assume that* M *satisfies conditions (3.2). For every* $T > 0$,

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} \Big| \int_0^T dt \int N^{1-d} \sum_{z\in\mathbb{T}_N^d} G(t,z/N)\tau_z (L^* - L^*_{\Lambda_\ell})(A_\ell \mathfrak{f})(\eta^M(0),\eta)\, f_t^N \, d\nu_\alpha^N \Big| \;=\; 0 \,.$$

Notice that in $L^*_{\Lambda_\ell}(A_\ell\mathfrak{f})(\eta^M(0),\eta)$, the generator is acting only on the second coordinate because $\ell \le M$.

Proof. Let $\mathfrak{f}_1(\alpha,\eta) = (\partial_\alpha \mathfrak{f})(\alpha,\eta)$. Since $\mathfrak{f}(\alpha,\cdot)$ is a smooth function, the contribution of $(L^* - L^*_{\Lambda_\ell})(A_\ell \mathfrak{f}_i)(\eta^M(0),\eta)$ is equal to

$$N^{1-d} M^{-d} \sum_{z\in\mathbb{T}_N^d} G(t,z/N)\tau_z \sum_{\substack{x\in\Lambda_M\\ x+y\not\in\Lambda_M}} \eta(x)[1-\eta(x+y)]p^*(y)(A_\ell\mathfrak{f}_1)(\eta^M(0),\eta) \;+\; o_N(1) \tag{6.1}$$

plus a similar term with a negative sign and $x+y$ in Λ_M, x not in Λ_M. Here the remainder $o_N(1)$ is of order N/M^{d+1}. From this point, the proof is divided in several steps.

Step 1. The first one consists in translating the local functions $\eta(x)[1-\eta(x+y)]$, which lies at the boundary of Λ_M, by few steps in order to have their support contained in Λ_M. For this purpose, it is enough to show that for every fixed y,

$$N^{1-d} M^{-d} \sum_{z\in\mathbb{T}_N^d} G(t,z/N)\tau_z \sum_{\substack{x\in\Lambda_M\\ x+y\not\in\Lambda_M}} \tau_x W(A_\ell\mathfrak{f}_1)(\eta^M(0),\eta) \tag{6.2}$$

is negligible if $W = h - \tau_{e_1} h$ for some local function h. Here and below, a function $H_{N,\ell}(t,\eta)$ is said to be negligible if

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} \Big| \int_0^T dt \int H_{N,\ell}(t,\eta)\, f_t^N\, d\nu_\alpha^N \Big| \;=\; 0$$

for all $T > 0$. Since there exists a finite constant C_0 such that $H_N(\mu^N \,|\, \nu_\alpha^N) \le C_0 N^d$ for all measure μ^N, by the entropy inequality, Feynman-Kac formula and the variational formula for the largest eigenvalue of a symmetric operator, to prove that a function is negligible, it is enough to show that

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} \int_0^T dt \sup_f \Big\{ \int H(t,\eta)\, f\, d\nu_\alpha^N - \varepsilon N^{2-d} D_N(f) \Big\} \;\le\; 0 \tag{6.3}$$

for every $\varepsilon > 0$. Here, the supremum is carried over all densities f and $D_N(f)$ is the Dirichlet form given by $D_N(f) = < -L_N\sqrt{f}, \sqrt{f} >$, where $< \cdot,\cdot >$ stands for the inner product in $L^2(\nu_\alpha^N)$.

Since the local function W has mean zero with respect to all canonical invariant states, $W = L^s_\Lambda w$ for some finite set Λ and some local function w, where L^s_Λ stands for the symmetric part of the generator L restricted to the set Λ. In particular, we need only to show that

$$N^{1-d} M^{-d} \sum_{z\in\mathbb{T}_N^d} G(t,z/N)\tau_z \sum_{\substack{x\in\Lambda_M\\ x+y\not\in\Lambda_M}} \tau_x(\nabla_b w)(A_\ell\mathfrak{f}_1)(\eta^M(0),\eta)$$

is negligible for a fixed bond $b = (b_1, b_2)$ and a fixed local function w. Fix $0 \le t \le T$, a density f with respect to ν_α^N and consider the linear term in variational formula (6.3):

$$N^{1-d} M^{-d} \sum_{z \in \mathbb{T}_N^d} G(t, z/N) \sum_{\substack{x \in \Lambda_M \\ x+y \notin \Lambda_M}} \int \tau_x (\nabla_b w)(A_\ell \mathfrak{f}_1)(\eta^M(0), \eta)\, \tau_{-z} f \, d\nu_\alpha^N \ ,$$

where we performed a change of variables $\xi = \tau_z \eta$. Since $\tau_x \nabla_b = \nabla_{b+x} \tau_x$, performing a change of variables $\xi = \sigma^{b+x}\eta$, we may rewrite the previous expression as

$$N^{1-d} M^{-d} \sum_{z \in \mathbb{T}_N^d} G(t, z/N) \sum_{\substack{x \in \Lambda_M \\ x+y \notin \Lambda_M}} \int \tau_x w (A_\ell \mathfrak{f}_1)(\eta^M(0), \eta)\, \nabla_{b+x} \tau_{-z} f \, d\nu_\alpha^N$$

plus a term of order NM^{-d-1}. This term appears when taking the difference $\nabla_{b+x}(A_\ell \mathfrak{f}_1)(\eta^M(0), \eta)$ which is absolutely bounded by CM^{-d}.

Rewrite the difference $a - b = \tau_{-z} f(\sigma^b \eta) - \tau_{-z} f(\eta)$ as $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$ and apply the elementary inequality $2ab \le \gamma a^2 + \gamma^{-1} b^2$, which holds for every $\gamma > 0$ to estimate the previous expression by $C\varepsilon^{-1} M^{-2} + \varepsilon N^{2-d} D_N(f)$. This proves that (6.2) is negligible, concluding the first step.

Step 2. Once that all functions have been translated to have its support contained in Λ_M, we take advantage of the fact that each function which appears in (6.1) at one side of the boundary, appears also at the other side with reversed sign. In particular, adding the intermediary terms to complete a telescopic sum, after (6.2), (6.1) can be rewritten as

$$N^{1-d} M^{1-d} \sum_{j=1}^{m} \sum_{z \in \mathbb{T}_N^d} G(t, z/N) \tau_z \sum_{x \in \Lambda_{M-A}} (\tau_x h_j)(\eta)\, (A_\ell \mathfrak{f}_1)(\eta^M(0), \eta)$$

for a family of local functions $h_j = g_j - \tau_{e_i} g_j$ for some $1 \le i \le d$. Here m is a finite integer which depends on $p(\cdot)$ only. In particular, the local functions h_j have mean zero with respect to all canonical invariant measures. Here again, A is taken large enough for the support of each local function $\tau_x h_j$ to be contained in Λ_M. We claim that such a term is negligible.

Since all local functions h which have mean zero with respect to all canonical invariant measures can be expressed as $L_\Lambda^s h_0$ for some finite set Λ and some local function h_0, fix a bond b, a local function h_0 and consider the linear term in (6.3):

$$N^{1-d} M^{-d} \sum_{z \in \mathbb{T}_N^d} G(t, z/N) \sum_{x \in \Lambda_{M-A}} \int \tau_x (\nabla_b h_0)\, (A_\ell \mathfrak{f}_1)(\eta^M(0), \eta)\, \tau_{-z} f \, d\nu_\alpha^N \ .$$

Since $\tau_x \nabla_b = \nabla_{b+x} \tau_x$, a change of variables $\xi = \sigma^{b+x}\eta$, similar to the one performed in the first part of the proof, permits to write the previous expression as

$$\begin{aligned} &\frac{N^{1-d}}{M^d} \sum_{z \in \mathbb{T}_N^d} G(t, z/N) \sum_{x \in \Lambda_{M-A}} \int \tau_x h_0\, (A_\ell \mathfrak{f}_1)(\eta^M(0), \sigma^{b+x}\eta)\, \nabla_{b+x} \tau_{-z} f \, d\nu_\alpha^N \qquad (6.4) \\ &+ \frac{N^{1-d}}{M^d} \sum_{z \in \mathbb{T}_N^d} G(t, z/N) \sum_{x \in \Lambda_{M-A}} \int \tau_x h_0\, \nabla_{b+x} (A_\ell \mathfrak{f}_1)(\eta^M(0), \eta)\, \tau_{-z} f \, d\nu_\alpha^N \ . \end{aligned}$$

We claim that both terms can be estimated by $\varepsilon N^{2-d} D_N(f)$ and an expression which vanishes as $N \uparrow \infty$ and then $\ell \uparrow \infty$. Notice that in the second term, the gradient ∇_{b+x} is acting only on the second coordinate.

Consider the first line of (6.4). Repeating the arguments presented at the end of the first step, we may bound this integral by the sum of $\varepsilon N^{2-d} D_N(f)$ and

$$\frac{C\varepsilon^{-1}}{N^d M^d} \sum_{z \in \mathbb{T}_N^d} \sum_{x \in \Lambda_{M-A}} \int (A_\ell \mathfrak{f}_1)(\eta^M(0), \sigma^{b+x}\eta)^2 \Big\{ \tau_{-z} f(\eta) + \tau_{-z} f(\sigma^{b+x}\eta) \Big\} d\nu_\alpha^N$$

for some finite constant C. Notice that we got an extra factor N^{-1} in this passage and that we included G and h_0 in the constant. We perform a change of variables $\xi = \sigma^{b+x}\eta$ and denote by $\bar{f}$ the average of the translations of f: $\bar{f} = N^{-d} \sum_{z \in \mathbb{T}_N^d} \tau_z f$ to rewrite the previous sum as

$$C\varepsilon^{-1} \int (A_\ell \mathfrak{f}_1)(\eta^M(0), \eta)^2 \, \bar{f}(\eta) \, d\nu_\alpha^N \;+\; O(\ell^d M^{-d}) \; .$$

Here we took advantage of the fact that $(A_\ell \mathfrak{f}_1)(\eta^M(0), \sigma^{b+x}\eta) = (A_\ell \mathfrak{f}_1)(\eta^M(0), \eta)$ unless x belongs to Λ_ℓ. Since $\mathfrak{f}_1(\cdot, \eta)$ is a smooth function, uniformly in η, the integral in the previous expression is less than or equal to

$$C\varepsilon^{-1} \int (A_\ell \mathfrak{f}_1)(\eta^\ell(0), \eta)^2 \, \bar{f}(\eta) \, d\nu_\alpha^N \;+\; C\varepsilon^{-1} \int \Big\{ \eta^M(0) - \eta^\ell(0) \Big\}^2 \bar{f}(\eta) \, d\nu_\alpha^N \; .$$

The usual proof of the two blocks estimate permits to show that the second integral can be estimated by $\varepsilon N^{2-d} D_N(f)$ and an expression which vanishes as $N \uparrow \infty$ and then $\ell \uparrow \infty$. We leave the details to the reader. In contrast, the usual proof of the one block estimate permits to show that the limit, as $N \uparrow \infty$, of the first integral minus $\varepsilon N^{2-d} D_N(f)$ is bounded by

$$C\varepsilon^{-1} \sup_K \int \Big\{ \frac{1}{|\Lambda_{\ell-A}|} \sum_{y \in \Lambda_{\ell-A}} \tau_y \mathfrak{f}_1(K/|\Lambda_\ell|, \eta) \Big\}^2 d\mu_{\Lambda_\ell, K} \; .$$

In this formula, $\mu_{\Lambda_\ell, K}$ stands for the canonical measure on Λ_ℓ concentrated on configurations with K particles and the supremum is carried over all integers $0 \le K \le |\Lambda_\ell|$. Divide the average in Λ_ℓ in two averages and recall from Lemma A.7 in [8] that the Radon-Nikodym derivative $d\mu_{\Lambda_\ell,K}/d\nu^{\Lambda_\ell}_{K/|\Lambda_\ell|}$ is bounded, uniformly in K, provided $\nu_\beta^{\Lambda_\ell}$ stands for the grand canonical measure on Λ_ℓ with density β. The previous expression is thus less than or equal to

$$C\varepsilon^{-1} \sup_{0 \le \beta \le 1} \int \Big\{ \frac{1}{|\Lambda_{\ell,1}|} \sum_{y \in \Lambda_{\ell,1}} \tau_y \mathfrak{f}_1(\beta, \eta) \Big\}^2 d\nu_\beta^{\Lambda_\ell} \; .$$

In this formula, $\Lambda_{\ell,1}$ stands for one half of the cube Λ_ℓ. Since $\mathfrak{f}_1(\alpha, \cdot)$ is local function, with uniform support and which has mean zero with respect to ν_α^N, the previous expression is of order ℓ^{-d} because ν_α^N is a product measure. This conclude the estimation of the first term in (6.4).

We turn now to the second term of (6.4). Notice that the gradient $\nabla_{b+x}(A_\ell \mathfrak{f}_1)$ $(\eta^M(0), \eta)$ vanishes if x does not belong to $\Lambda_{\ell+A}$. In particular,

$$\sum_{x \in \Lambda_{M-A}} \tau_x h_0 \, \nabla_{b+x}(A_\ell \mathfrak{f}_1)(\eta^M(0), \eta) \;=\; \sum_{x \in \Lambda_{\ell+A}} \tau_x h_0 \, \nabla_{b+x}(A_\ell \mathfrak{f}_1)(\eta^M(0), \eta) \quad (6.5)$$

is bounded by a constant which does not depend on N. On the other hand, for every $0 \le K \le |\Lambda_M|$, repeating the computation presented in the second paragraph of the second step, from the end to the beginning, we obtain that

$$\begin{aligned}\sum_{x \in \Lambda_{\ell+A}} \int \tau_x h_0 \, \nabla_{b+x} (A_\ell \mathfrak{f}_1)(\eta^M(0), \eta) \, d\mu_{\Lambda_M, K} \\ = \sum_{x \in \Lambda_{\ell+A}} \int (\tau_x \nabla_b h_0) \, (A_\ell \mathfrak{f}_1)(\eta^M(0), \eta) \, d\mu_{\Lambda_M, K} \; .\end{aligned}$$

Summing over all bonds b, we recover $L^s h_0 = h = g - \tau_{e_i} g$, for some local function g and some $1 \le i \le d$. The previous expression is thus equal to

$$\begin{aligned}\sum_{x \in \partial_i^- \Lambda_{\ell+A}} \int (\tau_x g) \, (A_\ell \mathfrak{f}_1)(\eta^M(0), \eta) \, d\mu_{\Lambda_M, K} \\ - \sum_{x \in \partial_i^+ \Lambda_{\ell+A}} \int (\tau_x g) \, (A_\ell \mathfrak{f}_1)(\eta^M(0), \eta) \, d\mu_{\Lambda_M, K} \; ,\end{aligned}$$

where $\partial_i^- \Lambda_{\ell+A}$ stands for the lower boundary in the i-th direction of $\Lambda_{\ell+A}$ and $\partial_i^+ \Lambda_{\ell+A}$ for the upper boundary. In particular, x belongs to $\partial_i^\pm \Lambda_{\ell+A}$ if it belongs to $\Lambda_{\ell+A}$ and $\pm x_i = \ell + A$. Since the measure $\mu_{\Lambda_M, K}$ is uniform,

$$E_{\mu_{\Lambda_M, K}} [(\tau_x g) g'] \; = \; E_{\mu_{\Lambda_M, K}} [(\tau_y g) g']$$

if the support of $\tau_x g$ and the one of $\tau_y g$ do not intersect the one of g'. Therefore, choosing A large enough, the previous sum vanishes. This proves that the function (6.5) has mean zero with respect to all canonical invariant measures.

At this point, we follow the classical approach of nongradient systems (cf. [6], Chapter 7) to estimate the second term of (6.4) using the standard Rayleigh-Schroedinger perturbation theorem for the largest eigenvalue of a symmetric operator. After a few steps we bound the difference of the second term of (6.4) with $\varepsilon N^{2-d} D_N(f)$ by

$$\frac{N^{2-d} \varepsilon}{M^d} \sum_{z \in \mathbb{T}_N^d} \sup_K \Big\{ \frac{G(t, z/N)}{N \varepsilon^2} \int B \, f \, d\mu_{\Lambda_M, K} - \; < -L^s_{\Lambda_M} \sqrt{f}, \sqrt{f} >_{\mu_{\Lambda_M, K}} \Big\} \; .$$

In this formula, B stands for the function (6.5), the supremum is carried over all integers $0 \le K \le |\Lambda_M|$ and $< \cdot, \cdot >_{\mu_{\Lambda_M, K}}$ is the inner product in $L^2(\mu_{\Lambda_M, K})$. Since the spectral gap of the generator of the symmetric exclusion process in Λ_M is of order M^2 and $M^2 N^{-1}$ vanishes as $N \uparrow \infty$, by the perturbation theorem for the largest eigenvalue of a symmetric operator, the previous expression is less than or equal to

$$\frac{C}{M^d \varepsilon^3} \sup_K < (-L^s_{\Lambda_M})^{-1} B, B >_{\mu_{\Lambda_M, K}} \; .$$

Consider the linear term in the variational formula for the H_{-1} norm of B. It is given by $2 < B, f >_{\mu_{\Lambda_M, K}}$ for some function f in $L^2(\mu_{\Lambda_M, K})$. Since B has mean zero with respect to all canonical invariant measures, this is in fact a covariance that we estimate by $C_0(\ell) M^2 + C_1 M^{-2} < f, f >_{\mu_{\Lambda_M, K}}$. By the spectral gap for the symmetric exclusion process, the second term is bounded by $< (-L^s_{\Lambda_M}) f, f >_{\mu_{\Lambda_M, K}}$ if we choose C_0 sufficiently small. Therefore, $< (-L^s_{\Lambda_M})^{-1} B, B >_{\mu_{\Lambda_M, K}}$ is bounded by $C(\ell)M^2$. Since we are in dimension $d \ge 3$, the last displayed equation vanishes as $N \uparrow \infty$. This proves that the second term in (6.4) may be estimated by $\varepsilon N^{2-d} D_N(f)$ and an expression which vanishes as $N \uparrow \infty$. $\square$

We have just proved that we may replace L^* by $L^*_{\Lambda_\ell}$ in (4.3). We show now that we can replace the average $\eta^M(0)$ by the average $\eta^\ell(0)$.

Lemma 6.2. *Fix a function $\mathfrak{f}$ in $\mathfrak{F}$, a smooth function $G : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}$ and assume that M satisfies the conditions (3.2). For every $T > 0$,*

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} \Big| \int_0^T dt \int N^{1-d} \sum_{z\in\mathbb{T}^d_N} G(t,z/N) \tau_z \Big\{ L^*_{\Lambda_\ell}(A_\ell \mathfrak{f})(\eta^M(0),\eta) - L^*_{\Lambda_\ell}(A_\ell \mathfrak{f})(\eta^\ell(0),\eta) \Big\} f_t^N \, d\nu_\alpha^N \Big| \;=\; 0\,.$$

Proof. We have seen in the proof of the previous theorem that it is enough to show that

$$N^{1-d} \sum_{z\in\mathbb{T}^d_N} G(t,z/N) \tau_z \Big\{ L^*_{\Lambda_\ell}(A_\ell \mathfrak{f})(\eta^M(0),\eta) - L^*_{\Lambda_\ell}(A_\ell \mathfrak{f})(\eta^\ell(0),\eta) \Big\}$$

is negligible.

Consider a class of function $B(\beta,\eta)$, $0 \le \beta \le 1$, whose support is contained in Λ_ℓ. Repeating the well known steps of the proof of the one block estimate we obtain that

$$\int B(\eta^M(0),\eta) f(\eta)\, d\nu_\alpha^N \;=\; \sum_{K=0}^{|\Lambda_M|} C_K(f) \int B(K/|\Lambda_M|,\eta) f_{M,K}(\eta)\, d\mu_{\Lambda_M,K}\ ,$$

where,

$$C_K(f) \;=\; \int \mathbf{1}\{ \sum_{x\in\Lambda_M} \eta(x) = K\} f \, d\nu_\alpha^N\ , \quad f_{M,K}(\eta) \;=\; \frac{f_M}{\int f_M(\eta)\, d\mu_{\Lambda_M,K}}$$

and f_M is the conditional expectation $E_{\nu_\alpha^N}[f \,|\, \mathcal{F}_M]$. Here, for a set Λ, $\mathcal{F}_\Lambda$ stands for the σ-algebra generated by $\{\eta(z),\, z\in\Lambda\}$. At this point, $B(K/|\Lambda_M|,\cdot)$ is a local function with support in Λ_ℓ and we repeat the procedure for $f_{M,K}$, $\mu_{\Lambda_M,K}$ in place of f, ν_α^N. We obtain in this way that the previous sum is equal to

$$\sum_{K=0}^{|\Lambda_M|} C_K(f) \sum_{k=0}^{|\Lambda_\ell|} C_k(f_{M,K}) \int B(K/|\Lambda_M|,\eta) f_{M,K,\ell,k}(\eta)\, d\mu_{\Lambda_\ell,k}$$

with the obvious definitions for $C_k(f_{M,K})$, $f_{M,K,\ell,k}$.

Using that the Dirichlet form is convex, we may estimate

$$\int N^{1-d} \sum_{z\in\mathbb{T}^d_N} G(t,z/N) B(\eta^M(0),\eta)\, (\tau_{-z} f)\, d\nu_\alpha^N \;-\; \varepsilon N^{2-d} D_N(f)$$

by

$$N^{-d} \sum_{z\in\mathbb{T}^d_N} \sum_{K=0}^{|\Lambda_M|} C_K(f^z) \sum_{k=0}^{|\Lambda_\ell|} C_k(f^z_{M,K}) \tag{6.6}$$

$$\Big\{ G(t,z/N) N \int B(K/|\Lambda_M|,\eta) f^z_{M,K,\ell,k}(\eta)\, d\mu_{\Lambda_\ell,k} - \frac{\varepsilon N^2}{|\Lambda_\ell|} D_{\Lambda_\ell}(f^z_{M,K,\ell,k}, \mu_{\Lambda_\ell,k}) \Big\}\ .$$

In this formula, $f^z = \tau_{-z} f$ and $D_{\Lambda_\ell}(\cdot, \mu_{\Lambda_\ell,k})$ is the Dirichlet form associated to the generator $L^s_{\Lambda_\ell}$ and the reversible measure $\mu_{\Lambda_\ell,k}$. Assume that $B(K/|\Lambda_M|, \eta)$ has mean zero with respect to all invariant states $\mu_{\Lambda_\ell,k}$, which is the case of the function we are considering in this lemma. By the Rayleigh-Schroedinger perturbation theorem for the largest eigenvalue of a symmetric operator, the expression inside braces in the previous formula is less than or equal to

$$\frac{C|\Lambda_\ell|}{\varepsilon} < (-L^s_{\Lambda_\ell})^{-1} B(K/|\Lambda_M|, \eta), B(K/|\Lambda_M|, \eta) >_{\mu_{\Lambda_\ell,k}} . \tag{6.7}$$

We claim that in the particular case of this lemma, the previous expression is bounded by $C\varepsilon^{-1}(K/|\Lambda_M| - k/|\Lambda_\ell|)^2$. Indeed, let h be the local function $\mathfrak{f}(K/|\Lambda_M|, \eta) - \mathfrak{f}(k/|\Lambda_\ell|, \eta)$. In the case where B is the function which appears in the statement of the lemma, the linear term of the variational formula for the H_{-1} norm is

$$\frac{2}{|\Lambda_{\ell'}|} \sum_{y \in \Lambda_{\ell'}} \int (L^* \tau_y h)\, f \, d\mu_{\Lambda_\ell,k} ,$$

where f is in $L^2(\mu_{\Lambda_\ell,k})$. Since $L^*\tau_y h$ is a local function which has mean zero with respect to all invariant measures, we may localize f around y, replace the scalar product by a covariance, use the spectral gap of the symmetric exclusion process, restricted to a cube whose length depend only on the support of h, and apply Schwarz inequality to bound $< (\nabla_b E[f|\mathcal{F}_\Lambda])^2 >$ by $< (\nabla_b f)^2 >$. At the end we obtain that the previous expression is less than or equal to

$$\frac{C}{|\Lambda_{\ell'}|^2} \sum_{y \in \Lambda_{\ell'}} < (L^* \tau_y h)^2 >_{\mu_{\Lambda_\ell,k}} \; + \; < -L^s f, f >_{\mu_{\Lambda_\ell,k}} .$$

Since $\mathfrak{f}(\cdot, \eta)$ is smooth, uniformly in η, $L^*\tau_y h$ is absolutely bounded by $|K/|\Lambda_M| - k/|\Lambda_\ell||$. This proves that (6.7) is bounded above by $C\varepsilon^{-1}(K/|\Lambda_M| - k/|\Lambda_\ell|)^2$.

Up to this point we proved that the expression inside braces in (6.6) is bounded above by $C\varepsilon^{-1}(K/|\Lambda_M| - k/|\Lambda_\ell|)^2$. Recalling the definition of the constants appearing in (6.6), we have that this sum is in fact

$$\int \frac{C}{\varepsilon N^d} \sum_{z \in \mathbb{T}^d_N} \Big\{ \eta^M(0) - \eta^\ell(0) \Big\}^2 (\tau_{-z} f) \, d\nu^N_\alpha .$$

It remains to apply the two blocks estimate to conclude the proof. □

6.3. **Replacement of $H_{i,j}(\eta)$ by $\sigma_{i,j} F(\eta^\ell(0)) + J_{i,j}(\eta^\ell(0))$.** Fix a smooth function $G : \mathbb{T}^d \times \mathbb{R}_+ \to \mathbb{R}$ and two function $\mathfrak{f}$, $\mathfrak{g}$ in $\mathfrak{F}$. Since the local functions $\mathfrak{f}(\beta, \cdot)$ have a common finite support, for each fixed y, there exists a finite integer A such that

$$\nabla_{0,y} \Gamma_{\mathfrak{f}(\eta^M(0), \cdot)} \; = \; \nabla_{0,y} \sum_{z \in \Lambda_A} \mathfrak{f}(\eta^M(0), \tau_z \eta) .$$

Since $\mathfrak{f}(\cdot, \eta)$ are smooth functions, the difference between the previous expression and

$$\nabla_{0,y} \sum_{z \in \Lambda_A} \mathfrak{f}(\eta^\ell(0), \tau_z \eta)$$

is absolutely bounded by $C(A, \mathfrak{f})\, |\eta^M(0) - \eta^\ell(0)|$, for some finite constant $C(A, \mathfrak{f})$. By the two blocks estimate, the average over $\mathbb{T}^d_N$ of this absolute value is negligible.

After this replacement, the third line of (4.3) is seen to be composed of three different types of terms:

$$\sum_{x\in\mathbb{T}_N^d} G(t,x/N)\,\tau_x \sum_{y\in\mathbb{Z}^d} p^*(y)\,y_i\,y_j\,\eta(0)[1-\eta(y)]\ ,$$

$$\sum_{x\in\mathbb{T}_N^d} G(t,x/N)\,\tau_x \sum_{y\in\mathbb{Z}^d} p^*(y)\,y_i\,\eta(0)[1-\eta(y)]\,\Gamma^{A,\ell}_{y,\mathfrak{f}}(\eta)\ ,$$

$$\sum_{x\in\mathbb{T}_N^d} G(t,x/N)\,\tau_x \sum_{y\in\mathbb{Z}^d} p^*(y)\,\eta(0)[1-\eta(y)]\,\Gamma^{A,\ell}_{y,\mathfrak{f}}(\eta)\,\Gamma^{A,\ell}_{y,\mathfrak{g}}(\eta)\ ,$$

where, for some function $\mathfrak{h}$ in $\mathfrak{F}$,

$$\Gamma^{A,\ell}_{y,\mathfrak{h}}(\eta) \ = \ \nabla_{0,y} \sum_{z\in\Lambda_A} \mathfrak{h}(\eta^\ell(0),\tau_z\eta)\ .$$

By the one block estimate, the first sum can be replaced by

$$\sum_{x\in\mathbb{T}_N^d} G(t,x/N)\,\sigma_{i,j}\,F(\eta^\ell(x))\ .$$

We claim that the second sum is negligible because $\eta(0)[1-\eta(y)]\Gamma^{A,\ell}_{y,\mathfrak{f}}$ has mean zero with respect to all canonical invariant measures. Indeed, repeating the steps of the one block estimate, we are reduced to estimate

$$\sup_K \int \eta(0)[1-\eta(y)]\,\nabla_{0,y} \sum_{z\in\Lambda_A} \mathfrak{f}(K/|\Lambda_\ell|,\tau_z\eta)\,d\mu_{\Lambda_\ell,K}\ ,$$

where the supremum is carried over all $0\le K\le|\Lambda_\ell|$. A change of variables $\xi=\sigma^{0,y}\eta$ permits to rewrite the previous expression as

$$\sup_K \int \big\{\eta(y)-\eta(0)\big\} \sum_{z\in\Lambda_A} \mathfrak{f}(K/|\Lambda_\ell|,\tau_z\eta)\,d\mu_{\Lambda_\ell,K}\ .$$

The integral vanishes for each fixed K because $\mu_{\Lambda_\ell,K}$ is a uniform measure.

The third type of term requires some notation. For a function $\mathfrak{h}(\beta,\eta)$, smooth in the first coordinate and with a common finite support in the second, let

$$\tilde{\mathfrak{h}}(\alpha,\beta) \ = \ E_{\nu_\beta}[\mathfrak{h}(\alpha,\eta)]\ .$$

For $1\le i,j\le d$ and y in $\mathbb{Z}^d$, let

$$\mathfrak{h}^{i,j}_y(\beta,\eta) \ = \ \eta(0)[1-\eta(y)]\,\nabla_{0,y} \sum_{z\in\Lambda_A} \mathfrak{f}_i(\beta,\tau_z\eta)\,\nabla_{0,y} \sum_{z\in\Lambda_A} \mathfrak{f}_j(\beta,\tau_z\eta)\ .$$

Notice that $\mathfrak{h}$ is smooth in the first coordinate and have a common finite support on the second coordinate. Moreover, an elementary computation shows that

$$\sum_{y\in\mathbb{Z}^d} p^*(y)\tilde{\mathfrak{h}}^{i,j}_y(\beta,\beta) \ = \ 2\sum_{x\in\mathbb{Z}^d} <\mathfrak{f}_i(\beta,\cdot),(-L^s)\tau_x\mathfrak{f}_j(\beta,\cdot)>_\beta\ \ .$$

In this formula, $<\cdot,\cdot>_\beta$ stands for the inner product in $L^2(\nu_\beta)$. Denote the right hand side by $J_{\mathfrak{f}_i,\mathfrak{f}_j}(\beta)$. Lemma 6.3 below shows that we may replace in (4.3) the third type of term by

$$\sum_{x\in\mathbb{T}_N^d} G(t,x/N)\,J_{\mathfrak{f}_i,\mathfrak{f}_j}(\eta^\ell(x))\ .$$

Up to this point, we proved that the third line of (4.3) is equal to

$$(1/2) \sum_{i,j=1}^{d} \sum_{x \in \mathbb{T}_N^d} (\partial_{u_i} \lambda)(t,x/N)(\partial_{u_j} \lambda)(t,x/N) \Big\{ \sigma_{i,j} F(\eta^\ell(x)) + J_{n,i,j}(\eta^\ell(x)) \Big\}$$

plus a term of order $o(N^d)$, where

$$J_{n,i,j}(\beta) \;=\; 2 \sum_{x \in \mathbb{Z}^d} < \mathfrak{f}_{i,n}(\beta,\cdot), (-L^s)\tau_x \mathfrak{f}_{j,n}(\beta,\cdot) >_\beta \ .$$

Recall the definition of the function $J_{i,j}(\beta)$ given in (4.6). By Theorem 5.1, with the notation introduced in section 3, the previous sum is equal to

$$(1/2) \sum_{i,j=1}^{d} \sum_{x \in \mathbb{T}_N^d} (\partial_{u_i} \lambda)(t,x/N)(\partial_{u_j} \lambda)(t,x/N) \Big\{ \sigma_{i,j} F(\eta^\ell(x)) + J_{i,j}(\eta^\ell(x)) \Big\}$$

plus a term of order $o(\mathfrak{f}, N^d)$. To conclude this subsection, it remains to prove the next result.

Lemma 6.3. *Fix a function $\mathfrak{h}(\beta,\eta)$ smooth in the first coordinate and with finite common support in the second. For positive integers ℓ, m, let*

$$V_{\ell,m}^{\mathfrak{h}}(\eta) \;=\; \Big| \frac{1}{|\Lambda_m|} \sum_{y \in \Lambda_m} \mathfrak{h}(\eta^\ell(0), \tau_y \eta) - \tilde{\mathfrak{h}}(\eta^\ell(0), \eta^\ell(0)) \Big| \ .$$

Then,

$$\lim_{m\to\infty} \limsup_{\ell\to\infty} \sup_f \Big\{ \int \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \tau_x V_{\ell,m}^{\mathfrak{h}}(\eta)\, f\, d\nu_\alpha^N \; - \; \varepsilon N^{2-d} D_N(f) \Big\} \;=\; 0$$

for all $\varepsilon > 0$.

Proof. Since ν_γ is the Bernoulli product measure, for each β, an elementary computation shows that $(\partial_\gamma)\tilde{\mathfrak{h}}(\beta,\gamma) = \sum_{x\in\Lambda} < h(\beta,\eta); \eta(x) >_\gamma$, where $< \cdot; \cdot >_\gamma$ stands for the covariance with respect to ν_γ and Λ for a finite set which contains the common support of the function $\mathfrak{h}(\beta,\cdot)$. In particular, the derivative $(\partial_\gamma \tilde{\mathfrak{h}})(\beta,\gamma)$ is uniformly bounded. Hence,

$$\Big| \tilde{\mathfrak{h}}(\eta^\ell(0), \eta^\ell(0)) - \tilde{\mathfrak{h}}(\eta^\ell(0), \eta^m(0)) \Big| \;\le\; C(\mathfrak{h}) \big| \eta^m(0) - \eta^\ell(0) \big|$$

for some finite constant $C(\mathfrak{h})$. It follows from the two blocks estimate that we may replace $\tilde{\mathfrak{h}}(\eta^\ell(0), \eta^\ell(0))$ by $\tilde{\mathfrak{h}}(\eta^\ell(0), \eta^m(0))$ in the definition of $V_{\ell,m}^{\mathfrak{h}}$.

Following the classical proof of the one block estimate, we are reduced to estimate

$$\sup_K \int \Big| \frac{1}{|\Lambda_m|} \sum_{y \in \Lambda_m} \mathfrak{h}(K/|\Lambda_\ell|, \tau_y \eta) - \tilde{\mathfrak{h}}(K/|\Lambda_\ell|, \eta^m(0)) \Big| \, d\mu_{\Lambda_\ell, K} \ ,$$

where the supremum is carried over all $0 \le K \le |\Lambda_\ell|$. For each fixed ℓ, denote by K_ℓ the integer which maximizes the previous variational formula. There exists a subsequence ℓ' such that $K_{\ell'}/|\Lambda_{\ell'}|$ converges to some density β in $[0,1]$. In particular, the limsup, as $\ell \uparrow \infty$, of the previous expression is less than or equal to

$$\sup_{\beta\in[0,1]} \int \Big| \frac{1}{|\Lambda_m|} \sum_{y \in \Lambda_m} \mathfrak{h}(\beta, \tau_y \eta) - \tilde{\mathfrak{h}}(\beta, \eta^m(0)) \Big| \, d\nu_\beta$$

because the finite marginals of the canonical measure converges to the grand canonical measures. Since $\tilde{\mathfrak{h}}(\beta, \cdot)$ is a smooth function,

$$\tilde{\mathfrak{h}}(\beta, \eta^m(0)) \;=\; \tilde{\mathfrak{h}}(\beta,\beta) \pm C(\eta^m(0) - \beta) \;=\; E_{\nu_\beta}[\mathfrak{h}(\beta,\eta)] \pm C(\eta^m(0) - \beta)\ .$$

In particular, the previous variational formula is bounded above by

$$\sup_{\beta\in[0,1]} \int \Big| \frac{1}{|\Lambda_m|} \sum_{y\in\Lambda_m} \mathfrak{h}(\beta, \tau_y\eta) - E_{\nu_\beta}[\mathfrak{h}(\beta,\eta)] \Big| \, d\nu_\beta \;+\; C \sup_{\beta\in[0,1]} \int \big| \eta^m(0) - \beta \,\big| \, d\nu_\beta\ .$$

This expression vanishes as $m\uparrow\infty$ because ν_β is a product measure and $\mathfrak{h}(\beta,\cdot)$ are local functions with a finite common support. This concludes the proof of the lemma. □

6.4. **Estimation of the current.** Fix $i \le i \le d$ and recall the definition of $V_i^\ell(\eta)$ given just after (4.5). Let

$$A_{i,N,\ell,\mathfrak{f}}(t,\eta) \;=\; N^{1-d} \sum_{x\in\mathbb{T}_N^d} (\partial_{u_i}\lambda)(t, x/N) \tau_x V_i^\ell(\eta)\ .$$

By the nongradient estimates, for every $T\ge 0$,

$$\begin{aligned}
&\limsup_{\ell\to\infty} \limsup_{N\to\infty} \int_0^T dt \int \nu_\alpha^N(d\eta)\, f_t^N(\eta)\, A_{i,N,\ell,\mathfrak{f}}(t,\eta)\\
&\quad \le\; C_0 \sup_{\alpha\in[0,1]} \Big\vert\!\Big\vert\!\Big\vert w_i^*(\alpha,\eta) + \sum_{1\le j\le d} D_{i,j}(\alpha)[\eta(e_j) - \eta(0)] - L^*\mathfrak{f}_i(\alpha,\eta) \Big\vert\!\Big\vert\!\Big\vert_\alpha^2
\end{aligned}$$

for some finite constant C_0. Here $\vert\!\vert\!\vert \cdot \vert\!\vert\!\vert$ is the norm introduced at the beginning of section 5. We refer to section 6 of [8] for the proof. Note that we don't need in the present context the multiscale analysis of [8]. By Theorem 5.1 this expression vanishes if we replace $\mathfrak{f}_i$ by $\mathfrak{f}_{i,n}$ and let $n\uparrow\infty$.

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