Once More on Interaction of An Oscillator with An One-Dimensional Scalar Field.

Simple Exactly Solvable Models based on Finite Rank Perturbations Methods.

I: D'Alembert-Kirchhoff-like formulae

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Abstract

This paper is an electronic application to my set of lectures, subject:'Formal methods in solving differential equations and constructing models of physical phenomena'. Addressed, mainly: postgraduates and related readers. Content: a very detailed discussion of the simple model of interaction based on the equation array:

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2\Big(q(t) - Q(t)\Big) + f_0(t) \\
\frac{\partial^2 u(t,x)}{\partial t^2} &= c^2\frac{\partial^2 u(t,x)}{\partial x^2} - 4\gamma c\delta(x-x_0)\Big(Q(t) - q(t)\Big) + f_1(t,x) \\
Q(t) &= u(t,x_0)
\end{aligned}$$

Besides, less detailed discussion of related models. Central mathematical points: d'Alembert-Kirchhoff-like formulae. Central physical points: phenomena of Radiation Reaction, Braking Radiation and Resonance.

Introduction.

A Harmonic Oscillator Coupled to an One-Dimensional Scalar Field.

In this paper I will discuss several models of an one-dimensional harmonic oscillator coupled to an one-dimensional scalar field. Primarily I am interested in the model described by the equation array [1]

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2\Big(q(t) - Q(t)\Big) + f_0(t) \\
\frac{\partial^2 u(t,x)}{\partial t^2} &= c^2\frac{\partial^2 u(t,x)}{\partial x^2} - 4\gamma c\delta(x-x_0)\Big(Q(t) - q(t)\Big) + f_1(t,x) \\
Q(t) &= u(t,x_0)
\end{aligned}$$

[1] $4\gamma c\rho = k$, $\Omega^2 = k/M$; the constants of the model mean, e.g.: c = propagating waves velocity, ρ = 'a density' of the field, k = elasticity constant, M = mass of the particle. Of course, we assume $c > 0$.

One can say that this is a model of a point interaction. The considered model is not standard, if one means classical or quantum field models. In that field, a standard looks rather like this:

$$\begin{aligned}
\frac{\partial^2 q_0(t)}{\partial t^2} &= -\Omega^2 q_0(t) + \gamma_1 Q_\phi(t) + f_0(t) \\
\frac{\partial^2 \phi(t,x)}{\partial t^2} &= c^2 \frac{\partial^2 \phi(t,x)}{\partial x^2} + 4\gamma_2 c\delta(x-x_0)q_0(t) + f_1(t,x) \\
Q_\phi(t) &= \phi(t,x_0)
\end{aligned}$$

After indicating **d'Alembert-Kirchhoff-like formulae** for solutions to these systems I briefly compare them and discuss phenomena of **Radiation Reaction**, **Braking Radiation** and **Resonance**.

After that I will discuss an abstract analogue of the former system: [2]

$$\begin{aligned}
\ddot{q} &= -\Omega^2(q-Q) + f_0(t) \\
\ddot{u} &= Bu - 4\gamma_c\Big(\delta_{\alpha,t,x_0}\Big)\cdot\Big(Q-q\Big) + f_1(t) \\
Q &= Q(t) =< l|u(t) >
\end{aligned}$$

where I use a P.A.M. Dirac's "bra-ket" syntax and suppose that q and Q are usual (one-dimensional) functions of t :

$$q = q(t)\,, \quad Q = Q(t)\,,$$

B is an abstract linear operator, l is a linear functional, $\{u(t)\}_t$ and $\{\delta_{\alpha,t,x_0}\}_t$ are families of abstract elements; of course the type of δ_{α,t,x_0} must be the same as one of $u(t)$.

Normally I will suppose that δ_{α,t,x_0} does not depend on t, i.e. is constant in t. In that case I will write δ_{α,x_0} instead of δ_{α,t,x_0}.

In the next paper I will suppose that δ_{α,t,x_0} is constant in t. Then one can rewrite the above equations as follows:

$$\begin{pmatrix} \ddot{q} \\ \ddot{u} \end{pmatrix} = \begin{pmatrix} -\Omega^2 & \Omega^2 < l|_2 \\ 4\gamma_c\delta_{\alpha,x_0} < 1|_1 & B - 4\gamma_c\delta_{\alpha,x_0} < l|_2 \end{pmatrix} \begin{pmatrix} q \\ u \end{pmatrix} + \begin{pmatrix} f_0(t) \\ f_1(t) \end{pmatrix}$$

$$\begin{pmatrix} \ddot{q}_0 \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} -\Omega^2 & \gamma_1 < l|_2 \\ 4\gamma_{2,c}\delta_{\alpha,x_0} < 1|_1 & B \end{pmatrix} \begin{pmatrix} q_0 \\ \phi \end{pmatrix} + \begin{pmatrix} f_0(t) \\ f_1(t) \end{pmatrix}$$

where the subscribts 1 and 2 in $< \cdots|_1$ and $< \cdots|_2$ mean that arguments of $< \cdots|_1$ are elements of the first component of the vector $\begin{pmatrix} q \\ u \end{pmatrix}$ resp. $\begin{pmatrix} q_0 \\ \phi \end{pmatrix}$ and arguments of $< \cdots|_2$ are elements of the second component of the suitable vector. Of course,

$$< 1|q >=< 1|_1 q >= q\,, \quad < 1|q_0 >=< 1|_1 q_0 >= q_0$$

The subject will primarily be **resolvents formulae** i.e. the formulae that resolve the equation array:

$$z\begin{pmatrix} q \\ u \end{pmatrix} - \begin{pmatrix} -\Omega^2 & \Omega^2 < l|_2 \\ 4\gamma_c\delta_{\alpha,x_0} < 1|_1 & B - 4\gamma_c\delta_{\alpha,x_0} < l|_2 \end{pmatrix} \begin{pmatrix} q \\ u \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

[2] the proper modifications for the latter system will be evident

resp.

$$z\begin{pmatrix} q_0 \\ \phi \end{pmatrix} - \begin{pmatrix} -\Omega^2 & \gamma_1 < l|_2 \\ 4\gamma_{2,c}\delta_{\alpha,x_0} < 1|_1 & B \end{pmatrix}\begin{pmatrix} q_0 \\ \phi \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Of course, here all quantities, $q, u, w_1, w_2, \delta_{\alpha,x_0}, l$, etc. are supposed to be constant in t.

I will discuss Donoghue-Friedrichs-like solutions to the system [3] and resp. the phenomenon of **Resonance** and notion of the **Second Sheet**.

[3]i.e., I will discuss resolvents formulae of the system

1 Models of a Point Interaction of an only Oscillator with an only one-dimensional Scalar Field

In this section we fix measure units and let x be dimensionless position parameter, i.e.,

$$\text{physical position coordinate } = [\text{ length unit }] \times x + const\,.$$

Otherwise a confusion can ocurr, in relating to the definition

$$\int_{-\infty}^{\infty} \delta(x-x_0)f(x)dx = f(x_0)\,.$$

We assume the standard foramalism, where

$$\delta(x-x_0) = \frac{\partial 1_+(x-x_0)}{\partial x}$$

and where 1_+ stands for a unit step function (Heaviside function):

$$1_+(\xi) = \begin{cases} 1 & , \quad \text{if } \xi > 0\,, \\ 0 & , \quad \text{if } \xi < 0\,, \end{cases}$$

1.1 First Model. D'Alembert-Kirchhoff-like formulae

Recall that standard D'Alembert-Kirchhoff formulae read: if

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + f\,, \quad u = u(t,x)\,, \quad f = f(t,x) \tag{$*$}$$

then

$$u = u(t,x) \quad = \quad \frac{1}{2c}\int_s^t \Big(\tilde f(\tau, x+c(t-\tau)) - \tilde f(\tau, x-c(t-\tau))\Big)d\tau + u_0(t,s,x)$$

where s is arbitrarily fixed [4],

$$\begin{aligned} u_0(t,s,x) \quad &= \quad c_+(x+ct) + c_-(x-ct) \\ &= \quad \frac{1}{2}\Big(u(s,x+c(t-s)) + u(s,x-c(t-s))\Big) + \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} (\frac{\partial u(t,\xi)}{\partial t})\Big|_{t=s} d\xi \end{aligned}$$

and where $\tilde f$ stands for any Primitive function of f, i.e.

$$\frac{\partial \tilde f(t,x)}{\partial x} = f(t,x)$$

Note that

$$\tilde f(\tau, x+c(t-\tau)) - \tilde f(\tau, x-c(t-\tau))$$

does not depend on what primitive is one has chosen!!! Moreover, we need only $\tilde f$ and not f itself.

Recall that standard relation for an one-dimensional harmonic oscillator under an external force $f_0(t)$ is this:

$$\frac{\partial^2 q(t)}{\partial t^2} = -\Omega^2 q(t) + f_0(t)$$

[4] s is playing a role of an initial value of time if $u(s,\cdot)$ and $\frac{\partial u(t,\xi)}{\partial t})\Big|_{t=s}$ (the initial data) are given.

In this section I will firstly discuss a system described by

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2\Big(q(t)-Q(t)\Big)+f_0(t)\\
\frac{\partial^2 u(t,x)}{\partial t^2} &= c^2\frac{\partial^2 u(t,x)}{\partial x^2}-4\gamma c\delta(x-x_0)\Big(Q(t)-q(t)\Big)+f_1(t,x)\\
Q(t) &= u(t,x_0)
\end{aligned}$$

It means in patricular that I will take

$$f=-4\gamma c\delta(x-x_0)\Big(Q(t)-q(t)\Big)+f_1(t,x)$$

For such an f I conclude that

$$\tilde{f}=-4\gamma c 1_+(x-x_0)\Big(Q(t)-q(t)\Big)+\tilde{f}_1(t,x),$$

and then I infer that

$$\begin{aligned}
u(t,x) &= -2\gamma\int_s^t\Big(1_+(x+c(t-\tau)-x_0)-1_+(x-c(t-\tau)-x_0)\Big)\Big(Q(\tau)-q(\tau)\Big)d\tau\\
&\quad+\frac{1}{2c}\int_s^t\Big(\tilde{f}_1(\tau,x+c(t-\tau))-\tilde{f}_1(\tau,x-c(t-\tau))\Big)d\tau+u_0(t,s,x)\\
u(t,x) &= -2\gamma\int_s^t\Big(1_+(x+c(t-\tau)-x_0)-1_+(x-c(t-\tau)-x_0)\Big)\Big(u(\tau,x_0)-q(\tau)\Big)d\tau\\
&\quad+\frac{1}{2c}\int_s^t\Big(\tilde{f}_1(\tau,x+c(t-\tau))-\tilde{f}_1(\tau,x-c(t-\tau))\Big)d\tau+u_0(t,s,x)
\end{aligned}$$

Denote now, to be more concise,

$$u_{01}(t,s,x):=\frac{1}{2c}\int_s^t\Big(\tilde{f}_1(\tau,x+c(t-\tau))-\tilde{f}_1(\tau,x-c(t-\tau))\Big)d\tau+u_0(t,s,x)$$

and then rewrite the recent relation as following:

$$\begin{aligned}
u(t,x) &= -2\gamma\int_s^t\Big(1_+(x+c(t-\tau)-x_0)-1_+(x-c(t-\tau)-x_0)\Big)\Big(u(\tau,x_0)-q(\tau)\Big)d\tau\\
&\quad+u_{01}(t,s,x)
\end{aligned}$$

I have now seen: given q and u_{01}, then,

$$\boxed{\textbf{in order to obtain } u(t,x) \textbf{ I need to obtain ONLY } u(t,x_0)}$$

After this observation use the last formula for $u(t,x)$ and then obtain

$$\begin{aligned}
u(t,x_0) &= -2\gamma\int_s^t\Big(1_+(c(t-\tau))-1_+(c(\tau-t))\Big)\Big(u(\tau,x_0)-q(\tau)\Big)d\tau\\
&\quad+u_{01}(t,s,x_0)
\end{aligned}$$

Therefore, because

$$1_+(c(t-\tau))=1\,,\ 1_+(c(\tau-t))=0 \text{ for } t>\tau$$

one can obtain

$$u(t,x_0) \quad = \quad -2\gamma\int_s^t \Big(u(\tau,x_0)-q(\tau)\Big)d\tau + u_{01}(t,s,x_0)$$

i.e.,

$$Q(t) \quad = \quad -2\gamma\int_s^t \Big(Q(\tau)-q(\tau)\Big)d\tau + u_{01}(t,s,x_0)$$

Write it as

$$Q(t)+2\gamma\int_s^t Q(\tau)d\tau \quad = \quad 2\gamma\int_s^t q(\tau)d\tau + u_{01}(t,s,x_0)$$

Now recall that

$$\frac{\partial^2 q(t)}{\partial t^2} = -\Omega^2\Big(q(t)-Q(t)\Big)+f_0(t)$$

and for the moment denote

$$Q_0(t) := u_0(t,s,x_0) = c_+(x_0+ct)) + c_-(x_0-ct))$$

$$Q_{01}(t) := u_{01}(t,s,x_0) = \frac{1}{2c}\int_s^t \Big(\tilde f_1(\tau,x_0+c(t-\tau)) - \tilde f_1(\tau,x_0-c(t-\tau))\Big)d\tau + Q_0(t)$$

Then obtain

$$\begin{aligned} Q(t)+2\gamma\int_s^t Q(\tau)d\tau &= 2\gamma\int_s^t q(\tau)d\tau + Q_{01}(t) \\ \frac{\partial^2 q}{\partial t^2} &= -\Omega^2(q-Q)+f_0(t) \end{aligned}$$

We have now obtained an equation array for Q and q. The next step is to find an **insulated** equation for Q and one for q. We begin to search for the equation for q. For this purpose, we apply, at first, the operator

$$\Big(I+2\gamma\int_s^t \cdot d\tau\Big)$$

to the latter equation, i.e. to the equation

$$\frac{\partial^2 q}{\partial t^2} \quad = \quad -\Omega^2(q-Q)+f_0(t)$$

Then we infer

$$\begin{aligned} \frac{\partial^2 q}{\partial t^2}+2\gamma\int_s^t \frac{\partial^2 q(\tau)}{\partial \tau^2}d\tau &= -\Omega^2\Big(q+2\gamma\int_s^t q(\tau)d\tau\Big) \\ &\quad +\Omega^2\Big(Q+2\gamma\int_s^t Q(\tau)d\tau\Big) \\ &\quad +\Big(f_0+2\gamma\int_s^t f_0(\tau)d\tau\Big) \\ \frac{\partial^2 q}{\partial t^2}+2\gamma\int_s^t \frac{\partial^2 q(\tau)}{\partial \tau^2}d\tau &= -\Omega^2\Big(q+2\gamma\int_s^t q(\tau)d\tau\Big) \\ &\quad +\Omega^2\Big(2\gamma\int_s^t q(\tau)d\tau + Q_{01}(t)\Big) \\ &\quad +\Big(f_0+2\gamma\int_s^t f_0(\tau)d\tau\Big) \\ \frac{\partial^2 q}{\partial t^2}+2\gamma\int_s^t \frac{\partial^2 q(\tau)}{\partial \tau^2}d\tau &= -\Omega^2\Big(q+Q_{01}(t)\Big)+\Big(f_0+2\gamma\int_s^t f_0(\tau)d\tau\Big) \end{aligned}$$

The most recent equation for q rewrite as

$$\frac{\partial^2 q}{\partial t^2} = -2\gamma \int_s^t \frac{\partial^2 q(\tau)}{\partial \tau^2} d\tau - \Omega^2 q + \Omega^2 Q_{01}(t) + f_0(t) + 2\gamma \int_s^t f_0(\tau) d\tau$$

We can stop at this equation, or, observing that

$$\int_s^t \frac{\partial^2 q(\tau)}{\partial \tau^2} d\tau = \frac{\partial q(t)}{\partial t} - \frac{\partial q(t)}{\partial t}\Big|_{t=s},$$

we can stop at that:

$$\boxed{\frac{\partial^2 q(t)}{\partial t^2} = -2\gamma \frac{\partial q(t)}{\partial t} - \Omega^2 q(t) + 2\gamma \frac{\partial q(t)}{\partial t}\Big|_{t=s} + \Omega^2 Q_{01}(t) + f_0(t) + 2\gamma \int_s^t f_0(\tau) d\tau}$$

Some people prefer to write such an equation as following:

$$\boxed{\Big(\frac{\partial^2}{\partial t^2} + 2\gamma \frac{\partial}{\partial t} + \Omega^2\Big) q(t) = 2\gamma \frac{\partial q(t)}{\partial t}\Big|_{t=s} + \Omega^2 Q_{01}(t) + f_0(t) + 2\gamma \int_s^t f_0(\tau) d\tau}$$

Remark, whatever equation we take, we see an interesting detail: the equation is **not ordinary** differential equation, if we follow standard terminology. The case is because of the term

$$2\gamma \frac{\partial q(t)}{\partial t}\Big|_{t=s}$$

In the next subsections, we will return to this factor. However, the machinary of the **ordinary** differential equations does here quite for. For the time being, we turn to describing $Q(t)$ and $u(t,x)$.

After $q(t)$ is found, we can determine $Q(t)$, at least formally, by solving

$$Q(t) \quad = \quad -2\gamma \int_s^t \Big(Q(\tau) - q(\tau)\Big) d\tau + Q_{01}(t)$$

or

$$\frac{\partial Q(t)}{\partial t} \quad = \quad -2\gamma \Big(Q(\tau) - q(\tau)\Big) + \frac{\partial Q_{01}(t)}{\partial t}, \quad Q(s) = Q_{01}(s)$$

Thus we have already reduced our model, a model of an oscillator coupled the a scalar field, to a pair of linear 'ordinary' differential equations. Nevertheless we want to continue to analyse the matter and we now go seeking another relationships, which would simplify calculations of q, Q and u.

At first, we will obtain another insulated equation for Q, differently and in a different form.

We have

$$\begin{aligned} Q(t) + 2\gamma \int_s^t Q(\tau) d\tau &= 2\gamma \int_s^t q(\tau) d\tau + Q_{01}(t) \\ \frac{\partial^2 q}{\partial t^2} + \Omega^2 q &= \Omega^2 Q + f_0(t) \end{aligned}$$

i.e.,

$$\begin{aligned}\Big(Q(t)-Q_{01}(t)\Big)+2\gamma\int_s^t\Big(Q(\tau)-Q_{01}(\tau)\Big)d\tau &= 2\gamma\int_s^t\Big(q(\tau)-Q_{01}(\tau)\Big)d\tau\\ \Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)q &= \Omega^2 Q+f_0(t)\end{aligned}$$

Then we infer

$$\begin{aligned}&\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(\Big(Q(t)-Q_{01}(t)\Big)+2\gamma\int_s^t\Big(Q(\tau)-Q_{01}(\tau)\Big)d\tau\Big)\\ &\qquad= \Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(2\gamma\int_s^t\Big(q(\tau)-Q_{01}(\tau)\Big)d\tau\Big)\end{aligned}$$

and

$$\begin{aligned}&\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(\Big(Q(t)-Q_{01}(t)\Big)+2\gamma\int_s^t\Big(Q(\tau)-Q_{01}(\tau)\Big)d\tau\Big)\\ &\qquad= \Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(2\gamma\int_s^t q(\tau)d\tau\Big)-\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(2\gamma\int_s^t Q_{01}(\tau)d\tau\Big)\end{aligned}$$

Note

$$\frac{\partial^2}{\partial t^2}\int_s^t q(\tau)d\tau=\frac{\partial}{\partial t}q(t)=\int_s^t\frac{\partial^2}{\partial\tau^2}q(\tau)d\tau+\frac{\partial}{\partial t}q(t)\Big|_{t=s}$$

Then

$$\begin{aligned}&\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(\Big(Q(t)-Q_{01}(t)\Big)+2\gamma\int_s^t\Big(Q(\tau)-Q_{01}(\tau)\Big)d\tau\Big)\\ &\quad= 2\gamma\int_s^t\Big(\frac{\partial^2}{\partial\tau^2}+\Omega^2\Big)q(\tau)d\tau+2\gamma\frac{\partial}{\partial t}q(t)\Big|_{t=s}-\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(2\gamma\int_s^t Q_{01}(\tau)d\tau\Big)\end{aligned}$$

Then

$$\begin{aligned}&\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(\Big(Q(t)-Q_{01}(t)\Big)+2\gamma\int_s^t\Big(Q(\tau)-Q_{01}(\tau)\Big)d\tau\Big)\\ &\quad= 2\gamma\int_s^t\Big(\Omega^2Q(\tau)+f_0(\tau)\Big)d\tau+2\gamma\frac{\partial}{\partial t}q(t)\Big|_{t=s}-\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(2\gamma\int_s^t Q_{01}(\tau)d\tau\Big)\end{aligned}$$

Then

$$\begin{aligned}&\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(\Big(Q(t)-Q_{01}(t)\Big)+2\gamma\int_s^t\Big(Q(\tau)-Q_{01}(\tau)\Big)d\tau\Big)\\ &\quad= 2\gamma\int_s^t\Big(\Omega^2Q(\tau)-\Omega^2Q_{01}(\tau)\Big)d\tau+2\gamma\frac{\partial}{\partial t}q(t)\Big|_{t=s}-\frac{\partial^2}{\partial t^2}\Big(2\gamma\int_s^t Q_{01}(\tau)d\tau\Big)\\ &\qquad\qquad+2\gamma\int_s^t f_0(\tau)d\tau\end{aligned}$$

Then

$$\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(Q(t)-Q_{01}(t)\Big)+\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)2\gamma\int_s^t\Big(Q(\tau)-Q_{01}(\tau)\Big)d\tau$$

$$= 2\gamma\int_s^t \Big(\Omega^2 Q(\tau) - \Omega^2 Q_{01}(\tau)\Big)d\tau + 2\gamma\frac{\partial}{\partial t}q(t)\Big|_{t=s} - \frac{\partial^2}{\partial t^2}\Big(2\gamma\int_s^t Q_{01}(\tau)d\tau\Big) + 2\gamma\int_s^t f_0(\tau)d\tau$$

Then

$$\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(\Big(Q(t)-Q_{01}(t)\Big)+\frac{\partial^2}{\partial t^2}2\gamma\int_s^t\Big(Q(\tau)-Q_{01}(\tau)\Big)d\tau\Big) = 2\gamma\frac{\partial}{\partial t}q(t)\Big|_{t=s} - \frac{\partial^2}{\partial t^2}\Big(2\gamma\int_s^t Q_{01}(\tau)d\tau\Big) + 2\gamma\int_s^t f_0(\tau)d\tau$$

Then, finally,

$$\boxed{\Big(\frac{\partial^2}{\partial t^2}+2\gamma\frac{\partial}{\partial t}+\Omega^2\Big)\Big(Q(t)-Q_{01}(t)\Big) = 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=s} - 2\gamma\frac{\partial Q_{01}(t)}{\partial t} + 2\gamma\int_s^t f_0(\tau)d\tau}$$

On the surface, this equation appears to be a second order **ordinary** differential equation. It is not exactly the case. We may not arbitrary take the initial data for $Q(t)$. The proper ones are these:

$$\Big(Q(s)-Q_{01}(s)\Big)=0\,,\quad \frac{\partial\Big(Q(t)-Q_{01}(t)\Big)}{\partial t}\Big|_{t=s} = 2\gamma\Big(q(s)-Q_{01}(s)\Big)$$

Thus, we have already obtained two simple 'ordinary' differential equations for

$$q(t)\,,\quad Q(t)$$

Now let us analyse the expression

$$u = u(t,x) = -2\gamma\int_s^t \Big(1_+(x+c(t-\tau)-x_0) - 1_+(x-c(t-\tau)-x_0)\Big)\Big(Q(\tau)-q(\tau)\Big)d\tau + u_{01}(t,s,x)$$

We have: if $c\tau \neq ct+(x-x_0)$ and $c\tau \neq ct-(x-x_0)$ then

$$\begin{aligned}
&1_+(x+c(t-\tau)-x_0) - 1_+(x-c(t-\tau)-x_0)\\
&= \begin{cases} 1 \;, & c\tau < ct+(x-x_0) \\ 0 \;, & ct+(x-x_0) < c\tau \end{cases} - \begin{cases} 1 \;, & ct-(x-x_0) < \tau \\ 0 \;, & c\tau < ct-(x-x_0) \end{cases}\\
&= \begin{cases} 1 \;, & c\tau < ct-|x-x_0| \\ 0 \;, & ct-|x-x_0| < c\tau < ct+|x-x_0| \\ -1 \;, & ct+|x-x_0| < c\tau \end{cases}\\
&= \begin{cases} 1 \;, & \tau < t-|x-x_0|/c \\ 0 \;, & t-|x-x_0|/c < \tau < t+|x-x_0|/c \\ -1 \;, & t+|x-x_0|/c < \tau \end{cases}
\end{aligned}$$

Hence

$$u(t,x) \quad = \quad \left\{ \begin{array}{ccl} -2\gamma \displaystyle\int_s^{t-|x-x_0|/c} \Big(Q(\tau) - q(\tau)\Big) d\tau & , & \text{if } s \le t - |x-x_0|/c \\ 0 & , & \text{if } t - |x-x_0|/c < s \le t \\ 0 & , & \text{if } t \le s < t + |x-x_0|/c \\ +2\gamma \displaystyle\int_s^{t+|x-x_0|/c} \Big(Q(\tau) - q(\tau)\Big) d\tau & , & \text{if } t + |x-x_0|/c \le s \end{array} \right\}$$

$$+ \, u_{01}(t,s,x)$$

Besides, remember that

$$Q(t) + 2\gamma \int_s^t Q(\tau) d\tau \quad = \quad 2\gamma \int_s^t q(\tau) d\tau + Q_{01}(t)$$

If we take this factor into account, then we deduce, finally,

$$u(t,x) \quad = \quad \left\{ \begin{array}{ccl} Q(t - |x-x_0|/c) - Q_{01}(t - |x-x_0|/c) & , & \text{if } s \le t - |x-x_0|/c \\ 0 & , & \text{if } t - |x-x_0|/c < s \le t \\ 0 & , & \text{if } t \le s < t + |x-x_0|/c \\ -Q(t + |x-x_0|/c) + Q_{01}(t + |x-x_0|/c) & , & \text{if } t + |x-x_0|/c \le s \end{array} \right\}$$

$$+ \, u_{01}(t,s,x)$$

Now substitute anywhere $u_0(t,s,x_0)$ for $Q_0(t)$, and $u_{01}(t,s,x_0)$ for $Q_{01}(t)$, and resume.

1.1.1 Putting It Together

We have discussed the system

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2\Big(q(t)-Q(t)\Big)+f_0(t)\\
\frac{\partial^2 u(t,x)}{\partial t^2} &= c^2\frac{\partial^2 u(t,x)}{\partial x^2}-4\gamma c\delta(x-x_0)\Big(Q(t)-q(t)\Big)+f_1(t,x)\\
Q(t) &= u(t,x_0)
\end{aligned}$$

and concluded that:

$$\begin{aligned}
\Big(\frac{\partial^2}{\partial t^2}+2\gamma\frac{\partial}{\partial t}+\Omega^2\Big)q(t) &= 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=s}+\Omega^2 u_{01}(t,s,x_0)+f_0(t)+2\gamma\int_s^t f_0(\tau)d\tau\\
Q(t) &= -2\gamma\int_s^t\Big(Q(\tau)-q(\tau)\Big)d\tau+u_{01}(t,s,x_0)
\end{aligned}$$

An insulated equation for $Q(t)$ is this:

$$\begin{aligned}
&\Big(\frac{\partial^2}{\partial t^2}+2\gamma\frac{\partial}{\partial t}+\Omega^2\Big)\Big(Q(t)-u_{01}(t,s,x_0)\Big)\\
&\qquad = 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=s}-2\gamma\frac{\partial u_{01}(t,s,x_0)}{\partial t}+2\gamma\int_s^t f_0(\tau)d\tau
\end{aligned}$$

The proper initial relations are these:

$$\Big(Q(s)-u_{01}(s,s,x_0)\Big)=0\,,\ \frac{\partial\Big(Q(t)-u_{01}(t,s,x_0)\Big)}{\partial t}\Big|_{t=s}=2\gamma\Big(q(s)-u_{01}(s,s,x_0)\Big)$$

For $u=u(t,x)$, we have concluded:

$$\begin{aligned}
u(t,x) &= -2\gamma\int_s^t\Big(1_+(x+c(t-\tau)-x_0)-1_+(x-c(t-\tau)-x_0)\Big)\Big(Q(\tau)-q(\tau)\Big)d\tau\\
&\quad+u_{01}(t,s,x)
\end{aligned}$$

Other expressions for $u=u(t,x)$ are these:

$$u(t,x) = \left\{\begin{array}{ccl} Q(t-|x-x_0|/c)-u_{01}(t-|x-x_0|/c,s,x_0) & , & \text{if } s\le t-|x-x_0|/c\\ 0 & , & \text{if } t-|x-x_0|/c<s\le t\\ 0 & , & \text{if } t\le s<t+|x-x_0|/c\\ -Q(t+|x-x_0|/c)+u_{01}(t+|x-x_0|/c,s,x_0) & , & \text{if } t+|x-x_0|/c\le s\end{array}\right\}$$

$$+u_{01}(t,s,x)$$

where u_{01} is defined by the relations

$$\begin{aligned}
u_0(t,s,x) &:= c_+(x+ct)+c_-(x-ct)\\
&= \frac{1}{2}\Big(u(s,x+c(t-s))+u(s,x-c(t-s))\Big)+\frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)}\Big(\frac{\partial u(t,\xi)}{\partial t}\Big)\Big|_{t=s}d\xi\\
u_{01}(t,s,x) &:= \frac{1}{2c}\int_s^t\Big(\tilde f_1(\tau,x+c(t-\tau))-\tilde f_1(\tau,x-c(t-\tau))\Big)d\tau+u_0(t,s,x)
\end{aligned}$$

In the following subsections, we will mostly discuss the case where

$$c = 1\,, s = 0\,, x_0 = 0\,, f_1 = 0\,.$$

it will be convenient to have rewritten some of the recent formulae in the proper way.

$$\Big(\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \Omega^2\Big)q(t) = 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0} + \Omega^2 u_0(t,0,0) + f_0(t) + 2\gamma\int_0^t f_0(\tau)d\tau$$

$$Q(t) \quad = \quad -2\gamma\int_0^t \Big(Q(\tau) - q(\tau)\Big)d\tau + u_0(t,0,0)$$

An insulated equation for $Q(t)$ is this:

$$\Big(\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \Omega^2\Big)\Big(Q(t) - u_0(t,0,0)\Big) \quad = \quad 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0} - 2\gamma\frac{\partial u_0(t,0,0)}{\partial t} + 2\gamma\int_0^t f_0(\tau)d\tau$$

The proper initial conditions are these:

$$\Big(Q(0) - u_0(0,0,0)\Big) = 0\,, \ \frac{\partial\Big(Q(t) - u_0(t,0,0)\Big)}{\partial t}\Big|_{t=0} = 2\gamma\Big(q(0) - u_0(0,0,0)\Big)$$

Expressions for $u = u(t,x)$, are these:

$$u(t,x) \quad = \quad \left\{ \begin{array}{ccl} Q(t-|x|) - u_0(t-|x|,0,0) & , & \text{if } 0 \le t - |x| \\ 0 & , & \text{if } t - |x| < 0 \le t \\ 0 & , & \text{if } t \le 0 < t + |x| \\ -Q(t+|x|) + u_0(t+|x|,0,0) & , & \text{if } t + |x| \le 0 \end{array} \right\}$$

$$+ u_0(t,0,x)$$

where

$$\begin{aligned} u_0(t,0,x) \quad &:= \quad c_+(x+t) + c_-(x-t) \\ &= \quad \frac{1}{2}\Big(u(0,x+t) + u(0,x-t)\Big) + \frac{1}{2}\int_{x-t}^{x+t}\big(\frac{\partial u(t,\xi)}{\partial t}\big)\Big|_{t=0}d\xi \end{aligned}$$

1.2 A Standard Model. D'Alembert-Kirchhoff-like formulae

The considered in the previous subsection model is not standard, if one means classical or quantum field models. In that field, a standard looks rather like this:

$$\begin{aligned}
\frac{\partial^2 q_0(t)}{\partial t^2} &= -\Omega^2 q_0(t) + \gamma_1 Q_\phi(t) + f_0(t) \\
\frac{\partial^2 \phi(t,x)}{\partial t^2} &= c^2 \frac{\partial^2 \phi(t,x)}{\partial x^2} + 4\gamma_2 c\delta(x-x_0) q_0(t) + f_1(t,x) \\
Q_\phi(t) &= \phi(t, x_0)
\end{aligned}$$

In the latter situation one can obtain that

$$Q_\phi(t) = 2\gamma_2 \int_s^t q_0(\tau) d\tau + \phi_{01}(t, s, x_0)$$

for the proper $\phi_{01}(t,x)$. Next, with the reasons of the previous subsections:

$$\boxed{\frac{\partial^2 q_0(t)}{\partial t^2} = -\Omega^2 q_0(t) + 2\gamma_2\gamma_1 \int_s^t q_0(\tau) d\tau + \gamma_1 \phi_{01}(t, s, x_0) + f_0(t)}$$

and

$$\begin{aligned}
&\Big(\frac{\partial^2}{\partial t^2} + \Omega^2\Big)\Big(Q_\phi(t) - \phi_{01}(t, s, x_0)\Big) \\
&\qquad = 2\gamma_2 \Big(\frac{\partial^2}{\partial t^2} + \Omega^2\Big) \int_s^t q_0(\tau) d\tau \\
&\qquad = 2\gamma_2 \int_s^t \Big(\frac{\partial^2}{\partial \tau^2} + \Omega^2\Big) q_0(\tau) d\tau + 2\gamma_2 \frac{\partial}{\partial t} q(t)\Big|_{t=s} \\
&\qquad = 2\gamma_2 \int_s^t \Big(\gamma_1 Q_\phi(\tau) + f_0(\tau)\Big) d\tau + 2\gamma_2 \frac{\partial}{\partial t} q(t)\Big|_{t=s} \\
&\qquad = 2\gamma_2\gamma_1 \int_s^t \Big(Q_\phi(\tau) - \phi_{01}(\tau, s, x_0)\Big) d\tau \\
&\qquad\quad + 2\gamma_2 \int_s^t \Big(\gamma_1 \phi_{01}(\tau, s, x_0) + f_0(\tau)\Big) d\tau + 2\gamma_2 \frac{\partial}{\partial t} q(t)\Big|_{t=s}
\end{aligned}$$

Thus

$$\boxed{\begin{aligned}
&\Big(\frac{\partial^2}{\partial t^2} + \Omega^2\Big)\Big(Q_\phi(t) - \phi_{01}(t, s, x_0)\Big) \\
&\qquad = 2\gamma_2\gamma_1 \int_s^t \Big(Q_\phi(\tau) - \phi_{01}(\tau, s, x_0)\Big) d\tau \\
&\qquad\quad + 2\gamma_2 \int_s^t \Big(\gamma_1 \phi_{01}(\tau, s, x_0) + f_0(\tau)\Big) d\tau + 2\gamma_2 \frac{\partial}{\partial t} q(t)\Big|_{t=s}
\end{aligned}}$$

We can reduce these equations to **ordinary** differential equations, but these ones become **third** order equations,

$$\begin{aligned}
\Big(\frac{\partial^3}{\partial t^3} + \Omega^2 \frac{\partial}{\partial t} - 2\gamma_2\gamma_1\Big) q_0(t) &= \gamma_1 \frac{\partial \phi_{01}(t, s, x_0)}{\partial t} + \frac{\partial f_0(t)}{\partial t} \\
\Big(\frac{\partial^3}{\partial t^3} + \Omega^2 \frac{\partial}{\partial t} + 2\gamma_2\gamma_1\Big)\Big(Q_\phi(t) - \phi_{01}(t, s, x_0)\Big) &= 2\gamma_2 \Big(\gamma_1 \phi_{01}(t, s, x_0) + f_0(t)\Big)
\end{aligned}$$

subject to the supplementary initial relations

$$\begin{aligned}\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)q_0(t)\Big|_{t=s} &= f_0(s)\\ \Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(Q_\phi(t)-\phi_{01}(t,s,x_0)\Big)\Big|_{t=s} &= 2\gamma_2\frac{\partial}{\partial t}q(t)\Big|_{t=s}\end{aligned}$$

As for $\phi(t,x)$, one can obtain that

$$\begin{aligned}\phi(t,x) &= 2\gamma_2\int_s^t\Big(1_+(x+c(t-\tau)-x_0)-1_+(x-c(t-\tau)-x_0)\Big)q_0(\tau)d\tau\\ &\quad+\phi_{01}(t,s,x)\end{aligned}$$

Other expressions for $\phi(t,x)$ will be these:

$$\phi(t,x) = \left\{\begin{array}{ccl} 2\gamma_2\int_s^{t-|x-x_0|/c} q_0(\tau)d\tau & , & \text{if } s\le t-|x-x_0|/c\\ 0 & , & \text{if } t-|x-x_0|/c<s\le t\\ 0 & , & \text{if } t\le s<t+|x-x_0|/c\\ -2\gamma_2\int_s^{t+|x-x_0|/c} q_0(\tau)d\tau & , & \text{if } t+|x-x_0|/c\le s\end{array}\right\}$$

$$+\phi_{01}(t,s,x)$$

$$\phi(t,x) = \left\{\begin{array}{ccl} Q_\phi(t-|x-x_0|/c)-\phi_{01}(t-|x-x_0|/c,s,x_0) & , & \text{if } s\le t-|x-x_0|/c\\ 0 & , & \text{if } t-|x-x_0|/c<s\le t\\ 0 & , & \text{if } t\le s<t-|x-x_0|/c\\ -Q_\phi(t+|x-x_0|/c)+\phi_{01}(t+|x-x_0|/c,s,x_0) & , & \text{if } t+|x-x_0|/c\le s\end{array}\right\}$$

$$+\phi_{01}(t,s,x)$$

where ϕ_{01} is defined of course in the same manner as in the previous subsection, by the relations:

$$\begin{aligned}\phi_0(t,s,x) &:= c_{\phi,+}(x+ct)+c_{\phi,-}(x-ct)\\ &= \frac{1}{2}\Big(\phi(s,x+c(t-s))+\phi(s,x-c(t-s))\Big)+\frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)}\Big(\frac{\partial\phi(t,\xi)}{\partial t}\Big)\Big|_{t=s}d\xi\\ \phi_{01}(t,s,x) &:= \frac{1}{2c}\int_s^t\Big(\tilde f_1(\tau,x+c(t-\tau))-\tilde f_1(\tau,x-c(t-\tau))\Big)d\tau+\phi_0(t,s,x)\end{aligned}$$

The case where

$$c = 1\,, s = 0\,, x_0 = 0\,, f_1 = 0\,.$$

is this:

$$\boxed{\frac{\partial^2 q_0(t)}{\partial t^2} = -\Omega^2 q_0(t) + 2\gamma_2\gamma_1 \int_0^t q_0(\tau)d\tau + \gamma_1\phi_0(t,0,0) + f_0(t)}$$

$$\boxed{\begin{aligned}&\Big(\frac{\partial^2}{\partial t^2} + \Omega^2\Big)\Big(Q_\phi(t) - \phi_0(t,0,0)\Big)\\ &\qquad = 2\gamma_2\gamma_1 \int_0^t \Big(Q_\phi(\tau) - \phi_0(\tau,0,0)\Big)d\tau\\ &\qquad\quad + 2\gamma_2 \int_0^t \Big(\gamma_1\phi_0(\tau,0,0) + f_0(\tau)\Big)d\tau + 2\gamma_2\frac{\partial}{\partial t}q(t)\Big|_{t=0}\end{aligned}}$$

Reducing to ordinary differential equations:

$$\begin{aligned}\Big(\frac{\partial^3}{\partial t^3} + \Omega^2\frac{\partial}{\partial t} - 2\gamma_2\gamma_1\Big)q_0(t) &= \gamma_1\frac{\partial\phi_0(t,0,0)}{\partial t} + \frac{\partial f_0(t)}{\partial t}\\ \Big(\frac{\partial^3}{\partial t^3} + \Omega^2\frac{\partial}{\partial t} + 2\gamma_2\gamma_1\Big)\Big(Q_\phi(t) - \phi_0(t,0,0)\Big) &= 2\gamma_2\Big(\gamma_1\phi_0(t,0,0) + f_0(t)\Big)\end{aligned}$$

supplementary initial relations:

$$\begin{aligned}\Big(\frac{\partial^2}{\partial t^2} + \Omega^2\Big)q_0(t)\Big|_{t=0} &= f_0(0)\\ \Big(\frac{\partial^2}{\partial t^2} + \Omega^2\Big)\Big(Q_\phi(t) - \phi_0(t,0,0)\Big)\Big|_{t=0} &= 2\gamma_2\frac{\partial}{\partial t}q(t)\Big|_{t=0}\end{aligned}$$

As for $\phi(t,x)$,

$$\begin{aligned}\phi(t,x) &= \left\{\begin{array}{ccl} Q_\phi(t-|x|) - \phi_0(t-|x|,0,0) & , & \text{if } 0 \le t-|x| \\ 0 & , & \text{if } t-|x| < 0 \le t \\ 0 & , & \text{if } t \le 0 < t+|x| \\ -Q_\phi(t+|x|) + \phi_0(t+|x|,0,0) & , & \text{if } t+|x| \le 0 \end{array}\right\}\\ &\qquad + \phi_0(t,0,x)\end{aligned}$$

where ϕ_0 is defined by the relations:

$$\begin{aligned}\phi_0(t,0,x) &:= c_{\phi,+}(x+t) + c_{\phi,-}(x-t)\\ &= \frac{1}{2}\Big(\phi(0,x+t) + \phi(0,x-t)\Big) + \frac{1}{2}\int_{x-t}^{x+t}\Big(\frac{\partial\phi(t,\xi)}{\partial t}\Big)\Big|_{t=0}d\xi\end{aligned}$$

1.3 Particular Cases. 1. Radiation Reaction, Braking Radiation

We are all familiar with the fact that any solution to any linear inhomogeneous equation, whatever its nature, is a sum of a solution to the associated homogeneous equation plus arbitrarily taken and fixed solution to the former linear inhomogeneous equation, isn't it? Just let us now consider these two cases separately, the case where equations are homogeneous and then inhomogeneous.

We start out emphasising that the homogeneous equation array, connected to

$$\Big(\frac{\partial^2}{\partial t^2}+2\gamma\frac{\partial}{\partial t}+\Omega^2\Big)q(t)=2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}+\Omega^2 u_0(t,0,0)+f_0(t)+2\gamma\int_0^t f_0(\tau)d\tau$$

$$\Big(\frac{\partial^2}{\partial t^2}+2\gamma\frac{\partial}{\partial t}+\Omega^2\Big)\Big(Q(t)-u_0(t,0,0)\Big) = 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}-2\gamma\frac{\partial u_0(t,0,0)}{\partial t}+2\gamma\int_0^t f_0(\tau)d\tau$$

is exactly

$$\begin{aligned}\Big(\frac{\partial^2}{\partial t^2}+2\gamma\frac{\partial}{\partial t}+\Omega^2\Big)q(t) &= 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}\\ \Big(\frac{\partial^2}{\partial t^2}+2\gamma\frac{\partial}{\partial t}+\Omega^2\Big)\Big(Q(t)-u_0(t,0,0)\Big) &= 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}\end{aligned}$$

and NOT

$$\begin{aligned}\Big(\frac{\partial^2}{\partial t^2}+2\gamma\frac{\partial}{\partial t}+\Omega^2\Big)q(t) &= 0\\ \Big(\frac{\partial^2}{\partial t^2}+2\gamma\frac{\partial}{\partial t}+\Omega^2\Big)\Big(Q(t)-u_0(t,0,0)\Big) &= 0\end{aligned}$$

The difference is a **rank one** term $2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}$ [5]. This detail allows us to apply usual machinary of finite rank perturbations theory. Thus, having put

$$\Omega_\gamma := \sqrt{\Omega^2-\gamma^2}$$

and having taken into account the supplementary initial relations:

$$Q(0)=0\,,\ \frac{\partial Q(t)}{\partial t}\Big|_{t=0}=2\gamma q(0)$$

one can show that

$$\begin{aligned}q(t) &= e^{-\gamma t}\Big(cos(\Omega_\gamma t)+\gamma\frac{sin(\Omega_\gamma t)}{\Omega_\gamma}\Big)\Big(q(0)-\frac{2\gamma}{\Omega^2}\frac{\partial q(t)}{\partial t}\Big|_{t=0}\Big)+e^{-\gamma t}\frac{sin(\Omega_\gamma t)}{\Omega_\gamma}\frac{\partial q(t)}{\partial t}\Big|_{t=0}\\ &\quad+\frac{2\gamma}{\Omega^2}\frac{\partial q(t)}{\partial t}\Big|_{t=0}\end{aligned}$$

$$\begin{aligned}Q(t) &= e^{-\gamma t}\Big(cos(\Omega_\gamma t)+\gamma\frac{sin(\Omega_\gamma t)}{\Omega_\gamma}\Big)\Big(-\frac{2\gamma}{\Omega^2}\frac{\partial q(t)}{\partial t}\Big|_{t=0}\Big)+e^{-\gamma t}\frac{sin(\Omega_\gamma t)}{\Omega_\gamma}2\gamma q(0)\\ &\quad+\frac{2\gamma}{\Omega^2}\frac{\partial q(t)}{\partial t}\Big|_{t=0}\end{aligned}$$

[5] This term, as a function of t , is a fixed function of t, e.g. 1, multiplied by a CONSTANT depended on q, i.e., by a fixed functional of q. Using the Dirac's syntax, this term can be written as $|a><b|$ with $|a>=1$ and $<b|q>=2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}$.

and then

$$\begin{aligned} u(t,x) &= e^{-\gamma(t-|x|)}\Big(cos\Omega_\gamma(t-|x|)+\gamma\frac{sin\Omega_\gamma(t-|x|)}{\Omega_\gamma}\Big)\Big(-\frac{2\gamma}{\Omega^2}\frac{\partial q(t)}{\partial t}\Big|_{t=0}\Big) \\ &\quad + e^{-\gamma(t-|x|)}\frac{sin\Omega_\gamma(t-|x|)}{\Omega_\gamma}2\gamma q(0) \\ &\quad + \frac{2\gamma}{\Omega^2}\frac{\partial q(t)}{\partial t}\Big|_{t=0}, \text{ if } 0\le t-|x| \\ u(t,x) &= 0\,, \text{ if } 0 > t-|x| \end{aligned}$$

Notice,

$$q(t)-Q(t) \;=\; e^{-\gamma t}cos(\Omega_\gamma t)q(0)+e^{-\gamma t}\frac{sin\Omega_\gamma t}{\Omega_\gamma}\Big(\frac{\partial q(t)}{\partial t}\Big|_{t=0}-2\gamma q(0)\Big)$$

and then

$$q(t)-Q(t)\to 0 \text{ as } t\to+\infty\,,\ \gamma>0$$

If we concentrate now on the system's behaviour at large t , a mathematical detail calls attention. We observe:

$$q(t)\to 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0} \text{ as } t\to+\infty\,.$$

Nevertheless, the limit function

$$q_\infty(t)=2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}$$

is NO solution to

$$\frac{\partial^2 q(t)}{\partial t^2}=-2\gamma\frac{\partial q(t)}{\partial t}-\Omega^2 q(t)+2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}$$

every time that

$$2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}\neq 0\,,$$

because the $q=q_\infty$ is a constant, hence its derivative is zero but we have need of

$$\frac{\partial q(t)}{\partial t}\Big|_{t=0}=\frac{\partial q_\infty}{\partial t}=\frac{\partial}{\partial t}\Big(2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}\Big)=0\,.$$

Similar phenomena, one can detect them in the electrodynamics of moving charges.

Now, we are directing our attention to the fact that the oscillator moves with damped amplitude:

$$|q(t)-q_\infty|\le e^{-\gamma t}\cdot const\,.$$

The reason is very plain. When the particle (oscillator) begins to move, it makes some region of the field 'move': the oscillator generates, emits waves, –one is used to saying: 'the oscillator radiates'.

Now then, the oscillator radiates. It entails some energy expenses and thus the field, made move, eventually makes the conduct of the oscillator change. This phenomenon is said to be the **radiation reaction**.

Next, since the oscillator has emitted waves, waves run away and brings away some portions of the oscillator energy, braking the moving of the oscillator. If something had reflected the waves, more precise, if the emitted waves had returned to the oscillator, we could wait for that the these waves would stimulate an increase of the amplitude of the oscillator. But if no wave returns to the oscilator, –it is exactly our case–, the damping dominates and the amplitude decreases [6]. This phenomenon is said to be the **braking radiation** or **damping radiation**.

[6] as we have seen, it decreases to the zero value

1.4 Particular Cases. 2. Resonance

We begin to analyse the inhomogeneous equations with the most simple case, where there is an only rigt-hand incident wave

$$u_{01}(t,0,x) = u_0(t,0,x) = c_+(x+t)$$

and where one can easy compute the solution: don't deviate from standards, let

$$u_{01}(t,0,x) = u_0(t,0,x) = c_+(x+t) := A\sin(k(x+t))$$

i.e.,

$$\Big(\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \Omega^2\Big)q(t) \quad = \quad 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0} + \Omega^2 A\sin(kt)$$

$$\Big(\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \Omega^2\Big)\Big(Q(t) - A\sin(kt)\Big) \quad = \quad 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0} - 2\gamma Ak\cos(kt)$$

According to the standard reading of the universiy course to the theory of ordinary differential equations, if

$$\Big(\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \Omega^2\Big)y(t) \quad = \quad const + \Omega^2 A\sin(kt)$$

$$\Big(\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \Omega^2\Big)Y(t) \quad = \quad const - 2\gamma Ak\cos(kt)$$

then a partial solution to these equations is given by:

$$y(t) \quad = \quad \frac{\Omega^2 A}{(-k^2+\Omega^2)^2 + (2\gamma k)^2}\big((-k^2+\Omega^2)\sin(kt) - 2\gamma k\cos(kt)\big) + \frac{const}{\Omega^2}$$

$$Y(t) \quad = \quad \frac{-2\gamma Ak}{(-k^2+\Omega^2)^2 + (2\gamma k)^2}\big(2\gamma k\sin(kt) + (-k^2+\Omega^2)\cos(kt)\big) + \frac{const}{\Omega^2}$$

On the surface, there is no essential deflecting from the situation in the university course of ordinary differential equation, but it is only on the surface.

Through the presence of the term $\frac{\partial q(t)}{\partial t}\Big|_{t=0}$ the forms

$$q(t) \quad = \quad A_s\sin(kt) + A_c\cos(kt) + Const$$

need not contain a particular solution to the former equation. Although, we have, of course,

$$\begin{aligned} q(t) \quad = \quad & \frac{\Omega^2 A}{(-k^2+\Omega^2)^2 + (2\gamma k)^2}\big((-k^2+\Omega^2)\sin(kt) - 2\gamma k\cos(kt)\big) + const_1 \\ & + \frac{2\gamma}{\Omega^2}\frac{\partial q(t)}{\partial t}\Big|_{t=0} \\ & + e^{-\gamma t}\Big(const_1\cos(\Omega_\gamma t) + const_2\sin(\Omega_\gamma t)\Big) \end{aligned}$$

As for $Q(t) - A\sin(kt)$, on referring to the equation to $Q(t) - A\sin(kt)$ we see, on the contrary, that the term $2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}$ is there exterior and does not hinder for immediate displaying a particular solution to the lonely equation to $Q(t) - A\sin(kt)$. It is this:

$$Q(t) - A\sin(kt) \quad = \quad \frac{-2\gamma Ak}{(-k^2+\Omega^2)^2+(2\gamma k)^2}\big(2\gamma k\sin(kt) + (-k^2+\Omega^2)\cos(kt)\big) + \frac{2\gamma}{\Omega^2}\frac{\partial q(t)}{\partial t}\Big|_{t=0}$$

Nevertheless, this form is improper, because of the initial relations

$$\Big(Q(0) - c_+(0)\Big) = 0\,, \ \frac{\partial\Big(Q(t) - c_+(t)\Big)}{\partial t}\Big|_{t=0} = 2\gamma\Big(q(0) - c_+(0)\Big)$$

which in our case read

$$Q(0) = 0\,, \ \frac{\partial\Big(Q(t) - A\sin(kt)\Big)}{\partial t}\Big|_{t=0} = 2\gamma q(0)$$

Once again, we need to take into account the terms containing $e^{-\gamma t}\Big(\cdots\Big)$:

$$\begin{aligned} Q(t) - A\sin(kt) \quad &= \quad \frac{-2\gamma Ak}{(-k^2+\Omega^2)^2+(2\gamma k)^2}\big(2\gamma k\sin(kt) + (-k^2+\Omega^2)\cos(kt)\big) + \frac{2\gamma}{\Omega^2}\frac{\partial q(t)}{\partial t}\Big|_{t=0} \\ &\quad + e^{-\gamma t}\Big(const_3\cos(\Omega_\gamma t) + const_4\sin(\Omega_\gamma t)\Big) \end{aligned}$$

Let us restrict ourselves to the case where $\gamma t >> 1$. In this case $e^{-\gamma t}\Big(\cdots\Big)\approx 0$ and

$$\begin{aligned} q(t) \quad &\approx \quad \frac{\Omega^2 A}{(-k^2+\Omega^2)^2+(2\gamma k)^2}\big((-k^2+\Omega^2)\sin(kt) - 2\gamma k\cos(kt)\big) + const_1 \\ &\quad + \frac{2\gamma}{\Omega^2}\frac{\partial q(t)}{\partial t}\Big|_{t=0} \end{aligned}$$

$$Q(t) - A\sin(kt) \quad \approx \quad \frac{-2\gamma Ak}{(-k^2+\Omega^2)^2+(2\gamma k)^2}\big(2\gamma k\sin(kt) + (-k^2+\Omega^2)\cos(kt)\big) + \frac{2\gamma}{\Omega^2}\frac{\partial q(t)}{\partial t}\Big|_{t=0}$$

If $k^2 = \Omega^2$, then the amplitude of $q(t)$ becomes maximal [7]: This phenomenonon is said to be the **resonance** .

Next, if in addition $\frac{\partial q(t)}{\partial t}\Big|_{t=0} = 0$, then

$$\begin{aligned} Q(t) - A\sin(kt) \quad &\approx \quad \frac{-2\gamma Ak}{(2\gamma k)^2}\left(2\gamma k\sin(kt)\right) \\ &= \quad - A\sin(kt) \end{aligned}$$

[8]

Let us now recall, that

$$u(t,x) \quad = \quad \left\{ \begin{array}{ccl} Q(t-|x|) - c_+(t-|x|) & , & \text{if } 0 \le t-|x| \\ 0 & , & \text{if } t-|x| < 0 \le t \\ 0 & , & \text{if } t \le 0 < t+|x| \\ -Q(t+|x|) + c_+(t+|x|) & , & \text{if } t+|x| \le 0 \end{array} \right\} + c_+(x+t)$$

and hence, if $k^2 = \Omega^2$, $\frac{\partial q(t)}{\partial t}\Big|_{t=0} = 0$, $\gamma t >> 1$, then

$$u(t,x) \quad \approx \quad \left\{ \begin{array}{ccl} -A\sin(t-|x|) & , & \text{if } 0 \le t-|x| \\ 0 & , & \text{if } t-|x| < 0 \le t \\ 0 & , & \text{if } t \le 0 < t+|x| \\ -A\sin(t+|x|) & , & \text{if } t+|x| \le 0 \end{array} \right\} + A\sin(x+t)$$

[7] up to assumption made just now

[8] hence, $Q(t) \approx 0$

In particular,

$$u(t,x) \quad \approx \quad \left\{ \begin{array}{ccc} 0 & , & \text{if } 0 \le t-|x|, x<0, \gamma t >> 1 \\ 2A\sin(x+t) & , & \text{if } 0 \le t-|x|, x>0, \gamma t >> 1 \end{array} \right\}$$

We see, in the case of resonance, the incident wave is essentially completely reflected by the oscillator!!!

1.5 Comments. Discussion.

In this section, we have considered two models of interaction and obtained for oscillators the corresponding equations:

$$\frac{\partial^2 q(t)}{\partial t^2} = -2\gamma\frac{\partial q(t)}{\partial t} - \Omega^2 q(t) + 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0} + \Omega^2 u_0(t,0,0) + f_0(t) + 2\gamma\int_0^t f_0(\tau)d\tau$$

$$\frac{\partial^2 q_0(t)}{\partial t^2} = -\Omega^2 q_0(t) + 2\gamma_2\gamma_1\int_0^t q_0(\tau)d\tau + \gamma_1\phi_{01}(t,0,0) + f_0(t)$$

These equations are not literally **ordinary** differential equations , **ordinary** differential equations, because of 'unordinary' terms

$$+2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0} \text{ and, resp., } 2\gamma_1\gamma_2\int_0^t q_0(\tau)d\tau$$

We can reduce these equations to **ordinary** differential equations, but the latter turn out to be **third** order equations:

$$\frac{\partial^3 q(t)}{\partial t^3} = -2\gamma\frac{\partial^2 q(t)}{\partial t^2} - \Omega^2\frac{\partial q(t)}{\partial t} + \frac{\partial}{\partial t}\Big(\Omega^2 u_0(t,0,0) + f_0(t) + 2\gamma\int_0^t f_0(\tau)d\tau\Big)$$

$$\frac{\partial^3 q_0(t)}{\partial t^3} = -\Omega^2\frac{\partial q_0(t)}{\partial t} + \gamma_1\Big(2\gamma_2 q_0(t) + \frac{\partial u_0(t,0,0)}{\partial t}\Big) + \frac{\partial f_0(t)}{\partial t}$$

We concentrate now on the system's behaviour at large t . In order to estimate the asymptotic behaviour of $q(t)$ and $q_0(t)$, as $t \to +\infty$, let us handle with the characteristic polynomials. They are:

$$\lambda^3 + 2\gamma\lambda^2 + \Omega^2\lambda \quad , \quad \lambda_0^3 + \Omega^2\lambda_0 - 2\gamma_1\gamma_2$$

The first polynomial has the roots

$$\lambda_1 = 0, \lambda_2 = -\gamma + \sqrt{\gamma^2 - \Omega^2}, \lambda_3 = -\gamma - \sqrt{\gamma^2 - \Omega^2},$$

and we see that first root is zero and two other roots have non-positive real parts. Hence, exept the unique case, the case of

$$+2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0} \neq 0\,,$$

we have exponentially decaying $q(t)$ as $t \to +\infty$. Whatever the case, $q(t)$ is bounded as $t \to +\infty$. Recall, if f_0 and u_0 both are identically zero, then

$$q(t) \to 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0} \text{ as } t \to +\infty\,.$$

However, the limit function

$$q_\infty(t) = 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}$$

is NO solution to

$$\frac{\partial^2 q(t)}{\partial t^2} = -2\gamma\frac{\partial q(t)}{\partial t} - \Omega^2 q(t) + 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}$$

every time that

$$2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0} \neq 0\,.$$

As for the second polynomial, and the second equation, the situation is more dramatic, much more. The polynomial

$$\lambda_0^3 + \Omega^2\lambda_0 - 2\gamma_1\gamma_2$$

has one pure real root λ_{01} and two complex-conjugated ones: $\lambda_{02}\,, \lambda_{03}\,, \lambda_{03} = \overline{\lambda_{02}}$. Since

$$\lambda_{01} + \lambda_{02} + \lambda_{03} = 0\,,\ \lambda_{01}\lambda_{02}\lambda_{03} = 2\gamma_1\gamma_2$$

we have

$$\lambda_{01} + 2Re\lambda_{02} = 0\,,\ \lambda_{01}|\lambda_{02}|^2 = 2\gamma_1\gamma_2$$

and hence

$$\lambda_{01} > 0\ \ (!!!)\ \ , Re\lambda_{02} = Re\lambda_{03} < 0\,.$$

Thus, we observe an EXPONENTIAL GROWTH of the oscillator amplitude, as $t \to +\infty$, the fact used to perplexing physicist's mind. We cannot here hope we have simply confused the 'time direction'. If we had, we would have **two** roots with strictly positive real parts! We defer the more detailed discussion on this subject and notice only, that a related phenomenon is known in electrodynimics, see Abraham-Lorentz-Dirac equations.

2 A little more general Model with Finite Rank Interaction. The Case of non-local Interaction and arbitrary Interaction Forces.

We will now slightly change the model. We will not suppose that δ shall be Dirac's δ-function [9]and B be $\frac{\partial^2}{\partial t^2}$ on the whole line. At the beginning of this section, we will consider an abstract d'Álambert-like equation and recall the associated d'Alembert-like solution formulae. Then we introduce an abstract analogue of the interaction discussed in the previous section. Then we discuss some partial cases, imitating the reasons of the same section, and finally indicate how the abstract formulae correlate with the formulae obtained in the previous section.

Now then. When solving the equation

$$\frac{\partial^2 u}{\partial t^2} = Bu + f$$

denote

$$u_1 := \frac{\partial u}{\partial t}.$$

Write the equation to be solved as

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ u_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ Bu + f \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ B & 0 \end{pmatrix}\begin{pmatrix} u \\ u_1 \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

Let

$$V_{t,s}$$

denote the propagator for

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ B & 0 \end{pmatrix}\begin{pmatrix} v \\ v_1 \end{pmatrix}$$

Then

$$\begin{pmatrix} u(t) \\ u_1(t) \end{pmatrix} = V_{t,s}\begin{pmatrix} u(s) \\ u_1(s) \end{pmatrix} + \int_s^t V_{t,\tau}\begin{pmatrix} 0 \\ f(\tau) \end{pmatrix} d\tau$$

Apply usual matrix -form representation

$$V_{t,s} = \begin{pmatrix} V_{11}(t,s) & V_{12}(t,s) \\ V_{21}(t,s) & V_{22}(t,s) \end{pmatrix}$$

Then obtain

$$u(t) = V_{11}(t,s)u(s) + V_{12}(t,s)\Big(\frac{\partial u(t)}{\partial t}\Big|_{t=s}\Big) + \int_s^t V_{12}(t,\tau)f(\tau)d\tau$$

$$u_0(t) := V_{11}(t,s)u(s) + V_{12}(t,s)\Big(\frac{\partial u(t)}{\partial t}\Big|_{t=s}\Big)$$

$$u(t) = u_0(t,s) + \int_s^t V_{12}(t,\tau)f(\tau)d\tau$$

I will now take

$$\begin{aligned} f(t) &= -\delta_{\alpha,t,x_0}U_0(t,Q) + f_1(t) \\ Q(t) &= < l|u(t) > \end{aligned}$$

[9]In order to no confusion can ocurr we replace the symbol δ by δ_α

where:

1) $U_0(t,Q)$ is such that its values must be **numbers**, and

2) l is a **linear** functional.

In addition I will take

$$M\frac{\partial^2 q}{\partial t^2} \quad = \quad F_{pf}(q,Q,t)$$

In the previous section δ_{α,t,x_0} was a function of x and it did not depend on t

$$\delta_{\alpha,t}(x) = \delta(x - x_0)\,;$$

As for l, it was such that for any function F of x one had set

$$< l|F >= F(x_0)$$

In addition we had there taken

$$U_0(t,Q) = 4\gamma_c\Big(Q(t) - q(t)\Big)\,, \gamma_c = \gamma c\,,$$

$$F_{pf}(q,Q,t)/M = -\Omega^2\Big(q(t) - Q(t)\Big) + f_0(t)\,.$$

Now we are mimicking, imitating arguments of the previous section:

$$u(t) = u_0(t) + \int_s^t V_{12}(t,\tau)f(\tau)d\tau$$

$$u(t) = u_0(t) + \int_s^t V_{12}(t,\tau)\Big(-\delta_{\alpha,\tau,x_0}U_0(\tau,Q) + f_1(\tau)\Big)d\tau$$

$$u(t) = u_0(t) + \int_s^t V_{12}(t,\tau)f_1(\tau)d\tau - \int_s^t V_{12}(t,\tau)\Big(\delta_{\alpha,\tau,x_0}U_0(\tau,Q)\Big)d\tau$$

$$u_{01}(t) := u_0(t) + \int_s^t V_{12}(t,\tau)f_1(\tau)d\tau$$

$$u(t) = u_{01}(t) - \int_s^t V_{12}(t,\tau)\Big(\delta_{\alpha,\tau,x_0}U_0(\tau,Q)\Big)d\tau$$

$$< l|u(t) >=< l|u_{01}(t) > - < l|\int_s^t V_{12}(t,\tau)\Big(\delta_{\alpha,\tau,x_0}U_0(\tau,Q)\Big)d\tau >$$

We will suppose

$$< l|\int_s^t V_{12}(t,\tau)\Big(\delta_{\alpha,\tau,x_0}U_0(\tau,Q)\Big)d\tau >= \int_s^t < l|V_{12}(t,\tau)\Big(\delta_{\alpha,\tau,x_0}U_0(\tau,Q)\Big) > d\tau$$

Hence

$$< l|u(t) >=< l|u_{01}(t) > - \int_s^t < l|V_{12}(t,\tau)\delta_{\alpha,\tau,x_0} > U_0(\tau,Q)d\tau$$

Recall

$$Q(t) :=< l|u(t) >$$

and put

$$K_0(t,\tau) :=< l|V_{12}(t,\tau)\delta_{\alpha,\tau} >\,, \qquad Q_0(t) :=< l|u_0(t) >\,,$$

$$Q_{01}(t) := < l|u_{01}(t) > = < l|u_0(t) > + \int_s^t < l|V_{12}(t,\tau)f_1(\tau) > d\tau$$

Then obtain

$$Q(t) = Q_0(t) - \int_s^t K_0(t,\tau)U_0\Big(\tau, Q\Big)d\tau$$

Let us now define an operator $\hat{K}_0$:

$$(\hat{K}_0 h)(t) := \int_s^t K_0(t,\tau)h(\tau)d\tau$$

Then the recent relation becomes as following:

$$Q = Q_{01} - \hat{K}_0\Big(U_0\Big(\cdot, Q\Big)\Big)$$

If we take into account that

$$M\frac{\partial^2 q}{\partial t^2} = F_{pf}(q, Q, t)$$

then the equations of motion become

$$\begin{aligned} Q &= Q_{01} - \hat{K}_0\Big(U_0\Big(\cdot, Q\Big)\Big) \\ M\frac{\partial^2 q}{\partial t^2} &= F_{pf}(q, Q, t) \end{aligned}$$

The character of this relation array is too general. Little can be said about its properties without any specifying. So, let us restrict ourselves: First, we will assume

$$\begin{aligned} U_0(t, Q) &= 4\gamma_c\Big(Q(t) - q(t)\Big) \\ F_{pf}(q, Q, t) &= -K(q - Q) + F_0(t) = M\Big(-\Omega^2(q - Q) + f_0(t)\Big) \end{aligned}$$

and denote

$$\ddot{q} := \frac{\partial^2 q}{\partial t^2}$$

Then the equations of motion become

$$\begin{aligned} Q &= Q_{01} - 4\gamma_c\hat{K}_0\Big(Q - q\Big) \\ \ddot{q} &= -\Omega^2(q - Q) + f_0 \end{aligned}$$

We infer:

1st step,

$$\begin{aligned} Q + 4\gamma_c\hat{K}_0 Q &= Q_{01} + 4\gamma_c\hat{K}_0 q \\ \ddot{q} &= -\Omega^2(q - Q) + f_0 \end{aligned}$$

2nd step,

$$\begin{aligned} \Big(1 + 4\gamma_c\hat{K}_0\Big)Q &= Q_{01} + 4\gamma_c\hat{K}_0 q \\ \ddot{q} &= -\Omega^2 q + \Omega^2 Q + f_0 \end{aligned}$$

3rd step,

$$
\begin{aligned}
\Big(1 + 4\gamma_c \hat{K}_0\Big) Q &= Q_{01} + 4\gamma_c \hat{K}_0 q \\
\Big(1 + 4\gamma_c \hat{K}_0\Big) \ddot{q} &= \Big(1 + 4\gamma_c \hat{K}_0\Big)\Big(- \Omega^2 q + \Omega^2 Q + f_0\Big)
\end{aligned}
$$

4th step,

$$
\begin{aligned}
\Big(1 + 4\gamma_c \hat{K}_0\Big) Q &= Q_{01} + 4\gamma_c \hat{K}_0 q \\
\Big(1 + 4\gamma_c \hat{K}_0\Big) \ddot{q} &= -\Big(1 + 4\gamma_c \hat{K}_0\Big)\Omega^2 q + \Big(1 + 4\gamma_c \hat{K}_0\Big)\Omega^2 Q + \Big(1 + 4\gamma_c \hat{K}_0\Big) f_0 \\
&= -\Omega^2\Big(1 + 4\gamma_c \hat{K}_0\Big) q + \Omega^2\Big(1 + 4\gamma_c \hat{K}_0\Big) Q + \Big(1 + 4\gamma_c \hat{K}_0\Big) f_0 \\
& \qquad \left\{ \begin{array}{r} \text{replace here } \Big(1 + 4\gamma_c \hat{K}_0\Big) Q \\ \text{by } \Big(Q_{01} + 4\gamma_c \hat{K}_0 q\Big) \end{array} \right\} \\
&= -\Omega^2\Big(1 + 4\gamma_c \hat{K}_0\Big) q + \Omega^2\Big(Q_{01} + 4\gamma_c \hat{K}_0 q\Big) + \Big(1 + 4\gamma_c \hat{K}_0\Big) f_0 \\
&= -\Omega^2 q + \Omega^2 Q_{01} + \Big(1 + 4\gamma_c \hat{K}_0\Big) f_0
\end{aligned}
$$

We have now seen:

$$
\Big(1 + 4\gamma_c \hat{K}_0\Big) \ddot{q} = -\Omega^2 q + \Omega^2 Q_{01} + \Big(1 + 4\gamma_c \hat{K}_0\Big) f_0
$$

It means

$$
\begin{aligned}
& \frac{\partial^2 q(\tau)}{\partial \tau^2} + 4\gamma_c \int_s^t K_0(t,\tau) \frac{\partial^2 q(\tau)}{\partial \tau^2} d\tau \\
& \quad = \quad -\Omega^2 q(t) + \Omega^2 Q_{01}(t) + f_0(t) + 4\gamma_c \int_s^t K_0(t,\tau)\Big(f_0(\tau)\Big) d\tau
\end{aligned}
$$

We rewrite this equation as following:

$$
\frac{\partial^2 q}{\partial t^2} = -4\gamma_c \int_s^t K_0(t,\tau) \frac{\partial^2 q(\tau)}{\partial \tau^2} d\tau - \Omega^2 q + \Omega^2 Q_{01}(t) + f_0(t) + 4\gamma_c \int_s^t K_0(t,\tau) f_0(\tau) d\tau
$$

and then, replacing $Q_{01}(t)$ by $< l|u_{01}(t) >$, as

$$
\begin{aligned}
\frac{\partial^2 q}{\partial t^2} = -4\gamma_c \int_s^t K_0(t,\tau) \frac{\partial^2 q(\tau)}{\partial \tau^2} d\tau - \Omega^2 q & \\
+ \Omega^2 < l|u_{01}(t) > + f_0(t) + 4\gamma_c \int_s^t K_0(t,\tau) f_0(\tau) d\tau &
\end{aligned}
$$

Thus, we have obtained an **insulated** equation for q, and the equation resembles ones in the previous section. To continue, we must specify the nature of the abstract terms, of $u(t)$.

If they, $u(t)$, all are elements of an functional space, say simple, if $u(t)$ are functions of a 'spacial' variable x, our constructions and formulae appear as follows:

$$
u(t,x) = u_0(t,s,x) + \int_s^t \Big(V_{12}(t,\tau) f(\tau)\Big)(x) d\tau
$$

$$
u(t,x) = u_0(t,s,x) + \int_s^t \int_X V_{12}(t,\tau,x,\xi) f(\tau,\xi) d\xi d\tau
$$

In this case we have

$$f(t,x) := -\delta_{\alpha,t,x_0}(x)U_0(t,Q) + f_1(t,x)$$

(in the previous section, $\delta_{\alpha,t,x_0}(x,x_0) = \delta(x-x_0)$) Then

$$u(t,x) = u_0(t,s,x) - \int_s^t \int_X V_{12}(t,\tau,x,\xi)\Big(\delta_{\alpha,\tau,x_0}(\xi)U_0(\tau,Q) - f_1(\tau,\xi)\Big)d\xi d\tau$$

Hence

$$\begin{aligned}
< l|u(t) > &= < l|u(t,\cdot) > \\
&= < l|u_0(t,s,\cdot) - \int_s^t \int_X V_{12}(t,\tau,\cdot,\xi)\Big(\delta_{\alpha,\tau,x_0}(\xi)U_0(\tau,Q) - f_1(\tau,\xi)\Big)d\xi d\tau > \\
&= < l|u_0(t,s,\cdot) > - < l|\int_s^t \int_X V_{12}(t,\tau,\cdot,\xi)\delta_{\alpha,\tau,x_0}(\xi)U_0(\tau,Q)d\xi d\tau > \\
&\quad + < l|\int_s^t \int_X V_{12}(t,\tau,\cdot,\xi)f_1(\tau,\xi)d\xi d\tau > \\
&= < l|u_0(t,s,\cdot) > - \int_s^t < l|\int_X V_{12}(t,\tau,\cdot,\xi)\delta_{\alpha,\tau,x_0}(\xi)d\xi > U_0(\tau,Q)d\tau \\
&\quad + < l|\int_s^t \int_X V_{12}(t,\tau,\cdot,\xi)f_1(\tau,\xi)d\xi d\tau >
\end{aligned}$$

Hence

$$\begin{aligned}
< l|u(t) > &= < l|u(t,\cdot) > \\
&= < l|u_0(t,s,\cdot) > - \int_s^t < l|\int_X V_{12}(t,\tau,\cdot,\xi)\delta_{\alpha,\tau,x_0}(\xi)d\xi > U_0(\tau,Q)d\tau
\end{aligned}$$

i.e.

$$Q(t) = Q_{01}(t) - \int_s^t < l|\int_X V_{12}(t,\tau,\cdot,\xi)\delta_{\alpha,\tau,x_0}(\xi)d\xi > U_0(\tau,Q)d\tau$$

$$Q(t) = Q_{01}(t) - \int_s^t K_0(t,\tau)U_0(\tau,Q)d\tau$$

Of course

$$K_0(t,\tau) := < l|\int_X V_{12}(t,\tau,\cdot,\xi)\delta_{\alpha,\tau,x_0}(\xi)d\xi >$$

$$Q_{01} = < l|u_0(t,s,\cdot) > + < l|\int_s^t \int_X V_{12}(t,\tau,\cdot,\xi)f_1(\tau,\xi)d\xi d\tau >$$

If we presume B, l to be the same as before, in the previous section, i.e., if B is $c^2\frac{\partial^2}{\partial x^2}$ on the whole line, and $< l|F >= F(x_0)$, then

$$\begin{aligned}
&\int_s^t \int_X V_{12}(t,\tau,x,\xi)f(\tau,\xi)d\xi d\tau \\
&\qquad = \frac{1}{2}\int_s^t \Big(\tilde{f}(\tau,x+c(t-\tau)) - \tilde{f}(\tau,x-c(t-\tau))\Big)d\tau
\end{aligned}$$

$$\begin{aligned}
&\int_s^t \int_X V_{12}(t,\tau,x,\xi)\delta_{\alpha,\tau,x_0}(\xi)U_0(\tau,Q)d\xi d\tau \\
&\quad = \frac{1}{2c}\int_s^t \Big(\tilde{\delta}_{\alpha,\tau,x_0}(x+c(t-\tau)) - \tilde{\delta}_{\alpha,\tau,x_0}(x-c(t-\tau))\Big)U_0(\tau,Q)d\tau
\end{aligned}$$

Thus we obtain

$$K_0(t,\tau) = \frac{1}{2c}\Big(\tilde{\delta}_{\alpha,\tau,x_0}(x_0 + c(t-\tau)) - \tilde{\delta}_{\alpha,\tau,x_0}(x_0 - c(t-\tau))\Big)$$

Here $\tilde{\delta}_{\alpha,\tau,x_0}$ stands for any primitive of δ_{α,τ,x_0}, i.e.,

$$\frac{\partial\tilde{\delta}_{\alpha,\tau,x_0}(x)}{\partial x} = \delta_{\alpha,\tau,x_0}(x)$$

Note once again that

$$\tilde{\delta}_{\alpha,\tau,x_0}(x + c(t-\tau)) - \tilde{\delta}_{\alpha,\tau,x_0}(x - c(t-\tau))$$

does not depend on whatever primitive of δ_{α,τ,x_0} which one has chosen!!!

Suppose, $\delta_{\alpha,t,x_0}(x)$ is of the form

$$\delta_{\alpha,t,x_0}(x) = \delta_\alpha(x - x_0)$$

Then we obtain

$$\begin{aligned}
K_0(t,\tau) &= \frac{1}{2c}\Big(\tilde{\delta}_{\alpha,\tau,x_0}(x_0 + c(t-\tau)) - \tilde{\delta}_{\alpha,\tau,x_0}(x_0 - c(t-\tau))\Big) \\
&= \frac{1}{2c}\Big(\tilde{\delta}_\alpha(c(t-\tau)) - \tilde{\delta}_\alpha(c(\tau-t))\Big)
\end{aligned}$$

In this case, we have finally seen,

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -2\frac{\gamma_c}{c}\int_s^t \Big(\tilde{\delta}_\alpha(c(t-\tau)) - \tilde{\delta}_\alpha(c(\tau-t))\Big)\frac{\partial^2 q(\tau)}{\partial\tau^2}d\tau - \Omega^2 q + \Omega^2 Q_{01}(t) \\
&\qquad + f_0(t) + 2\frac{\gamma_c}{c}\int_s^t \Big(\tilde{\delta}_\alpha(c(t-\tau)) - \tilde{\delta}_\alpha(c(\tau-t))\Big) f_0(\tau)d\tau \\
&= -2\gamma\int_s^t \Big(\tilde{\delta}_\alpha(c(t-\tau)) - \tilde{\delta}_\alpha(c(\tau-t))\Big)\frac{\partial^2 q(\tau)}{\partial\tau^2}d\tau - \Omega^2 q + \Omega^2 Q_{01}(t) \\
&\qquad + f_0(t) + 2\gamma\int_s^t \Big(\tilde{\delta}_\alpha(c(t-\tau)) - \tilde{\delta}_\alpha(c(\tau-t))\Big) f_0(\tau)d\tau
\end{aligned}$$

In any case,

$$\frac{\partial^2 q(t)}{\partial t^2} - f_0(t) + 2\frac{\gamma_c}{c}\int_s^t 2K_0(t,\tau)\Big(\frac{\partial^2 q(\tau)}{\partial\tau^2} - f_0(\tau)\Big)d\tau = -\Omega^2\Big(q(t) - Q_{01}(t)\Big)$$

or, more generally,

$$\begin{aligned}
Q(t) &= Q_{01}(t) - \int_s^t K_0(t,\tau)U_0(\tau,Q)d\tau\,, \\
M\frac{\partial^2 q(t)}{\partial t^2} &= F_{pf}(q,Q,t)\,.
\end{aligned}$$

3 APPENDIX

3.1 APPENDIX A. Abstract Linear Response Formula

We recall some abstract linear response formulae.

First, we recall linear response formula for the **first** order differential equation. We indicate only formula itself, minimum of details. We say only, a deriving of this formula is based on an abstract modification of the method of **variation of constant**. As for detailed description, how one produces such a formula, as well as for proofs, background and all that, see suitable standard manuals.

Let $\{A(t)\}_t$ stand for a family of linear operators, so that the **homogeneous** linear equation

$$\frac{\partial v(t)}{\partial t} = A(t)v(t), \quad v(t)\Big|_{t=s} = v_{initial}, \qquad (t \geq s)$$

has a unique solution $v(t)$, whatever 'initial time' s and initial data $v_{initial}$ may be.

Let

$$V_{t,s}$$

denote the propagator, alias evolution operator, to the former system, i.e.,

$$v(t) = V_{t,s}v_{initial} = V_{t,s}v(s).$$

One can show that

$$V_{t,r}V_{r,s} = V_{t,s}\ , \ \text{if } t \geq r \geq s, \ (\text{ consistency relation })$$

$$V_{t,t} = V_{r,r} = V_{s,s} = I\ .$$

The linear response formula reads: The formula

$$\boxed{u(t) = V_{t,s}u_{initial} + \int_s^t V_{t,\tau}f(\tau)d\tau}$$

gives a solution to the **inhomogeneous** equation

$$\boxed{\frac{\partial u(t)}{\partial t} = A(t)u + f(t), \quad u(t)\Big|_{t=s} \equiv u(s) = u_{initial}.}$$

We will write the response formula as following:

$$\boxed{u(t) = V_{t,s}u(s) + \int_s^t V_{t,\tau}f(\tau)d\tau.}$$

One can now produce the response formula for the **second** order differential equations. As befor, the response formula accommodates the formula, which describes solution to a **homogeneous** equation, to the case, where one need to solve the associated **inhomogeneous** one.

Now then. When solving the equation

$$\frac{\partial^2 u(t)}{\partial t^2} = K(t)\frac{\partial u(t)}{\partial t} + B(t)u(t) + f(t)$$

denote

$$u_1(t) := \frac{\partial u(t)}{\partial t}\,.$$

Write the equation to be solved as

$$\begin{aligned}\frac{\partial}{\partial t}\begin{pmatrix} u(t) \\ u_1(t) \end{pmatrix} &= \begin{pmatrix} u_1(t) \\ K(t)u_(t) + B(t)u(t) + f(t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ B(t) & K(t) \end{pmatrix}\begin{pmatrix} u(t) \\ u_1(t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}\end{aligned}$$

Let

$$V_{t,s}$$

denote the propagator to

$$\frac{\partial}{\partial t}\begin{pmatrix} v(t) \\ v_1(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ B(t) & K(t) \end{pmatrix}\begin{pmatrix} v(t) \\ v_1(t) \end{pmatrix}$$

Then

$$\begin{pmatrix} u(t) \\ u_1(t) \end{pmatrix} = V_{t,s}\begin{pmatrix} u(s) \\ u_1(s) \end{pmatrix} + \int_s^t V_{t,\tau}\begin{pmatrix} 0 \\ f(\tau) \end{pmatrix} d\tau$$

Apply usual matrix-form representation

$$V_{t,s} = \begin{pmatrix} V_{11}(t,s) & V_{12}(t,s) \\ V_{21}(t,s) & V_{22}(t,s) \end{pmatrix}$$

Then obtain finally

$$\boxed{u(t) \quad = \quad V_{11}(t,s)u(s) + V_{12}(t,s)\Big(\frac{\partial u(t)}{\partial t}\Big|_{t=s}\Big) + \int_s^t V_{12}(t,\tau)f(\tau)d\tau}$$

Notice,

$$\begin{pmatrix} V_{11}(t,s) & V_{12}(t,s) \\ V_{21}(t,s) & V_{22}(t,s) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

and the formula

$$v(t) = V_{11}(t,s)v(s) + V_{12}(t,s)\Big(\frac{\partial v(t)}{\partial t}\Big|_{t=s}\Big)$$

is a formula which describes solutions to the homogeneous equation. Sometimes, it is instructive to notice that

$$\frac{\partial}{\partial t}v(t) = \frac{\partial}{\partial t}V_{11}(t,s)v(s) + \frac{\partial}{\partial t}V_{12}(t,s)\Big(\frac{\partial v(t)}{\partial t}\Big|_{t=s}\Big)$$

and hence

$$V_{21}(t,s) = \frac{\partial}{\partial t}V_{11}(t,s)\,, \quad V_{22}(t,s) = \frac{\partial}{\partial t}V_{12}(t,s)\,.$$

3.2 APPENDIX B.

Response Formula for $\ddot{y}(t)+2\gamma\dot{y}(t)+\Omega^2y(t)=f(t)$

If one applies the response formula to the equation in the title, then one finds:

$$\begin{aligned} y(t) &= e^{-\gamma(t-s)}\Big(\cos\Omega_\gamma(t-s)+\gamma\frac{\sin\Omega_\gamma(t-s)}{\Omega_\gamma}\Big)y(s) \\ &\quad + e^{-\gamma(t-s)}\frac{\sin\Omega_\gamma(t-s)}{\Omega_\gamma}\Big(\frac{\partial y(t)}{\partial t}\Big|_{t=s}\Big) \\ &\quad + \int_s^t e^{-\gamma(t-\tau)}\frac{\sin\Omega_\gamma(t-\tau)}{\Omega_\gamma}f(\tau)d\tau \end{aligned}$$

where $\Omega_\gamma^2=\Omega^2-\gamma^2$ [10].

Actually, the solution to the joined homogeneous equation

$$\ddot{x}(t)+2\gamma\dot{x}(t)+\Omega^2x(t)=0$$

is this:

$$\begin{aligned} x(t) &= e^{-\gamma t}\Big(\cos\Omega_\gamma t+\gamma\frac{\sin\Omega_\gamma t}{\Omega_\gamma}\Big)x(0)+e^{-\gamma t}\frac{\sin\Omega_\gamma t}{\Omega_\gamma}\dot{x}(0) \\ \dot{x}(t) &= e^{-\gamma t}\Big(-{\Omega_\gamma}^2-\gamma^2\Big)\Big(\frac{\sin\Omega_\gamma t}{\Omega_\gamma}\Big)x(0)+e^{-\gamma t}\Big(\cos\Omega_\gamma t-\gamma\frac{\sin\Omega_\gamma t}{\Omega_\gamma}\Big)\dot{x}(0) \end{aligned}$$

Since $\Omega_\gamma^2=\Omega^2-\gamma^2$, we can besides write

$$\begin{aligned} x(t) &= e^{-\gamma t}\Big(\cos\Omega_\gamma t+\gamma\frac{\sin\Omega_\gamma t}{\Omega_\gamma}\Big)x(0)+e^{-\gamma t}\frac{\sin\Omega_\gamma t}{\Omega_\gamma}\dot{x}(0) \\ \dot{x}(t) &= e^{-\gamma t}\Big(-\Omega^2\Big)\Big(\frac{\sin\Omega_\gamma t}{\Omega_\gamma}\Big)x(0)+e^{-\gamma t}\Big(\cos\Omega_\gamma t-\gamma\frac{\sin\Omega_\gamma t}{\Omega_\gamma}\Big)\dot{x}(0) \end{aligned}$$

The coefficients of the latter equation for $x(t)$ are constant. Hence

$$\begin{aligned} x(t) &= e^{-\gamma(t-s)}\Big(\cos\Omega_\gamma(t-s)+\gamma\frac{\sin\Omega_\gamma(t-s)}{\Omega_\gamma}\Big)x(s) \\ &\qquad +e^{-\gamma(t-s)}\frac{\sin\Omega_\gamma(t-s)}{\Omega_\gamma}\dot{x}(s) \\ \dot{x}(t) &= e^{-\gamma(t-s)}\Big(-\Omega^2\Big)\frac{\sin\Omega_\gamma(t-s)}{\Omega_\gamma}x(s) \\ &\qquad +e^{-\gamma(t-s)}\Big(\cos\Omega_\gamma(t-s)-\gamma\frac{\sin\Omega_\gamma(t-s)}{\Omega_\gamma}\Big)\dot{x}(s) \end{aligned}$$

Thus we have seen, that the formula for the propagator $V_{t,s}$, which associates with the homogeneous equation $\ddot{x}(t)+2\gamma\dot{x}(t)+\Omega^2x(t)=0$, is this:

$$\begin{aligned} V_{t,s} &= \begin{pmatrix} V_{11}(t,s) & V_{12}(t,s) \\ V_{21}(t,s) & V_{22}(t,s) \end{pmatrix} \\ &= \begin{pmatrix} e^{-\gamma(t-s)}\Big(\cos\Omega_\gamma(t-s)+\gamma\frac{\sin\Omega_\gamma(t-s)}{\Omega_\gamma}\Big) & e^{-\gamma(t-s)}\frac{\sin\Omega_\gamma(t-s)}{\Omega_\gamma} \\ e^{-\gamma(t-s)}\Big(-\Omega^2\Big)\frac{\sin\Omega_\gamma(t-s)}{\Omega_\gamma} & e^{-\gamma(t-s)}\Big(\cos\Omega_\gamma(t-s)-\gamma\frac{\sin\Omega_\gamma(t-s)}{\Omega_\gamma}\Big) \end{pmatrix} \end{aligned}$$

[10] if $\Omega_\gamma=0$, then read $\frac{\sin\Omega_\gamma t}{\Omega_\gamma}=t$, of course

Repeat, finally, the abstract response formula

$$\boxed{u(t) \quad = \quad V_{11}(t,s)u(s) + V_{12}(t,s)\Big(\frac{\partial u(t)}{\partial t}\Big|_{t=s}\Big) + \int_s^t V_{12}(t,\tau)f(\tau)d\tau}$$

and replace there the proper terms according to

$$V_{11}(t,s) = e^{-\gamma(t-s)}\Big(\cos\Omega_\gamma(t-s) + \gamma\frac{\sin\Omega_\gamma(t-s)}{\Omega_\gamma}\Big)$$

$$V_{12}(t,s) = e^{-\gamma(t-s)}\frac{\sin\Omega_\gamma(t-s)}{\Omega_\gamma}$$

3.3 APPENDIX C.

Particular Solution to

$\ddot{y}(t) + 2\gamma\dot{y}(t) + \Omega^2 y(t) = A_s \sin(kt+\phi) + A_c \cos(kt+\phi)$

We have

$$
\begin{aligned}
\Big(\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \Omega^2\Big)\Big(a_s \sin(kt+\phi) + a_c \cos(kt+\phi)\Big) & \\
= \quad & \Big(-k^2+\Omega^2\Big) a_s \sin(kt+\phi) + 2\gamma k a_c \cos(kt+\phi) \\
& - 2\gamma k a_s \sin(kt+\phi) + \Big(-k^2+\Omega^2\Big) a_c \cos(kt+\phi)
\end{aligned}
$$

In order to
$$\begin{pmatrix} -k^2+\Omega^2 & 2\gamma k \\ -2\gamma k & -k^2+\Omega^2 \end{pmatrix} \begin{pmatrix} a_s \\ a_c \end{pmatrix} = \begin{pmatrix} A_s \\ A_c \end{pmatrix}$$
it is sufficient that
$$\Big((-k^2+\Omega^2)^2 + (2\gamma k)^2\Big) \begin{pmatrix} a_s \\ a_c \end{pmatrix} = \begin{pmatrix} -k^2+\Omega^2 & -2\gamma k \\ 2\gamma k & -k^2+\Omega^2 \end{pmatrix} \begin{pmatrix} A_s \\ A_c \end{pmatrix}$$
Thus, if
$$(-k^2+\Omega^2)^2 + (2\gamma k)^2 \neq 0$$
then
$$\begin{pmatrix} a_s \\ a_c \end{pmatrix} = \frac{1}{(-k^2+\Omega^2)^2 + (2\gamma k)^2} \begin{pmatrix} -k^2+\Omega^2 & -2\gamma k \\ 2\gamma k & -k^2+\Omega^2 \end{pmatrix} \begin{pmatrix} A_s \\ A_c \end{pmatrix}$$

3.4 APPENDIX D. Helpful formulae

Derivatives. 1)

$$\begin{aligned}\frac{\partial e^{-\gamma t}\Big(A\cos(\omega t+\varphi)+B\sin(\omega t+\varphi)\Big)}{\partial t} \\ &= -\gamma e^{-\gamma t}\Big(A\cos(\omega t+\varphi)+B\sin(\omega t+\varphi)\Big) \\ &\quad + e^{-\gamma t}\Big(-\omega A\sin(\omega t+\varphi)+\omega B\cos(\omega t+\varphi)\Big) \\ &= e^{-\gamma t}\Big((-\gamma A+\omega B)\cos(\omega t+\varphi)+(-\omega A-\gamma B)\sin(\omega t+\varphi)\Big)\end{aligned}$$

2)

$$\frac{\partial e^{-\gamma t}\Big(-\gamma\cos(\omega t+\varphi)+\omega\sin(\omega t+\varphi)\Big)}{\partial t} = e^{-\gamma t}(\gamma^2+\omega^2)\cos(\omega t+\varphi)$$

3)

$$\frac{\partial e^{-\gamma t}\Big(\omega\cos(\omega t+\varphi)+\gamma\sin(\omega t+\varphi)\Big)}{\partial t} = -e^{-\gamma t}(\gamma^2+\omega^2)\sin(\omega t+\varphi)$$

2a)

$$\begin{aligned}&\frac{\partial e^{-\gamma(t-\tau)}\Big(-\gamma\cos(\omega(t-\tau)+\varphi)+\omega\sin(\omega(t-\tau)+\varphi)\Big)}{\partial\tau} \\ &\qquad = -e^{-\gamma(t-\tau)}(\gamma^2+\omega^2)\cos(\omega(t-\tau)+\varphi)\end{aligned}$$

3a)

$$\begin{aligned}&\frac{\partial e^{-\gamma t}\Big(\omega\cos(\omega(t-\tau)+\varphi)+\gamma\sin(\omega(t-\tau)+\varphi)\Big)}{\partial\tau} \\ &\qquad = e^{-\gamma(t-\tau)}(\gamma^2+\omega^2)\sin(\omega(t-\tau)+\varphi)\end{aligned}$$

Other.

$$2\sin\alpha\sin\beta = cos(\alpha-\beta) - cos(\alpha+\beta)$$

$$\begin{aligned}\omega(t-\tau)-\omega_s(\tau-t_0) &= (\omega+\omega_s)(t-\tau)-\omega_s(t-\tau)-\omega_s(\tau-t_0) \\ &= (\omega+\omega_s)(t-\tau)-\omega_s(t-t_0)\end{aligned}$$

$$\begin{aligned}\omega(t-\tau)+\omega_s(\tau-t_0) &= (\omega-\omega_s)(t-\tau)+\omega_s(t-\tau)+\omega_s(\tau-t_0) \\ &= (\omega-\omega_s)(t-\tau)+\omega_s(t-t_0)\end{aligned}$$

$$\begin{aligned}&2\sin\omega(t-\tau)\sin\omega_s(\tau-t_0) \\ &\quad = \cos\Big((\omega+\omega_s)(t-\tau)-\omega_s(t-t_0)\Big) - \cos\Big((\omega-\omega_s)(t-\tau)+\omega_s(t-t_0)\Big)\end{aligned}$$

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