GLOBAL CONTINUITY OF THE INTEGRATED DENSITY OF STATES FOR RANDOM LANDAU HAMILTONIANS

DEDICATED TO LEONID PASTUR ON THE OCCASION OF HIS 65^{TH} BIRTHDAY

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Abstract. We prove that the integrated density of states (IDS) for the randomly perturbed Landau Hamiltonian is Hölder continuous at all energies with any Hölder exponent $0 < q < 1$. The random Anderson-type potential is constructed with a nonnegative, compactly supported single-site potential u. The distribution of the *iid* random variables must be absolutely continuous with a bounded, compactly supported density. This extends a previous result [4] that was restricted to magnetic fields having rational flux through the unit square. Furthermore, we prove that the IDS is Hölder continuous as a function of the nonzero magnetic field strength, and that the density of states is in $L^p_{loc}(\mathbb{R})$, for any $1 \leq p < \infty$.

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1. Introduction

In this note, we prove a new result on the Hölder continuity of the integrated density of states (IDS) for Landau Hamiltonians with random, Anderson-type potentials. The unperturbed Landau Hamiltonian $H_L(B)$ on $L^2(\mathbb{R}^2)$ has the form

$$H_L(B) = (-i\nabla - A)^2, \text{ where } A(x_1, x_2) = \frac{B}{2}(-x_2, x_1), \tag{1.1}$$

where $B \neq 0$ is the magnetic field strength. The spectrum of $H_L(B)$ is pure point and consists of an increasing sequence of degenerate, isolated eigenvalues $\{E_j(B) = (2j+1)|B| \mid j = 0, 1, \ldots, \}$, each of infinite multiplicity. As the spectrum is independent of the sign of B, we will always assume $B > 0$. The perturbed family of Landau Hamiltonians is given by

$$H_\omega(B) = H_L(B) + V_\omega. \tag{1.2}$$

The random potential V_ω is the Anderson-type random perturbation of the form

$$V_\omega(x) = \sum_{j \in \mathbb{Z}^2} \omega_j u(x - j). \tag{1.3}$$

The single-site potential function u, and the random coupling constants $\{\omega_j \mid j \in \mathbb{Z}^2\}$, satisfy the following two hypotheses:

(H1): The single-site potential u is a nonnegative, compactly-supported function with $u \in L^\infty(\mathbb{R}^2)$, satisfying $u > u_0 > 0$ on some nonempty open set, for some constant $u_0 > 0$.

(H2): The random coupling constants $\{\omega_j \mid j \in \mathbb{Z}^2\}$, are independent and identically distributed. The distribution has a compactly-supported density $h_0 \in L^\infty(\mathbb{R})$ with $\operatorname{supp} h_0 \subset [-m, M]$, for $0 \leq m, M < \infty$.

We note that there is no limitation in taking u normalized so that $\|u\|_\infty \leq 1$, that we assume from now on, as the results are uniform with respect to $\|V_\omega\|_\infty$.

The traditional manner to define the integrated density of states (IDS) $N_0(E; B)$ for $H_0 = H_L(B)$, and $N(E; B)$ for H_ω, is to use the local Hamiltonians associated with finite-area regions $\Lambda \subset \mathbb{R}^2$. For a regular, bounded region $\Lambda \subset \mathbb{R}^2$, we define the local Hamiltonians H_0^Λ and H_ω^Λ to be the self-adjoint operators on

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$L^2(\Lambda)$ obtained by restricting H_0 and H_ω to Λ, with self-adjoint boundary conditions. We assume that these local operators are lower bounded with a uniform, finite lower bound independent of Λ. The IDS is defined using the eigenvalue counting function for the local Hamiltonian. Let $N_0^\Lambda(E;B)$, respectively, $N_\omega^\Lambda(E;B)$, be the number of eigenvalues of H_0^Λ, respectively H_ω^Λ, less than or equal to E. The function $N_\omega^\Lambda(E;B)$ depends on the realization ω. The integrated density of states for H_ω is defined by

$$N(E;B) = \lim_{|\Lambda|\to\infty} \frac{N_\omega^\Lambda(E;B)}{|\Lambda|}, \tag{1.4}$$

when this limit exists. Similarly, we define $N_0(E;B)$ using the counting function $N_0^\Lambda(E;B)$ for H_0^Λ. Because $N_0(E;B)$ and $N(E;B)$ are monotonic functions of E, each has at most a countable number of discontinuities. We assume that each has been defined to be right continuous.

It will be convenient for us to use an equivalent formulation of the IDS as presented, for example, in [25]. Let $\Lambda \subset \mathbb{R}^2$ be a finite-area region as above. Let χ_Λ be the characteristic function for the region Λ. For any bounded function f of compact support, the operator $\chi_\Lambda f(H_\omega)$ is a trace class operator. Let $J \subset \mathbb{R}$ be a bounded Borel set. The density of states measure ν is defined by

$$\nu(J) \equiv \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \, Tr(\chi_\Lambda E_\omega(J)\chi_\Lambda), \tag{1.5}$$

when this limit exists. The IDS is given by $N(E;B) = \nu((-\infty, E])$, when E is a point of continuity of N.

The IDS for both H_0 and H_ω exist under the hypotheses given above, and we refer to [10, 16, 24] for proofs of these facts (we mention [2, 19, 25] for discussions of the IDS without magnetic fields). Because of $\mathbb{Z}^2$-ergodicity with respect to the magnetic translation group (see section 2), the IDS $N(E;B)$ is almost surely independent of the realization ω. The IDS $N_0(E;B)$ for the unperturbed Landau Hamiltonian $H_L(B)$ (1.1) is a piecewise constant, monotone increasing function (cf. the example in [24]) given by

$$N_0(E;B) = \frac{B}{2\pi} \begin{cases} 0 & \text{if } E < B \\ \ell & \text{if } (2\ell-1)B \leq E < (2\ell+1)B \end{cases} \tag{1.6}$$

Consequently, it is not locally Hölder continuous at all energies. The effect of the random potential V_ω is to improve the regularity of the IDS almost surely. Let us also note that because of gauge invariance, the IDS for H_0 and H_ω are independent of the choice of the vector potential A provided $B = \partial_1 A_2 - \partial_2 A_1$. *We will often write $N_0(E)$ and $N(E)$, for $N_0(E;B)$ and $N(E;B)$, respectively, when the explicit dependence on B is not needed.*

1.1. **The Main Results on the IDS.** It is known that $N(E)$ for the random family H_ω is locally Lipschitz continuous in the following sense. Given an $N > 0$, there is a $B_N > 0$ so that for $B > B_N$, the IDS $N(E)$ is Lipschitz continuous on $(0, 2(N+1)B)\backslash\{E_j(B) \mid j = 0, 1, \ldots, N\}$ almost surely [3, 17, 32]. Under some additional conditions, Wang [33] also proved that $N(E)$ is smooth outside of a given Landau level for sufficiently large magnetic field strength (depending on the level). There has been some discussion as to the behavior of the IDS at the Landau energies $E_j(B)$, when B is not necessarily large relative to the level j. If the single-site potential u in (1.3) satisfies $u|\Lambda_1(0) > \epsilon\chi_{\Lambda_1(0)} > 0$, for some $\epsilon > 0$, then the IDS is locally Lipschitz continuous at all energies for any $B \neq 0$. More recently, some of us in [4] extended these results and proved global Hölder continuity of the IDS at all energies, provided the magnetic field strength B satisfies $B \in 2\pi\mathbb{Q}$, the *rational flux condition* for the unit square $\Lambda_1(0)$. This condition allows the problem to be treated using the method of Floquet decomposition. The case of irrational flux was not resolved in that paper because of lack of explicit control on how a constant that enters from an application of the unique continuation principle depends on B. In this paper, we compute this constant explicitly, and, by using a new method, avoid the use of the Floquet decomposition. These two advances allow us to modify the argument of [4] and prove the following general theorem that extends these known results. This more general technique will be thoroughly explored in [15].

Theorem 1.1. *Let $H_L(B)$ be the Landau Hamiltonian (1.1) with magnetic field $B \neq 0$, and assume hypotheses (H1)–(H2). Then the IDS $N(E)$ is locally uniformly Hölder continuous at all energies with any* *exponent $0 < q < 1$. That is, for any exponent $0 < q < 1$, and for any bounded interval $I \subset \mathbb{R}$, there is a finite constant $C_{I,q} > 0$, locally uniformly bounded in B for $B \neq 0$, so that for any $E, E' \in I$, we have*

$$|N(E) - N(E')| \leq C_{I,q} |E - E'|^q. \tag{1.7}$$

This result follows from a regularity property of the density of states (DOS) measure ν defined in (1.5) above.

Theorem 1.2. *Under the hypotheses (H1)–(H2), the DOS measure ν defined in (1.5) for H_ω exists and is globally absolutely continuous with respect to Lebesgue measure. Furthermore, for any exponent $0 < q < 1$, and for any bounded interval $I \subset \mathbb{R}$, there is a finite constant $C_{I,q} > 0$ so that for any Borel $J \subset I$, we have*

$$\nu(J) \leq C_{I,q} |J|^q. \tag{1.8}$$

The density of the measure, i.e. the Radon-Nikodym derivative $(d\nu/d\lambda)$, belongs to $L^p_{loc}(\mathbb{R})$, for any $1 \leq p < \infty$.

We also prove a Hölder continuity result on the IDS as a function of B. We do not believe that the Hölder exponent in the following theorem is optimal.

Theorem 1.3. *Under hypotheses (H1)–(H2), the IDS $N_B(E)$ is a Hölder continuous function of B, for $B \neq 0$. That is, for any exponent $0 < \alpha < 1/4$, and any bounded interval $J \subset \mathbb{R} \backslash \{0\}$, there is a finite constant $C_{(\alpha,J,E)} > 0$ so that for any $B, B_0 \in J$, we have*

$$|N(E;B) - N(E;B_0)| \leq C_{(\alpha,J,E)} |B - B_0|^\alpha. \tag{1.9}$$

In the proof of this theorem, we use the regularity of $N(E;B)$ as a function of E. This idea was used by B. Simon [30] who proved that the IDS for $H_L(B) + V_{per}$, where V_{per} is a $\mathbb{Z}^2$-periodic potential, is continuous in the field strength B at any energy E that is a point of continuity for the IDS. As a consequence of Theorems 1.1 and 1.3, we have the following corollary on joint continuity in E and B.

Corollary 1.1. *Under hypotheses (H1)–(H2), the IDS is jointly Hölder continuous on any compact subset of $\mathbb{R} \times \mathbb{R}\backslash\{0\}$ in the (E, B) energy-magnetic field plane.*

1.2. **Contents of this Article.** In section 2, we review the group of magnetic translations and the ergodicity of the perturbed Landau Hamiltonians. In section 3 we prove a Wegner estimate that will imply the main result, Theorem 1.1, given a certain positivity estimate. This key positivity estimate is proved for Landau Hamiltonians in section 4. In section 5, we prove the continuity of the IDS with respect to the magnetic field. In section 6, the appendix, we collect various technical estimates needed in the proofs.

2. The Magnetic Translation Group and Ergodicity

The magnetic translations for the constant magnetic field problem in two-dimensions are defined as follows. Much of this discussion applies to Schrödinger operators with constant magnetic fields in higher dimensions (cf. [31]). For any field strength $B \in \mathbb{R}$, any vector $\alpha \in \mathbb{R}^2$, and $f \in C_0^\infty(\mathbb{R}^2)$, we define the magnetic translation by α as

$$U_\alpha^B f(x) \equiv e^{\frac{iB}{2}(x_1\alpha_2 - x_2\alpha_1)} f(x + \alpha). \tag{2.1}$$

For $\alpha, \beta \in \mathbb{R}^2$, we have the relations:

$$U_\alpha^B U_\beta^B = e^{iB(\alpha_1\beta_2 - \alpha_2\beta_1)} U_\beta^B U_\alpha^B. \tag{2.2}$$

In a standard way, the family $\{U_\alpha^B \mid \alpha \in \mathbb{R}^2\}$, defined in (2.1)–(2.2), extends to a projective unitary representation of $\mathbb{R}^2$ on $L^2(\mathbb{R}^2)$. We note that for any magnetic field strength B,

$$U_\alpha^B H_L(B) U_{-\alpha}^B = H_L(B). \tag{2.3}$$

We define two fundamental vectors in $\mathbb{R}^2$,

$$\alpha_1 = (L_1, 0), \text{ and } \alpha_2 = (0, L_2), \tag{2.4}$$

for any $L_1, L_2 \in \mathbb{R}$, and write U_j^B, for $j = 1, 2$, for the corresponding unitaries as in (2.1). It follows from (2.2) that we have the commutation rules:

$$U_1^B U_2^B = e^{iBL_1L_2} U_2^B U_1^B. \tag{2.5}$$

It is clear that $\{U_1^B, U_2^B\}$ generate a two-parameter Abelian group provided the following *integer flux condition* for the square $[0, L_1] \times [0, L_2]$ is satisfied:

$$BL_1L_2 \in 2\pi\mathbb{Z}. \tag{2.6}$$

More generally, suppose that only the *rational flux condition* for the square $[0, L_1] \times [0, L_2]$ is satisfied:

$$BL_1L_2 = 2\pi p/q, \text{ for } p, q \in \mathbb{Z}, \tag{2.7}$$

with p, q having no common divisors. Then, assuming (2.7), the unitary operators $\{(U_1^B)^q, U_2^B\}$ satisfy the commutation relations

$$(U_1^B)^q U_2^B = e^{iqBL_1L_2} U_2^B (U_1^B)^q = U_2^B (U_1^B)^q, \tag{2.8}$$

so the pair $\{(U_1^B)^q, U_2^B\}$ generate an Abelian group. Under the rational flux condition, we used Floquet theory relative to this Abelian group in order to study the IDS for H_ω in [4].

Let us now consider the transformation properties of H_ω with respect to the magnetic translations by $\mathbb{Z}^2$. The mapping $\gamma = (n_1, n_2) \in \mathbb{Z}^2 \to U_\gamma^B$ of $\mathbb{Z}^2$ into a family of unitary operators defined in (2.1) is a strongly continuous, unitary ray representation of the additive group $\mathbb{Z}^2$. For $\gamma, \gamma' \in \mathbb{Z}^2$, we have

$$U_\gamma^B U_{\gamma'}^B = e^{i\frac{B}{2}\gamma\wedge\gamma'} U_{\gamma+\gamma'}^B, \tag{2.9}$$

where $\gamma \wedge \gamma' \equiv n_1 n_2' - n_2 n_1'$. In order to compute the action of U_γ^B on H_ω, we recall the invariance of H_0 in (2.3), and consider a related action by $\mathbb{Z}^2$ on the probability space Ω. For each $\gamma = (n_1, n_2) \in \mathbb{Z}^2$, we define a map $T_\gamma : \Omega \to \Omega$ whose action on the random coupling constants satisfies

$$(T_\gamma \omega)_{(j_1, j_2)} = \omega_{(j_1+n_1, j_2+n_2)}. \tag{2.10}$$

It is clear, under (H2), that this action is the usual ergodic group of translations on the probability space Ω. It follows from this remark, the identity (2.3), and the form of the random potential (1.3), that

$$U_\gamma^B H_\omega U_{-\gamma}^B = H_{T_\gamma \omega}, \tag{2.11}$$

for all $\gamma \in \mathbb{Z}^2$. From this identity, the general theory of ergodic Schrödinger operators developed by Kirsch and Martinelli [20, 21] states that there is a set $\Sigma \subset \mathbb{R}$ so that $\sigma(H_\omega) = \Sigma$ with probability one. Furthermore, the IDS exists and is independent of ω almost surely (for the details, cf. [10, 16, 24]).

3. Proof of a Wegner Estimate

The proof of the Hölder continuity of the DOS measure, Theorem 1.2, follows from a locally uniform estimate on the following quantity

$$\frac{1}{|\Lambda|} I\!E\{Tr(\chi_\Lambda E_\omega(\cdot)\chi_\Lambda)\}, \tag{3.1}$$

where we denote the spectral family of H_ω by $E_\omega(\cdot)$, and $I\!E(\cdot)$ is the expectation with respect to the random variables ω_j, $j \in \mathbb{Z}^2$. Such an estimate is related to the Wegner estimate [34], except that the Hamiltonian H_ω in (3.1) is the infinite-area operator (1.2). We will prove the following result in this section, and then use it to prove Theorem 1.2.

Theorem 3.1. *Let q satisfy $0 < q < 1$, and let $I \subset \mathbb{R}$ be a bounded interval. There is a finite constant $C_{I,q} > 0$ so that for any subinterval $\mathcal{O} \subset I$, there is a length scale $L_{\mathcal{O}}$, such that for all $L > L_{\mathcal{O}}$, we have*

$$I\!E\{Tr(\chi_L E_\omega(\mathcal{O})\chi_L)\} \le C_{I,q}|\mathcal{O}|^q L^2, \tag{3.2}$$

where χ_L is the characteristic function of the square Λ_L of side length L.

The proof of this estimate will follow the general ideas of the proof of the Wegner estimate given in [4]. One difference is that the length scale $L_{\mathcal{O}}$ depends on the interval $\mathcal{O}$. Of course, this does not affect the infinite area limit, and, consequently, the properties of the DOS, as will be shown in the proof of Theorem 1.2 below. The main positivity estimate that we need to prove Theorem 3.1 is the following. We will prove this estimate in section 4.

Theorem 3.2. *Let $H_L(B)$ be the Landau Hamiltonian in (1.1) and let Π_n be the projector onto the infinite-dimensional eigenspace for $H_L(B)$ corresponding to the eigenvalue $E_n(B)$. Let $u \geq 0$ be a single-site potential satisfying hypothesis (H1), and define the potential $\tilde{V}$ by*

$$\tilde{V}(x) \equiv \sum_{j \in \mathbb{Z}^2} u(x-j). \tag{3.3}$$

Then, there exists a finite constant $0 < C_n(B,u) < \infty$, so that

$$\Pi_n \tilde{V} \Pi_n \ \geq \ C_n(B,u) \Pi_n. \tag{3.4}$$

Proof of the Wegner Estimate (3.2) given Theorem 3.2.
1. We begin with the observation that it suffices to prove the theorem for bounded intervals $\Delta \subset \mathbb{R}$ sufficiently small so that they contain *at most one Landau level for $H_L(B)$*. Since the separation of the eigenvalues of $H_0 \equiv H_L(B)$ is $2B$, this can be done provided $|\Delta| < 2B$. We note that it follows from the proof that the constant $C_{I,q}$ in (3.2) is uniform with respect to any subintervals on I. Suppose that Δ is centered on an arbitrary, but fixed, energy E_0. For $\Delta \subset \mathbb{R}$, let $f_\Delta(\lambda) \in C_0^\infty(\mathbb{R})$ be a smooth, nonnegative function of compact support with $f_\Delta|\Delta = 1$, $0 \leq f_\Delta \leq 1$, and $|\text{supp } f_\Delta| < 2|\Delta|$, so that supp f_Δ contains at most one Landau level, the same as Δ. We then have the operator inequality $E_\omega(\Delta) \leq f_\Delta(H_\omega)$. We begin with an estimate on

$$I\!\!E\{Tr(\chi_\Lambda E_\omega(\Delta)\chi_\Lambda)\} \leq I\!\!E\{Tr(\chi_\Lambda f_\Delta(H_\omega)^2 \chi_\Lambda)\}. \tag{3.5}$$

First, if Δ contains one Landau level, $E_n(B)$, we write a decomposition of the trace on the right in (3.5) as

$$\begin{aligned} Tr\chi_\Lambda f_\Delta(H_\omega)^2\chi_\Lambda &= Tr\chi_\Lambda f_\Delta(H_\omega)\Pi_n^\perp f_\Delta(H_\omega)\chi_\Lambda \\ &\quad + Tr\chi_\Lambda f_\Delta(H_\omega)\Pi_n f_\Delta(H_\omega)\chi_\Lambda. \end{aligned} \tag{3.6}$$

We will use the fact that $H_0 - E_0$ is boundedly invertible on the range of $\Pi_n^\perp \equiv 1 - \Pi_n$. Second, if Δ does not contain a Landau level, we know directly that $H_0 - E_0$ is boundedly invertible. Let $d_0 \equiv \min\{\text{dist } (E_0, E_q(B)) \mid E_q(B) \notin \Delta\}$, so $d_0 > 0$. We begin with the first case when $E_n(B) \in \Delta$. It is easy to estimate the term in (3.6) involving $\Pi_n^\perp$. We write

$$\begin{aligned} Tr\chi_\Lambda f_\Delta(H_\omega)\Pi_n^\perp f_\Delta(H_\omega)\chi_\Lambda &= Tr\chi_\Lambda f_\Delta(H_\omega)(H_\omega - E_0)(H_0-E_0)^{-1}\Pi_n^\perp f_\Delta(H_\omega)\chi_\Lambda \\ &\quad - Tr\chi_\Lambda f_\Delta(H_\omega)V_\omega \Pi_n^\perp (H_0 - E_0)^{-1} f_\Delta(H_\omega)\chi_\Lambda. \end{aligned} \tag{3.7}$$

To estimate the second term in (3.7), we expand the trace in the eigenfunctions ψ_j of the nonnegative, compact operator $f_\Delta(H_\omega)\chi_\Lambda f_\Delta(H_\omega)$. Applying a quadratic inequality, this results in the bound,

$$\begin{aligned} &|Tr\chi_\Lambda f_\Delta(H_\omega)V_\omega\Pi_n^\perp(H_0-E_0)^{-1}f_\Delta(H_\omega)\chi_\Lambda| \\ &\leq \tfrac{1}{2d_0^2} Tr\chi_\Lambda f_\Delta V_\omega^2 f_\Delta \chi_\Lambda + \tfrac{1}{2} Tr\chi_\Lambda f_\Delta \Pi_n^\perp f_\Delta \chi_\Lambda, \end{aligned} \tag{3.8}$$

where we write f_Δ for $f_\Delta(H_\omega)$, when convenient. The first term in (3.7) is easily seen to be bounded above by

$$|Tr\chi_\Lambda f_\Delta(H_\omega)(H_\omega - E_0)(H_0-E_0)^{-1}\Pi_n^\perp f_\Delta(H_\omega)\chi_\Lambda| \leq \frac{|\Delta|}{d_0} Tr\chi_\Lambda f_\Delta^2 \chi_\Lambda. \tag{3.9}$$

Bringing the second term on the right in (3.8) to the left in (3.7), we obtain an upper bound for the complementary term

$$Tr\chi_\Lambda f_\Delta(H_\omega)\Pi_n^\perp f_\Delta(H_\omega)\chi_\Lambda \leq \frac{1}{d_0^2} Tr\chi_\Lambda f_\Delta V_\omega^2 f_\Delta\chi_\Lambda + \frac{2|\Delta|}{d_0} Tr\chi_\Lambda f_\Delta^2\chi_\Lambda. \tag{3.10}$$

2. We use the lower bound (3.4) in order to estimate the second term on the right in (3.6). Substituting this into the trace for Π_n, we obtain

$$\begin{aligned} Tr\chi_\Lambda f_\Delta \Pi_n f_\Delta \chi_\Lambda &\leq \frac{1}{C_n(B,u)} Tr\chi_\Lambda f_\Delta \Pi_n \tilde{V} \Pi_n f_\Delta \chi_\Lambda \\ &\leq \frac{1}{C_n(B,u)} \Big\{ Tr\chi_\Lambda f_\Delta \tilde{V} f_\Delta \chi_\Lambda \\ &\quad - Tr\chi_\Lambda f_\Delta \Pi_n^\perp \tilde{V} \Pi_n f_\Delta \chi_\Lambda - Tr\chi_\Lambda f_\Delta \Pi_n \tilde{V} \Pi_n^\perp f_\Delta \chi_\Lambda \Big\}, \end{aligned} \tag{3.11}$$

where we dropped the positive term $Tr\chi_\Lambda f_\Delta \Pi_n^\perp \tilde{V} \Pi_n^\perp f_\Delta \chi_\Lambda$ since it occurs with a negative sign. As in [4], the two terms on the last line of (3.11) can be estimated in the same manner. We note that it follows from hypothesis (H1) that there exists a finite constant $D_0 > 0$, depending only on u, so that $\tilde{V}^2 \leq D_0 \tilde{V}$. Using this constant, the second term on the last line in (3.11) is bounded above by

$$\begin{aligned} |Tr\chi_\Lambda f_\Delta \Pi_n^\perp \tilde{V} \Pi_n f_\Delta \chi_\Lambda| &\leq \frac{D_0^2}{C_n} Tr\chi_\Lambda f_\Delta \Pi_n^\perp f_\Delta \chi_\Lambda \\ &\quad + \frac{C_n}{4} Tr\chi_\Lambda f_\Delta \Pi_n f_\Delta \chi_\Lambda, \end{aligned} \tag{3.12}$$

where $C_n \equiv C_n(B,u)$ is the constant occurring in (3.4). We then obtain for the direct term

$$Tr\chi_\Lambda f_\Delta \Pi_n f_\Delta \chi_\Lambda \leq \frac{2}{C_n} Tr\chi_\Lambda f_\Delta \tilde{V} f_\Delta \chi_\Lambda + \frac{4D_0^2}{C_n^2} Tr\chi_\Lambda f_\Delta \Pi_n^\perp f_\Delta \chi_\Lambda. \tag{3.13}$$

3. We now combine (3.10) and (3.13) and obtain

$$\begin{aligned} Tr\chi_\Lambda f_\Delta (H_\omega)^2 \chi_\Lambda &= Tr\chi_\Lambda f_\Delta(H_\omega) \Pi_n f_\Delta(H_\omega) \chi_\Lambda + Tr\chi_\Lambda f_\Delta(H_\omega) \Pi_n^\perp f_\Delta(H_\omega) \chi_\Lambda \\ &\leq \frac{2|\Delta|}{d_0} \left\{ 1 + \frac{4D_0^2}{C_n^2} \right\} Tr\chi_\Lambda f_\Delta^2 \chi_\Lambda \\ &\quad + \frac{2}{C_n} Tr\chi_\Lambda f_\Delta \tilde{V} f_\Delta \chi_\Lambda + \left\{ \frac{1}{d_0^2} + \frac{4D_0^2}{C_n^2 d_0^2} \right\} Tr\chi_\Lambda f_\Delta V_\omega^2 f_\Delta \chi_\Lambda. \end{aligned} \tag{3.14}$$

We first choose $|\Delta|$ sufficiently small so that the coefficient of $Tr\chi_\Lambda f_\Delta (H_\omega)^2 \chi_\Lambda$ in (3.14) satisfies

$$\frac{2|\Delta|}{d_0} \left\{ 1 + \frac{4D_0^2}{C_n^2} \right\} < 1/2. \tag{3.15}$$

Hence, this term can be moved to the left in (3.14). Second, in order to evaluate the last term in (3.14), we note that hypotheses (H1) and (H2) imply that there exists a finite constant $C_1(u, h_0) > 0$ so that $V_\omega^2 \leq C_1(u, h_0) \tilde{V}$. Using this inequality, we can combine the two remaining terms on the right in (3.14). Upon choosing $|\Delta|$ smaller, if necessary, we find that there exists a finite constant $C_2 > 0$ so that

$$Tr\chi_\Lambda E_\omega(\Delta) \chi_\Lambda \leq C_2 Tr\chi_\Lambda f_\Delta \tilde{V} f_\Delta \chi_\Lambda, \tag{3.16}$$

where

$$C_2 \equiv 2 \left\{ \frac{2}{C_n} + C_1 \left(\frac{1}{d_0^2} + \frac{4D_0^2 C_1}{C_n^2 d_0^2} \right) \right\}. \tag{3.17}$$

In the second case, when Δ does not contain a Landau level, we find by an analogous, but simpler, computation

$$Tr\chi_\Lambda E_\omega(\Delta) \chi_\Lambda \leq \frac{C_1}{d_0} Tr\chi_\Lambda f_\Delta \tilde{V} f_\Delta \chi_\Lambda. \tag{3.18}$$

4. In order to estimate the trace on the right in (3.16) and (3.18), we must replace the potential $\tilde{V}$ by a local potential using the localization due to the characteristic function χ_Λ. Let us suppose that $\Lambda = \Lambda_L = [-L/2, L/2] \times [-L/2, L/2]$ is a square of side length $L > 0$. For any $0 < \epsilon < 1$, we denote by $\Lambda_\epsilon \subset \Lambda_L$ the square of side length $(1-\epsilon)L$. We write $\partial\Lambda_\epsilon$ for the annular region $\Lambda_{(1+\epsilon)L} \backslash \Lambda_{(1-\epsilon)L}$, so that the area of this annular region is $4\epsilon L^2$. Finally, we denote the complement $\mathbb{R}^2 \backslash \Lambda_{(1+\epsilon)L}$ by Λ_ϵ^c. As above,

we use the notation χ_X for the characteristic function of the set $X \subset \mathbb{R}^2$. The sets defined here provide a partition of $\mathbb{R}^2$ as

$$1_{\mathbb{R}^2} = \chi_{\Lambda_\epsilon} + \chi_{\partial\Lambda_\epsilon} + \chi_{\Lambda_\epsilon^c}. \tag{3.19}$$

Using this partition of unity, we write

$$\chi_\Lambda f_\Delta \tilde{V} f_\Delta \chi_\Lambda = \chi_\Lambda f_\Delta \tilde{V} \chi_{\Lambda_\epsilon} f_\Delta \chi_\Lambda + \chi_\Lambda f_\Delta \tilde{V} \chi_{\partial\Lambda_\epsilon} f_\Delta \chi_\Lambda + \chi_\Lambda f_\Delta \tilde{V} \chi_{\Lambda_\epsilon^c} f_\Delta \chi_\Lambda. \tag{3.20}$$

The first term on the right in (3.20) is the main term. We estimate the expectation of the trace of this term using the estimate on the spectral shift function proved in [7]. The two remainder terms in (3.20) are dealt with using either the smallness of the boundary term or the decay of the kernel of the operator $\chi_\Lambda f_\Delta(H_\omega)\chi_{\Lambda_\epsilon^c}$ that follows from the fact that the regions Λ and Λ_ϵ^c are disjoint with a positive distance $\epsilon L/2$ between them.

5. We calculate the trace of the first term on the right in (3.20) as follows. Let $g(\lambda) \in \mathbb{C}^\infty(\mathbb{R})$ be a smooth function with support of g' containing Δ, and so that $f_\Delta(\lambda) \leq -2|\Delta| g'(\lambda)$. We can then write

$$\begin{aligned} Tr \chi_\Lambda f_\Delta \tilde{V} \chi_{\Lambda_\epsilon} f_\Delta \chi_\Lambda &\leq Tr f_\Delta \tilde{V} \chi_{\Lambda_\epsilon} \\ &\leq -2|\Delta| Tr g'(H_\omega) \tilde{V} \chi_{\Lambda_\epsilon}. \end{aligned} \tag{3.21}$$

A key formula in this setting is

$$\sum_{j \in \tilde{\Lambda}_\epsilon} \frac{\partial}{\partial \omega_j} Tr(g(H_\omega) - g(H_0))\chi_{\Lambda_\epsilon} = Tr g'(H_\omega) \tilde{V} \chi_{\Lambda_\epsilon}. \tag{3.22}$$

We can now follow the calculation in [7] and [4] using the spectral shift function. As a result, we find that for each $0 \leq q < 1$, there exists a finite, positive constant $C_{q,1} > 0$, so that

$$I\!E\{Tr \chi_\Lambda f_\Delta \tilde{V} \chi_{\Lambda_\epsilon} f_\Delta \chi_\Lambda\} \leq C_{q,1} |\Delta|^q (1-\epsilon)^2 L^2. \tag{3.23}$$

6. We now calculate the two remainder terms in (3.20). The boundary term is easily seen to be bounded above by

$$Tr \chi_\Lambda f_\Delta \tilde{V} \chi_{\partial\Lambda_\epsilon} f_\Delta \chi_\Lambda \leq 4\epsilon L^2 \left(\sup_{j \in \partial\Lambda_\epsilon} \| \chi_\Lambda f_\Delta(H_\omega) u_j^{1/2} \|_{HS}^2 \right). \tag{3.24}$$

The term involving the unbounded complementary region is bounded above by

$$Tr \chi_\Lambda f_\Delta \tilde{V} \chi_{\Lambda_\epsilon^c} f_\Delta \chi_\Lambda \leq \| \chi_\Lambda f_\Delta(H_\omega) \chi_{\Lambda_\epsilon^c} \|_{HS}^2. \tag{3.25}$$

In the appendix, section 6, we prove two estimates on the Hilbert-Schmidt norms of the localized operators appearing in (3.24) and (3.25). We assume that f_Δ is obtained from a function f by scaling, so the derivatives of f are order one. In this way, the k^{th}-derivative f_Δ is of order $|\Delta|^{-k}$. For the unscaled function f, we introduce the following notation that is used in the estimates of the appendix:

$$|||f|||_k \equiv \sum_{l=0}^{k} d_l \int_{\mathbb{R}} |f^{(l)}(\xi)| \, d\xi, \tag{3.26}$$

where the finite constants $d_j > 0$ depend only on the function σ used in the construction of the almost analytic extension given in (6.1). First, we prove that there is a finite constant $C_0 > 0$, depending on B and independent of j, so that

$$\| \chi_\Lambda f_\Delta(H_\omega) u_j^{1/2} \|_{HS} \leq C_0 |||f|||_3 \, |\Delta|^{-2}. \tag{3.27}$$

Consequently, for the boundary term (3.24), we obtain

$$Tr \chi_\Lambda f_\Delta \tilde{V} \chi_{\partial\Lambda_\epsilon} f_\Delta \chi_\Lambda \leq C_2 \epsilon L^2 |\Delta|^{-4}. \tag{3.28}$$

Second, we consider $f_\Delta(H_\omega)$ localized between two disjoint regions. Let χ_j, for $j = 1, 2$ be characteristic functions on two disjoint regions, at least one of which is bounded, and let $d_{12} \equiv \text{dist}(\text{supp}\chi_1, \text{supp}\chi_2)$. For any $M \in \mathbb{N}$, and for any $f \in C_0^{k+1}(\mathbb{R})$, with $k \geq M+1$, there is a constant $0 < B_M < \infty$, independent of f and d_{12}, so that

$$\| \chi_1 f_\Delta(H_\omega) \chi_2 \|_{HS} \leq B_M d_{12}^{-M} |||f|||_{k+1}. \tag{3.29}$$

Hence, setting $k = M + 1$, and taking into account the scaling of f_Δ, we obtain a bound for the term involving the interior and the disjoint exterior (3.25),

$$Tr\chi_\Lambda f_\Delta \tilde{V} \chi_{\Lambda_\epsilon^c} f_\Delta \chi_\Lambda \leq B_M^2 |||f|||_{M+2}^2 (\epsilon L)^{-2M} |\Delta|^{-2M-2}. \tag{3.30}$$

7. To complete the proof, we combine the estimates (3.23), (3.28), and (3.30), to obtain

$$I\!E\{Tr\chi_\Lambda E_\Delta \chi_\Lambda\} \leq C_{q,1}|\Delta|^q (1-\epsilon)^2 L^2 + C_2 \epsilon L^2 |\Delta|^{-4} + C_3 (\epsilon L)^{-2M} |\Delta|^{-2M-2}, \tag{3.31}$$

where the constants depend on f and its derivatives, but are independent of L and $|\Delta|$. For any given Δ, we choose $\epsilon = |\Delta|^5$ and $M = 2$. Then, for all $L > L_\Delta \equiv |\Delta|^{-9/2}$, we have

$$I\!E\{Tr\chi_\Lambda E_\Delta \chi_\Lambda\} \leq C_q |\Delta|^q |\Lambda|, \tag{3.32}$$

proving the first part of the theorem. We next need to check the uniformity of the constants with respect to subintervals of Δ. To see this, notice that the constant C_2 in (3.17) depends on Δ only through d_0 and if $U \subset \Delta$, then d_0 increases, so that C_2 is uniform with respect to all subintervals of Δ. The same result holds for the constant in (3.18). Similarly, the constant $C_{q,1}$ in (3.23) depends only on the uniform norm $\|g\|_\infty$ of the function g in (3.21). This norm is is independent of Δ. In the same manner, one checks that the constants C_2 and C_3 in (3.31) depend only on the norm $|||f|||_3$ of the *unscaled* function f. Thus, these constants are independent of Δ. It follows that the constant $C_{I,q}$ in (3.2) is uniform with respect to all subintervals of I. This completes the proof of Theorem 3.1. □

We now give a proof of Theorem 1.2, given Theorem 3.1. The key fact we use is the uniformity of constant $C_{I,q}$ in the Wegner estimate with respect to subintervals.

Proof of Theorem 1.2 given Theorem 3.1.
We fix a bounded interval $I \subset \mathbb{R}$. Let $U \subset I$ be an open interval. From Theorem 3.1, there is a length scale L_U so that for $L > L_U$, we have

$$I\!E\{Tr(\chi_L E_\omega(U) \chi_L)\} \leq C_{I,q} L^2 |U|^q, \tag{3.33}$$

where χ_L is the characteristic function on the square Λ_L. Dividing by L^2 and taking the limit $L \to \infty$, we find that the DOS measure ν satisfies

$$\nu(U) \leq C_{I,q} |U|^q. \tag{3.34}$$

Next, let $\mathcal{O} \subset I$ be an arbitrary open subset. There is a countable collection $\{\mathcal{O}_n\}$ of disjoint, open intervals so that $\mathcal{O} = \cup_n \mathcal{O}_n$. From (3.34), it follows that

$$\nu(\mathcal{O}) \leq C_{I,q} \sum_n |\mathcal{O}_n|^q = C_{I,q} |\mathcal{O}|^q. \tag{3.35}$$

We now consider an arbitrary Borel subset $J \subset I$. For any $\epsilon > 0$, there is an open set $\mathcal{O}_\epsilon$ containing J so that $|\mathcal{O}_\epsilon \sim J| \leq \epsilon$. It follows from part 1 of the proof that the constant constant $C_{I,q} > 0$ is independent of ϵ, and we have

$$\nu(J) \leq \nu(\mathcal{O}_\epsilon) \leq C_{I,q} |\mathcal{O}_\epsilon|^q \leq C_{I,q} (|J| + \epsilon)^q. \tag{3.36}$$

Since $\epsilon > 0$ is arbitrary, we find that

$$\nu(J) \leq C_{I,q} |J|^q, \tag{3.37}$$

which is the main estimate of Theorem 1.2. The fact that (3.37) implies that the density $(d\nu/d\lambda)$ is in $L^p_{loc}(\mathbb{R})$, for $1 \leq p < \infty$ is given, for example, in [6]. □

4. Positivity Estimates for Landau Projectors

The key local positivity estimate (3.4) follows from an explicit calculation of a lower bound for the left side of (3.4) employing explicit formulas for the infinite-dimensional projectors for the Landau levels. In this case, we are able to make a fairly explicit calculation of the constant appearing in (3.4). This provides a quantitative version of the unique continuation principle, as used in [4]. Recall that $\Pi_n = \Pi_n(B)$, $n \in \mathbb{Z}_+$ is the orthogonal projection on eigenspace corresponding to the n-th Landau level for the Landau Hamiltonian $H_0 = H_L(B)$ with constant magnetic field $B > 0$, and that the potential $\tilde{V}$ is defined by

$$\tilde{V}(x) \equiv \sum_{j \in \mathbb{Z}^2} u(x - j), \tag{4.1}$$

for a single-site potential u satisfying (H1).

Theorem 4.1. *For any $n \in \mathbb{Z}_+$, for a single-site potential u satisfying (H1), and for any $B > 0$, there is a finite, positive constant $C_n(B, u)$, so that the potential $\tilde{V}$ satisfies*

$$\Pi_n \tilde{V} \Pi_n \geq C_n(B, u) \Pi_n, \quad n \in \mathbb{Z}_+. \tag{4.2}$$

In order to prove Theorem 4.1, we need several auxiliary assertions. The basic technical observation comes from the work of Raikov and Warzel [26]. Lemma 4.1 summarizes Lemmas 3.2 and 3.3, and Remark 3.2 of this paper [26].

Lemma 4.1. *Let $V \in L^p(\mathbb{R}^2)$, $p > 1$. Then, the operator $\Pi_n V \Pi_n$, $n \in \mathbb{Z}_+$, is selfadjoint and compact in $\Pi_n L^2(\mathbb{R}^2)$. Moreover, if V is radially symmetric, then the eigenvalues $\mu_{n,k}(V)$, $k \in \mathbb{Z}_+$, of $\Pi_n V \Pi_n$ are equal to $\langle V \varphi_{n,k}, \varphi_{n,k} \rangle$, where $\langle ., . \rangle$ denotes the scalar product in $L^2(\mathbb{R}^2)$, and the functions $\varphi_{n,k}$ are given by*

$$\varphi_{n,k}(x) := \sqrt{\frac{n!}{\pi k!}} \left(\frac{B}{2} \right)^{(k-n+1)/2} (x_1 + ix_2)^{k-n} \mathrm{L}_n^{(k-n)} \left(B|x|^2/2 \right) e^{-B|x|^2/4}, \quad x \in \mathbb{R}^2,$$

where $\mathrm{L}_n^{(k-n)}$ are the generalized Laguerre polynomials given by

$$\mathrm{L}_n^{(k-n)}(\xi) := \sum_{l=\max\{0,n-k\}}^{n} \binom{k}{n-l} \frac{(-\xi)^l}{l!}, \quad \xi \geq 0, \quad n \in \mathbb{Z}_+, \quad k \in \mathbb{Z}_+.$$

Finally, the normalized eigenfunctions of $\Pi_n V \Pi_n$ corresponding to the eigenvalues $\mu_{n,k}(V)$, $k \in \mathbb{Z}_+$, are equal to $\varphi_{n,k}$. In particular, the eigenfunctions are independent of V.

Remark: Note that in [26] the eigenvalues $\langle V \varphi_{n,k}, \varphi_{n,k} \rangle$ and the eigenfunctions $\varphi_{n,k}$ are enumerated starting with $k = -n$, and not with $k = 0$, as is done here.

We will apply this result by noting that hypothesis (H1) guarantees that the single-site potential u is strictly positive on some open set. We denote by $D(x, R)$ the disk of radius $R > 0$, centered at $x \in \mathbb{R}^2$. We set $\nu_{n,k}(R) := \mu_{n,k}(\chi_{D(0,R)})$.

Corollary 4.1. *The eigenvalues $\nu_{n,k}(R)$, $k \in \mathbb{Z}_+$, of the operator $\Pi_n \chi_{D(0,R)} \Pi_n$, $n \in \mathbb{Z}_+$, can be written as*

$$\nu_{n,k}(R) = \frac{n!}{k!} \int_0^{\varrho} \xi^k \, \mathrm{L}_n^{(k-n)}(\xi)^2 \, e^{-\xi} \, d\xi, \tag{4.3}$$

where $\varrho = \varrho(R) := BR^2/2$. Moreover, the asymptotic relation

$$\nu_{n,k}(R) = \frac{e^{-\varrho} \varrho^{-n+1}}{n!} \frac{k^{2n-1} \varrho^k}{k!} (1 + o(1)), \quad k \to \infty, \tag{4.4}$$

holds uniformly with respect to $R \in [R_1, R_2]$, $0 < R_1 < R_2 < \infty$.

Proof. Using Lemma 4.1, passing to polar coordinates (r, θ) in the integral $\langle \chi_{D(0,R)} \varphi_{n,k}, \varphi_{n,k} \rangle$, and changing the variable $Br^2/2 = \xi$, we immediately obtain (4.3). In order to check (4.4), we assume first of all that $k \geq n$. In this case, we have

$$\nu_{n,k}(R) = \frac{n!}{k!} \sum_{l=0}^{n} \sum_{m=0}^{n} (-1)^{l+m} \frac{1}{m! l!} \binom{k}{n-l} \binom{k}{n-m} \int_0^{\varrho(R)} e^{-\xi} \xi^{k-n+m+l} d\xi. \tag{4.5}$$

We note that

$$\lim_{k \to \infty} k^{s-n} \binom{k}{n-s} = \frac{1}{(n-s)!}, \quad s \in \{0, \ldots, n\}, \tag{4.6}$$

(see e.g. [1, Eq. 6.1.46]). Moreover, the asymptotic relation

$$\int_0^\varrho e^{-\xi}\xi^{k+\alpha}d\xi = e^{-\varrho}\varrho^{\alpha+1}\frac{\varrho^k}{k}(1+o(1)), \quad k\to\infty, \tag{4.7}$$

holds for any fixed $\alpha\in\mathbb{R}$ uniformly with respect to $\varrho\in[\varrho_1,\varrho_2]$ with $0<\varrho_1<\varrho_2<\infty$. Combining (4.6) and (4.7), we find that the leading asymptotic term of the sum in (4.5) corresponds to $m=l=0$. Employing (4.5), (4.6) with $s=0$, and (4.7) with $\alpha=-n$, we get (4.4). □

Lemma 4.2. *Let $n\in\mathbb{Z}_+$. For each $0<\varepsilon<R$, $\lambda>1$, and $\eta>0$, there exists a constant $C_0 = C_0(n,\varepsilon,R,\eta)>0$, such that*

$$\Pi_n\chi_{D(0,\varepsilon)}\Pi_n \geq C_0\left(\Pi_n\chi_{D(0,R)}\Pi_n - \eta\,\Pi_n\chi_{D(0,\lambda R)}\Pi_n\right). \tag{4.8}$$

Proof. We fix $\delta\in(0,1)$, and, bearing in mind (4.4), pick K such that $k\geq K$ implies

$$(1-\delta)\frac{e^{-\varrho(R_0)}\varrho(R_0)^{k-n+1}}{n!}\frac{k^{2n-1}}{k!} \leq \nu_{n,k}(R_0) \leq (1+\delta)\frac{e^{-\varrho(R_0)}\varrho(R_0)^{k-n+1}}{n!}\frac{k^{2n-1}}{k!}, \quad R_0\in[R/2,2\lambda R]. \tag{4.9}$$

We will show that if $0<\varepsilon<R$, $\lambda>1$, and $k_0\geq K$, then the operator inequality

$$\Pi_n\chi_{D(0,\varepsilon)}\Pi_n \geq C_1\left(\Pi_n\chi_{D(0,R)}\Pi_n - C_2\Pi_n\chi_{D(0,\lambda R)}\Pi_n\right) \tag{4.10}$$

holds with

$$C_1 = C_1(n,\varepsilon,R,k_0) := \min_{k\in\{0,\dots,k_0\}}\frac{\nu_{n,k}(\varepsilon)}{\nu_{n,k}(R)}, \tag{4.11}$$

and

$$C_2 := \frac{1+\delta}{1-\delta}\lambda^{-2(k_0-n+1)}e^{-\varrho(R)+\varrho(\lambda R)}. \tag{4.12}$$

By Lemma 4.1, the operators $\Pi_n\chi_{D(0,\varepsilon)}\Pi_n$, $\Pi_n\chi_{D(0,R)}\Pi_n$, and $\Pi_n\chi_{D(0,\lambda R)}\Pi_n$, are diagonalizable in the same basis $\{\varphi_{n,k}\}_{k\in\mathbb{Z}_+}$. Hence, in order to prove (4.10), it suffices to check that the numerical inequalities

$$\nu_{n,k}(\varepsilon)\geq C_1\left(\nu_{n,k}(R) - C_2\,\nu_{n,k}(\lambda R)\right) \tag{4.13}$$

hold for each $k\in\mathbb{Z}_+$. If $k\leq k_0$, then (4.13) is valid because in this case (4.11) implies $\nu_{n,k}(\varepsilon)\geq C_1\nu_{n,k}(R)$ which obviously entails (4.13). If $k>k_0$, by (4.11) and (4.12) we have

$$\nu_{n,k}(R) - C_2\nu_{n,k}(\lambda R)$$

$$\leq (1+\delta)\frac{e^{-\varrho(R)}\varrho(R)^{-n+1}}{n!}\frac{k^{2n-1}\varrho(R)^k}{k!} - \frac{1+\delta}{1-\delta}\lambda^{-2(k_0-n+1)}e^{-\varrho(R)+\varrho(\lambda R)}(1-\delta)\frac{e^{-\varrho(\lambda R)}\varrho(\lambda R)^{-n+1}}{n!}\frac{k^{2n-1}\varrho(\lambda R)^k}{k!}$$

$$= (1+\delta)\frac{e^{-\varrho(R)}}{n!}\frac{k^{2n-1}\varrho(\lambda R)^{k-n+1}}{k!}\lambda^{2(n-1)}\left(\lambda^{-2k}-\lambda^{-2k_0}\right).$$

Since $\lambda>1$, we find that $\nu_{n,k}(R) - C_2\nu_{n,k}(\lambda R)\leq 0$, if $k>k_0$, which again implies (4.13). To complete the proof of the lemma, we fix $\eta>0$. Taking into account (4.12), we choose $k_0 = k_0(\eta)\geq K$ so large that $C_2\leq\eta$. The main estimate (4.8), with $C_0 = C_1(n,\varepsilon,R,k_0(\eta))$, now follows from (4.13). □

Remark: If $n=0$, then (4.3) implies

$$e^{-\varrho(R)}\frac{\varrho(R)^{k+1}}{(k+1)!}\leq \nu_{0,k}(R)\leq \frac{\varrho(R)^{k+1}}{(k+1)!}, \quad \forall k\in\mathbb{Z}_+.$$

Because of this improved estimate, we also have (4.13) with the constants

$$\tilde{C}_1 := e^{-\varrho(R)}\left(\frac{\varrho(\varepsilon)}{\varrho(R)}\right)^{k_0+1} = e^{-\varrho(R)}\left(\frac{\varepsilon}{R}\right)^{2(k_0+1)}, \quad \tilde{C}_2 := e^{\varrho(\lambda R)-\varrho(R)}\lambda^{-2(k_0+1)},$$

which are more explicit that the ones given by (4.11) – (4.12).

We now give the proof of the positivity result, Theorem 4.1.

Proof. For $j \in \mathbb{Z}^2$, the invariance of $H_L(B)$ with respect to group of magnetic translations (2.3) gives

$$U_j^B \Pi_n \chi_{D(0,\varepsilon)} \Pi_n U_{-j}^B = \Pi_n \chi_{D(j,\varepsilon)} \Pi_n. \tag{4.14}$$

Hypothesis (H1) on the single-site potential u guarantees that there exists an $\epsilon > 0$ so that

$$\tilde{V}(x) \geq \sum_{j \in \mathbb{Z}^2} u(x-j) \geq \sum_{j \in \mathbb{Z}^2} \chi_{D(j,\varepsilon)}(x). \tag{4.15}$$

We now choose $R > \sqrt{2}/2$ so that $\sum_{j \in \mathbb{Z}^2} \chi_{D(j,R)} \geq \chi_{\mathbb{R}^2}$. We fix $\lambda > 1$. Then, there is a finite constant $K_0 > 0$ so that $\sum_{j \in \mathbb{Z}^2} \chi_{D(j,\lambda R)} \leq K_0$. It now follows from Lemma 4.2 that for any $0 < \eta < 1$, there is a constant C_0 so that

$$\begin{aligned} \Pi_n \tilde{V} \Pi_n &\geq \sum_{j \in \mathbb{Z}^2} \Pi_n \chi_{D(j,\varepsilon)} \Pi_n \\ &\geq \sum_{j \in \mathbb{Z}^2} C_0 \left(\Pi_n \chi_{D(j,R)} \Pi_n - \eta \Pi_n \chi_{D(j,\lambda R)} \Pi_n \right) \\ &\geq C_0 (1 - \eta K_0) \Pi_n. \end{aligned} \tag{4.16}$$

We now choose $\eta = 1/2K_0$. The resulting constant depends on the Landau level index n, the function u, and the field strength B. □

5. Continuity of the IDS with respect to the Magnetic Field

The proof of the continuity of the IDS with respect to magnetic field strength follows from the vague convergence of the density of states measures with respect to the magnetic field strength. The proof of this theorem, given below, relies on the following representation of the density of states (DOS) measure:

$$\nu_B(f) \equiv \int f(\lambda) d\nu_B(\lambda) = I\!E\{Tr[\chi_0 f(H_\omega(B))\chi_0]\}, \tag{5.1}$$

where χ_0 is the characteristic function on the unit cube. This representation follows from the definition of the DOS and the ergodicity of the Hamiltonian, expressed in (2.11). A proof can be found in [16]. Using this representation, we have the following proposition on the vague convergence of the density of states measures with respect to the magnetic field.

Proposition 5.1. *For any $f \in C_0^5(\mathbb{R})$, there is a finite constant $C(f^{(5)}) > 0$, depending on $f^{(j)}$, with $j = 1, \ldots 5$, so that for any $B, B' \in \mathbb{R}^+$, we have*

$$\left| \int f(\lambda)(d\nu_B(\lambda) - d\nu_{B'}(\lambda)) \right| \leq C(f^{(5)}) \tilde{\delta}(B, B'), \tag{5.2}$$

where

$$\tilde{\delta}(B, B') = max\,(|B - B'|, |(B')^2 - B^2|). \tag{5.3}$$

If we use a scaled function $f(s/\lambda)$, for $\lambda > 0$, so that $|f^{(j)}| \sim \lambda^{-j}$, for $j = 1, \ldots, 5$, then $C(f^{(5)}) = \mathcal{O}(\lambda^{-4})$.

Proof.
1. We prove the theorem using the Helffer-Sjöstrand formula (6.2) of the appendix, section 6, and the representation formula (5.1). We write the left side of (5.2) using the representation formula (5.1) as

$$\int f(\lambda)(d\nu_B(\lambda) - d\nu_{B'}(\lambda)) = I\!E\{Tr[\chi_0(f(H_\omega(B)) - f(H_\omega(B')))\chi_0]\}. \tag{5.4}$$

We fix any λ_0 satisfying $-\infty < \lambda_0 < 2\min\,(0, \inf \sigma(H_\omega))$, and define a function $F(s) = (-\lambda_0 + s)f(s)$. This function F is in $C_0^5(\mathbb{R})$ like f (note that it is sufficient to consider those f with support in $[\min\,(0, \lambda_0/2), \infty)$). Let $\tilde{F}$ be a compactly supported almost analytic extension of order 5 of F, as constructed in (6.1). We write

$z = \xi + i\eta \in \mathbb{C}$. An extension $\tilde{F}$ of order 5, as in (6.1)), satisfies the estimate $|\partial_{\overline{z}} \tilde{F}(\xi + i\eta)| \sim |\eta|^4 |F^{(5)}(\xi)|$, as $|\eta| \to 0$. With these choices, the Helffer-Sjöstrand formula for the operator $f(H)$ has the form

$$f(H) = \frac{1}{\pi} \int_{\mathbb{C}} \partial_{\overline{z}} \tilde{F}(z)(H - \lambda_0)^{-1}(H - z)^{-1} \; d\xi \; d\eta. \tag{5.5}$$

The extra power of the resolvent helps in the calculation of the trace norm in our case. Let us denote by $R^B_\omega(z)$ the resolvent $(H_\omega(B) - z)^{-1}$, and by $R^B_0(z)$ the resolvent $(H_L(B) - z)^{-1}$. We write the right side of (5.4) using the Helffer-Sjöstrand formula (5.5) as

$$f(H_\omega(B)) - f(H_\omega(B')) = \frac{1}{\pi} \int_{\mathbb{C}} \partial_{\overline{z}} \tilde{F}(z)[R^B_\omega(\lambda_0) R^B_\omega(z) - R^{B'}_\omega(\lambda_0) R^{B'}_\omega(z)] \; d\xi \; d\eta. \tag{5.6}$$

This integral is *a priori* convergent because of the vanishing of $\partial_{\overline{z}} \tilde{F}$ as $|\eta| \to 0$, and since $\tilde{F}$ has compact support.

2. From (1.1), the difference of the two perturbed Landau Hamiltonians is

$$\delta(B, B') \equiv H_\omega(B) - H_\omega(B') = (B' - B)\hat{A} \cdot p + (1/4)(B^2 - (B')^2)\hat{A} \cdot \hat{A}, \tag{5.7}$$

where the normalized vector potential $\hat{A}$ is given by $\hat{A} = (-x_2, x_1)$, and $p = -i\nabla$. The operators $(p_k - A_k)^j$, for $j = 1, 2$, are relatively-$H_L(B)$ bounded, and satisfy the norm estimate,

$$\|(p_k - A_k)^j R^B_0(z)\| \; \leq \; C_0 |\eta|^{-1}. \tag{5.8}$$

In Lemma 6.1 of the appendix, we prove the two following the basic estimates, using the boundedness of the potential V_ω, and the relative-bound estimate (5.8). Let B^* denote either B or B', and let χ be any smooth function χ of compact support. Then, we have

$$\|\delta(B, B') R^{B^*}_\omega(z) \chi\| \; \leq \; C_0 \tilde{\delta}(B, B') |\eta|^{-3}, \tag{5.9}$$

and

$$\|\delta(B, B') R^{B^*}_\omega(z) R^{B^*}_\omega(\lambda_0)[H_\omega(B^*), \chi]\| \; \leq \; C_1 \tilde{\delta}(B, B') |\eta|^{-3}, \tag{5.10}$$

where the constants C_0 and C_1 depend on B^*.

3. We expand the difference of operators on the right side of (5.6) as follows:

$$\begin{aligned} & \chi_0 (R^B_\omega(\lambda_0) R^B_\omega(z) - R^{B'}_\omega(\lambda_0) R^{B'}_\omega(z)) \chi_0 \\ = & \; -\chi_0 R^B_\omega(\lambda_0) R^B_\omega(z) \delta(B, B') R^{B'}_\omega(z)) \chi_0 \\ & \; -\chi_0 R^B_\omega(\lambda_0) \delta(B, B') R^{B'}_\omega(\lambda_0) R^{B'}_\omega(z)) \chi_0 \\ = & \; I + II. \end{aligned} \tag{5.11}$$

To compute the trace norm of I, we use commutators to push the cut-off function through to the left so that we can use estimate (5.6) and the Hölder inequality for trace ideals (cf. [29]) in order to express the trace norm in terms of Hilbert-Schmidt norms. In what follows, $\chi_{0,j}$ will denote smooth functions with compact support so that $\chi_{0,j-1}\chi_{0,j} = \chi_{0,j}$. After standard manipulations, we find

$$\begin{aligned} \|I\|_1 \; \leq \; & \{ \|\chi_0 R^B_\omega(\lambda_0)\|_{HS} \; \|\chi_{0,1} R^B_\omega(z)\|_{HS} \\ & + \|\chi_0 R^B_\omega(\lambda_0)\|_{HS} \|\chi_{0,1} R^B_\omega(z)\|_{HS} \; \|[H_\omega(B), \chi_{0,2}] R^B_\omega(\lambda_0)\| \\ & + \|\chi_0 R^B_\omega(\lambda_0)\|_{HS} \; \|\chi_{0,3} R^B_\omega(\lambda_0)\|_{HS} \; \|R^B_\omega(z)\| \; \|[H_\omega(B), \chi_{0,1}] R^B_\omega(\lambda_0)[H_\omega(B), \chi_{0,2}]\| \} \\ & \times \|\delta(B, B') R^{B'}_\omega(z) \chi_0\|. \end{aligned} \tag{5.12}$$

Similarly, the second term II in (5.11) can be bounded as

$$\begin{aligned} \|II\|_1 \; \leq \; & \|\chi_0 R^B_\omega(\lambda_0)\|_{HS} \; \|R^{B'}_\omega(\lambda_0) \chi_0\|_{HS} \left\{ \|\delta(B, B') R^{B'}_\omega(z) \chi_0\| \right. \\ & \left. + \; \|\delta(B, B') R^{B'}_\omega(z) R^{B'}_\omega(\lambda_0)[H_\omega(B'), \chi_{0,1}]\| \right\}. \end{aligned} \tag{5.13}$$

We use the estimates (5.9) and (5.10) in order to estimate the terms involving the operator norm in (5.12) and (5.13). We see that it suffices to estimate the Hilbert-Schmidt norms $\|\chi_0 R^B_\omega(\lambda_0)\|_{HS}$ and $\|\chi_0 R^B_\omega(z)\|_{HS}$. For the latter, we use the resolvent formula and write

$$\|\chi_0 R^B_\omega(z)\|_{HS} \leq \frac{C_0}{|\eta|} \|\chi_0 R^B_\omega(\lambda_0)\|_{HS}. \tag{5.14}$$

4. Hence, it suffices to estimate the Hilbert-Schmidt norm of $\chi_0 R^B_\omega(\lambda_0)$. Squaring this norm, we write $\|R^B_\omega(\lambda_0)\chi_0\|^2_{HS} = \|\chi_0 |R^B_\omega(\lambda_0)|^2 \chi_0\|_1$ (cf. [29]). In order to estimate this trace norm, we need some facts about the resolvent $R^B_L(z) \equiv (H_L(B) - z)^{-1}$ of the Landau Hamiltonian presented in the appendix, section 6. It follows from (6.3)–(6.4) that

$$\|\chi_0 R^B_L(z)\|^2_{HS} = \|\chi_0 |R^B_L(z)|^2 \chi_0\|_1 = \frac{B}{2\pi} \sum_{n \geq 0} \frac{1}{|E_n(B) - z|^2}. \tag{5.15}$$

Note that this is summable for any $z \neq E_m(B)$, for some $m \in \mathbb{N}$. We will use this for $z = \lambda_0$, so there is no divergence coming from this term. Now, returning to the estimate of $\|R^B_\omega(\lambda_0)\chi_0\|_{HS}$, as the random potential V_ω is bounded, we can replace $R^B_\omega(\lambda_0)$ by $R^B_L(\lambda_0)$ using the second resolvent formula:

$$R^B_\omega(\lambda_0) = R^B_L(\lambda_0) - R^B_L(\lambda_0) V_\omega R^B_\omega(\lambda_0), \tag{5.16}$$

and $\|V_\omega R^B_\omega(\lambda_0)\| \leq C(h_0, u, \lambda_0)$, for some finite constant $C(h_0, u, \lambda_0) > 0$, depending only on supp h_0, the energy λ_0, and the single-site potential u. Consequently, we can write

$$\begin{aligned} \chi_0 (R^B_\omega(\lambda_0))^2 \chi_0 \;&=\; \chi_0 (R^B_L(\lambda_0))^2 \chi_0 \\ &\quad - \chi_0 R^B_L(\lambda_0) R^B_\omega(\lambda_0) V_\omega R^B_L(\lambda_0) \chi_0 \\ &\quad - \chi_0 R^B_L(\lambda_0) V_\omega R^B_\omega(\lambda_0) R^B_L(\lambda_0) \chi_0 \\ &\quad + \chi_0 R^B_L(\lambda_0) V_\omega (R^B_\omega(\lambda_0))^2 V_\omega R^B_L(\lambda_0) \chi_0. \end{aligned} \tag{5.17}$$

Using the estimate (5.15), and this identity, we obtain

$$\|\chi_0 R^B_\omega(\lambda_0)\|^2_{HS} = \|\chi_0 |R^B_\omega(\lambda_0)|^2 \chi_0\|_1 \leq C_1. \tag{5.18}$$

5. We return to the right side of (5.4). The trace of (5.6) is estimated by writing (5.11) and using estimate (5.9) and (5.18). We see that the singularity of the trace on the real axis behaves like $|\eta|^{-4}$. As the function $\partial_{\overline{z}} \tilde{F}(\xi + i\eta)| \sim |\eta|^4$, as η vanishes, we see that the integral over η is convergent. This completes the proof. □

Theorem 5.1. *For any $E > 0$, the IDS $N(E; B)$ is a Hölder continuous function of B, for $B \neq 0$. That is, for any exponent $0 < \alpha < 1/4$, there is a finite constant $C_{(\alpha, B_0, E)} > 0$, locally uniform on compact subsets of E, so that for $B_0, B \neq 0$, with $|B_0 - B|$ sufficiently small,*

$$|N(E; B) - N(E; B_0)| \; \leq C_{(\alpha, B_0, E)} |B - B_0|^\alpha. \tag{5.19}$$

Proof.
1. Let $\Sigma_0 \equiv \; \inf \sigma(H_\omega(B))$, almost surely. For any $K \subset \mathbb{R}$, let χ_K be the characteristic function for the set K. For any $E \geq \Sigma_0$, and for any $0 < \eta < 1$, let $f^+_E \in C^5_0(\mathbb{R})$ be a nonnegative function with $0 \leq f^+_E \leq 1$, and such that f^+_E is monotone decreasing on $[\Sigma_0 - 1, E + \eta]$, with $\chi_{[\Sigma_0, E]} f^+_E = \chi_{[\Sigma_0, E]}$. We define $f^-_E(x) = f^+_E(x - \eta)$ so that $\chi_{[\Sigma_0 - 1, E - \eta]} f^-_E = \chi_{[\Sigma_0 - 1, E - \eta]}$. In this way, we have $0 \leq f^+_E - f^-_E \leq 1$, and the difference $(f^+_E - f^-_E)\chi_{[\Sigma_0, \infty]}$ is supported in $[E - \eta, E + \eta]$.
2. We fix $E > 0$ and take any $B_0 > 0$, without loss of generality, since $N(E; B)$ is an even function of B. We consider any $B > 0$ with $|B - B_0| << 1$. If $N(E; B) > N(E; B_0)$, then monotonicity implies that

$$0 \leq N(E; B) - N(E; B_0) \leq \nu_B(f^+_E) - \nu_{B_0}(f^-_E), \tag{5.20}$$

whereas, if $N(E; B) < N(E; B_0)$, we have

$$0 \leq N(E; B_0) - N(E; B) \leq \nu_{B_0}(f^+_E) - \nu_B(f^-_E). \tag{5.21}$$

In the case of (5.20), we have

$$
\begin{aligned}
0 \leq N(E;B) - N(E;B_0) &\leq \nu_B(f_E^+) - \nu_{B_0}(f_E^-) \\
&\leq |\nu_B(f_E^+) - \nu_{B_0}(f_E^+)| + \nu_{B_0}(f_E^+ - f_E^-) \\
&\leq C((f_E^+)^{(5)})\tilde{\delta}(B,B_0) + \nu_{B_0}(f_E^+ - f_E^-),
\end{aligned} \tag{5.22}
$$

where we used Proposition 5.1 to evaluate the first term on the right in (5.22). To study the second term on the right of the last line in (5.22), we recall that supp $(f_E^+ - f_E^-) = [E-\eta, E+\eta]$, and that $0 \leq f_E^+ - f_E^- \leq 1$. We know from Theorem 4.3 that the IDS is Hölder continuous in energy at B_0, so for any $0 < q < 1$, there is a constant $C_{(q,B_0,E)} > 0$, locally uniform in the energy, so that

$$
|\nu_{B_0}(f_E^+ - f_E^-)| \leq C_{(q,B_0,E)}\eta^q. \tag{5.23}
$$

We recall from Proposition 5.1 that $C((f_E^+)^{(5)}) \sim \eta^{-4}$. We take $\eta = |B - B_0|^\sigma$, for an appropriate small constant $\sigma > 0$. Since we assume that $|B - B_0| < 1$, we have $\tilde{\delta}(B,B_0) = |B - B_0|$. Hence, for Hölder continuity, we want $1 - 4\sigma > 0$, or $0 < \sigma < 1/4$. This implies that $0 < q\sigma < 1/4$, since $0 < q < 1$ is arbitrary. If we have the case in (5.21), we find a similar calculation yields

$$
\begin{aligned}
0 < N(E;B_0) - N(E;B) &\leq \nu_{B_0}(f_E^+) - \nu_B(f_E^-) \\
&\leq \nu_{B_0}(f_E^+ - f_E^-) + |\nu_{B_0}(f_E^-) - \nu_B(f_E^-)| \\
&\leq \nu_{B_0}(f_E^+ - f_E^-) + C((f_E^-)^{(j)})\tilde{\delta}(B,B_0) \\
&\leq C_{(\alpha,B_0,E)}|B - B_0|^\alpha,
\end{aligned} \tag{5.24}
$$

for some $0 < \alpha < 1/4$, as above. We used $\eta = |B - B_0|^\sigma$, for any $0 < \sigma < 1/4$, Proposition 5.1, and (5.23). For both cases, we find that for $|B - B_0| < 1$, and for any $0 < \alpha < 1/4$, there is a finite constant $C_{(\alpha,B_0,E)} > 0$ so that

$$
|N(E;B) - N(E;B_0)| \leq C_{(\alpha,B_0,E)}|B - B_0|^\alpha. \tag{5.25}
$$

This proves the result. □

6. Appendix: Some Technical Estimates

We briefly recall the Helffer-Sjöstrand formula for functions of self-adjoint operators, and refer to Davies' book [9] for more details. Let $f \in C_0^{(k+1)}$ be a real-valued function. We construct an almost analytic extension of f of order k with the aid of a smooth cut-off function $\sigma(\eta)$ satisfying supp $\sigma = [-1,1]$ and $\sigma|[-1/2,1/2] = 1$. An almost analytic extension of f, denoted by $\tilde{f}(z)$, with $z = \xi + i\eta$, is constructed as

$$
\tilde{f}(z) = \sum_{j=0}^{k} \frac{f^{(j)}(\xi)}{j!}(i\eta)^j \sigma(\eta). \tag{6.1}
$$

This compactly-supported function satisfies $\tilde{f}(\xi) = f(\xi)$, and $|\partial_{\bar{z}}\tilde{f}(\xi + i\eta)| \sim |\eta|^k |f^{(k+1)}(\xi)|$, as $|\eta| \downarrow 0$. The Helffer-Sjöstrand formula for $f(H)$, where H is a self-adjoint operator and function f is as above, is

$$
f(H) = \frac{i}{2\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{f}(z)(H - z)^{-1}\, d\xi\, d\eta. \tag{6.2}
$$

The resolvent of the Landau Hamiltonian $R_0(z) = (H_L(B) - z)^{-1}$ has an integral kernel given by

$$
R_0(z)(x,y) = \sum_{n \geq 0} \frac{\Pi_n(x,y)}{E_n(B) - z}, \tag{6.3}
$$

for $(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$. This sum converges in the strong operator sense for $\Im z \neq 0$. Let Π_n be the projection onto the infinitely-degenerate subspace corresponding to the Landau level $E_n(B) = (2n+1)B$. The projector Π_n has an explicit kernel given by

$$
\Pi_n(x,y) = L_n(\frac{B}{2}|x - y|^2)\Pi_0(x,y), \tag{6.4}
$$

where L_n is the n^{th}-Laguerre polynomial normalized so that $L_n(0) = 1$ (see the definition of $L_n = L_n^{(0)}$ in Lemma 4.1). The projectors satisfy

$$\Pi_n(x,x) = L_n(0)\Pi_0(x,x) = \frac{B}{2\pi}. \tag{6.5}$$

The ground state projector Π_0 has the explicit kernel

$$\Pi_0(x,y) = \left(\frac{B}{2\pi}\right) e^{-\frac{B}{4}|x-y|^2} e^{-i\frac{B}{2}x\wedge y}. \tag{6.6}$$

Proposition 6.1. *Let χ_j, for $j = 1, 2$, be two bounded, real-valued functions of compact support, and suppose that $f \in C_0^3(\mathbb{R})$ is real-valued. Then, the operator $\chi_1 f(H_\omega)\chi_2$ is in the Hilbert-Schmidt class, and there exists a finite, positive constant $C_2 > 0$, independent of B and f, so that*

$$\begin{aligned} \|\chi_1 f(H_\omega)\chi_2\|_{HS} &\leq C_2(1+B^2)^{1/2}B^{-1/2}\|\chi_2\|_\infty\, |supp\ \chi_2|^{1/2} \\ &\times \left(\|\chi_1\|_\infty |supp\ \chi_1|^{1/2}|supp\ f|^{1/2}\|f\|_\infty + \sum_{j=0}^{3} d_j \int |f^{(j)}(\xi)| d\xi \right), \end{aligned} \tag{6.7}$$

where the finite constants $d_j > 0$, for $j = 0, 1, 2, 3$ depend only on σ appearing in (6.1).

Proof.

1. From the resolvent formula for $R_\omega(z)$ and $R_0(z)$, and Helffer-Sjöstrand formula (6.2), we have

$$f(H_\omega) = f(H_L(B)) - \frac{1}{\pi}\int\int \partial_{\bar{z}}\tilde{f}(z)R_\omega(z)V_\omega R_0(z)\ d\xi\ d\eta, \tag{6.8}$$

where we write $z = \xi + i\eta$. It follows that it suffices to estimate the Hilbert-Schmidt norm of

$$\chi_1 f(H_L(B))\chi_2, \tag{6.9}$$

and of

$$\chi_1 R_\omega(z)V_\omega R_0(z)\chi_2. \tag{6.10}$$

2. To estimate the Hilbert-Schmidt (HS) norm of (6.9), the compactness of the support of f means that it contains at most finitely-many Landau levels $E_k(B)$, for $k = i_1, \ldots, i_L$. If supp f does not contain a Landau level, the operator $f(H_L(B))$ vanishes. In the former case, we have $f(H_L(B)) = \sum_{k=i_1}^{i_L} f(E_k(B))\Pi_k$, and we can use the the explicit form of the projector Π_k given in (6.4) to calculate the HS norm,

$$\begin{aligned} \|\chi_1 f(H_L(B))\chi_2\|_{HS}^2 &\leq \sum_{k=i_1}^{i_L} |f(E_k(B))|^2 \int\int \chi_1(x)^2|L_k(\frac{B}{2}|x-y|^2)\Pi_0(x,y)|^2\chi_2(y)^2\ d^2x\ d^2y \\ &\leq \left(\frac{B}{2\pi}\right)\sum_{k=i_1}^{i_L} |f(E_k(B))|^2\ |\text{supp}\ \chi_1|\|\chi_1\|_\infty^2\|\chi_2\|_\infty^2, \end{aligned} \tag{6.11}$$

where we used the fact that $\int_0^\infty e^{-w}L_k^2(w)\ dw = 1$ (cf. [1]).

3. As for the main term (6.10), we use the fact that the random potential satisfies $\|V_\omega\| \leq M_0 < \infty$, and the norm bound

$$\|R_\omega(z)\| \leq 1/|\eta|, \tag{6.12}$$

with $z = \xi + i\eta$. Since $\|A\|_{HS}^2 = \|A^*A\|_1$, we have to estimate

$$\|R_0(z)\chi_2\|_{HS}^2 = \|\chi_2|R_0(z)|^2\chi_2\|_1. \tag{6.13}$$

The operator $\chi_2|R_0(z)|^2\chi_2$ is a positive operator with an explicit kernel $K_0(x,y;z)$ given by

$$K_0(x,y;z) = \sum_{n=0}^{\infty}\frac{\chi_2(x)\Pi_n(x,y)\chi_2(y)}{|E_n(B)-z|^2} = \sum_{n=0}^{\infty}\frac{\chi_2(x)\Pi_n(x,y)\chi_2(y)}{|(2n+1)B-z|^2}, \tag{6.14}$$

that converges in the strong operator sense for $\Im z \neq 0$. For the diagonal part of the kernel, it follows from (6.5) that

$$K_0(x,x;z) = \frac{B\chi_2(x)^2}{2\pi} \sum_{n=0}^{\infty} \frac{1}{((2n+1)B-\xi)^2+\eta^2}. \tag{6.15}$$

It is easy to see that

$$K_0(x,x;z) \leq \left(\frac{B\|\chi_2\|_\infty^2}{2\pi} \right) \left\{ \frac{1}{\eta^2} + \frac{\pi^2}{4B^2} \right\}. \tag{6.16}$$

Consequently, the trace norm is bounded as

$$\|\chi_2 |R_0(z)|^2 \chi_2\|_1 \leq \|\chi_2\|_\infty^2 \ |\text{supp } \chi_2| \left(\frac{B}{2\pi} \right) \left\{ \frac{1}{\eta^2} + \frac{\pi^2}{4B^2} \right\}. \tag{6.17}$$

We substitute this estimate into the second term of (6.8) and obtain

$$\begin{aligned} & \| \int \int \partial_{\bar{z}} \tilde{f}(z) \chi_1 R_\omega(z) V_\omega R_0(z) \chi_2 \ d\xi \ d\eta \|_{HS} \\ \leq & \ C_1 M_0 \int \int \frac{|\partial_{\bar{z}} \tilde{f}(z)|}{|\eta|} \left\{ \frac{1}{\eta^2} + \frac{\pi^2}{4B^2} \right\}^{1/2} \ d\eta d\xi, \end{aligned} \tag{6.18}$$

where the constant C_1 is

$$C_1 = \|\chi_2\|_\infty |\text{supp } \chi_2|^{1/2} (B/2\pi)^{1/2}. \tag{6.19}$$

4. In order to evaluate the integral (6.18), we take an almost analytic extension $\tilde{f}$ of order 3, as in (6.1). We note that $\partial_{\bar{z}} \tilde{f}$ has one term depending on $f^{(3)}$, and another term that is multiplied by $\sigma'(\eta)$. This latter term vanishes in a neighborhood of the real axis $\eta = 0$, so it can be estimated directly. As for the contribution of the former term to the integral (6.18), as the coefficient of this term vanishes as $|\eta|^2$, the integral is finite. Combining (6.11) and (6.18), we obtain the bound (6.7), proving the proposition. □

Proposition 6.2. *Let χ_j, for $j = 1, 2$, be two bounded, real-valued functions with disjoint supports, so that dist (supp χ_1, supp χ_2) $\equiv d_{12} > 0$, and suppose that χ_1 has compact support. For any positive integer M, let $f \in C_0^{k+1}(\mathbb{R})$, for $k \geq M+1$, be a real-valued function. Then, there exists a constant $C_M > 0$, independent of f and d_{12}, so that*

$$\|\chi_1 f(H_\omega) \chi_2\|_{HS} \ \leq \ \left(\frac{C_M}{d_{12}^M} \right) \left(\sum_{j=0}^{k+1} d_j \int_{\mathbb{R}} |f^{(j)}(\xi)| \ d\xi \right), \tag{6.20}$$

where the finite constants d_j depend only on σ appearing in (6.1).

Proof.
1. The proof of this proposition is based on the polynomial decay of the operator $f(H_\omega)$ when localized between two functions of disjoint support. This result depends on the decay of the localized resolvent $\chi_1 R_\omega(z) \chi_2$ that in turn follows from the Combes-Thomas method [8]. These results are well-known, see, for example, [13] or [22]. Here, we note that we can obtain the estimate in the Hilbert-Schmidt norm under the above hypotheses. The basic Combes-Thomas estimate for resolvent of H_ω has the form

$$\|\chi_1 R_\omega(z) \chi_2\|_{HS} \leq \frac{C_1}{(\text{dist } (z, \sigma(H_\omega)))} e^{-C_2 \text{dist } (z,\sigma(H_\omega)) d_{12}}, \tag{6.21}$$

where the constants can be chosen independent of z. For our purposes, it suffices to use the following simple estimate,

$$e^{-t} \leq \frac{b_M}{t^M}, \ \text{for } t \geq 0, \ \text{with } b_M = e^{-M} M^M, \tag{6.22}$$

so that

$$\|\chi_1 R_\omega(z) \chi_2\|_{HS} \leq \left(\frac{1}{d_{12}^M} \right) \left(\frac{C_1 b_M}{C_2^M} \right) \left(\frac{1}{\text{dist } (z, \sigma(H_\omega))^{1+M}} \right). \tag{6.23}$$

2. We use the estimate (6.23) and the Helffer-Sjöstrand formula (6.2) to calculate the norm (6.20). For any integer M, there is a finite constant $C_M > 0$ so that if the order k of the almost analytic extension of f satisfies $k \geq M + 1$, then

$$
\begin{aligned}
\|\chi_1 f(H_\omega)\chi_2\|_{HS} &\leq C_1 \int\int |\partial_{\overline{z}}\tilde{f}(z)| \, \|\chi_1 R_\omega(z)\chi_2\|_{HS} \; d\xi \; d\eta \\
&\leq \frac{C_M}{d_{12}^M} \int\int |\partial_{\overline{z}}\tilde{f}(z)||\eta|^{k-(M+1)}|f(\xi)^{(k+1)}(\xi)| \; d\xi \; d\eta \\
&\quad + (\text{finite terms depending on } f^{(j)}, j = 1, \ldots, k), \qquad (6.24)
\end{aligned}
$$

giving the main result. □

Finally, we prove two technical bounds used in section 5. Let χ be a smooth function with compact support.

Lemma 6.1. *For any $B, B' \neq 0$, we define $\delta(B, B') \equiv H_\omega(B) - H_\omega(B')$, and define $\tilde{\delta}(B, B') =$ max $(|B - B'|, |(B')^2 - B^2|)$. We then have*

$$\|\delta(B, B')R_\omega^{B^*}(\xi + i\eta)\chi_0\| \leq C_0(B^*)\tilde{\delta}(B, B')|\eta|^{-3}, \qquad (6.25)$$

where B^* is either B or B', the constant $C_0(B^*) > 0$ depends on B^* and χ. Furthermore, we have

$$\|\delta(B, B')R_\omega^{B^*}(z)R_\omega^{B^*}(\lambda_0)[H_\omega(B^*), \chi]\| \;\leq\; C_1(B^*)\tilde{\delta}(B, B')|\eta|^{-3}, \qquad (6.26)$$

where the constant $C_1(B^*)$ depends on B^* and χ.

Proof. It is slightly more convenient to do the computation in the Landau gauge $A = B(0, x_1)$. We write the difference $\delta(B, B')$ as

$$
\begin{aligned}
\delta(B, B') &= (B^2 - (B')^2)x_1^2 + 2(B' - B)x_1 p_2 \\
&= \{(B^2 - (B')^2) + 2B^*(B' - B)\}x_1^2 + 2(B' - B)x_1 V_2^*, \qquad (6.27)
\end{aligned}
$$

where $V_2^* = p_2 - B^* x_1$. To prove (6.25), it suffices to estimate $x_1^2 R_\omega^{B^*}\chi$, and $x_1 V_2^* R_\omega^{B^*}\chi$. Since $\|p_1 R_\omega^{B^*}\| = \|V_2^* R_\omega^{B^*}\| = \mathcal{O}(|\eta|^{-1})$, and $\|x_1^2\chi\|$ is bounded, we easily obtain the estimates by commuting the powers of x_1 to the right. The proof of (6.26) is similar. Here, we make use of the fact that one of the resolvents is evaluated at the fixed value λ_0, and hence does not contribute to the singularity in η. The proof is again obtained by commuting the powers of x_1 to the right. □

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