Existence of the Bogoliubov $S(g)$ Operator for the $(:\phi^4:)_2$ Quantum Field Theory

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Abstract

We prove the existence of the Bogoliubov $S(g)$ operator for the $(:\phi^4:)_2$ quantum field theory. The construction is nonperturbative and relies on a theorem of Kisyński. It implies almost automatically the properties of unitarity and causality for disjoint supports in the time variable.

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I. INTRODUCTION AND SUMMARY

Recent progress in perturbative quantum field theory for the Stückelberg-Bogoliubov-Epstein-Glaser $S(g)$ operator [1, 2] in nonabelian gauge theories [3] (see also [4]), revived interest in a long-standing problem: is it possible to construct $S(g)$ *nonperturbatively* in quantum field theory? This question is of obvious relevance to theories where the (dimensionless) coupling constant is large ($\gtrsim 1$) – e.g. strong interactions – for which perturbation theory is not expected to be asymptotic.

For certain superrenormalizable theories – the $(: P(\phi) :)_2$ theories – there exists, for weak coupling, a construction of the true (LSZ-Haag-Ruelle) scattering operator, due to Osterwalder and Séneor [5] and Eckmann, Epstein and Fröhlich [6], one of the crowning achievements of constructive quantum field theory. The method of proof was, however, perturbative: the perturbation series for the scattering operator was shown to be asymptotic.

In contrast to the true scattering operator, $S(g)$ is, in perturbation theory, the generating functional for the time-ordered products of Wick polinomials. However, on the basis of [7] one might expect that, in the present massive case, defining

$$g_\varepsilon(x) \equiv g(\varepsilon x) \; ; \qquad g \in \mathcal{S}(\mathbb{R}^2)$$

the (adiabatic) limit

$$S\Psi \equiv \lim_{\varepsilon \to 0} \; S(g_\varepsilon)\Psi \tag{1.1}$$

exists, $\forall \; \Psi \in \mathcal{D}$, where $\mathcal{D}$ is a Poincaré-invariant dense set in Fock space $\mathcal{F}$. Thus we expect that the physical S-matrix elements are obtainable as

$$(\Phi, S\Psi) \equiv \lim_{\varepsilon \to 0} \; (\Phi, S(g_\varepsilon)\Psi) \; , \tag{1.2}$$

with $\Phi \in \mathcal{F}$, $\Psi \in \mathcal{D}$, where $g(0) > 0$ should be identified with the coupling constant. In [4] an algebraic construction of the adiabatic limit was performed for perturbative QED.

A natural nonperturbative approach to construct $S(g)$ for the $(: \phi^4 :)_2$ theory (and hopefully for any super-renormalizable QFT) consists in proving the existence of a (unique) solution of the evolution (propagator) equation ($\hbar = 1$)

$$i\frac{\partial U(t,s)}{\partial t}\Psi = \tilde{H}(t)U(t,s)\Psi \ , \tag{1.3}$$

with

$$\tilde{H}(t) \equiv H_g(t) + M\mathbf{1} \ , \tag{1.4}$$

where M is a constant introduced in order to make $\tilde{H}(t)$ a positive operator (see section II) and

$$H_g(t) \equiv H_0 + V_g(t) \ . \tag{1.5}$$

In (1.3) $U(t,s)$ is a two-parameter family of unitary operators on (symmetric) Fock space $\mathcal{F}$. H_0 is the free field Hamiltonian corresponding to a zero-time scalar field $\phi(x,0)$ of mass m [8, 9], and, formally, for

$$g \in \mathcal{D}(\mathbb{R}^2) \tag{1.6}$$

let

$$V_g(t) = \int dx\ g(x,t) : \phi^4(x,0) : \ . \tag{1.7}$$

Above, $\mathcal{D}$ denotes the Schwartz space of infinitely differentiable functions of compact support. The operators in (1.3) are expected to satisfy the propagator conditions:

$$\begin{aligned} &U(t,s)U(s,r) = U(t,r) \ , && -\infty < r \leq s \leq t < \infty \ , \\ &U(t,t) = \mathbf{1} \ , && \forall\, t \in \mathbb{R} \ . \end{aligned} \tag{1.8}$$

The vector Ψ is supposed to belong to the domain $D(\tilde{H}(s))$ (dense in $\mathcal{F}$) such that

$$U(t,s)D(\tilde{H}(s)) \subset D(\tilde{H}(t)) \tag{1.9}$$

Above and elsewhere in this paper $D(A)$ denotes the domain of an operator A.

Under assumptions (1.3) and (1.9), defining the "Dirac (or interaction) picture propagator" by

$$U^D(t,s) \equiv e^{i(H_0+M)t}U(t,s)e^{-i(H_0+M)s} \ , \tag{1.10}$$

it follows that

$$i\frac{\partial U^D(t,s)}{\partial t}\Psi = H_g^D(t)U^D(t,s)\Psi\ , \tag{1.11}$$

for $\Psi \in e^{i(H_0+M)s}D(H_g(s))$, which is a dense set in $\mathcal{F}$ for every s, where

$$H_g^D(t) \equiv e^{iH_0t}V_g(t)e^{-iH_0t}\ . \tag{1.12}$$

One may then define

$$S(g) \equiv s - \lim_{\substack{t\to+\infty \\ s\to-\infty}} U^D(t,s)\ , \tag{1.13}$$

if the above limit exists; $S(g)$ is expected to satisfy

(i) $S(g)^{-1} = S(g)^*$ (*unitarity*);

(ii) $S(g_1+g_2) = S(g_1)S(g_2)$ if

$(ii.a)$ supp $g_1 >$ supp g_2

and/or

$(ii.b)$ supp $g_1 \sim$ supp g_2 (*causality*)

where "$\sim$" means "spacelike to", i.e., $(x-y)^2 = (t_1-t_2)^2 - (x_1-x_2)^2 < 0$, $\forall\ (t_1,x_1) \in$ supp g_1 and $\forall\ (t_2,x_2) \in$ supp g_2;

(iii) There exists a unitary representation $U(a,\Lambda)$ of the Poincaré group on $\mathcal{F}$ – the scalar field representation of mass m –such that

$$U(a,\Lambda)S(g)U(a,\Lambda)^{-1} = S(\{a,\Lambda\}g)\ ,$$

where

$$(\{a,\Lambda\}g)(x) = g(\Lambda^{-1}(x-a))$$

(*Lorentz covariance*).

The basic problem to prove (1.3)-(1.9) is the fact that $D(H_g(t))$ is, for each $g \in \mathcal{D}(\mathbb{R}^2)$, time-dependent. In section II we state the basic existence theorem we employ, which is due to Kisyński [11] (see also [12]). In section III we prove our central existence theorem for $S(g)$, as well as properties (i) and $(ii.a)$. In section IV we provide a brief summary of the remarkable results of [11], establishing a concrete link between them and our conditions in section III. We leave the conclusion and open problems to section V. Appendix A summarizes some of the basics elements of the construction of [11] and [15] for the convenience of the reader.

II. THE BASIC EXISTENCE THEOREM

The Hamiltonian of the $(:\phi^4:)_2$ theory [10] is given by (1.5), where

$$H_0 = \int \omega(k) a^*(k) a(k) dk \ , \tag{2.1}$$

with

$$\omega(k) = \left(k^2 + m^2\right)^{\frac{1}{2}} \ , \tag{2.2}$$

is the free field Hamiltonian on symmetric Fock space $\mathcal{F}$, with

$$[a(k), a^*(k')] = \delta(k - k') \ . \tag{2.3}$$

The self-interaction V_g is given by (1.7), with the $t = 0$ scalar free field of mass m:

$$\phi(x) = \frac{1}{(4\pi)^{\frac{1}{2}}} \int e^{-ikx} \left[a^*(k) + a(-k)\right] \omega(k)^{-\frac{1}{2}} dk \tag{2.4}$$

Thus V_g may be written [10]

$$\begin{aligned} V_g(t) \ &= \sum_{j=0}^{4} \binom{4}{j} \int a^*(k_1) \cdots a^*(k_j) a(-k_{j+1}) \cdots a(-k_4) \\ &\times \ \tilde{g}(\textstyle\sum_{i=1}^{4} k_i, t) \prod_{i=1}^{4} \omega(k_i)^{-\frac{1}{2}} dk_i \ , \end{aligned} \tag{2.5}$$

where

$$\tilde{g}(k, t) \equiv \int dx \ e^{ikx} g(x, t) \tag{2.6}$$

The number operator N is defined by

$$N = \int dk \ a^*(k) a(k) \ , \tag{2.7}$$

By [10] (Lemma 2.2)

$$\left\|(N+\mathbf{1})^{-\frac{j}{2}}\, V_g(t)\, (N+\mathbf{1})^{-\frac{4-j}{2}}\right\| \leq \text{ const. } \|W\|_{L^2}\ , \qquad |j| \leq 4 \tag{2.8}$$

where

$$W(k,t) \equiv \tilde{g}(\sum_{i=1}^{4} k_i, t) \prod_{i=1}^{4} \omega(k_i)^{-\frac{1}{2}} \tag{2.9}$$

The above mentioned lemma just uses the Fock space definitions of the creation and annihilation operators and the Schwartz inequality. We need two theorems due to Glimm and Jaffe, which we state as adapted to our case:

Theorem II.1 [10] **(a)** *$H(t)$ is self-adjoint on the domain*

$$D(H(t)) = D(H_0) \cap D(V_g(t))\ , \tag{2.10}$$

where $D(V_g(t))$ is the domain of the unique self-adjoint closure of $V_g(t)$ on the domain

$$D_0 = \bigcup_{n=0}^{\infty} D(H_0^n). \tag{2.11}$$

(b) *$H(t)$ is essentially self-adjoint on D_0.*

Theorem II.2 [9] *For each $g \in \mathcal{D}(\mathbb{R}^2)$, there exists $0 < M_g < \infty$ such that*

$$H_g(t) \geq -M_g \mathbf{1} \tag{2.12}$$

as a bilinear form on $D_0 \times D_0$.

By theorem II.2 and **(b)** of theorem II.1, $H(t)$ is a semi-bounded self-adjoint operator, and thus defining

$$M = M_g + c, \tag{2.13}$$

for some $c > 0$, then

$$\tilde{H}(t) = H_g(t) + M\mathbf{1} \geq c\mathbf{1} \tag{2.14}$$

is a positive self-adjoint operator. Let $\mathcal{F}_{+2} = D(H_0)$ endowed with the Hilbert space structure given by

$$f_{+2}(x,y) = \langle (H_0+1)\,x, (H_0+1)\,y \rangle \tag{2.15}$$

and denote $\sqrt{f_{+2}(x,x)}$ by $||x||_{+2}$. By the Riesz lemma we may associate $\mathcal{F}_{+2}$ and the space $\mathcal{F}_{-2}$ of continuous conjugate linear functions on $\mathcal{F}_{+2}$. While we consider $\mathcal{F}$ isomorphic to its conjugate dual space $\mathcal{F}^*$, the isomorphism being the identity, the isomorphism of $\mathcal{F}_{+2}$ with $\mathcal{F}_{-2}$ is given by the operator $(H_0+1)^2$, because

$$||v||_{-2} = \sup\{|\langle w, v\rangle| \ : \ ||w||_{+2} \leq 1\}.$$

Since $f_{+2}(x,y) = \langle x, (H_0+1)^2 y\rangle$, we have

$$\begin{aligned} \left\|(H_0+1)^2 y\right\|_{-2} &= \sup\left\{|\langle w, (H_0+1)^2 y\rangle| \ : \ ||w||_{+2} = \sqrt{\langle w, (H_0+1)^2 w\rangle} \leq 1\right\} \\ &= \left\|(H_0+1)^2 y\right\| = ||y||_{+2}, \end{aligned}$$

from which also, for $y \in \mathcal{F}$:

$$||y||_{-2} = ||\,(H_0+1)^{-1} y||, \tag{2.16}$$

which explains the notation $\mathcal{F}_{-2}$. Clearly $||x|| \leq ||x||_{+2}$ for $x \in \mathcal{F}_{+2}$, and by (2.16), $||y||_{-2} \geq ||y||$ for $y \in \mathcal{F}$. Thus, under the above conditions:

$$\mathcal{F}_{+2} \subset \mathcal{F} \subset \mathcal{F}_{-2}. \tag{2.17}$$

A bounded operator $\mathbf{B}$ from $\mathcal{F}_{+2}$ to $\mathcal{F}_{-2}$ is thus such that, for some constant c,

$$||\mathbf{B}\psi||_{-2} \leq c||\psi||_{+2} \qquad \psi \in \mathcal{F}_{+2}, \tag{2.18}$$

or, by (2.15) and (2.16),

$$||\,(H_0+1)^{-1}\mathbf{B}\psi|| \leq c||\,(H_0+1)\,\psi|| \qquad \psi \in \mathcal{F}_{+2}, \tag{2.19}$$

or

$$||\,(H_0+1)^{-1}\mathbf{B}\,(H_0+1)^{-1}\phi|| \leq c||\phi|| \qquad \phi \in \mathcal{F}. \tag{2.20}$$

Now, by (2.14), we may define $\tilde{H}(t)^{1/2}$, and, by (2.8) for $x \in \mathcal{F}_{+2}$, the closed sesquilinear form

$$S(x,y) = \langle \tilde{H}(t)^{1/2}x, \tilde{H}(t)^{1/2}y\rangle \tag{2.21}$$

which is, by the form representation theorem [14], the form of the operator $\tilde{H}(t)$. In section III we show the explicit connection of (2.21) to the basic theorem of Kisyński [11], which we state in the form of theorems II.23 and II.24 of [12], with slight changes:

Theorem II.3 *Let (2.17) hold and $\tilde{H}(t)$ $(-T \leq t \leq S)$ be a one-parameter family of strictly positive (i.e. satisfying (2.14)) self-adjoint operators on $\mathcal{F}$. Suppose that $\tilde{H}(t) : \mathcal{F}_{+2} \to \mathcal{F}_{-2}$ are bounded and twice differentiable, with a continuous second derivative, in the $||\cdot||_{-2,2}-$norm (2.18). Then there exists a two-parameter family $U(t,s)$ of unitary propagators satisfying (1.3), (1.8) and (1.9).*

III. THE CENTRAL EXISTENCE THEOREM

We now use theorem II.3 in order to prove our main

Theorem III.1 *The $(:\phi^4:)_2$ theory, as defined by (1.5), (1.6), (1.7), (2.1) and (2.2) satisfies a stronger condition than the hypothesis of theorem II.3: $H_g(\cdot)$ is infinitely differentiable as an operator from $\mathcal{F}_{+2}$ to $\mathcal{F}_{-2}$.*

In order to prove theorem III.1 we first show a useful auxiliary result.

Lemma III.1 *Let W be defined by (2.9). Then there exists $r > 1$ such that*

$$||W(\cdot,t)||_2 \leq ||g(\cdot,t)||_r \tag{3.1}$$

where

$$||g(\cdot,t)||_r = \left(\int_{-\infty}^{+\infty} dk|\tilde{g}(k,t)|^r\right)^{1/r} . \tag{3.2}$$

Proof. We have

$$\begin{aligned} ||W(\cdot,t)||_2^2 \; = & \int_{-\infty}^{+\infty} dk_1\omega(k_1)^{-1} \cdot \int_{-\infty}^{+\infty} dk_2\omega(k_2)^{-1} \\ & \cdot \int_{-\infty}^{+\infty} dk_3\omega(k_3)^{-1} \cdot \int_{-\infty}^{+\infty} dk'|\tilde{g}(k',t)|^2\omega\left(k' - \sum_{i=1}^{3} k_i\right)^{-1} \end{aligned} \tag{3.3}$$

by the change of variable $k' = \sum_{i=1}^{3} k_i$. Introducing further the variables K_1, K_2, K_3 such that

$$\begin{aligned} K_1 &= k_1 + k_2 + k_3 \\ K_2 &= k_1 + k_2 \\ K_3 &= k_1 \end{aligned}$$

so that $k_3 = K_1 - K_2$ and $k_2 = K_2 - K_3$, we write (3.3) as

$$||W(\cdot,t)||_2^2 = \left(\omega^{-1} * \left(\omega^{-1} * \left(\omega^{-1} * \left(\omega^{-1} * |\tilde{g}|^2\right)\right)\right)\right)(0), \tag{3.4}$$

where the convolution is defined as usual by

$$(f * g)(k) = \int_{-\infty}^{+\infty} dk_1 f(k-k_1) g(k_1).$$

Consider, now, the quantity associated to the right-hand side of (3.3):

$$\begin{aligned} I(q,t) \;\equiv\; & \int_{-\infty}^{+\infty} dk_1\, \omega(k_1-q)^{-1} \cdot \int_{-\infty}^{+\infty} dk_2\, \omega(k_2)^{-1} \\ \cdot\; & \int_{-\infty}^{+\infty} dk_3\, \omega(k_3)^{-1} \cdot \int_{-\infty}^{+\infty} dk'\, |\tilde{g}(k',t)|^2 \omega\left(k' - \sum_{i=1}^{3} k_i\right)^{-1} \end{aligned} \tag{3.5}$$

Since $g \in \mathcal{D}(\mathbb{R})$ this function is differentiable, hence continuous, in q for any compact subset containing the origin, which implies that $I(0,t) \le \|I(\cdot,t)\|_\infty$ (where $\|\cdot\|_\infty$-norm is with respect to the q-variable).

We now apply Young's inequality [13]

$$||f * g||_r \le C_{rpq} ||f||_p ||g||_q$$

with C_{rpq} a constant and

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

to (3.4), starting with $r = \infty$. Above,

$$||f||_p = \left(\int_{-\infty}^{+\infty} dk |f(k)|^p\right)^{1/p}.$$

We thus obtain

$$\|W(\cdot,t)\|_2^2 \le C_{2r_1r_2} \left\|\omega^{-1}\right\|_{r_1} \left\|\left(\omega^{-1} * \left(\omega^{-1} * \left(\omega^{-1} * |\tilde{g}|^2\right)\right)\right)\right\|_{r_2}$$

with $r_1^{-1} + r_2^{-1} = 1$, and so on, up to (indicating all the constants resulting from the Young's inequality by C')

$$||W(\cdot,t)||_2^2 \le C' \left\|\omega^{-1}\right\|_{r_1} \left\|\omega^{-1}\right\|_{r_3} \left\|\omega^{-1}\right\|_{r_5} \left\|\omega^{-1}\right\|_{r_7} \left\||\tilde{g}|^2\right\|_{r_8} \tag{3.6}$$

with $r_3^{-1} + r_4^{-1} = 1 + r_2^{-1}$, $r_5^{-1} + r_6^{-1} = 1 + r_4^{-1}$, $r_7^{-1} + r_8^{-1} = 1 + r_6^{-1}$. We require $r_i > 1$, for $i = 1, 3, 5, 7$, so that $||\omega^{-1}||_{r_i} < \infty$, the choice $r_1 = r_2 = 2$, $r_3 = r_4 = \frac{4}{3}$, $r_5 = r_6 = \frac{8}{7}$, $r_7 = r_8 = \frac{16}{15}$ is, for instance, possible. By (3.6)

$$||W(\cdot, t)||_2^2 \leq C \left\| |\tilde{g}|^2 \right\|_r \tag{3.7}$$

with

$$r > 1 \; . \tag{3.8}$$

Above

$$\| \, |\tilde{g}|^2 \, \|_r = \left(\int_{-\infty}^{+\infty} dk \, |\tilde{g}(k, t)|^{2r} \right)^{1/r} \; . \tag{3.9}$$

obtaining finally, (3.7). $\square$

Proof of III.1 By (2.8),

$$\left\| (N + \mathbf{1})^{-1} \, V_g(t) \, (N + \mathbf{1})^{-1} \right\| \leq \text{const.} ||W||_{L^2} \tag{3.10}$$

and, by (2.2), $\omega(k) \geq m\mathbf{1}$; hence

$$\left\| (H_0 + \mathbf{1})^{-1} \, (N + \mathbf{1}) \right\| \leq d_1 \qquad \left\| (N + \mathbf{1}) \, (H_0 + \mathbf{1})^{-1} \right\| \leq d_2,$$

for constants d_1 e d_2. Hence, by (3.10) and (3.1),

$$\left\| (H_0 + \mathbf{1})^{-1} \, V_g(t) \, (H_0 + \mathbf{1}) \right\| \leq \text{const.} ||g(\cdot, t)||_r \tag{3.11}$$

with $r > 1$: a fortiori this holds for $H_g(\cdot)$ by (1.5), hence

$$\left\| (H_0 + \mathbf{1})^{-1} \, H_g(t) \, (H_0 + \mathbf{1}) \right\| \leq \text{const.} ||g(\cdot, t)||_r. \tag{3.12}$$

By (2.20) and theorem II.3 we need only prove that the l.h.s. of (3.12) is three times differentiable. We shall prove that

$$\left\| (H_0 + \mathbf{1})^{-1} \left(\frac{H_g(t + h) - H_g(t)}{h} - {H'}_g(t) \right) (H_0 + \mathbf{1})^{-1} \right\| \longrightarrow 0 \quad \text{as} \quad h \to 0, \tag{3.13}$$

where

$$H_g'(t) = H_0 + V_{g'}(t) \tag{3.14}$$

with

$$V_{g'}(t) = \int dx : \phi^4(x,0) : g'(x,t) \tag{3.15}$$

and

$$g'(x,t) \equiv \frac{\partial g(x,t)}{\partial t}.$$

We now prove (3.13). By (3.12)

$$\left\| (H_0 + \mathbf{1})^{-1} \left(\frac{H_g(t+h) - H_g(t)}{h} - {H'}_g(t) \right) (H_0 + \mathbf{1})^{-1} \right\|$$
$$\leq \text{const.} \int_{-\infty}^{\infty} dk \left| \int dx\, e^{-ikx} \left(\frac{g(x,t+h) - g(x,t)}{h} - g'(x,t) \right) \right|^{2r}. \tag{3.16}$$

We now write the integral on the right-hand side of (3.16) as

$$\int_{-\infty}^{\infty} dk \cdots = \int_{-\infty}^{1} dk \cdots + \int_{-1}^{1} dk \cdots + \int_{1}^{\infty} dk \cdots$$

and estimate only the last integral above; the others are similar. Thus

$$\begin{aligned} J \equiv & \int_1^{\infty} dk \left| \int dx e^{-ikx} \left(\frac{g(x,t+h) - g(x,t)}{h} - g'(x,t) \right) \right|^{2r} \\ \leq & \int_1^{\infty} \frac{dk}{k^2} \left| \int dx e^{-ikx} \left(\frac{D_x^2 g(x,t+h) - D_x^2 g(x,t)}{h} - D_x^2 g'(x,t) \right) \right|^{2r} \end{aligned} \tag{3.17}$$

where we have used two partial integrations and $D_x \equiv \frac{\partial}{\partial x}$. Let now

$$V(x,t) \equiv D_x^2 g(x,t). \tag{3.18}$$

Now V is also an infinitely differentiable function of compact support and

$$V(x,t+h) = V(x,t) + hV'(x,t) + \frac{h^2}{2!} V''(x,t+t_h^*(x)) \tag{3.19}$$

by Taylor's formula with remainder, where $0 < t_h^*(x) < h$. Putting (3.19) into (3.17) we get

$$J \leq c \frac{h}{2} \left(\int_{-\infty}^{\infty} dx \left| V''(x,t+t_h^*(x)) \right| \right)^{1/2} \leq c\, h (\sup_{x,t} |V''(x,t)|).$$

The other estimates are similar and yield (3.13). We now notice that the bounds (3.12) <u>continue to hold</u> for $H_g'(t)$ with $||g(\cdot,t)||_r$ replaced by $||g'(\cdot,t)||_r$ on the right-hand side of (3.12). Thus the same proof applies to $H_g'(t)$, $H_g''(t)$, ... and in fact $H_g(t)$ is infinitely differentiable as an operator from $\mathcal{F}_{+2}$ to $\mathcal{F}_{-2}$. □

Proposition III.1 *The $S(g)$ matrix for the $(:\phi^4)_2$ theory, as defined in (1.13), is unitary and it satisfies the causality condition for disjoint supports [condition (ii.a) – section I].*

Proof. The unitarity follows directly from the existence theorems. For the proof of causality it is convenient explicitly dispose the dependence of the propagators on the function g. Let $\mathrm{supp}_t\, g_1 > \mathrm{supp}_t\, g_2$ and suppose $\mathrm{supp}_t\, g_1 \subset (r, +\infty)$ and $\mathrm{supp}_t\, g_2 \subset (-\infty, r)$, where supp_t stands for the support in the time variable. Then, for $t > r > s$ we have

$$U^D_{(g_1+g_2)}(t,s) = U^D_{(g_1+g_2)}(t,r)U^D_{(g_1+g_2)}(r,s) \tag{3.20}$$

but

$$\begin{aligned} i\frac{\partial}{\partial t}U^D_{(g_1+g_2)}(t,r)\Psi &= H^D_{(g_1+g_2)}(t)U^D_{(g_1+g_2)}(t,r)\Psi \\ &= H^D_{g_1}(t)U^D_{(g_1+g_2)}(t,r)\Psi \end{aligned}$$

and, by the uniqueness of the solutions of the above equation, we have $U^D_{(g_1+g_2)}(t,r) = U^D_{g_1}(t,r)$. Analogously, we have $U^D_{(g_1+g_2)}(r,s) = U^D_{g_2}(r,s)$. This, together with (3.20) imply that

$$U^D_{(g_1+g_2)}(t,s) = U^D_{g_1}(t,r)U^D_{g_2}(r,s)$$

from this equation and the fact that $U^D_{g_1}(t,s) = U^D_{g_1}(t,r)$ and $U^D_{g_2}(r,s) = U^D_{g_2}(t,s)$ due to the support properties of g_1 and g_2, we finally have

$$U^D_{(g_1+g_2)}(t,s) = U^D_{g_1}(t,s)U^D_{g_2}(t,s) \tag{3.21}$$

Then, by (3.21) and the definition (1.13), we obtain

$$S(g_1+g_2) = S(g_1)S(g_2)\ ,$$

□

IV. THE RELATION BETWEEN KISYŃSKI'S THEORY AND THEOREM III.1

Let us now briefly summarize (without proof) some steps in Kisyński's proof of the theorem II.3. First of all, we will state a crucial auxiliary theorem. Let X be a

Banach space with the norm $\|\cdot\|$ and $A(t)$, $t \in [-T_1, T_2]$ $(T_1, T_2 > 0)$, a family of linear operators in X. Consider the following conditions:

(a) there exists a family $\|\cdot\|_t$, , $t \in [-T_1, T_2]$, of norms in X equivalent to $\|\cdot\|$ such that $|\,\|\Psi\|_t - \|\Psi\|_s\,| \leq k\,\|\Psi\|_s\,|t-s|$ with $k =$ const., $-T_1 \leq s, t \leq T_2$ and $\Psi \in X$;

(b) for all $t \in [-T_1, T_2]$ the set $D(A(t))$ is dense in X;

(c) there exists a constant $\lambda_0 \geq 0$ such that $R(\lambda - \epsilon A(t)) = X$ and $\|(\lambda - \epsilon A(t))\Psi\|_t \geq (\lambda - \lambda_0)\|\Psi\|_t$ for $\epsilon = \pm 1$, $\lambda > \lambda_0$, $t \in [-T_1, T_2]$ and $\Psi \in D(A(t))$;

(d) there exists a family $R(t)$, $t \in [-T_1, T_2]$, of invertible bounded linear operators in X, such that $R(t)$ is twice weakly continuously differentiable in $[-T_1, T_2]$ and $(R(t))^{-1} D(A(t)) = Y =$ const. $\forall t \in [-T_1, T_2]$;

(e) $(R(t))^{-1} A(t) R(t)$ is weakly continuously differentiable.

Above $R(A)$ stands for the range of the operator A. Then we have:

Theorem IV.1 ([11], Theorem 4.4) *Let the conditions (a) - (e) be satisfied. Then there exists a two-parametrics family of propagators $U(t,s)$, $-T_1 \leq s, t \leq T_2$, such that*

$$\Psi(t) \equiv U(t,s)\Psi(s)\ , \qquad \Psi(s) \in D(A(s))\ ,$$

is the unique solution of the problem

$$\frac{d}{dt}\Psi(t) = A(t)\Psi(t) \tag{4.1}$$

with initial data $\Psi(s)$. The bounded propagators $U(t,s)$ are strongly continuous on $-T_1 \leq s, t \leq T_2$ and satisfy:

$$U(t,t) = 1\ , \qquad \forall\, t \in [-T_1, T_2]\ ; \tag{4.2}$$

$$U(t,s)U(s,r) = U(t,r)\ , \qquad \text{for } -T_1 \leq r, s, t \leq T_2\ ; \tag{4.3}$$

$$U(t,s)D(A(s)) = D(A(t))\ , \qquad \text{for } -T_1 \leq s, t \leq T_2\ ; \tag{4.4}$$

besides, $\forall\ s \in [-T_1, T_2]$ *and* $\Psi \in D(A(s))$ *the function* $U(t,s)\Psi$ *is continuously differentiable (in the sense of the norm) in* X*, satisfying:*

$$\frac{d}{dt}U(t,s)\Psi = A(t)U(t,s)\Psi\ . \tag{4.5}$$

The method of proof of this theorem is to reduce the problem to the case where we have an operator with constant domain by making use of the properties of $R(t)$ [for an outline of Kisyński's solution of the problem (4.1) with $D(A(t)) = \text{const.}$ see Appendix A].

Let us now consider Kisyński's approach to the abstract Schrödinger equation

$$\frac{d}{dt}\Psi(t) = -iA(t)\Psi(t)\ , \qquad -T_1 \le t \le T_2 \tag{4.6}$$

where $\Psi \in \mathcal{H}$, with $\mathcal{H}$ a Hilbert space and $A(t)$ an operator in $\mathcal{H}$ defined as follows. Consider the condition:

- (i) Let $\mathcal{H}$ be a Hilbert space, $\mathcal{H}_+$ a dense subset of $\mathcal{H}$ and, $\forall t \in [-T_1, T_2]$, let $\langle \cdot, \cdot \rangle_t^+$ be a scalar product defined on $\mathcal{H}_+$ which makes it a Hilbert space $\mathcal{H}_+^t$ algebraically and topologically contained in $\mathcal{H}$. Assume that $\langle \cdot, \cdot \rangle_t^+$ is n times $(n \ge 1)$ continuously differentiable on $[-T_1, T_2]$.

If the condition (i) is satisfied we have

Lemma IV.1 ([11], Lemma 7.2) *The equality*

$$\langle \Phi, \Psi \rangle_t^+ = \langle \Phi, Q(t)\Psi \rangle_{-T_1}^+\ , \qquad \Phi, \Psi \in \mathcal{H}_+\ , \quad t \in [-T_1, T_2] \tag{4.7}$$

defines a bounded n *times weakly continuously differentiable operator* $Q(t)$ *on* $\mathcal{H}_+^{-T_1}$*. For all fixed* $t \in [-T_1, T_2]$*,* $Q(t)$ *is hermitian with* $\inf Q(t) > 0$ *in* $\mathcal{H}_+^{-T_1}$*.*

Other consequences of the condition (i) are that we can define another operator $J_{-T_1}(t)$ by means of the equality ([11], Lemma 7.4)

$$\langle \Phi, \Psi \rangle = \langle \Phi, J_{-T_1}(t)\Psi \rangle_t^+ \qquad \Phi \in \mathcal{H}_+\ , \ \Psi \in \mathcal{H} \tag{4.8}$$

with $J_{-T_1}(t)$ a positive hermitian operator in $\mathcal{L}(\mathcal{H})$ such that $J_{-T_1}(t)\mathcal{H}_+$ is a dense subset of $\mathcal{H}_+^t$. Then, defining

$$\|\Psi\|_t^- \equiv \|J_{T_1}(t)\Psi\|_t^+\ , \qquad \Psi \in \mathcal{H}\ , \tag{4.9}$$

it follows that the completion $\mathcal{H}_t^- = \mathcal{H}_{T_1}^- \equiv \mathcal{H}^-$ of $\mathcal{H}$ in the norm $\|\cdot\|_t^-$ contains $\mathcal{H}$ algebraically and topologically ([11], Lemma 7.5).

Finally, we can define an operator $A(t)$ by means of the form $\langle\cdot,\cdot\rangle_t^+$ according to the following lemma:

Lemma IV.2 ([11], Lemma 7.7) *For all* $t \in [-T_1, T_2]$

$$D(A(t)) = \left\{ \Psi \in \mathcal{H}^+ : \sup_{\Phi\in\mathcal{H}^+,\, \|\Phi\|\leq 1} \{ \left| \langle \Phi, \Psi \rangle_t^+ \right| \} < +\infty \right\} \tag{4.10}$$

$$\langle \Phi, A(t)\Psi \rangle \equiv \langle \Phi, \Psi \rangle_t^+ \,, \qquad \Psi \in D(A(t)) \tag{4.11}$$

define an inversible self-adjoint positive operator $A(t)$ *in* $\mathcal{H}$*, with*

$$D(A(t)) = (Q(t))^{-1}\, D(A(-T_1)) \tag{4.12}$$

and

$$A(t) = (J_{-T_1}(t))^{-1} = A(-T_1)Q(t)\ . \tag{4.13}$$

Then the operator $A(t)$ is showed to satisfy the Schrödinger equation (4.6) and the propagators of the problem (4.6) satisfy the properties enumerated in the theorem II.3 ([11], Theorem 8.1). In order to proof his Theorem 8.1 for the operator $A(t)$, as defined above, Kisyński made use of the theorem IV.1 identifying $R(t) = (Q(t))^{-1}$.

Let us now show that the $(:\ \phi^4\ :)_2$ theory satisfies the necessary conditions for the theorem II.3. In fact, all we need to show is that the condition (i) is satisfied. However in benefit of clarity we will explicitly display the main operators introduced in Kisyński's proof and some of its properties.

As defined in section II, $\mathcal{F}$ is the symmetric Fock space and $\mathcal{F}_{+2} = D(H_0)$ is a dense subset of $\mathcal{F}$. Then, taking the closure $\mathcal{F}_{+2}^t$ of $\mathcal{F}_{+2}$ in the norm induced by the scalar product $\langle\cdot,\cdot\rangle_t^+$, which is related to the operator $\tilde{H}(t)$ [see equation (2.14)] by means of the form (2.21), i.e.,

$$\langle \Phi, \Psi \rangle_t^+ \equiv S(\Phi, \Psi) = \langle \tilde{H}(t)^{1/2}\Phi, \tilde{H}(t)^{1/2}\Psi \rangle \tag{4.14}$$

we can show the following:

Proposition IV.1 $\mathcal{F}_{+2}^{t}$ *is a Hilbert space such that*

$$\mathcal{F}_{+2}^{t} \subset \mathcal{F} \tag{4.15}$$

algebraically and topologically.

Proof. That $\mathcal{F}_{+2}^{t}$ is a Hilbert space follows immediately from the fact that the form defined in (4.14) is closed (see, e.g., [14]). The property that $\mathcal{F}_{+2}^{t} \subset \mathcal{F}$ algebraically is trivial. So, it remains to show that (4.15) holds topologically. This is achieved by showing that for $\{\Psi_n\}_{n=1}^{\infty} \in \mathcal{F}_{+2}$ and $\{\Psi_n\} \in \mathcal{F}_{+2}$ such that

$$\|f_n - f\| \longrightarrow 0 \tag{4.16}$$

we have

$$\|f_n - f\|_{+}^{t} \longrightarrow 0.$$

To show this, set

$$\begin{aligned}
\left(\|f_n - f\|_{+}^{t}\right)^2 &= \langle (f_n - f)\,, (f_n - f)\rangle_{+}^{t} \\
&= \langle (f_n - f)\,, \tilde{H}(t)\,(f_n - f)\rangle \\
&= \langle (H_0 + \mathbf{1})\,(f_n - f)\,, (H_0 + \mathbf{1})^{-1}\,\tilde{H}(t)\,(H_0 + \mathbf{1})^{-1} \\
&\times\ (H_0 + \mathbf{1})\,(f_n - f)\rangle
\end{aligned}$$

The Schwartz inequality applied to the last term above yields

$$\|f_n - f\|_{+}^{t} \leq \|\,(H_0 + \mathbf{1})^{-1}\,\tilde{H}(t)\,(H_0 + \mathbf{1})^{-1}\,\|\;\|\,(H_0 + \mathbf{1})\,(f_n - f)\,\|^2$$

The first term on the right-hand side is bounded due to (3.12). The second term on the right-hand side converges since $H_0 + \mathbf{1}$ is a self-adjoint operator (hence closed) and, by hypothesis, (4.16) holds. Then the proof of the proposition is complete. $\square$

In addition, it follows straightforwardly from (4.14) and theorem III.1 that $\langle \cdot, \cdot \rangle_{t}^{+}$ is n times (infinitely, in fact) continuously differentiable. Then it is proved that the condition (i) is satisfied and the theorem II.3 follows as proved in [11] and summarized above.

Now we turn to explicitly show the properties of $Q(t)$ in our case. From (4.14) and the definition

$$\langle \Phi, \Psi \rangle_{+}^{t} \equiv \langle \Phi, Q(t)\Psi \rangle_{-T_1}^{+}$$

we obtain that $Q(t)$ is the operator

$$Q(t) = \left(\tilde{H}(-T_1)\right)^{-1} \tilde{H}(t) \tag{4.17}$$

Proposition IV.2 *$Q(t)$, as defined in (4.17), is a (strictly) positive hermitian operator in $\mathcal{F}_{+2}$ and it is infinitely weakly differentiable.*

Proof. It follows directly from the properties of the scalar product $\langle \cdot, \cdot \rangle_t^+$ that $Q(t)$ is infinitely weakly differentiable.

For Φ, $\Psi \in \mathcal{F}_{+2}$, we have

$$\begin{aligned} \left(\langle \Phi, Q(t)\Psi \rangle_{-T_1}^{+}\right)^* &= \langle Q(t)\Psi, \Phi \rangle_{-T_1}^{+} \\ &= \langle \tilde{H}(-T_1)^{-1}\tilde{H}(t)\Psi, \tilde{H}(-T_1)\Phi \rangle = \langle \tilde{H}(t)\Psi, \Phi \rangle, \end{aligned} \tag{4.18}$$

where we have used (4.17) . We then have that

$$\begin{aligned} \left(\langle \Phi, Q(t)\Psi \rangle_{-T_1}^{+}\right)^* &= \langle \Psi, \tilde{H}(-T_1) \left(\tilde{H}(-T_1)\right)^{-1} \tilde{H}(t)\Phi \rangle \\ &= \langle \Psi, \tilde{H}(-T_1)Q(t)\Phi \rangle = \langle \Psi, Q(t)\Phi \rangle_{-T_1}^{+}, \end{aligned} \tag{4.19}$$

which proves that $Q(t)$ is hermitian.

In order to prove that $Q(t)$ is strictly positive on $\mathcal{F}_{+2}$, we must remember that, since $\mathcal{F}_{+2}^t \subset \mathcal{F}_{+2}\ \forall\, t$ algebraically and topologically, it follows that the norms $\| \cdot \|_{-T_1}^{+}$ and $\| \cdot \|_t^{+}$ are equivalent, i.e., there exists $a_t \geq 1$ such that $a_t^{-1}\| \cdot \|^{+} \leq \| \cdot \|_t^{+} \leq a_t \| \cdot \|_{-T_1}^{+}$. Then, for $\Psi \in \mathcal{F}_{+2}$,

$$\begin{aligned} \langle \Psi, Q(t)\Psi \rangle_{-T_1}^{+} &= (\|\Psi\|_{+}^{t})^2 \\ &\geq a_t^{-2}(\|\Psi\|_{+}^{-T_1})^2 \end{aligned} \tag{4.20}$$

from which follows that $\inf Q(t) > 0$ and the proof is complete. □

V. CONCLUSION: OPEN PROBLEMS

The problem of the nonperturbative construction of $S(g)$ for the $(:\phi^4:)_2$ quantum field theory was addressed in [16] using Yosida's approach, which requires that the domain of $H_g(t)$ be time-independent. For test functions $g(x,t) = h_1(x) \cdot f_1(t)$, i.e., of the product form, this condition is satisfied, but already for a sum of two products, e.g., $g(x,t) = h_1(x) \cdot f_1(t) + h_2(x) \cdot f_2(t)$, with f_1 and f_2 having disjoint supports, this is no longer true, and thus the results of [16] are incomplete. The present approach does not suffer from this inconvenience, and g is allowed to be an arbitrary infinitely differentiable function of compact support. Moreover, the use of a scale of spaces makes the theory very flexible, being applicable to more singular super-renormalizable theories, as well as to four-dimensional theories with an ultra-violet cutoff. It is a very challenging problem to discover a possibility of "renormalization" of the exponentials of the type (A.7) in the latter, in analogy to the interesting approach of Barata [17] and Gentile [18] to the study of certain two-level systems.

There are, however, open problems even to finish this program for the present $(:\phi^4:)_2$ theory: proof of causality for space-like supports (*ii. b)* and proof of Lorentz covariance (*iii*). For this purpose, the method outlined in [16] seems natural: the above properties would follow from a proof of Faris's product formula [19] under the assumptions of Theorem IV 1. We hope to return to this problem in the future.

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Appendix A

Let us consider the problem (4.1) for the case in which $D(A(t)) = \text{const.}$. The notation is as in the first part of section IV.

Consider the following conditions (in that follows $t \in [-T_1, T_2]$, unless otherwise specified):

(i) there exists a family $\|\cdot\|_t$, of norms in X such that $a^{-1}\|\Psi\| \leq \|\Psi\|_t \leq \|\Psi\|_s \leq a\|\Psi\|$, $a \geq 1$, for $-T_1 \leq s \leq t \leq T_2$ and $\Psi \in X$;

(ii) Y is a dense subset of X with $D(A(t)) = Y$;

(iii) for all $\lambda > 0$ and $\Psi \in Y$ we have $R(\lambda - A(t)) = X$ and $\|(\lambda - A(t))\Psi\|_t \geq \lambda\|\Psi\|_t$;

(iv) $A(t)$ is weakly continuously differentiable.

Theorem A.1 ([11], theorem 3.0) *Let the conditions* (i) –(iv) *be satisfied. Then, there exists an unique solution of the problem (4.1) and the corresponding propagator $U(t, s)$ is strongly continuous in $-T_1 \leq s \leq t \leq T_2$ and satisfies the properties (4.2), (4.3), (4.4) and (4.5).*

Now we shall explain some aspects of Kisyński's proof of this theorem. Consider the family of equations

$$\frac{d}{dt}\Phi(t) = A_n(t)\Phi(t) \,, \qquad \Phi(0) = \Phi_0 \,, \qquad n = 1, 2, \cdots \,, \tag{A.1}$$

with

$$A_n(t) = nA(t)\left(n - A(t)\right)^{-1} \,. \tag{A.2}$$

The set Y supplied with the norm $|\!|\!|\cdot|\!|\!|_t = \|((1 - A(t))\cdot\|$ is a Banach space algebraically and topologically contained in X. Then, from (i) and (ii) it follows that $A(t) \in \mathcal{L}(Y, X)$ is a weakly continuously differentiable operator, which, by the Banach-Steinhaus theorem, implies $\|A(t)\Phi\| \leq C|\!|\!|\Phi|\!|\!|_0$ for $\Phi \in Y$ and some constant C (the equivalence of the norms $|\!|\!|\cdot|\!|\!|_t$ was used). So, by using (i) and (iii), it follows that

$$\left\|\Phi - n(n - A(t))^{-1}\Phi\right\| = \frac{1}{n}\left\|(1 - A(t)/n)^{-1}\left(A(t)\Phi\right)\right\| \leq \frac{Ca^2}{n}|\!|\!|\Phi|\!|\!|_0 \,, \tag{A.3}$$

which implies that $n(n-A(t))^{-1}$ converges strongly and uniformly to 1. Therefore, the sequence of bounded operators $A_n(t)$ converges strongly to $A(t)$. The operators $A_n(t)$ are weakly continuosly differentiable, therefore they satisfy a Lipschitz condition in the sense of the norm. Hence, it follows that $A_n(t)$ is continuous in the sense of the norm and Yosida's method [15] guarantee the existence and uniqueness of the evolution operators $U_n(t,s)$ of the equation (A.1) satisfying the properties equivalent to (4.2) – (4.5). Besides, $U_n(t,s)$ satisfy [11]

$$\|U_n(t,s)\| \leq M \ . \tag{A.4}$$

Before to proceed we will consider the equation (A.1) perturbed by the bounded (in X) weakly continuous operator $B(t) = -\frac{dA(t)}{dt}(1-A(t))^{-1}$, that is,

$$\frac{d}{dt}\Phi(t) = (A_n(t) + B(t))\,\Phi(t) \ , \qquad \Phi(0) = \Phi_0 \tag{A.5}$$

The evolution operator of (A.5), denoted $H_n(t,s)$, is given by

$$H_n(t,s) = (1-A(t))U_n(t,s)(1-A(s))^{-1} \ .$$

Then, it follows that $H_n(t,s) \in \mathcal{L}(X)$ is weakly continuously differentiable in $-T_1 \leq s,t \leq T_2$, satisfying

$$\|H_n(t,s)\| \leq D \ . \tag{A.6}$$

Now, we subdivide the segment $[-T_1, T_2]$ into K equal intervals. Then, the conditions $(T \equiv T_1 + T_2)$

$$U_{nK}(t,s) = \exp\left\{(t-s)A_n(-T_1 + \tfrac{i-1}{K}T)\right\} \ , \tag{A.7}$$

$-T_1 + \frac{i-1}{K}T \leq s,t \leq -T_1 + \frac{i}{K}T$, $i = 1, \ldots, K$, and

$$U_{nK}(t,s)U_{nK}(s,r) = U_{nK}(t,s) \ , \qquad -T_1 \leq r,s,t \leq T_2 \ , \tag{A.8}$$

define a unique family of operators $U_{nK}(t,s) \in \mathcal{L}(X)$ continuous in the sense of the norm such that

$$\|U_{nK}(t,s)\| \leq a^2 \ . \tag{A.9}$$

The operators $U_{nK}(t,s)$ satisfy

$$\frac{\partial}{\partial s} U_{nK}(t,s) = -U_{nK}(t,s) A_n(-T_1 + \tfrac{T}{K}[\tfrac{Ks}{T}]) \ ,$$

where $[(Ks)/T]$ stands for the integer part of $(Ks)/T$. Besides, for fixed K, $U_{nK}(t,s)$, $n = 1, 2, \ldots$, is a sequence uniformly strongly convergent in $-T_1 \le s \le t \le T_2$.

Then, by integrating $\frac{\partial}{\partial \tau} U_{nK}(t,\tau) U_n(\tau, s)$ we obtain

$$U_n(t,s) - U_{nK}(t,s) = \int_s^t U_{nK}(t,\tau) \left(A_n(\tau) - A_n(-T_1 + \tfrac{T}{K}[\tfrac{K\tau}{T}]) \right) U_n(\tau, s) d\tau \ . \quad \text{(A.10)}$$

We have [11]

$$\left\| A_n(\tau)\Phi - A_n(-T_1 + \tfrac{T}{K}[\tfrac{K\tau}{T}])\Phi \right\| \le \frac{\text{const.}}{K} \|\!|\Phi|\!\|_0 \ . \quad \text{(A.11)}$$

Then, since

$$U_n(t,s) = (1 - A(t))^{-1} H_n(t,s)(1 - A(s))$$

and $(1 - A(s)) \in \mathcal{L}(Y, X)$ and $(1 - A(t))^{-1} \in \mathcal{L}(X, Y)$ are weakly differentiable, we obtain, by using (A.6),

$$\|\!|U_n(t,s)\Phi|\!\|_0 \le \text{const.} \|\!|\Phi|\!\|_0 \ , \quad \text{(A.12)}$$

for $\Phi \in Y$ and $-T_1 \le s \le t \le T_2$. Then, from (A.9), (A.10), (A.11) and (A.12), it follows that

$$\|U_n(t,s)\Phi - U_{nK}(t,s)\Phi\| \le \frac{\mathrm{L}}{K} \|\!|\Phi|\!\|_0 \ , \quad \text{(A.13)}$$

with $L =$ constant.

Now, for $\Phi \in Y$ we have

$$\begin{aligned} \|U_n(t,s)\Phi - U_m(t,s)\Phi\| &\le \|U_n(t,s)\Phi - U_{nK}(t,s)\Phi\| \\ &+ \|U_{mK}(t,s)\Phi - U_m(t,s)\Phi\| \\ &+ \|U_{nK}(t,s)\Phi - U_{mK}(t,s)\Phi\| \\ &\le 2\frac{L}{K} \|\!|\Phi|\!\|_0 + \|U_{nK}(t,s)\Phi - U_{mK}(t,s)\Phi\| \quad \text{(A.14)} \end{aligned}$$

The first term in the r.h.s. may be made arbitrarily small for large K. After this, one choose n and m so large that the second term becomes arbitrarily small for all

$-T_1 \le s \le t \le T_2$, since the sequence $U_{nK}(t,s)$ is uniformly strongly convergent. Since Y is dense in X, and from (A.4), (A.14) implies that the convergence is in all of X, in the triangle $-T_1 \le s \le t \le T_2$. Then, it follows directly from the properties of $U_n(t,s)$ that $U(t,s) = s - \lim_{n\to\infty} U_n(t,s)$ is the evolution operator of (4.1) for constant domain [11].

Remark. The proof outlined above is valid for $-T_1 \le s \le t \le T_2$. However, by substituting the conditions (i) and (iii) above by the conditions (a) and (c) in the theorem IV.1 the proof can be extended for the square $-T_1 \le s, t \le T_2$.

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