CRITICAL MANIFOLDS AND STABILITY IN HAMILTONIAN SYSTEMS WITH NON-HOLONOMIC CONSTRAINTS

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Abstract. We explore a particular approach to the analysis of dynamical and geometrical properties of autonomous, Pfaffian non-holonomic systems in classical mechanics. The method is based on the construction of a certain auxiliary constrained Hamiltonian system, which comprises the non-holonomic mechanical system as a dynamical subsystem on an invariant manifold. The embedding system possesses a completely natural structure in the context of symplectic geometry, and using it in order to understand properties of the subsystem has compelling advantages. We discuss generic geometric and topological properties of the critical sets of both embedding and physical system, using Conley-Zehnder theory and by relating the Morse-Witten complexes of the 'free' and constrained system to one another. Furthermore, we give a qualitative discussion of the stability of motion in the vicinity of the critical set. We point out key relations to sub-Riemannian geometry, and a potential computational application.

Introduction

We introduce and explore a particular approach to the analysis of autonomous, Pfaffian non-holonomic systems in classical mechanics, which renders them naturally accessible to the methods of symplectic and sub-Riemannian geometry. We note that typical examples of systems encountered in sub-Riemannian geometry emerge from optimal control, or 'vakonomic' problems, which are derived from a different variational principle (minimization of the Carnot-Caratheodory distance) than the Euler-Lagrange equations of classical mechanical systems with non-holonomic constraints (the Hölder variational principle, cf. [3], and section III in this paper). The strategy is based on the introduction of an artificial Hamiltonian system with constraints that are compatible with the symplectic structure, constructed in a manner that it comprises the non-holonomic mechanical system as a dynamical subsystem on an invariant manifold. The main focus of the discussion in this paper aims at the geometrical and topological properties of the critical sets of both embedding and mechanical system, on the stability of equilibria, and an application of the given analysis to a computational problem.

There exists a multitude of different approaches to the description and analysis of non-holonomic systems in classical mechanics, stemming from various subareas of application. The geometrical approach given here has been strongly influenced by [37] and [11, 12, 32].

A construction for the Lagrangian case, which is closely related to what will be presented in section III, has been given in [13]. A different approach in the Hamiltonian picture is dealt with in [35]. A geometrical theory of non-holonomic systems with a strong influence of network theory has been developed in [38]. The geometrical structure of non-holonomic systems with symmetries and the associated reduction theory, as well as aspects of their stability theory has been at the focus in the important works [7, 23, 24, 40], and other papers by the same authors.

This paper is structured as follows. In section I, we introduce a class of Hamiltonian systems with non-integrable constraints. Given a symplectic manifold (M, ω) and a non-integrable, symplectic distribution V, we focus on the flow $\tilde{\Phi}_t$ generated by the component X_H^V of the Hamiltonian vector field X_H in V. In section II, we study the geometry and topology of the critical set $\mathfrak{C}$ of the constrained Hamiltonian system. The main technical tool used for this purpose is a gradient-like flow ϕ_t, whose critical set $\mathfrak{C}$ is identical to that of $\tilde{\Phi}_t$. Assuming that the Hamiltonian $H : M \to \mathbb{R}$ is a Morse function, it is proved that generically, $\mathfrak{C}$ is a normal hyperbolic submanifold of M. Using Conley-Zehnder theory, we prove a topological formula for closed, compact $\mathfrak{C}$, that is closely related to the Morse-Bott inequalities. A second, alternative proof is given, based on the use of the Morse-Witten complex, to elucidate relations between the 'free' and the constrained system. In section III, we give a qualitative, partly non-rigorous discussion of the stability of the constrained Hamiltonian system, and conjecture a stability criterion for the critically stable case. A proof of the asserted criterion, which would involve methods of KAM and Nekhoroshev theory, is beyond the scope of the present work. We derive an expression for orbits in the vicinity of a critically stable equilibrium that is adapted to the flag of V, and point out relations to sub-Riemannian geometry.

In section IV, we consider Hamiltonian mechanical systems with Pfaffian constraints. We show that for any such system, there exists an auxiliary constrained Hamiltonian system of the type introduced in section I. We study the global topology of the critical manifold of the constrained mechanical system, and again discuss the stability of equilibria. Finally, we propose a computational application, a method to numerically determine the generic connectivity components of the critical manifold.

1. A NON-INTEGRABLE GENERALIZATION OF DIRAC CONSTRAINTS

Let (M, ω, H) be a Hamiltonian system, where M is a smooth, symplectic $2n$-manifold with C^∞ symplectic structure $\omega \in \Lambda^2(M)$, and where $H \in C^\infty(M)$ is the Hamiltonian function. For $p = 1, \dots, 2n$, let $\Lambda^p(M)$ denote the $C^\infty(M)$-module of p-forms on M. The Hamiltonian vector field $X_H \in \Gamma(TM)$ is determined by

$$i_{X_H}\omega = -dH \ ,$$

where i stands for interior multiplication. Given a smooth distribution $W \subset TM$, $\Gamma(W)$ will denote the $C^\infty(M)$-module of smooth sections of W. The Hamiltonian flow is the 1-parameter group $\Phi_t \in \mathrm{Diff}(M)$ generated by X_H, with $t \in \mathbb{R}$, and $\Phi_0 = \mathrm{id}$. Its orbits are solutions of

$$\partial_t \Phi_t(x) \ = \ X_H(\Phi_t(x)) \tag{1}$$

for all $x \in M$, and $t \in \mathbb{R}$.

Let us first recall some standard facts about Dirac constraints that will be useful in the subsequent construction. Let, for $f, g \in C^\infty$,

$$\{f, g\} = \omega(X_f, X_g) \tag{2}$$

denote the smooth, non-degenerate Poisson structure on M induced by ω. It is a derivative in both of its arguments, and satisfies the Jacobi identity $\{f, \{g, h\}\} + (\text{cyclic}) = 0$, thus $(C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra. Then, (1) translates into

$$\partial_t f(\Phi_t(x)) \ = \ \{H, f\}(\Phi_t(x)) \ , \tag{3}$$

for all $f \in C^\infty(M)$, and all $x \in M$, $t \in \mathbb{R}$.

Let $j : M' \hookrightarrow M$ be an embedded, smooth, $2k$-dimensional symplectic submanifold of M, endowed with the pullback symplectic structure $j^*\omega$. The Dirac bracket corresponds to the induced Poisson bracket on M',

$$\{f, g\}_D \ = \ (j^*\omega)(X_{\tilde{f}}, X_{\tilde{g}}) \ ,$$

defined for any pair of extensions $\tilde{f}, \tilde{g} \in C^\infty(M)$ of $f, g \in C^\infty(M')$. If M' is locally characterized as the locus of common zeros of some family of functions $G_i \in C^\infty(M)$, $i = 1, \dots, 2(n-k)$, the following explicit construction of the Dirac bracket can be given, [25]. Since $M' \subset M$ is symplectic, the $(n-k)^2$ functions locally given by $D_{ij} := \{G_i, G_j\}$ can be patched together to define a matrix-valued C^∞ function that is invertible everywhere on M'. The explicit formula for the Dirac bracket is locally given by

$$\{f,g\}_D \;=\; \{\tilde{f},\tilde{g}\} \;-\; \{\tilde{f},G_i\}\, D^{ij}\, \{G_j,\tilde{g}\}\;, \tag{4}$$

where D^{ij} denotes the components of the inverse of $[D_{ij}]$.

This construction can be put into the more general context of a symplectic distribution.

Definition 1.1. *A distribution V over the base manifold M is symplectic if V_x is a symplectic subspace of T_xM with respect to ω_x, for all $x \in M$. Its symplectic complement $V^\perp$ is the distribution which is fibrewise ω-skew orthogonal to V. Furthermore, an embedding $I \subset \mathbb{R} \hookrightarrow M$ that is tangent to V is called V-horizontal.*

Clearly, $V^\perp$ is by itself symplectic, and smoothness of V and ω implies smoothness of $V^\perp$. Furthermore, the Whitney sum bundle $V \oplus V^\perp$ is TM. Thus, let V denote an integrable, smooth, symplectic rank $2k$-distribution V over M. Clearly, any section $X \in \Gamma(TM)$ has a decomposition

$$X = X^V + X^{V^\perp}\;,$$

where $X^{V^{(\perp)}} \in \Gamma(V^{(\perp)})$, so that $\omega(X^V, X^{V^\perp}) = 0$. Furthermore, there exists an ω-skew orthogonal tensor $\pi_V : TM \longrightarrow TM$ with

$$\mathrm{Ker}(\pi_V) = V^\perp \quad , \quad \pi_V(X) = X \quad \forall X \in \Gamma(V)\;,$$

which satisfies

$$\omega(\pi_V(X), Y) = \omega(X, \pi_V(Y)) \tag{5}$$

for all $X, Y \in \Gamma(TM)$. It will be referred to as the ω-skew orthogonal projection tensor associated to V. Let $Y_1, \dots, Y_{2k}$ denote a local spanning family of vector fields for V. Then, V being symplectic is equivalent to the matrix $[C_{ij}] := [\omega(Y_i, Y_j)]$ being invertible.

Lemma 1.1. *Let $[C^{kl}]$ denote the inverse of $[C_{ij}]$, and let $\theta_j := i_{Y_j}\omega \in \Lambda^1(TM)$. Then, locally, $\pi_V = C^{ij} Y_i \otimes \theta_j$.*

Proof. The fact that $C^{ij} Y_i \otimes \theta_j$ is a projector, and that (5) holds, follows from $\theta_i(Y_j) = C_{ij}$, and $C_{ij}C^{jk} = \delta_i^l$. Its rank is clearly $2k$, and it is straightforward to see that its kernel is given by $\Gamma(V^\perp)$. $\square$

1.1. **Non-Integrable Constraints.** The quadruple (M, ω, H, V) naturally defines a dynamical system whose orbits are all V-horizontal. Its flow is simply the 1-parameter group of diffeomorphisms $\tilde{\Phi}_t$ generated by $X_H^V := \pi_V(X_H) \in \Gamma(V)$, with

$$\partial_t \tilde{\Phi}_t(x) = X_H^V(\tilde{\Phi}_t(x)) \tag{6}$$

for every $x \in M$. In a local Darboux chart, where ω is represented by

$$\mathcal{J} = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}, \tag{7}$$

and where $x(t)$ stands for the vector of coordinate components of $\tilde{\Phi}_t(a)$, (6) is given by

$$\partial_t x(t) = (P_x \, \mathcal{J} \, \partial_x H)(x(t)) = \big(\mathcal{J} \, P_x^\dagger \, \partial_x H\big)(x(t)) \, . \tag{8}$$

P denotes the matrix of π_V, and $P^\dagger$ is its transpose.

If the condition of integrability imposed on V is dropped, this dynamical system will allow for the description of non-holonomic mechanics. If V is integrable, M is foliated into $2k$-dimensional symplectic integral manifolds of V. On every leaf $j : M' \to M$, the induced dynamical system corresponds to the pullback Hamiltonian system $(M, j^*\omega, H \circ j)$. In this sense, (6) generalizes the Dirac constraints.

A new class of dynamical systems is obtained by discarding the requirement of integrability on V. Let $[\cdot, \cdot]$ denote the Lie bracket. We recall that the distribution V is non-integrable if there exists a filtration

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_r, \tag{9}$$

inductively defined by $V_0 = V$, and $V_i = V_{i-1} + [V_0, V_{i-1}]$, where $V_1 \neq V_0$. The sequence $\{V_i\}_1^r$ is called the *flag* of V. If the fibre ranks of all V_i are base point independent, V is called equiregular. The smallest number $\deg(V)$ at which the flag of V stabilizes, that is, for which $V_s = V_{\deg(V)}$, $\forall s \geq \deg(V)$, is called the degree of non-holonomy of V. If $V_{\deg(V)} = TM$, one says that V satisfies Chow's condition, or that it is totally non-holonomic.

Proposition 1.1. *(Frobenius condition) V is integrable if and only if locally,*

$$\Lambda_{ij}^k := (\pi_V)_i^r (\pi_V)_j^s \big(\partial_r (\pi_V)_s^k - \partial_s (\pi_V)_r^k\big) = 0 \tag{10}$$

everywhere on M.

Proof. V is integrable if and only if $\bar{\pi}_V\left([\pi_V(X), \pi_V(Y)]\right) = 0$ for all sections X, Y of TM, which is equivalent to $V_1 = V$. The asserted formula is the local coordinate representation of this condition. □

1.2. **An Auxiliary Almost Kähler Structure.** For the analysis in subsequent sections, it will be useful to introduce an almost Kähler structure on M that is adapted to V. To this end, let us briefly recall some basic definitions. Let g denote a Riemannian metric on M. An almost complex structure J is a smooth bundle isomorphism $J : TM \to TM$ with $J^2 = -\mathbf{1}$. Together with g, it defines a two form satisfying

$$\omega_{g,J}(X, Y) = g(JX, Y) \tag{11}$$

for all sections X, $Y \in \Gamma(TM)$. g is hermitean if $g(JX, JY) = g(X, Y)$, and Kähler if $\omega_{g,J}$ is closed. The triple $(g, J, \omega_{g,J})$ is called compatible. Every symplectic manifold admits an almost complex structure J, and a Kähler metric g, such that (g, J, ω) is compatible.

Proposition 1.2. *For any symplectic manifold (M, ω), together with a symplectic distribution V, there exists a compatible triple (g, J, ω), such that π_V is symmetric with respect to g, and $\pi_V JX = J\pi_V X$ for all $X \in \Gamma(TM)$.*

Proof. We pick a smooth Riemannian metric $\tilde{g}$ on M, relative to which π_V is symmetric, for instance by choosing an arbitrary Riemannian metric g' on M, and defining $\tilde{g}(X, Y) := g'(\pi_V X, \pi_V Y) + g'(\bar{\pi}_V X, \bar{\pi}_V Y)$, where $\bar{\pi}_V = \mathbf{1} - \pi_V$. We consider the non-degenerate, smooth bundle map K defined by $\omega(X, Y) = \tilde{g}(KX, Y)$, which is skew symmetric with respect to $\tilde{g}$, that is, its $\tilde{g}$-adjoint is $K^* = -K$. $K^*K = -K^2$ is smooth, positive definite, non-degenerate and $\tilde{g}$-symmetric, hence there is a unique smooth, positive definite, $\tilde{g}$-symmetric bundle map A defined by $A^2 = -K^2$, which commutes with K. Consequently, the bundle map $J = KA^{-1}$ satisfies $J^2 = -\mathbf{1}$, and defines an almost complex structure. Since A is positive definite and $\tilde{g}$-symmetric, $g(X, Y) := \tilde{g}(AX, Y)$ is a Riemannian metric with $\omega(X, Y) = g(JX, Y)$. Moreover, this metric is hermitean, since $g(JX, JY) = \tilde{g}(KX, A^{-1}KY) = -\tilde{g}(X, K^2A^{-1}Y) = \tilde{g}(X, AY) = g(X, Y)$. In fact, since ω is closed, g is Kähler.

To show that π_V is g-symmetric, we note that π_V commutes with K, since $\tilde{g}(K\pi_V X, Y) = \omega(\pi_V X, Y) = \omega(X, \pi_V Y) = \tilde{g}(KX, \pi_V Y) = \tilde{g}(\pi_V KX, Y)$ for all $X, Y \in \Gamma(TM)$, using that π_V is symmetric with respect to $\tilde{g}$. Hence, π_V commutes with $A^2 = -K^2$, and it straightforwardly follows from the $\tilde{g}$-orthogonality of A, π_V, and the positivity of A that π_V and A commute. Hence, π_V is g-symmetric, and it is also clear that π_V commutes with $J = KA^{-1}$. Thus, J in particular restricts to a bundle map $J : V \to V$. □

1.3. **Further Properties.** Some key properties of Hamiltonian systems concerning symmetries, Poisson brackets, energy conservation, and, to some degree, symplecticness, can be generalized to the constrained system.

1.3.1. *Symmetries.* Let us assume that the Hamiltonian system (M, ω, H) admits a symplectic G-action (G some Lie group) $\Psi : G \to \mathrm{Diff}(M)$, such that $\Psi_h^* \omega = \omega$ and $H \circ \Psi_h = H$ for all $h \in G$. Then, we will say that the constrained system (M, ω, H, V) admits a G-symmetry if $\Psi_{h*} V = V$ holds for all $h \in G$.

1.3.2. *Generalized Dirac bracket.* The smooth, $\mathbb{R}$-bilinear, antisymmetric pairing on $C^\infty(M)$ associated to (M, ω, V) given by

$$\{f, g\}_V \ := \ \omega(\pi_V(X_f), \pi_V(X_g)) \tag{12}$$

is a straightforward generalization of the Dirac and Poisson brackets. Along orbits of $\tilde{\Phi}_t$, one has

$$\partial_t f(\tilde{\Phi}_t(x)) \ = \ \{H, f\}_V(\tilde{\Phi}_t(x))$$

for all $x \in M$, in analogy to (3). However, the bracket (12) does not satisfy the Jacobi identity if V is non-integrable, but it satisfies a Jacobi identity on every (symplectic) integral manifold if V is integrable.

1.3.3. *Energy conservation.* This key conservation law also holds for the constrained system.

Proposition 1.3. *The energy H is an integral of motion of the dynamical system (6).*

Proof. This follows from the antisymmetry of the generalized Dirac bracket, which implies that $\partial_t H = \{H, H\}_V = 0$. □

1.3.4. *Symplecticness.* The flow $\tilde{\Phi}_t$ is not symplectic, but the following holds. Let us consider

$$\begin{aligned}
\partial_t \tilde{\Phi}_t^* \omega \ &= \ \tilde{\Phi}_t^* \mathcal{L}_{X_H^V} \omega = \tilde{\Phi}_t^* \big(d i_{X_H^V} \omega + i_{X_H^V} d\omega \big) \\
&= \ - \tilde{\Phi}_t^* \, d \left(\, (\pi_V)^i_k \, \partial_i H \, dx^k \, \right) = - \tilde{\Phi}_t^* \ \left(\, \partial_l (\pi_V)^i_k \, \partial_i H \, dx^l \wedge dx^k \, \right) \\
&= \ -\frac{1}{2} \tilde{\Phi}_t^* \Big(\big(\partial_l (\pi_V)^i_k - \partial_k (\pi_V)^i_l \big) \partial_i H dx^l \wedge dx^k \Big) \, .
\end{aligned}$$

Hence, the restriction of $\partial_t \tilde{\Phi}_t^* \omega$ to $X, Y \in \Gamma(V)$ is given by

$$\partial_t \tilde{\Phi}_t^* \omega(X, Y) \ = \ -\frac{1}{2} \tilde{\Phi}_t^* \Big(\Lambda^i_{rs} \partial_i H \, X^r Y^s \Big) \, ,$$

where Λ^i_{rs} is defined in lemma 1.1. Thus, the right hand side vanishes identically if and only if V is integrable. In the latter case, the restriction of $\tilde{\Phi}_t^* \omega$ to $V \times V$ equals its value for $t = 0$, given by $\omega(\pi_V(\cdot), \pi_V(\cdot))$. On every integral manifold $j : M' \to M$ of V, $\tilde{\Phi}_t$ is symplectic with respect to the pullback symplectic structure $j^*\omega$.

2. THE GEOMETRY AND TOPOLOGY OF THE CRITICAL MANIFOLD

In this main section, we address geometrical and topological properties of the critical set $\mathfrak{C}$ of the constrained Hamiltonian system (M, ω, H, V). The main result of the subsequent analysis is, for M compact and without boundary, the topological formula (18) that interrelates the Poincaré polynomials of M and $\mathfrak{C}_{gen}$ in a manner closely akin to the Morse-Bott inequalities. This result implies that the topology of M necessitates the existence of certain connectivity components of $\mathfrak{C}_{gen}$ of a prescribed index. The analysis is structured as follows.

In section 2.1, we prove that $\mathfrak{C}$ is, in the sense of Sard's theorem, generically a smooth $2(n-k)$-dimensional submanifold $\mathfrak{C}_{gen} \subset M$. For the special case in which V is integrable, it is shown that the intersection of any integral manifold of V with $\mathfrak{C}_{gen}$ is a discrete set, in agreement with the usual understanding that critical points in Hamiltonian systems - on every leaf of the foliation in the integrable case - are typically isolated.

In section 2.2, we introduce the main tool for the analysis of $\mathfrak{C}$, an auxiliary gradient-like flow $\phi_t \in \mathrm{Diff}(M)$ generated by the vector field $\pi_V \nabla_g H$, where g is the Kähler metric of the compatible quadruple introduced after proposition 1.2. From this point on, we assume that $H : M \to \mathbb{R}$ is a Morse function. Let $j : \mathfrak{C}_{gen} \hookrightarrow M$ denote the embedding. We show that $\mathfrak{C}_{gen}$ is normal hyperbolic with respect to ϕ_t, and that the critical points of j^*H on $\mathfrak{C}_{gen}$ are precisely those of H on M. The latter is quintessential for our discussion of the topology of $\mathfrak{C}_{gen}$ via comparison of the Morse-Witten complexes of $(\mathfrak{C}_{gen}, j^*H)$ and of (M, H) in section 2.4.

In section 2.3, we prove (18) by an application of Conley-Zehnder theory, [15], to the auxiliary gradient-like system. The assumptions on $\mathfrak{C}$ are slightly less strict than genericity. In particular, assuming that $\mathfrak{C} \setminus \mathfrak{C}_{gen}$ is a disjoint union of C^1 manifolds, we show that every connectivity component of $\mathfrak{C} \setminus \mathfrak{C}_{gen}$ is contained in a H-level surface, and that $\mathfrak{C} \setminus \mathfrak{C}_{gen}$ can be deformed away by an infinitesimal perturbation of the vector field.

In section 2.4, we assume that $\mathfrak{C} = \mathfrak{C}_{gen}$, and give a second proof based on the comparison of the Morse-Witten complexes of (M, H) and $(\mathfrak{C}_{gen}, H|_{\mathfrak{C}_{gen}})$. Our construction only uses the theory for non-degenerate Morse functions, not for Morse-Bott functions. The interest in this discussion is to elucidate the relationship between critical points of the 'free' system

(M, H), and of the critical manifold $\mathfrak{C}_{gen}$ of the constrained system (M, H, V). The special case of mechanical systems (where M is noncompact) will be analyzed in a later section.

2.1. **Generic Properties of the Critical Set.** Let us to begin with recall some basic definitions. Critical points of H are given by zeros of dH, and a corresponding value of H is called a critical value. A critical level surface Σ_E corresponds to a critical value E of H, whereas a regular level surface Σ_E contains no critical points of H (the corresponding value of E is then called regular). The critical set of the constrained Hamiltonian system (M, ω, H, V) is given by

$$\mathfrak{C} \;=\; \left\{x \in M \mid X_H^V(x) \;=\; 0\right\} \;\subset\; M\;.$$

The following theorem holds independently of the fact whether V is integrable or not.

Theorem 2.1. *In the generic case, the critical set is a piecewise smooth,* $2(n-k)$*-dimensional submanifold of* M*.*

Proof. Let $\{Y_i\}_{i=1}^{2k}$ denote a smooth, local family of spanning vector fields for V over an open neighborhood $U \subset M$. Since V is symplectic, the fact that X_H^V is a section of V implies that $\omega(Y_i, X_H^V)$ cannot be identically zero for all i and everywhere in U. Due to the ω-skew orthogonality of π_V, and $\pi_V Y_i = Y_i$,

$$\omega(Y_i, X_H^V) \;=\; \omega(\pi_V(Y_i), X_H) \;=\; \omega(Y_i, X_H) \;=\; Y_i(H)\;.$$

Thus, with $\underline{F} := (Y_1(H), \ldots, Y_{2k}(H)) \in C^\infty(U, \mathbb{R}^{2k})$, it is clear that $\mathfrak{C} \cap U = \underline{F}^{-1}(\underline{0})$. Since $\underline{F}$ is smooth, Sard's theorem implies that regular values, having smooth, $2(n-k)$-dimensional submanifolds of U as preimages, are dense in $\underline{F}(U)$ [28]. $\square$

For future technical convenience, we pick a local spanning family $\{Y_i \in \Gamma(V)\}_{i=1}^{2k}$ for V that satisfies

$$\omega(Y_i, Y_j) \;=\; \tilde{J}_{ij}\;,$$

with $\tilde{J} := \begin{pmatrix} 0 & \mathbf{1}_k \\ -\mathbf{1}_k & 0 \end{pmatrix}$. This choice is always possible.

Furthermore, introducing the associated family of 1-forms $\{\theta_i\}$ by $\theta_i(\cdot) := \omega(Y_i, \cdot)$, theorem 1.1 implies that

$$\pi_V \;=\; \tilde{J}^{ij} Y_i \otimes \theta_j\;,$$

where $\tilde{J}^{ij}$ are the components of $\tilde{J}^{-1} = -\tilde{J}$. Expanding X_H^V with respect to the basis $\{Y_i\}$ gives

$$X_H^V \;=\; \pi_V(X_H^V) \;=\; -\,Y_i(H)\,\tilde{J}^{ij}\,Y_j\;, \tag{13}$$

where one uses the relationship $\theta_j(X_H^V) = Y_j(H)$ obtained in the proof of theorem 2.1. Then, the following proposition holds, which is in the subsequent discussion interpreted as the property of normal hyperbolicity of $\mathfrak{C}$ with respect to a certain gradient-like flow if the genericity assumption is satisfied.

Proposition 2.1. *Under the genericity assumption of theorem 2.1, the $2k \times 2k$-matrix given by $[Y_k(Y_i(H))(a)]$ is invertible for all $a \in \mathfrak{C}$, and every local spanning family $\{Y_i \in \Gamma(V)\}$ of V.*

Proof. Let us pick a local basis $\{Y_i\}_1^{2k}$ for V, and $\{Z_j\}_1^{2(n-k)}$ for $V^\perp$, which together span TM. Let $a \in \mathfrak{C}_{gen}$, and assume the generic situation of theorem 2.1. Because $\mathfrak{C}_{gen}$ is defined as the set of zeros of the vector field (13), the kernel of the linear map

$$dF_i(\cdot)\tilde{J}^{ik}Y_k\Big|_a \;:\; T_aM \;\rightarrow\; V_a \;,$$

where $F_i := Y_i(H)$, is precisely $T_a\mathfrak{C}_{gen}$, and has a dimension $2(n-k)$.

In the basis $\{Y_1|_a, \ldots, Y_{2k}|_a, Z_1|_a, \ldots, Z_{2(n-k)}|_a\}$, its matrix is given by

$$A \;=\; [A_V \; A_{V^\perp}] \;,$$

where $A_V := [Y_i(F_j)\tilde{J}^{jk}Y_k|_a]$, and $A_{V^\perp} := [Z_i(F_j)\tilde{J}^{jk}Y_k|_a]$. Bringing A into upper triangular form, A_V is likewise transformed into upper triangular form. Because the rank of A is $2k$, and A_V is a $2k \times 2k$-matrix, its upper triangular form has $2k$ nonzero diagonal elements. Consequently, A_V is invertible, and due to the invertibility of $\tilde{J}$, one arrives at the assertion. □

Corollary 2.1. *Let $\{Y_i\}_{i=1}^{2k}$ denote a local spanning family for V, and let $\{X_i\}_{i=1}^{2k}$ be a local spanning family (of C^∞ sections) of $N\mathfrak{C}_{gen}$, interpreted as a vector bundle over $\mathfrak{C}_{gen}$ that is embedded in $\cup_{x\in\mathfrak{C}_{gen}} T_xM$. Then, the matrix $[g(Y_i, X_j)(x)]_{i,j=1}^{2k}$ is invertible for every $x \in \mathfrak{C}_{gen}$.*

Corollary 2.2. *Let $\mathfrak{C}_{gen}$ satisfy the genericity assumption of theorem 2.1. If V is integrable, the intersection of any integral manifold of V with $\mathfrak{C}_{gen}$ is a discrete set.*

Proof. The previous proposition implies that generically, integral manifolds of V intersect $\mathfrak{C}_{gen}$ transversely. Their dimensions are mutually complementary, hence the intersection set is zero-dimensional. □

2.2. **Normal Hyperbolicity and an Auxiliary Gradient-Like System.** In this section, we introduce the main tool necessary for the analysis of the topology of $\mathfrak{C}$, given by an auxiliary gradient-like flow on M whose same critical set is also $\mathfrak{C}$. Furthermore, we show that generically, $\mathfrak{C}$ is normal hyperbolic with respect to the latter.

2.2.1. *A Generalized Hessian.* To begin with, we define a generalized Hessian for $\mathfrak{C}$. The usual coordinate-free definition of the Hessian of H is ∇dH, evaluated at the critical points of H, where ∇ is the Levi-Civita connection of the Kähler metric g. Let $\pi_V^\dagger$ denote the dual projection tensor associated to π_V, which acts on sections of the cotangent bundle T^*M, so that for any 1-form θ, $\langle \pi_V^\dagger \theta, X\rangle = \langle \theta, \pi_V X\rangle$. The generalization of the Hessian in our context is the tensor $\nabla(\pi_V^\dagger dH)$, which acts as a bilinear form on $\Gamma(TM)\times\Gamma(TM)$ by way of

$$\begin{aligned}\nabla(\pi_V^\dagger dH)(X,Y) &:= \langle \nabla_X(\pi_V^\dagger dH), Y\rangle \\ &= (((\pi_V)_r^j H_{,j})_{,s} - \Gamma_{ri}^s (\pi_V)_s^j H_{,j}) X^r Y^s,\end{aligned}$$

where Γ_{ri}^s are the Christoffel symbols. Evaluating this quantity on $\mathfrak{C}$, the second term in the bracket on the lower line is zero. The non-vanishing term is determined by the matrix

$$[K_{rs}] := [((\pi_V)_r^j H_{,j})_{,s}]. \tag{14}$$

One straightforwardly verifies that $(\pi_V)_i^j K_{jk} = K_{ik}$ is satisfied everywhere on $\mathfrak{C}$, hence $\mathrm{rank}\{K\} \le \mathrm{rank}\{\pi_V\} = 2k = \mathrm{rank}\{V\}$. Clearly, the corank of $K|_a$ equals the dimension of the connectivity component of $\mathfrak{C}$ containing a.

2.2.2. *Definition of the Gradient-Like System.* A flow is gradient-like if there exists a function $f : M \to \mathbb{R}$ that decreases strictly along all of its non-constant orbits. The flow $\tilde{\Phi}_t$ of the constrained Hamiltonian system is not gradient-like, and hence turns out to be of limited use for the study of the global topology of $\mathfrak{C}$, because invariant sets of $\tilde{\Phi}_t$ do in general not only contain fixed points, but also periodic orbits.

Instead, we introduce the auxiliary dynamical system

$$\partial_t \gamma(t) = -(\pi_V \nabla_g H)(\gamma(t)) \; , \tag{15}$$

where $\gamma : I \subset \mathbb{R} \to M$, which turns out to be an extremely powerful tool for our purpose. Let us denote its flow by $\phi_t \in \mathrm{Diff}(M)$. The orbits of (15) are clearly V-horizontal, and both Φ_t^c and ϕ_t exhibit the same critical set $\mathfrak{C}$.

Proposition 2.2. *The flow ϕ_t is gradient-like.*

Proof. Since

$$\begin{aligned}\partial_t H(\gamma(t)) &= \langle dH(\gamma(t))\,,\,\partial_t\gamma(t)\rangle = -g\left(\nabla_g H\,,\,\pi_V\nabla_g H\right)(\gamma(t))\\ &= -g\left(\pi_V\nabla_g H\,,\,\pi_V\nabla_g H\right)(\gamma(t)) \leq 0,\end{aligned}$$

it follows that H decreases strictly along the non-constant orbits of ϕ_t. We have here used the fact that (g, J, ω, π_V) is a compatible quadruple. □

(g, J, ω, π_V) has been constructed for this precise reason. It is immediately clear that ϕ_t generates no periodic trajectories, hence $\mathfrak{C}$ comprises all invariant sets of ϕ_t.

2.2.3. *Morse Functions and Non-Degenerate Critical Manifolds.* Let us next recall some standard definitions from Morse- and Morse-Bott theory that will be needed in the subsequent discussion. The dimensions of the zero and negative eigenspaces of the Hessian of f at a critical point a are called the nullity and the index of the critical point a. If all critical points of $f : M \to \mathbb{R}$ have a zero nullity, f is called a Morse function, and the index is then called the Morse index of a. If the critical points of f are not isolated, but elements of critical manifolds that are non-degenerate in the sense of Bott, H is called a Morse-Bott function [8]. Throughout this section, we will assume that H is a Morse function.

Furthermore, we recall some standard definitions related to normal hyperbolicity, applied to the case of $\mathfrak{C}$. A connectivity component $\mathfrak{C}_i$ is locally normal hyperbolic at the point $a \in \mathfrak{C}$ with respect to ϕ_t if it is a manifold at a, and if the restriction of A_a to the normal space $N_a\mathfrak{C}$ is non-degenerate. A connectivity component $\mathfrak{C}_i$ is called non-degenerate if it is a manifold that is everywhere normal hyperbolic with respect to ϕ_t. The index of a non-degenerate connectivity component $\mathfrak{C}_i$ is the number of eigenvalues of the constrained Hessian A_a on $\mathfrak{C}_i$ that are contained in the negative half plane.

Proposition 2.3. *If $\mathfrak{C}$ is generic in the sense of Sard's theorem, it is normal hyperbolic with respect to the gradient-like flow ϕ_t.*

Proof. This follows straightforwardly from proposition 2.1. □

Let $\mathfrak{C}_i$, $i = 1, .., l$ denote the connectivity components of $\mathfrak{C} = \cup\mathfrak{C}_i$, and let $j_i : \mathfrak{C}_i \hookrightarrow M$ denote the embedding of the i-th components.

Proposition 2.4. *Assume that $\mathfrak{C}$ satisfies the genericity assumption in the sense of Sard's theorem, and that $H : M \to \mathbb{R}$ is a Morse function. Then, $H_i := H \circ j_i : \mathfrak{C}_i \to \mathbb{R}$ is a Morse function, and $x \in \mathfrak{C}_i$ is a critical point of H_i if and only if it is a critical point of H.*

Proof. It is trivially clear that every critical point of H is a critical point of H_i. To prove the opposite direction, suppose that a is an extremum of $H|\mathfrak{C}_i$. Then, $\nabla_g H|_a \in N_a\mathfrak{C}$, but also, by definition of $\mathfrak{C}$, $P_a\nabla_g H|_a = 0$. By corollary 2.1, this can only be true if $\nabla_g H_a = 0$. The Hessian of the restriction of H at any critical point of H_i is nondegenerate, thus H_i is a Morse function on $\mathfrak{C}_i$. □

Corollary 2.3. *The critical points of $H|_{\mathfrak{C}_{gen}} : \mathfrak{C}_{gen} \to \mathbb{R}$ are precisely the critical points of $H : M \to \mathbb{R}$. If $\mathfrak{C}_i$ is a non-generic connectivity component that is a normal hyperbolic submanifold of M, $\mathfrak{C}_i \subset \Sigma_{H(\mathfrak{C}_i)}$.*

Proof. The first assertion follows trivially from the previous proposition. Assuming that $\mathfrak{C}_i$ is a non-generic connectivity component of $\mathfrak{C}$ that is a manifold and normal hyperbolic, the previous proposition implies that there are no extrema of $H|_{\mathfrak{C}_i}$. Thus, $\mathfrak{C}_i$ is a submanifold of the level surface $\Sigma_{H(\mathfrak{C}_i)}$. □

2.3. **Approach via Conley-Zehnder Theory.** The goal in this and the next section is to derive the relationship (18) between the Poincaré polynomials of $\mathfrak{C}$, and M. We first approach this problem by use of Conley-Zehnder theory, [15], and under slightly less restrictive assumptions than genericity.

Let us for convenience first recall some of the key elements from Conley-Zehnder theory, [15]. Let $\mathfrak{C}_i$ be any compact component of $\mathfrak{C}$. An index pair associated to $\mathfrak{C}_i$ is a pair of compact sets $(N_i, \tilde{N}_i)$ that possesses the following properties. The interior of N_i contains $\mathfrak{C}_i$, and moreover, $\mathfrak{C}_i$ is the maximal invariant set under ϕ_t in the interior of N_i. $\tilde{N}_i$ is a compact subset of N_i that has empty intersection with $\mathfrak{C}_i$, and the trajectories of all points in N_i that leave N_i at some time under the gradient-like flow ϕ_t intersect $\tilde{N}_i$. $\tilde{N}_i$ is called the exit set of N_i.

The homotopy type of the pointed space $N_i/\tilde{N}_i$ only depends on $\mathfrak{C}_i$, by a result proven in [15], so that the relative cohomology $H^*(N_i, \tilde{N}_i)$ (with coefficients appropriately chosen) is independent of the particular choice of index pairs (the space $N_i/\tilde{N}_i$ is obtained from collapsing the subspace $\tilde{N}_i$ of N_i to a point). The equivalence class $[N_i/\tilde{N}_i]$ of pointed topological spaces under homotopy is called the *Conley index* of $\mathfrak{C}_i$.

Let I denote a compact invariant invariant set under ϕ_t. A Morse decomposition of I is a finite, disjoint family of compact, invariant subsets $\{M_1, \dots, M_n\}$ that satisfies the following requirement on the ordering. For every $x \in I \setminus \cup_i M_i$, there exists a pair of indices

$i < j$, such that $\lim_{t\to-\infty} \phi_t(x) \subset M_i$, and $\lim_{t\to\infty} \phi_t(x) \subset M_j$. Such an ordering, if it exists, is called admissible, and the M_i are called Morse sets of I.

For every compact invariant set I admitting an admissibly ordered Morse decomposition, there exists an increasing sequence of compact sets N_i with $N_0 \subset N_1 \subset \cdots \subset N_m$, such that (N_i, N_{i-1}) is an index pair for M_i, and (N_m, N_0) is an index pair for I, [15].

Consider compact manifolds $A \supset B \supset C$. It is a standard fact that the exact sequence of relative cohomologies

$$\cdots \stackrel{\delta^{k-1}}{\to} H^k(A,B) \to H^k(A,C) \to H^k(B,C) \stackrel{\delta^k}{\to} H^{k+1}(A,B) \to \ldots$$

implies that, with $r_{i,p}$ denoting the rank of $H^p(N_i, N_{i-1})$,

$$\sum_{i,p} \lambda^p r_{i,p} = \sum_p b_p \lambda^p + (1+\lambda)\mathcal{Q}(\lambda)$$

for the indicated Poincaré polynomials (cf. for instance [22]). b_j is the j-th Betti number of the index pair (N_m, N_0) of I, and $\mathcal{Q}(\lambda)$ is a polynomial in λ with non-negative integer coefficients. Due to the positivity of the coefficients of $\mathcal{Q}(\lambda)$, it is clear that $\sum_i r_{i,p} \geq b_p$.

If M is compact and closed, and if $\mathfrak{C}$ is non-degenerate, the following holds. The invariant set I can be chosen to be equal to M. We let $N_m = M$ and $N_0 = \emptyset$ denote the top and bottom elements of the sequence defined above, and order the connected elements of $\mathfrak{C}$ according to the descending values of the maximum of H attained on each $\mathfrak{C}_i$. Then, $\mathfrak{C}$ furnishes a Morse decomposition for M. The homology groups of M are isomorphic to the relative homology groups of the index pair (N_m, N_0). Hence the numbers b_p are the Betti numbers of M.

Proposition 2.5. *Let $\mathfrak{C}_i \subset \mathfrak{C}_{gen}$ be a generic connectivity component, compact and without boundary, and let $(N_i, \tilde{N}_i)$ denote any associated index pair. Then,*

$$H^{q+\mu_i}(N_i, \tilde{N}_i) \;\cong\; H^q(\mathfrak{C}_i)\ , \tag{16}$$

where μ_i is the index of $\mathfrak{C}_i$, and $q = 0, \ldots, dim(\mathfrak{C}_i)$.

Proof. We consider, for $\epsilon_0 > 0$ small, a compact tubular ϵ_0-neighborhood U of $\mathfrak{C}_i$ (of dimension $2n$), and let

$$W^{cu}_U(\mathfrak{C}_i) \;:=\; (W^-(\mathfrak{C}_i) \cup \mathfrak{C}_i) \cap U$$

denote the intersection of the center unstable manifold of $\mathfrak{C}_i$ with U. $W^-(\mathfrak{C}_i)$ denotes the unstable manifold of $\mathfrak{C}_i$. Pick some small, positive $\epsilon < \epsilon_0$, and let U_ϵ be the compact tubular ϵ-neighborhood of $W^{cu}_U(\mathfrak{C}_i)$ in U.

Letting ϵ continuously go to zero, we obtain a homotopy equivalence of tubular neighborhoods, for which $W_U^{cu}(\mathfrak{C}_i)$ is a deformation retract. Let

$$U_\epsilon^{out} \ := \ \partial U_\epsilon \cap \phi_{\mathbb{R}}(U_\epsilon)$$

denote the intersection of ∂U_ϵ with all orbits of the gradient-like flow that contain points in U_ϵ. Then, $(U_\epsilon, U_\epsilon^{out})$ is an index pair for $\mathfrak{C}_i$, and by letting ϵ continuously go to zero, U_ϵ^{out} is homotopically retracted to $\partial W_U^{cu}(\mathfrak{C}_i)$.

Thus, by homotopy invariance,

$$H^*(U_\epsilon, U_\epsilon^{out}) \cong H^*(W_U^{cu}(\mathfrak{C}_i), \partial W_U^{cu}(\mathfrak{C}_i)).$$

Since $\mathfrak{C}_i$ is normal hyperbolic with respect to the gradient-like flow, $W_U^{cu}(\mathfrak{C}_i)$ has a constant dimension $n_i + \mu(\mathfrak{C}_i)$ everywhere, where $n_i = \dim\mathfrak{C}_i$. Therefore, by Lefschetz duality, [16],

$$H^{n_i+\mu_i-p}(W_U^{cu}(\mathfrak{C}_i), \partial W_U^{cu}(\mathfrak{C}_i)) \cong H_p(W_U^{cu}(\mathfrak{C}_i) \setminus \partial W_U^{cu}(\mathfrak{C}_i)),$$

where $\mu_i = \mu(\mathfrak{C}_i)$, the index of $\mathfrak{C}_i$. Since $\mathfrak{C}_i$ is a deformation retract of the interior of $W_U^{cu}(\mathfrak{C}_i)$, the respective cohomology groups are isomorphic.

Due to $dim(\mathfrak{C}_i) = n_i$, we have by Poincaré duality

$$H_p(W_U^{cu}(\mathfrak{C}_i) \setminus \partial W_U^{cu}(\mathfrak{C}_i)) \cong H_p(\mathfrak{C}_i) \cong H^{n_i-p}(\mathfrak{C}_i),$$

so that with $q := n_i - p$,

$$H^{q+\mu_i}(U_\epsilon, U_\epsilon^{out}) \cong H^q(\mathfrak{C}_i), \tag{17}$$

which proves the claim. □

From (17), we deduce that $r_{i,p} = \dim H^{i,p-\mu_i}(\mathfrak{C}_i)$ (recalling that μ_i is the index of $\mathfrak{C}_i$), hence $r_{i,p} = b_{p-\mu_i}(\mathfrak{C}_i)$. Assuming that the number of connected components of $\mathfrak{C}$ is finite, one thus obtains

$$\sum_{i,p} \lambda^{p+\mu_i} b_p(\mathfrak{C}_i) = \sum_p \lambda^p b_p(M) + (1+\lambda)\mathcal{Q}(\lambda), \tag{18}$$

which in particular implies $\sum_i b_{p-\mu_i}(\mathfrak{C}_i) \geq b_p(M)$. Setting $\lambda = -1$,

$$\sum_p (-1)^{p+\mu_i} b_p(\mathfrak{C}_i) = \sum_i (-1)^{\mu_i} \chi(\mathfrak{C}_i) = \chi(M),$$

where χ denotes the Euler characteristic.

Remark. In the case of mechanical systems, the phase space of the relevant constrained Hamiltonian system is non-compact, and the critical manifold is generally unbounded. Therefore, the arguments used here do not apply. However, since in that case, M and $\mathfrak{C}$ are vector bundles, we are nevertheless able to prove results that are fully analogous to (18).

We can prove a slightly more general result by relaxing the assumption of genericity.

Proposition 2.6. *Assume that $\mathfrak{C} \setminus \mathfrak{C}_{gen}$ is a disjoint union of C^1-manifolds. Then,*

$$\sum_{\substack{i,p \\ \mathfrak{C}_i \in \mathfrak{C}_{gen}}} b_{i,p} \, \lambda^{p+\mu_i} \; = \; \sum_p b_p \, \lambda^p \; + \; (1+\lambda) \, \tilde{\mathcal{Q}}(\lambda) \; . \tag{19}$$

$b_{i,p}$ are the p-th Betti numbers of the connectivity components $\mathfrak{C}_i$ of $\mathfrak{C}_{gen}$, b_p are the Betti numbers of M, and $\tilde{\mathcal{Q}}$ is a polynomial with non-negative integer coefficients.

Proof. We show that X_H^V can be infinitesimally perturbed such that $\mathfrak{C} \setminus \mathfrak{C}_{gen}$ is removed. Consider, for $\epsilon > 0$, the compact tubular neighborhoods

$$U_\epsilon(\mathfrak{C}_i) = \{x \in M \mid dist_g(x, \mathfrak{C}_i) \leq \epsilon\} \tag{20}$$

of connectivity components $\mathfrak{C}_i \subset \mathfrak{C} \setminus \mathfrak{C}_{gen}$, where $dist_g$ denotes the Riemannian distance function induced by g. We introduce a vector field X_ϵ, given by $\pi_V \nabla_g H$ in $M \setminus U_\epsilon(\mathfrak{C}_i)$, and in the interior of every $U_\epsilon(\mathfrak{C}_i)$ with $\mathfrak{C}_i \subset \mathfrak{C} \setminus \mathfrak{C}_{gen}$, by

$$X_\epsilon|_x = \pi_V \nabla_g H|_x + \epsilon h(x) \nabla_g H|_x \; . \tag{21}$$

Here, $h \in C^1(U_\epsilon(\mathfrak{C}_i), [0,1])$, obeying $h|_{\mathfrak{C}_i} = 1$ and $h|_{\partial U_\epsilon(\mathfrak{C}_i)} = 0$ is strictly monotonic along the the flow lines generated by $\pi_V \nabla_g H$. h exists because $\mathfrak{C} \setminus \mathfrak{C}_{gen}$ is a disjoint union of C^1-manifolds.

For all $\mathfrak{C}_i \subset \mathfrak{C} \setminus \mathfrak{C}_{gen}$, $\nabla_g H$ is strictly non-zero in $U_\epsilon(\mathfrak{C}_i)$, as shown above. We have

$$g(X_\epsilon, \nabla_g H) = (\|\pi_V \nabla_g H\|_g^2)(x) + \epsilon h(x)(\|\nabla_g H\|_g^2)(x),$$

where we have used the g-symmetry of π_V, and $\|X\|_g^2 \equiv g(X, X)$. The first term on the right hand side is non-zero on the boundary of $U_\epsilon(\mathfrak{C}_i)$, while the second term vanishes. Moreover, the second term is non-zero everywhere in the interior of $U_\epsilon(\mathfrak{C}_i)$. Therefore, X_ϵ vanishes nowhere in $U_\epsilon(\mathfrak{C}_i)$. Hence, X_ϵ is a deformation of $\pi_V \nabla_g H$, with critical set $\mathfrak{C}_{gen}$. Notably, $\mathfrak{C}_{gen}$ cannot be removed in this manner, since it contains critical points of H.

Since $g(X_\epsilon, \nabla_g H)$ is strictly positive in every $U_\epsilon(\mathfrak{C}_i)$, X_ϵ generates a gradient-like flow. Since $\|X_\epsilon - \pi_V \nabla_g H\|_g \leq O(\epsilon)$ everywhere on M, one can pick X_ϵ arbitrarily close to $\pi_V \nabla_g H$

in the L^∞-norm on $\Gamma(TM)$ induced by $\|\cdot\|_g$. Carrying out the Conley-Zehnder construction with respect to the flow generated by X_ϵ yields (19). This result does not require the assumption of normal hyperbolicity on $\mathfrak{C}$. $\square$

2.4. **Approach via the Morse-Witten Complex.** We will next provide a different derivation of (19), based on the construction of the Morse-Witten differential complex. The motivation is to clarify the orbit structure of the gradient-like system, and to devise an explicit construction that relates the Morse-Witten complexes of the free and constrained system to one another. This in particular only involves the corresponding theory for *non*-degenerate Morse functions.

Let us to begin with briefly recall the basic framework of this construction. Let M be a compact, closed, orientable and smooth n-manifold, and let $f : M \to \mathbb{R}$ be a Morse function. Let C^p denote the free $\mathbb{Z}$-module generated by the critical points of f with a Morse index p. The set $C = \oplus_p C^p$ is the free $\mathbb{Z}$-module generated by the critical points of f, and graded by their Morse indices. There exists a natural coboundary operator $\delta : C^p \to C^{p+1}$, with $\delta \circ \delta = 0$, whose construction we recall next, cf. [4, 9, 17, 36].

Introducing an auxiliary Riemannian structure on M, we let W_a^- and W_a^+ denote the unstable and stable manifold of the critical point a of f under the gradient flow, respectively, and assign an arbitrary orientation to every W_a^-. The orientation of M, together with the orientation of W_a^- at every critical point a induces an orientation of W_a^+. Morse functions, for which all W_a^- and $W_{a'}^+$ intersect transversely, are dense in $C^\infty(M)$. The dimension of W_a^- equals the Morse index $\mu(a)$ of a, and the dimension of the intersection $M(a, a') := W_a^- \cap W_{a'}^+$ is given by $\max\{\mu(a) - \mu(a'), 0\}$. For pairs of critical points a and a' with relative Morse index 1, say $\mu(a) = \mu(a') + 1$, $M(a, a')$ is a finite collection of gradient lines that connect a with a'.

The intersection of $M(a, a')$ with any regular level surface Σ_c of f with $f(a) < f(\Sigma_c) = c < f(a')$ is transverse, and consists of a finite collection of isolated points. Then, one picks the orientation of Σ_c, which, combined with the section $\nabla_g f$ of its normal bundle, shall agree with the orientation of M. The submanifolds $W_{a,c}^- := W_a^- \cap \Sigma_c$ and $W_{a',c}^+ := W_{a'}^+ \cap \Sigma_c$ of Σ_c are smooth, compact and closed, with complementary dimensions in Σ_c, and orientations picked above. Hence, their intersection number, which is often in this context written as $\langle a, \delta a'\rangle := \sharp(W_{a,c}^-, W_{a',c}^+)$, is well-defined, [20]. The coboundary operator of the Morse-Witten complex is defined as the $\mathbb{Z}$-linear map $\delta : C^p \to C^{p+1}$, defined by

$$\delta a' = \sum_{\mu(a)=p+1} \langle a, \delta a' \rangle a.$$

Theorem 2.2. *The cohomology of the differential complex (C, δ) is isomorphic to the de Rham cohomology of M,* $\mathrm{ker}\delta/\mathrm{im}\delta \cong H^*(M, \mathbb{Z})$.

The proof can for instance be found in [4, 17, 29, 36]. If $\langle a, \delta a' \rangle \neq 0$ for a pair a and a' of critical points with a relative Morse index 1, we will say that they are effectively connected (by gradient lines).

As is well-known, the existence of the Morse-Witten complex implies the strong Morse inequalities in the following manner. Let $Z^p := \mathrm{ker}\delta \cap C^p$ denote the p-th cocyle group, $B^p \subset C^p$ the p-th coboundary group, and $H^p := Z^p \setminus B^p$ the p-th cohomology group under δ. Thus, $\mathrm{dim}H^p = b_p(M)$ by theorem 2.2. From

$$\mathrm{dim}C^p = b_p(M) + \mathrm{dim}B^p + \mathrm{dim}B^{p+1},$$

where $\mathrm{dim}C^p = N_p$ (the number of critical points of f with Morse index p), follows

$$\sum_{p=0}^{n} \lambda^p N_p = \sum_{p=0}^{n} \lambda^p b_p(M) + (1+\lambda) \sum_{p=1}^{n} \lambda^{p-1} \mathrm{dim}B^p \tag{22}$$

(both B^0 and B^{2n+1} are empty). The coefficients of the polynomial $\mathcal{Q}(\lambda) = \sum \lambda^{p-1}\mathrm{dim}B^p$ are evidently non-negative and integer. Clearly, $\mathrm{dim}B^p$ is the number of critical points of Morse index p that are effectively connected to critical points of Morse index $p-1$ via gradient lines of f.

2.4.1. **Comparing the Complexes for the Free and Constrained System.** The goal of our discussion here is to devise an explicit construction that relates the Morse-Witten complex of the free system (M, H) to the one on the critical manifold $(\mathfrak{C}_{gen}, H|_{\mathfrak{C}_{gen}})$, by a deformation of the gradient-like flow ϕ_t. This will yield (19).

Let $\mathfrak{C}_i$ denote the i-th connectivity component of $\mathfrak{C}_{gen}$, and $\mathcal{A}_i := \{a_{i,1}, \dots, a_{i,m}\}$ the critical points of H contained in $\mathfrak{C}_i$. Furthermore, let $\mu(a_{i,r})$ be the associated Morse indices of $H : M \to \mathbb{R}$, and $H_i := H|_{\mathfrak{C}_i}$ denote the restriction of H to $\mathfrak{C}_i$. By proposition 2.3 and corollary 2.3, $H_i : \mathfrak{C}_i \to \mathbb{R}$ is a Morse function, whose critical points are precisely the elements of $\mathcal{A}_i$. The index $\mu(\mathfrak{C}_i)$ of $\mathfrak{C}_i$ equals the number of negative eigenvalues of the Hessian of H at any $a_{i,r} \in \mathcal{A}_i$ whose eigenspace is normal to $\mathfrak{C}_i$.

The Morse index of $a_{i,r}$ with respect to H_i is thus $\mu(a_{i,r}) - \mu(\mathfrak{C}_i)$. To define the Morse-Witten complex associated to $\mathfrak{C}_i$, we introduce the free $\mathbb{Z}$-module generated by the elements of $\mathcal{A}_i$, graded by the Morse indices p of the critical points of H_i,

$$C_i = \oplus_p C_i^p.$$

To construct the coboundary operator $\delta_i : C_i^p \to C_i^{p+1}$, one uses the gradient flow on $\mathfrak{C}_i$ generated by H_i, thus obtaining

$$\ker\delta_i / \mathrm{im}\delta_i \cong H^*(\mathfrak{C}_i, \mathbb{Z}). \tag{23}$$

Application of (22) shows that for every $\mathfrak{C}_i \in \mathfrak{C}_{gen}$,

$$\sum_p \lambda^p N_{i,p} \;=\; \sum_p \lambda^p b_p(\mathfrak{C}_i) + (1+\lambda) \sum_p \lambda^{p-1} \mathrm{dim} B_i^p, \tag{24}$$

where B_i^p is the p-th coboundary group of the Morse-Witten complex of $\mathfrak{C}_i$, and $N_{i,p}$ is the number of critical points of H_i on $\mathfrak{C}_i$ of Morse index p.

Since every critical point of H lies on precisely one generic component $\mathfrak{C}_i$, the number N_q of critical points of H with a Morse index q is given by

$$N_p = \sum_i N_{i;p-\mu(\mathfrak{C}_i)}.$$

Thus, combining (24) with (22), one obtains

$$\begin{aligned} \sum_{i,p;\mathfrak{C}_i \in \mathfrak{C}_{gen}} \lambda^q b_{q-\mu(\mathfrak{C}_i)}(\mathfrak{C}_i) &= \sum_q \lambda^q b_q(M) \\ &+ (1+\lambda) \sum_q \lambda^{q-1} \Big(\mathrm{dim} B^q - \sum_{\mathfrak{C}_i \in \mathfrak{C}_{gen}} \mathrm{dim} B_i^{q-\mu(\mathfrak{C}_i)} \Big) . \end{aligned}$$

Hence, (19) is equivalent to the statement that the polynomial on the last line, which is multiplied by $(1+\lambda)$, has non-negative integer coefficients.

By a homotopy argument, we will now prove that for all q,

$$\mathrm{dim} B^q \geq \sum_{\mathfrak{C}_i \in \mathfrak{C}_{gen}} \mathrm{dim} B_i^{q-\mu(\mathfrak{C}_i)} \tag{25}$$

holds, thus obtaining an alternative proof of (19). The main motivation here is to give an explicit construction that geometrically elucidates this relation, noting that the left hand side is defined by the flow of the 'free' gradient-like system corresponding to (M, ω, H), while the right hand side is defined by the constrained gradient-like system corresponding to (M, ω, H, V). We note that $\mathrm{dim} B_i^{q-\mu(\mathfrak{C}_i)}$ denotes the number of critical points of H with a Morse index q in $\mathfrak{C}_i$, which are effectively connected to critical points of Morse index $p+1$ in

$\mathfrak{C}_i$ via gradient lines of the Morse function H_i on $\mathfrak{C}_i$. Therefore, the sum on the right hand side of (25) equals the number of those critical points of H with a Morse index q, which are effectively connected to critical points of Morse index $q+1$ via gradient lines of the functions $H \circ j_i$ on all generic $\mathfrak{C}_i$. Here, $j_i : \mathfrak{C}_i \to M$ denotes the corresponding inclusion maps.

2.4.2. *Proof of (25).* Our strategy consists of constructing a homotopy of vector fields v_s, with $s \in [0,1]$, whose zeros are hyperbolic and independent of s, which generate gradient-like flows. They interpolate between $v_1 := \nabla_g H$, and v_0, which is a vector field that is tangent to $\mathfrak{C}_{gen}$. For every $s \in [0,1]$, we construct a coboundary operator via the one-dimensional integral curves of v_s that connect its zeros. These coboundary operators are independent of s, and act on the free $\mathbb{Z}$-module C of the Morse-Witten complex associated to (M,H). (25) then follows from a simple dimensional argument.

Lemma 2.1. *There exists $v_0 \in \Gamma(TM)$, which is gradient-like, and tangent to $\mathfrak{C}_{gen}$. Furthermore, the zeros of v_0 are hyperbolic, and identical to the critical points of H. The dimension of any unstable manifold of the flow generated by $-v_0$ equals the Morse index of the critical point of H from which it emanates.*

Proof. We recall the vector field X_ϵ constructed in the proof of proposition 2.6, and consider the compact tubular ϵ-neighborhoods $U_\epsilon(\mathfrak{C}_{gen})$, defined similarly as in (20). Furthermore, let $\bar{Q} = \bar{Q}^2$ (resp. $Q = \mathbf{1} - \bar{Q}$) be g-orthogonal, smooth tensors on $TU_\epsilon(\mathfrak{C}_{gen})$ of fixed rank $2(n-k)$ (resp. $2k$), with $\mathrm{Ker}\{\bar{Q}(a)\} = N_a\mathfrak{C}_{gen}$ (resp. $\mathrm{Ker}\{Q(a)\} = T_a\mathfrak{C}_{gen}$) for every $a \in \mathfrak{C}_{gen}$.

We define v_0 as follows. In $M \setminus U_\epsilon(\mathfrak{C}_{gen})$, it shall equal X_ϵ, and that for x in $U_\epsilon(\mathfrak{C}_{gen})$, it shall be given by

$$v_0(x) := (\pi_V \nabla_g H)(x) + h(x)(\bar{Q}\nabla_g H)(x),$$

where $h : U_\epsilon(\mathfrak{C}_{gen}) \to [0,1]$ is a smooth function obeying $h|_{\mathfrak{C}_{gen}} = 1$ and $h|_{\partial U_\epsilon(\mathfrak{C}_{gen})} = 0$. In particular, h is assumed to be strictly monotonic along all non-constant trajectories of the flow generated by $\pi_V \nabla_g H$, and dh shall vanish on $\mathfrak{C}_{gen}$.

It can be easily verified that v_0 possesses all of the desired properties. It generates a gradient-like flow, since outside of $U_\epsilon(\mathfrak{C}_{gen})$, $g(\nabla_g H, v_0) = g(\nabla_g H, X_\epsilon) > 0$, as has been shown in the proof of proposition 2.6. In the interior of $U_\epsilon(\mathfrak{C}_{gen})$, one finds $g(\nabla_g H, v_0) = \|\pi_V \nabla_g H\|_g^2 + h\|\bar{Q}\nabla_g H\|_g^2$, due to the g-orthogonality both of π_V and $\bar{Q}$. The first term on the right hand side vanishes everywhere on $\mathfrak{C}_{gen}$, but at no other point in $U_\epsilon(\mathfrak{C}_{gen})$. The second term equals $\|\bar{Q}\nabla_g H\|_g^2$ on $\mathfrak{C}_{gen}$. Since evidently, $\bar{Q}\nabla_g H|_{\mathfrak{C}_{gen}}$ is the gradient field of the Morse function $H|_{\mathfrak{C}_{gen}} : \mathfrak{C}_{gen} \to \mathbb{R}$ relative to the Riemannian metric on $T\mathfrak{C}_{gen}$ induced by g, its zeros are precisely the critical points of H on $\mathfrak{C}_{gen}$, and it possesses no other zeros. Because $g(v_0, \nabla_g H) > 0$ except at the critical points of H, it is clear that $-v_0$ generates a gradient-like flow $\psi_{0,t}$, so that H is strictly decreasing along all non-constant orbits. Furthermore, it is clear from the given construction that v_0 is tangent to $\mathfrak{C}_{gen}$.

To prove the remaining statements of the lemma, we note that the Jacobian matrix of v_0 at a in a local chart is given by

$$Dv_0(a) = (D_a^2 H)^\sharp + (P_a - Q_a)(D_a^2 H)^\sharp. \tag{26}$$

There is no dependence on h because $dh|_{\mathfrak{C}_{gen}} = 0$. Furthermore, $(D_a^2 H)^\sharp$ is defined as the matrix $[g^{ij} H_{,jk}|_a]$ in the given chart, and P_a denotes the matrix of $\pi_V(a)$. Normal hyperbolicity follows from the invertibility of $Dv_0(a)$, which is verified in the proof of lemma 2.1 below. □

Lemma 2.2. *Let $v_s := s\nabla_g H + (1-s)v_0$ with $s \in [0,1]$. Then, the flow $\psi_{s,t}$ generated by $-v_s$ is gradient-like for any $s \in [0,1]$. The zeros of v_s are hyperbolic fixed points of $\psi_{s,t}$, and independent of s. Thus, the dimensions of the corresponding unstable manifolds equal the Morse indices of the critical points of H from which they emanate, for all s.*

Proof. We consider $g(\nabla_g H, v_s) = s\|\nabla_g H\|_g^2 + (1-s)g(\nabla_g H, v_0)$. The first term on the right hand side is obviously everywhere positive except at the critical points of H, and the same has been proved previously for the second term. Thus, H decreases strictly along all non-constant orbits of $\psi_{s,t}$, hence the latter is gradient-like. The Jacobian of v_s at a critical point of H is given by

$$Dv_s(a) = (\mathbf{1}_{2n} + (1-s)(P_a - Q_a))(D_a^2 H)^\sharp.$$

$Dv_s(a)$ is invertible for all $s \in [0,1]$, since $(D_a^2 H)^\sharp$ is invertible, and $\mathrm{spec}\{P_a - Q_a\} \subset (-1,1)$. To prove the latter, we first observe that $\mathrm{spec}\{P_a - Q_a\} \subset [-1,1]$ is trivial, because P_a and Q_a both have a spectrum $\{0,1\}$. $\{-1,1\}$ is not included, because otherwise, $\bar{P}_a Q_a$, respectively $P_a \bar{Q}_a$, would not have a full rank, in contradiction to corollary 2.1. □

By smoothness of v_s, it follows that $\psi_{s,t}$ is C^∞ in s. Thus, s smoothly parametrizes a homotopy of stable and unstable manifolds of $\psi_{s,t}$ emanating from the critical points of H. Since the fixed points of $\psi_{s,t}$ are independent of s, and the dimensions of the corresponding unstable manifolds are equal to the Morse indices of the critical points of H, we consider, for every value of $s \in [0,1]$, the free $\mathbb{Z}$-module $C = \oplus_p C^p$ that is generated by the critical points of H, and graded by their Morse indices. For every s, we define a coboundary operator on C, using $\psi_{s,t}$ as follows. Picking a pair of critical points of H with a relative Morse index 1, we consider the unstable manifold $W^-_{s,a}$ of a, and the stable manifold $W^+_{s,a'}$ of a' associated to $\psi_{s,t}$. Since s parametrizes a homotopy of such manifolds, they naturally inherit an orientation from the one picked for $s=1$ in the construction of the coboundary operator of the Morse-Witten complex for (M,H).

Let Σ_E denote a regular energy surface for $H(a) < E < H(a')$. $W^{\pm}_{s,a}$ intersects Σ_E transversely, because H is strictly decreasing along all non-constant orbits generated by $-v_s$. $W^-_{s,a} \cap \Sigma_E$ and $W^+_{s,a'} \cap \Sigma_E$ define two homotopies of oriented submanifolds of Σ_E. By homotopy invariance of their intersection number, the coboundary operators are independent of s, and thus identical to the δ-operator of the Morse-Witten complex given for $s = 1$.

The stable and unstable manifolds of $\psi_{0,t}$ are either confined to some $\mathfrak{C}_i$, or connect critical points lying on different $\mathfrak{C}_i$'s. Let us consider pairs of critical points of H with a relative Morse index 1 that lie on the same component $\mathfrak{C}_i \in \mathfrak{C}_{gen}$, and the corresponding stable and unstable manifolds of $\psi_{0,t}$ which are contained in $\mathfrak{C}_i$. Since $v_0|_{\mathfrak{C}_i}$ is the projection of $\nabla_g H|_{\mathfrak{C}_i}$ to $T\mathfrak{C}_i$, these stable and unstable manifolds are the same as those which were used to define the Morse-Witten complex on $(\mathfrak{C}_i, H_i)$. Using only stable and unstable manifolds of $\psi_{0,t}$ contained in $\mathfrak{C}_{gen}$, we construct an operator $\tilde{\delta}$ acting on C in the same manner in which the coboundary operator was defined, thus obtaining $\tilde{\delta} \equiv \oplus_i \delta_i$, where δ_i is the coboundary operator of the Morse-Witten complex associated to the pair $(\mathfrak{C}_i, H_i)$. Let $P_i : C \to C_i$ stand for the projection of the free $\mathbb{Z}$-module C generated by all critical points of H to the one generated by the critical points contained in $\mathfrak{C}_i$. Eliminating all integral lines of $-v_0$ that connect critical points on different connectivity components of $\mathfrak{C}_{gen}$ in the above construction, one sees that $\delta_i = P_i \delta P_i$, thus $\tilde{\delta} = P_i \delta P_i$. Hence, clearly,

$$\dim(\mathrm{im}\delta|_{C^p}) \geq \dim(\mathrm{im}\tilde{\delta}|_{C^p}) ,$$

which precisely corresponds to (25). This completes the proof.

3. QUALITATIVE ASPECTS RELATED TO CRITICAL STABILITY

So far, we have established that in the generic case, the connectivity components of $\mathfrak{C} = \mathfrak{C}_{gen}$ are embedded submanifolds of dimension $2(n-k)$ equal to the corank of V.

Furthermore, we have seen that the topology of the symplectic manifold M enforces the existence of connectivity components of $\mathfrak{C}_{gen}$ of certain prescribed indices with respect to the auxiliary gradient-like flow ϕ_t.

In this section, we focus on the physical dynamics, characterized by the flow $\tilde{\Phi}_t$ generated by X_H^V, of the constrained Hamiltonian system (M, ω, H, V) in a tubular ϵ-neighborhood of $\mathfrak{C}_{gen}$, and particularly on the issue of stability. Let g again denote the auxiliary Kähler metric introduced in section 2, with the induced Riemannian distance function given by $dist_R$ (in contrast to the Carnot-Caratheodory distance function d_{C-C} induced by g, which will also be considered). We recall that a point $a \in \mathfrak{C}_{gen}$ is stable if there exists $\delta(\epsilon) > 0$ for every $\epsilon > 0$, so that for all t, $dist_R(\tilde{\Phi}_t(x), a) < \epsilon$ holds for all x with $dist_R(x, a) < \delta(\epsilon)$.

To elucidate the key differences between the local dynamics in the vicinity of $\mathfrak{C}_{gen}$ for the cases of integrable and non-integrable V, let us first describe the situation where V is integrable. As proved in corollary 2.2, M is foliated into $2k$-dimensional symplectic submanifolds which intersect $\mathfrak{C}_{gen}$ transversely. Thus, on every leaf $\mathcal{N}$, the equilibrium solutions are generically isolated points. Let the linear operator Ω_a correspond to the linearization of X_H^V on T_aM for some $a \in \mathcal{N} \cap \mathfrak{C}_{gen}$, and restricted to the fibre $V_a = T_a\mathcal{N} \subset T_aM$. Its spectrum, if it is not purely imaginary, conclusively characterizes the stability of a; we refer to this as the asymptotically (un)stable case. If the spectrum of Ω_a is purely imaginary, which we refer to as the critically stable case, it is well-known that if there exists a local Lyapunov function $L_{\mathcal{N}} : U(a) \cap \mathcal{N} \to \mathbb{R}$ for a, then a is stable.

If V is non-holonomic, the situation is similar in the asymptotically (un)stable case, but drastically different in the critically stable situation. In the critically stable case, the presence of a local degenerate Lyapunov function for a single equilibrium $a \in \mathfrak{C}_{gen}$ is of limited use, since there is a whole submanifold $\mathfrak{C}_{gen} \cap \Sigma_E$ (the H-level set for the energy E of the initial condition) of valid equilibria for a given energy E. One may relax this condition to the existence of the following class of functions.

Definition 3.1. *Let $\nabla_g^{\perp}$ denote the component of the gradient ∇_g normal to $\mathfrak{C}_{gen}$ with respect to g at $\mathfrak{C}_{gen}$. Let $U(a)$ be a $dist_R$-small open neighborhood of $a \in \mathfrak{C}_{gen}$. A local, degenerate, almost Lyapunov function for a is a class C^1 function $L : U(a) \to \mathbb{R}$, which satisfies $(\nabla_g^{\perp} L)(a') = 0$ for all $a' \in \mathfrak{C}_{gen} \cap U(a)$, and $\|\nabla_g L\|_g > 0$ for all $x \in U(a) \setminus \mathfrak{C}_{gen}$. Furthermore, $(\nabla_g dL)|_{a'}$ is positive definite quadratic form on $N_{a'}\mathfrak{C}_{gen}$ for all $a' \in U(a) \cap \mathfrak{C}_{gen}$, and $L(\tilde{\Phi}_t(x_0)) \leq L(x_0)$ for all $x_0 \in U(x_0)$, and all t such that $\tilde{\Phi}_t(x_0) \in U(a)$.*

Notably, L defined here is not a local degenerate Lyapunov function, because $\mathfrak{C}_i \cap U(a)$ is not a critical level set (we remark that this would be equivalent to L being a Morse-Bott function in $U(a)$), on which L is extremal.

While the existence of L guarantees that the orbit $\tilde{\Phi}_t(x_0)$ remains within a tubular ϵ-neighborhood of $\mathfrak{C}_{gen} \cap U(a)$ for all t such that $\tilde{\Phi}_t(x_0) \in U(a)$, it does not imply stability of $a \in \mathfrak{C}_{gen} \cap U$. There is an additional, necessary condition on the rational independence of the frequencies of the oscillatory linear problem that must be imposed. Otherwise, an inner resonance, connected to the appearance of small divisors, occurs, and $\tilde{\Phi}_t(x_0) \in U(a)$ may evolve away from a, in a diffusive motion along the higher flag elements of V that are approximately tangent to $\mathfrak{C}_{gen}$, while along V, which is transverse to $\mathfrak{C}_{gen}$, the motion is bounded and oscillatory.

From the analysis in section 2, it is clear that for every connectivity component $\mathfrak{C}_i \subset \mathfrak{C}_{gen}$ of index $\mu(\mathfrak{C}_i) = 0$ (with respect to ϕ_t), the Hamiltonian H is a local degenerate, almost Lyapunov function for all of its points. The minimum a^* of $H|_{\mathfrak{C}_i}$ on $\mathfrak{C}_i$ is a local minimum of H, and hence stable (since H serves as a Lyapunov function for a^*). Hence, in particular, if V is integrable, all points on $\mathfrak{C}_i$ are stable if $\mu(\mathfrak{C}_i) = 0$. We also note that on the connectivity components $\mathfrak{C}_j$ with index $\mu(\mathfrak{C}_j) > 0$, H is never a local degenerate, almost Lyapunov function.

The main focus in this section will be to discuss issues of this type. However, an essential part of sections 3.1.2 and 3.2.2 will be in mathematically non-rigorous terms, since a rigorous treatment of the matters addressed there would fall into the domain of KAM and Nekhoroshev theory, and is beyond the scope of the present work.

A concrete aim in this discussion is to arrive at stability criteria for equilibria of the constrained Hamiltonian system (M, ω, H, V). From an instructive, despite elementary, application of averaging theory, we conjecture a condition for the critically stable case that involves an incommensurability condition imposed on the frequencies of the linearized problem, as remarked above. In order to elucidate its geometric content, we study the dynamics in the vicinity of a critically stable equilibrium in a geometrically invariant form that is adapted to the flag of V. Invoking a perturbation expansion based on this description, we argue that the incommensurability condition, which might merely correspond to an artefact of the averaging method, cannot be omitted. A rigorous proof of the conjectured stability criterion is left for future work.

3.1. **Stability Criteria.** Let $a \in \mathfrak{C}_{gen}$, and pick some small neighborhood $U(a) \subset M$ together with an associated Darboux chart, with its origin at a. The equations of motion are given by $\partial_t x_t = P(x_t)\mathcal{J}H_{,x}(x_t) = X_H^V(x_t)$, where the coordinates are given by $x = (x^1, \dots, x^n, x_{n+1}, \dots, x_{2n})$, and $\mathcal{J}$ is the symplectic standard matrix. Furthermore, $H_{,x}$ abbreviates $\partial_x H$, and P is the $2n \times 2n$-matrix representing the tensor π_V. ω-skew orthogonality of π_V translates into $P(x)\mathcal{J}X(x) = \mathcal{J}P^\dagger(x)X(x)$ for all vector fields X.

Proposition 3.1. *There exists a chart in which the equations of motion have the form*

$$\partial_t(y_t, z_t) = \big(\Omega_0 y_t + Y(z_t, y_t)\ ,\ Z(z_t, y_t)\big) \in \mathbb{R}^{2k} \times \mathbb{R}^{2(n-k)}\ . \tag{27}$$

In particular, Ω_0 corresponds to the restriction of $DX_H^V(0)$ to V_0, and $|Y(y,z)|$, $|Z(y,z)| = O(|y|\,|z|) + O(|y|^2)$.

Proof. In a sufficiently small vicinity $U \subset \mathbb{R}^{2n}$ of the origin (corresponding to a), one infers from corollary 2.1 that $T_a\mathfrak{C}_{gen} \oplus V_0 = \mathbb{R}^{2n}$, for $a \in U \cap \mathfrak{C}_{gen}$. Accordingly, we choose local coordinates $z \in U'(0) \subset \mathbb{R}^{2(n-k)}$ on $\mathfrak{C}_{gen}$, and $\tilde{y} \in V_0$, noting that the decomposition $x = a(z) + \tilde{y}$ for any $x \in U \subset \mathbb{R}^{2n}$ is unique, where $a : \mathbb{R}^{2n-k} \hookrightarrow U$ is the (smooth) embedding. Let y denote the coordinates of $\tilde{y}$ with respect to some family of basis vectors for V_0. Then, (27) evidently follows from Taylor expansion. $\square$

3.1.1. *Asymptotic (In)stability.* If $\mathrm{spec}\{\Omega_0\} \cup i\mathbb{R} = \emptyset$, there exists, by the center manifold theorem, a coordinate transformation $(y, z) \to (\bar{y}, \bar{z})$, such that (27) becomes

$$\partial_t(\bar{y}_t, z_t) = \big(\Omega_0 \bar{y}_t + \bar{Y}(\bar{y}_t, \bar{z}_t)\ ,\ 0\big) \tag{28}$$

[40], where $\bar{Y}(0, \bar{z}) = 0$ for all $\bar{z}$. Thus, $a \in \mathfrak{C}_{gen}$ is asymptotically unstable if there are eigenvalues with a positive real part, and asymptotically stable if all eigenvalues have a negative real part. If V is integrable, asymptotic stability is impossible, because the dynamics is Hamiltonian on every integral manifold. However, if V is non-integrable, there is, to the author's knowledge, no obstruction to the existence of asymptotically stable equilibria, since the flow map is not symplectic.

3.1.2. *An Elementary Application of Averaging Theory.* In the case of critical stability, one has $\mathrm{spec}\{\Omega_0\} = \{i\omega_1, \dots, i\omega_{2k}\}$, with $\omega_i \in \mathbb{R} \setminus \{0\}$ for $i = 1, \dots, 2k$. Let us for the context of an averaging analysis assume that the vector fields on the r.h.s. of (27) are real analytic with respect to (y, z). We apply a complex linear coordinate transformation that diagonalizes Ω_0, and denote the complexified, new coordinates and vector fields again by (y,z), and $Y(y,z)$, $Z(y,z)$, respectively, by which we find

$$\partial_t(y_t,z_t) = \big(\mathrm{diag}(i\omega)y_t + Y(y_t,z_t)\,,\, Z(y_t,z_t)\big) \in \mathbb{C}^{2k}\times\mathbb{C}^{2(n-k)}\,, \tag{29}$$

where $\omega := (\omega_1,\dots,\omega_{2k})$. Complexifying (27), the continuation of $\mathfrak{C}_{gen}$ into $\mathbb{C}^{2n}$ is defined as the common zeros of $Y(0,z)$ and $Z(0,z)$ for $z\in\mathbb{C}^{2(n-k)}$.

We next introduce polar coordinates $(I,\phi)\in\mathbb{R}^{2k}\times[0,2\pi)^{2k}$ and $(J,\theta)\in\mathbb{R}^{2(n-k)}\times[0,2\pi)^{2(n-k)}$ in terms of $y^r =: e^{i\phi_r}I^r$ and $z^s =: e^{i\theta_s}J^s$, with $r=1,\dots,2k$ and $s=1,\dots,2n-2k$. In particular, $I\in\mathbb{R}^{2k}$, $J\in\mathbb{R}^{2n-2k}$, $\phi\in[0,2\pi]^{2k}=\mathbb{T}^{2k}$ (the $2k$-dimensional torus), and $\theta\in[0,2\pi]^{2n-2k}=\mathbb{T}^{2n-2k}$. For brevity, let $e^{i\phi}v := (e^{i\phi_1}v^1,\dots,e^{i\phi_{2k}}v^{2k})$ and $e^{i\theta}w := (e^{i\theta_1}w^1,\dots,e^{i\theta_{2(n-k)}}w^{2(n-k)})$, for $v\in\mathbb{C}^{2k}$ and $w\in\mathbb{C}^{2n-2k}$. (29) is then easily seen to be equivalent to (the dot abbreviates ∂_t)

$$\dot I = \mathrm{Re}\{e^{-i\phi}Y(e^{i\phi}I,e^{i\theta}J)\}\quad,\quad \dot\phi = \omega + \mathrm{Im}\{e^{-i\phi}\mathrm{diag}(\partial_I)Y(e^{i\phi}I,e^{i\theta}J)\}. \tag{30}$$

$$\dot J = \mathrm{Re}\{e^{-i\theta}Z(e^{i\phi}I,e^{i\theta}J)\}\quad,\quad \dot\theta = \mathrm{Im}\{e^{-i\theta}\mathrm{diag}(\partial_J)Z(e^{i\phi}I,e^{i\theta}J)\}. \tag{31}$$

Let us assume that $\epsilon := |I(0)| \ll 1$, and $|J(0)|\le O(\epsilon^2)$. We then introduce rescaled variables $I\to\epsilon I$ and $J\to\epsilon^2 J$.

Analyticity of $Y(y,z)$ and $Z(y,z)$ with respect to (y,z) implies that the power series expansion with respect to $e^{i\phi}I$ and $e^{i\theta}J$ converges for ϵ sufficiently small. Accordingly, (30) and (31) yield

$$\dot I^r = \sum_{|m|+|p|\ge 2}\epsilon^{|m|+2|p|-1}F^r_{mp}(I,J)e^{i(\langle m,\phi\rangle-\phi_r)}e^{i\langle p,\theta\rangle} \tag{32}$$

$$\dot J^s = \sum_{|m|+|p|\ge 2}\epsilon^{|m|+2|p|-2}G^s_{mp}(I,J)e^{i\langle m,\phi\rangle}e^{i\langle p,\theta\rangle} \tag{33}$$

$$\dot\phi_r = \omega_r + \sum_{|m|+|p|\ge 2}\epsilon^{|m|+2|p|-1}\Phi_{r;mp}(I,J)e^{i(\langle m,\phi\rangle-\phi_r)}e^{i\langle p,\theta\rangle} \tag{34}$$

$$\dot\theta_s = \sum_{|m|+|p|\ge 2}\epsilon^{|m|+2|p|-2}\Theta_{s;mp}(I,J)e^{i\langle m,\phi\rangle}e^{i\langle p,\theta\rangle}, \tag{35}$$

where we introduced the multiindices $m\in\mathbb{Z}^{2k}$ and $p\in\mathbb{Z}^{2n-2k}$, with $|m| := \sum|m_r|$ and $|p| := \sum|p_s|$. In this Fourier expansion with respect to the 2π-periodic angular variables ϕ and θ, every Fourier coefficient labeled by a pair of indices (m,p), is a homogenous polynomial of degree $|m|$ in I, and of degree $|p|$ in J.

If the components of ω are all mutually rationally independent, one may consider the averaged quantities $f_t(\phi)\to\bar f_t := (2\pi)^{-n}\int_{\mathbb{T}^n}d\phi f_t(\phi)$. From (29), $Y(y,z)$ and $Z(y,z)$ are

$O(|y|)$, thus their power series involve terms $e^{i\langle m,\phi\rangle}$ with $|m| \geq 1$, but none with $|m| = 0$. Averaging (32) $\sim$ (35) with respect to ϕ thus gives

$$\dot{\bar{I}} = \epsilon^2 \tilde{F}(\bar{I}, \bar{J}, \bar{\theta}) \ , \ \ \dot{\bar{J}} = 0 \ , \ \ \dot{\bar{\theta}} = 0 \tag{36}$$

for some function $\tilde{F}$, where the bars account for averaged variables. Thus, if we in addition assume that there exists a local degenerate almost Lyapunov function with respect to $\mathfrak{C}_{gen} \cap U$, it follows for the averaged equations of motion that $|\bar{I}|$ is bounded for all t. In particular, if the incommensurability condition holds, $|\bar{J}|$ is then also bounded for all t, and a (respectively 0) is, for the averaged system, stable. Based on these insights, and on intuition stemming from KAM and Nekhoroshev theory, it is thus natural to conjecture the following stability criterion.

Conjecture 3.1. *Let $\mathfrak{C}_i \subset \mathfrak{C}_{gen}$ be a connectivity component of the critical manifold, and let $a \in \mathfrak{C}_i$, with $\mathrm{spec}\{DX_H^V(a)\} \setminus \{0\} = \{i\omega_1, \dots, i\omega_{2k}\}$, and $\omega_i \in \mathbb{R} \setminus \{0\}$ for $i = 1, \dots, 2k$. Assume that (1) the frequencies ω_r are rationally independent, and (2) that there exists a local degenerate, almost Lyapunov function with respect to $\mathfrak{C}_i \cap U(a)$, in the sense of definition 3.1. Then, a is stable in the sense of Nekhoroshev. In particular, condition (2) is always satisfied (by the Hamiltonian H) if the index of $\mathfrak{C}_i$ is $\mu(\mathfrak{C}_i) = 0$.*

3.2. **The Relationship to Sub-Riemannan Geometry.** To elucidate the geometric nature of the requirement of rationally independent frequencies, we will now approach the discussion of critical stability from a different point of view. This discussion involves issues that are central to sub-Riemannian geometry, [5, 18, 19, 34].

We study the time evolution map in a tubular ϵ-vicinity of $U(a) \cap \mathfrak{C}_{gen}$ by invoking a geometrically invariant Lie series that is adapted to the elements of the flag of V. By an asymptotic analysis, we explain the mechanism by which an instability can arise. The reason is that if the eigenfrequencies of the linear problem are not incommensurable, the problem of small divisors appears. This picture seems to be familiar from the perturbation theory of integrable Hamiltonian systems, but we note once more that the lack of integrability here originates from the non-holonomy of the constraints. A rigorous treatment of this last part of the analysis is beyond our current scope, and left for future work.

3.2.1. *Dynamics Along the Flag of V.* Let U denote a small open neighborhood U of $a \in \mathfrak{C}$, and assume that $\mathfrak{C}_{gen} := \mathfrak{C} \cap U$ satisfies the genericity condition of theorem 2.1.

Lemma 3.1. *Let $\mathfrak{C}_{gen} = \mathfrak{C} \cap U$ have the genericity property formulated in theorem 2.1. Then, there exists $\epsilon > 0$ such that every point $x \in U$ with $d_R(x, \mathfrak{C}_{gen}) < \epsilon$ is given by*

$$x = \exp_s Y(a) \quad , \quad |s| < \epsilon$$

for some $Y \in \Gamma(V)$ with $\|Y\|_{g_M} \leq 1$, $a \in \mathfrak{C}_{gen}$ ($\exp_s Y$ denotes the 1-parameter group of diffeomorphisms generated by Y, with $\exp_0 Y = id$).

Proof. We choose a spanning family $\{Y_i \in \Gamma(V)\}_1^{2k}$ of V, with $\|Y_i\|_{g_M} = 1$. If for all $a \in \mathfrak{C}_{gen}$, $T_a \mathfrak{C}_{gen}$ contains no subspace of V_a, then

$$\exp_1(t_1 Y_1 + \cdots + t_{2k} Y_{2k})(\mathfrak{C}_{gen}) \;\; \cap \;\; U$$

is an open tubular neighborhood of $\mathfrak{C}_{gen}$ in U, for $t_i \in (-\epsilon, \epsilon)$. Because the normal space $N_a \mathfrak{C}_{gen}$ is dual to the span of the 1-forms dF_i at a, this condition is satisfied if and only if the matrix $[dF_j(Y_i)] = [Y_i(Y_j(H))]$ is invertible everywhere on $\mathfrak{C}_{gen}$. According to proposition 2.1, this condition is indeed fulfilled. $\square$

Hence, there is an element $Y \in \Gamma(V)$ with $\|Y\|_{g_M} \leq 1$, so that $x = \Psi_\epsilon(a)$ for some $0 < \epsilon \ll 1$. Since $a \in \mathfrak{C}_{gen}$, it is clear that under the flow generated by X_H^V, $\tilde{\Phi}_{\pm t}(a) = a$, thus the solution of (6) with initial condition x is given by

$$\Psi_\epsilon^t(a) \;:=\; \tilde{\Phi}_t \circ \Psi_\epsilon(a) \;=\; \left(\tilde{\Phi}_t \circ \Psi_\epsilon \circ \tilde{\Phi}_{-t}\right)(a) \; .$$

Ψ_ϵ^t is, in particular, the 1-parameter group of diffeomorphisms with respect to the variable ϵ that is generated by the pushforward vector field

$$Y_t(x) \;:=\; \tilde{\Phi}_{t\,*}\, Y(x) \;=\; d\tilde{\Phi}_t \circ Y(\tilde{\Phi}_{-t}(x)) \; , \tag{37}$$

where $d\tilde{\Phi}_t$ denotes the tangent map associated to $\tilde{\Phi}_t$. From the group property $Y_{s+t} = \tilde{\Phi}_{s\,*} Y_t$ follows that

$$\partial_t Y_t \;=\; \partial_s\big|_{s=0} \tilde{\Phi}_{s\,*} Y_t \;=\; [X_H^V, Y_t] \; , \tag{38}$$

everywhere in U.

Next, we pick a local spanning family $\{Y_i \in \Gamma(V)\}_{i=1}^{2k}$ for V that satisfies $\omega(Y_i, Y_j) = \tilde{J}_{ij}$, with $\tilde{J} := \begin{pmatrix} 0 & \mathbf{1}_k \\ -\mathbf{1}_k & 0 \end{pmatrix}$. Furthermore, defining $\theta_i(\cdot) := \omega(Y_i, \cdot)$, $\pi_V = \tilde{J}^{ij} Y_i \otimes \theta_j$, where $\tilde{J}^{ij}$ are the components of $\tilde{J}^{-1} = -\tilde{J}$. In particular,

$$X_H^V \;=\; \pi_V(X_H^V) \;=\; -\, Y_i(H)\, \tilde{J}^{ij}\, Y_j$$

in the basis $\{Y_i\}_{i=1}^{2k}$.

The following proposition characterizes the orbit emanating from x in terms of nested commutators with respect to Y_t.

Proposition 3.2. *Let $f, F_i \in C^\infty(U)$, where $F_i := Y_i(H)$, $i = 1, \dots, 2k$, and assume that $F_i(\Psi^t_\epsilon(a))$, $f(\Psi^t_\epsilon(a))$ are real analytic in ϵ. For $X, Y \in \Gamma(TM)$, let*

$$\mathcal{L}^r_Y X \ = \ [Y, \dots, [Y, X]]$$

denote the r-fold iterated Lie derivative. Then, for sufficiently small ϵ,

$$\partial_t f(\Psi^t_\epsilon(a)) = -F_i(\Psi^t_\epsilon(a))\tilde{J}^{ik} \sum_{r\geq 0} \frac{\epsilon^r}{r!} (\mathcal{L}^r_{Y_t} Y_k)(f \circ \Psi^t_\epsilon)(a) \ . \tag{39}$$

Proof. Clearly,

$$\begin{aligned} \partial_t f(\Psi^t_\epsilon(a)) \ &= \ X^V_H(f)(\Psi^t_\epsilon(a)) \\ &= \ -F_i(\Psi^t_\epsilon(a))\tilde{J}^{ik} Y_k(f)(\Psi^t_\epsilon(a)) \\ &= \ -F_i(\Psi^t_\epsilon(a))\tilde{J}^{ik} \left(\Psi^t_{\epsilon\,*} Y_k\right)(f \circ \Psi^t_\epsilon)(a) \ . \end{aligned} \tag{40}$$

Using the Lie series $\Psi^t_{\epsilon\,*} Y_k = \sum_r \frac{\epsilon^r}{r!} \mathcal{L}^r_{Y_t} Y_k$, we arrive at the assertion. □

Proposition 3.3. *Assume that $Y_{t=0} \in \Gamma(V)$, and let $\{Y_j\}_1^{2k}$ be the given local spanning family of V. Then, $\mathcal{L}^i_{Y_t} Y_j \in \Gamma(V_i)$, where V_i is the i-th flag element of V.*

Proof. Since $\tilde{\Phi}_{t\,*} : \Gamma(V) \to \Gamma(V)$, Y_t is a section of V for all t if it is for $t = 0$. The claim immediately follows from the definition of the flag of V. □

Proposition 3.3 implies that there are functions $a^i(t, \cdot) \in C^\infty(U)$, $i = 1, \dots, 2k$, so that $Y_t(x) = a^i(t, x) Y_i$. The next proposition determines their time evolution.

Proposition 3.4. *Let $Y_{t=0} = a^i_0 Y_i$ define the initial condition, and introduce the matrix*

$$\Omega_x := [Y_l(F_i)(x)\tilde{J}^{ij}] \ .$$

Then, pointwise in x,

$$a^m(t, x) \ = \ (\exp(-t\Omega_x))^m_j \, a^j_0 + F_j(x) R^{jm}_i(t, x) a^i_0 \ , \tag{41}$$

where

$$R^{jm}_i(t, x) := \tilde{J}^{jl} \tilde{J}^{nk} \int_0^t ds (\exp(-(t-s)\Omega_x))^m_k \omega \left([Y_l, \tilde{\Phi}_{s\,*} Y_i], Y_n\right) \ .$$

Proof. The initial condition at $t = 0$ is given by $Y_0 = a_0^i Y_i$, that is, by $a^i(0,x) = a_0^i$. Thus, by the definition of Y_t in (37), one has $Y_t = a_0^i\, \tilde{\Phi}_{t\,*} Y_i$, so that $a^i(t,x)Y_i = a_0^i \tilde{\Phi}_{t\,*} Y_i$. From $\omega(Y_i, Y_j) = \tilde{J}_{ij}$, $\tilde{J}_{ik} = -\tilde{J}_{ki}$ and $\tilde{J}_{im}\tilde{J}^{ml} = -\delta_i^l$,

$$a^l(t,x) = -a_0^i \omega\left(\tilde{\Phi}_{t\,*} Y_i\,,\, Y_j\right) \tilde{J}^{jl}\ .$$

Now, taking the t-derivative on both sides of the equality sign, one finds

$$\begin{aligned}
\partial_t a^m(t,x) &= -a_0^i \omega\left([X_H^V, \tilde{\Phi}_{t\,*} Y_i]\,,\, Y_k\right) \tilde{J}^{km} \\
&= -a^i(t,x) Y_i(F_j)(x) \tilde{J}^{jm} \\
&\quad -a_0^i F_j(x) \tilde{J}^{jl} \tilde{J}^{km} \omega\Big([Y_l, \tilde{\Phi}_{t\,*} Y_i]\,,\, Y_k\Big)\ .
\end{aligned}$$

Using the variation of constants formula pointwise in x, one arrives at the assertion. $\square$

3.2.2. *Non-holonomy and small divisors.* Using the description of the dynamics in the vicinity of a derived above, we will here use the small parameter ϵ for an asymptotic expansion. The intention of this part of the discussion, which is not rigorous, is to explain the geometric origin of the incommensurability condition on frequencies in conjecture 3.1.

We consider the following simplified situation:

(1) $\Omega_x = \Omega$, constant for all x in U.
(2) $\mathrm{spec}\{\Omega\} = \{i\omega_1, \dots, i\omega_{2k}\}$, with $\omega_r \in \mathbb{R}$.
(3) $\|\Omega\| := \sup_r |\omega_r| \ll \frac{1}{\epsilon}$.

Let us briefly comment on the generic properties of $\{\omega_r\}$. Writing $\Omega = \tilde{J}A$, we decompose the matrix $A = [Y_i(Y_j(H))(a)]$ into its symmetric and antisymmetric parts A_+ and A_-, respectively. $A_- = [[Y_j, Y_i](H)(a)]/2$ vanishes if V is integrable, which one deduces from $X_H|_a \in V_a^\perp$ for all $a \in \mathfrak{C}_{gen}$, and the Frobenius condition. The linear system $\underline{\dot{a}} = \tilde{J}A_+\underline{a}$ is Hamiltonian, hence the spectrum of $\tilde{J}A_+$ consists of complex conjugate pairs of eigenvalues in $i\mathbb{R}$ if it is purely imaginary (here, $\underline{a} := (a^1, \dots, a^{2k})$). Considering $\tilde{J}A_-$ as a perturbation of $\tilde{J}A_+$, we may generically assume that all frequencies ω_r are distinct from one another, and that there are both negative and positive frequencies.

Under the simplifying assumptions at hand, let us compute (41) to leading order in ϵ. From (41), one infers

$$Y_t = a_0^j (\exp(-t\Omega))_j^i Y_i + \sum_i O(|x|) Y_i\ ,$$

since $|F_j(x)| = O(|x|) = O(\epsilon)$, which follows from $F_j(a) = 0$. Thus,

$$[Y_t, X] = a_0^j \exp(-t\Omega)_j^i [Y_i, X] + \sum_i O(\epsilon)[Y_i, X] + \sum_i O(1) Y_i$$

for all $X \in \Gamma(TM)$, and $x \in U_\epsilon(a)$. Assuming that all objects in question are C^∞, iterating the Lie bracket $\mathcal{L}_{Y_t}$ r times produces

$$\Big(\prod_{m=1}^{r} a_0^{j_m} \big(\exp(-t\Omega)\big)_{j_m}^{i_m} + O(\epsilon) \Big) [Y_{i_1}, [Y_{i_2}, \dots, [Y_{i_r}, Y_l] \cdots]] \ ,$$

plus a series of terms with less than r nested Lie commutators that contribute to higher order corrections.

Let us, for the discussion of the leading order terms along each flag element of V, omit the relative errors of order $O(\epsilon)$. By the assumption of smoothness, our considerations are valid for $t \le O(\epsilon^{-1})$. Let us consider the term

$$F_i(\Psi_\epsilon^t(a)) \tilde{J}^{ik} \left(\mathcal{L}_{Y_t}^r Y_k \right) (f \circ \Psi_\epsilon^t)(a) \ , \tag{42}$$

for fixed r. It is easy to see that

$$F_i(\Psi_\epsilon^t(a)) = Y_t(F_i)(a) + O(\epsilon^2) \ , \tag{43}$$

due to $F_i(a) = 0$. Therefore,

$$F_i(\Psi_\epsilon^t(a)) \tilde{J}^{ik} = \epsilon \exp\left(-t\Omega\right)_j^m a_0^j \Omega_m^k + O(\epsilon^2) \ , \tag{44}$$

from a straightforward calculation.

Hence, the terms with r nested commutators in (42) are

$$\frac{\epsilon^{r+1}}{r!} a_0^i \exp(-t\,\Omega)_i^j \Omega_j^l \Big(\prod_{m=1}^{r} a_0^{j_m} \ (\exp(-t\,\Omega))_{j_m}^{i_m} \Big) [Y_{i_1}, [\dots, [Y_{i_r}, Y_l] \cdots]](f)(a) \\ + O(\epsilon^{r+2}) \ ,$$

as long as $dist_R(\Psi_\epsilon^t(a), a) \le O(\epsilon)$. This implies that for $f \in C^\infty(U)$,

$$\begin{aligned} f(\Psi_\epsilon^t(a)) \ &\sim \ f(a) + \sum_{r \ge 0} \frac{\epsilon^{r+1}}{r!} \int_0^t ds a_0^i \exp(-s\Omega)_i^j \Omega_j^l \\ &\Big(\prod_{m=1}^{r} a_0^{j_m} \ (\exp(-s\ \Omega))_{j_m}^{i_m} \Big) [Y_{i_1}, \dots, [Y_{i_r}, Y_l] \cdots](f)(a) \ , \end{aligned} \tag{45}$$

up to relative errors of higher order in ϵ for every fixed r.

If f is chosen as the i-th coordinate function x^i, so that $f(\Psi_\epsilon^t(a)) = x_t^i$, the quantity $[Y_{i_1}, \dots, [Y_{i_r}, Y_l] \cdots](f)(a)$ is the i-th coordinate of the vector field defined by the brackets at a. Consequently, (45) is the component decomposition of x_t^i relative to the flag of V at a, to leading order in ϵ.

By the given simplifying assumptions, $\mathrm{spec}\{\Omega\} \subset i\mathbb{R} \setminus \{0\}$, and the norm of $\exp(-s\Omega)$ is 1, independently of s. Consequently, the integrand of (45) is bounded for all s. It follows that if the r-th integral in the sum diverges, it will become apparent only for $t \geq O\big(\frac{1}{\epsilon^r}\big)$. This would correspond to an instability along the direction of the flag element V_r. While the leading term with $r = 0$ is bounded for all t, terms with $r > 0$ can diverge.

We next write

$$\underline{a}(s) = \exp(-s\Omega)\underline{a}_0 = \sum_{\alpha=1}^{2k} A_\alpha \underline{e}_\alpha \exp(-i\omega_\alpha s) \ , \tag{46}$$

where $\{\underline{e}_\alpha\}$ is an orthonormal eigenbasis of Ω with respect to the standard scalar product in $\mathbb{C}^{2k}$, and $\mathrm{spec}\{\Omega\} = \{i\omega_\alpha\}$. The amplitudes $A_\alpha \in \mathbb{C}$ are determined by the initial condition $a^i(t = 0) = a_0^i$, which we assume to be nonzero. By linear recombination of the vector fields Y_i, one can set $e_\alpha^i = \delta_{i,\alpha}$. Then, (45) can be written as

$$\sum_{r\geq 0} \frac{\epsilon^{r+1}}{r!} \sum_{l;i_1,\dots,i_r} I_{l;i_1,\dots,i_r}(t)[Y_{i_1}, \dots, [Y_{i_r}, Y_l] \cdots](f)(a) \ , \tag{47}$$

where

$$\begin{aligned} I_{l;i_1,\dots,i_r}(t) \ &:= \ \int_0^t ds \omega_l A_l \Big(\prod_{m=1}^{r} A_{j_m} \Big) e^{-is(\omega_l + \sum_{m=1}^r \omega_{j_m})} \\ &= \ \frac{i\omega_l A_l}{\omega_l + \sum_{m=1}^r \omega_{j_m}} \Big(\prod_{m=1}^{r} A_{j_m} \Big) \Big(e^{-it(\omega_l + \sum_{m=1}^r \omega_{j_m})} - 1 \Big) \ . \end{aligned} \tag{48}$$

We note that the sum of frequencies (of generically indefinite signs) in the denominator on the last line raises the problem of small divisors. We also remark that evidently, the nested commutators vanish if all indices $i_1, \dots, i_r, l$ have equal values, as it should be (otherwise, the solutions would always diverge).

3.2.3. *Rational frequency dependence and blow-up of solutions.* Let us next discuss the situation in which the small divisors approach zero. To this end, we introduce the set

$$\mathfrak{I}^{(r)}(t) \ := \ \{I_{l;i_1,\dots,i_r}(t)\}_{l,i_j=1}^{2k} \ \setminus \ \{I_{l;l,\dots,l}(t)\}_{l=1}^{2k} \ , \tag{49}$$

which we endow with the norm $\|\mathfrak{I}^{(r)}(t)\| := \sup_{I(t)\in\mathfrak{I}^{(r)}(t)} |I(t)|$, and let $\|A\| := \sup_{i=1,\dots,2k}\{|A_i|\}$, where A_i are $\mathbb{C}$-valued amplitudes.

Furthermore, let $\mathfrak{A} := \{\omega_1, \dots, \omega_{2k}\}$, and let

$$\mathfrak{A}_r := \underbrace{\mathfrak{A} + \cdots + \mathfrak{A}}_{r \text{ times}} , \tag{50}$$

denote its r-fold sumset, which is the set containing all sums of r elements of $\mathfrak{A}$.

For two sets of real numbers $\mathfrak{A}$ and $\mathfrak{B}$, we define

$$d(\mathfrak{A}, \mathfrak{B}) := \inf_{i,j} \left\{ |a_i - b_j| \;\middle|\; a_i \in \mathfrak{A} \ , \ b_j \in \mathfrak{B} \right\} . \tag{51}$$

Then, it follows from (48) that if $d(\mathfrak{A}_r, -\mathfrak{A}) > 0$,

$$\|\mathfrak{I}^{(r)}(t)\| \le d(\mathfrak{A}_r, -\mathfrak{A})^{-1} \|\Omega\| \|A\|^r \tag{52}$$

(the sum over frequencies $\sum_{m=1}^{r} \omega_{j_m}$ in (48) is an element of $\mathfrak{A}_r$, and can only equal $-\omega_l$ if $d(\mathfrak{A}_r, -\mathfrak{A}) = 0$). However, if $d(\mathfrak{A}_r, -\mathfrak{A}) = 0$, there is a tuple of indices $\{l; i_1, \dots, i_r\}$ such that

$$I_{l;i_1,\dots,i_r}(t) = -t\omega_l A_l \prod_{m=1}^{r} A_{j_m} , \tag{53}$$

in case of which $\|\mathfrak{I}^{(r)}(t)\| \sim t$, that is, a divergence linear in t for large t (recalling that the present asymptotic considerations require $t \le \epsilon^{-1}$). Only if there are simultaneously positive and negative frequencies, $d(\mathfrak{A}_r, -\mathfrak{A}) = 0$ is possible, but due to the remark at the beginning of subsection 3.2.2, this situation must generically assumed to be given.

As an illustration, the following picture holds for $r \le 2$. The fact that for $r = 0$, $\|\mathfrak{I}^{(0)}(t)\|$ is bounded for all t is clear. For $r = 1$, the first flag element $V_1 = [V, V]$ is in question. The condition for the emergence of a divergence is that $d(\mathfrak{A}, -\mathfrak{A}) = 0$. This is precisely given if there is a pair of frequencies $\pm\omega_i$ of equal modulus, but opposite sign. For $r = 2$, assuming that $d(\mathfrak{A}, -\mathfrak{A}) > 0$, the condition $d(\mathfrak{A}_2, -\mathfrak{A}) = 0$ implies that there is a triple of frequencies such that $\omega_{i_1} + \omega_{i_2} = -\omega_{i_3}$, $i_j \in \{1, \dots, 2k\}$. If this occurs, the solution will diverge in the direction of the second flag element, $V_2 = [V, [V, V]]$. The discussion for $r > 2$ continues in the same manner.

Hence, our conclusion from this asymptotic analysis is that if $d(\mathfrak{A}_r, -\mathfrak{A}) = 0$ for some r, then $\|\mathfrak{I}^{(r)}(t)\| = O(t)$ for $t \to \infty$.

The physical insight gained from the above discussion can be summarized as follows. If the frequencies of the linearized problem fail to satisfy the incommensurability condition $d(\mathfrak{A}_r, -\mathfrak{A}) > 0$ for all r, the equilibrium a is unstable. However, the time required for an orbit to exit from a Riemannian ϵ-neighborhood $U_\epsilon(a)$ is very large. In fact, assuming that $d(\mathfrak{A}_r, -\mathfrak{A}) = 0$ for some $r \le \deg(V)$ (the degree of non-holonomy of V), a time $t \sim O(\frac{1}{\epsilon^r})$

is necessary to exit from $U_\epsilon(a)$ in the direction of the flag element V_r (due to the factor $\frac{\epsilon^r}{r!}$ in (45)). We note that the orbit does not drift out from $U_\epsilon(a) \cap \mathfrak{C}_{gen}$ in the direction of V_a owing to the existence of a local degenerate Lyapunov function required in the conjectured stability criterion. Therefore, this discussion suggests that the incommensurability condition imposed on the frequencies of the linearized system can indeed not be omitted.

3.2.4. *Instabilities in the Context of Carnot-Caratheodory Geometry.* The constrained Hamiltonian system (M, ω, H, V) shares many characteristics with systems typically encountered in sub-Riemannian geometry [5, 18, 19, 34]. The natural metric structure in this context is given by the Carnot-Caratheodory distance function $dist_{C-C}$ induced by the Riemannian metric g. It assigns to a pair of points $x, y \in M$ the length of the shortest V-horizontal g-geodesic.

If V satisfies the Chow condition, $dist_{C-C}(x, y)$ is finite for all $x, y \in M$, by the Rashevsky-Chow theorem [5, 19]. In this case, the Carnot-Caratheodory ϵ-ball

$$B_\epsilon^{C-C}(a) := \Big\{ x \in M \Big| dist_{C-C}(x, a) < \epsilon \Big\}$$

is open in M.

If V fails to satisfy the Chow condition, pairs of points that cannot be joined by V-horizontal g_M-geodesics are assigned a Carnot-Caratheodory distance ∞. Then, M is locally foliated into submanifolds N_λ of dimension $(2n - \mathrm{rank} V_{\deg(V)})$ (we recall that $\deg(V)$ denotes the degree of non-holonomy of V), with λ in some index set, which are integral manifolds of the (necessarily integrable) final element $V_{\deg(V)}$ of the flag of V. On every N_λ, the distribution $V_\lambda := j_\lambda^* V$ satisfies the Chow condition, where $j_\lambda : N_\lambda \to M$ is the inclusion. Therefore, all points $x, y \in N_\lambda$ have a finite distance with respect to the Carnot-Caratheodory metric induced by the Riemannian metric $j_\lambda^* g_M$. Every leaf N_λ is an invariant manifold of the flow $\tilde{\Phi}_t$.

Let $\{Y_{i_r}\}_{r=1}^{\deg(V)}$ denote a local spanning family of TM such that $\{Y_{i_r}\}$ spans the flag element V_r. Let the g-length of all Y_{i_r}'s be 1. Then, we define the 'quenched' box

$$\mathrm{Box}_\epsilon(x) := \Big\{ \exp_1 \Big(\sum_{r=1}^{\deg(V)} \epsilon^r \sum_{i_r=1}^{\dim V_r} t_{i_r} \, Y_{i_r} \Big)(x) \Big| \, t_{i_r} \in (-1, 1) \Big\}$$

in N_λ, where λ is suitably picked so that $x \in N_\lambda$. Evidently, if V satisfies Chow's condition, $N_\lambda = M$. According to the ball-box theorem [5, 19], there are constants $C > c > 0$, such that

$$\mathrm{Box}_{c\epsilon}(x) \subset B_\epsilon^{C-C}(x) \subset \mathrm{Box}_{C\epsilon}(x) \, .$$

Carnot-Caratheodory ϵ-balls can be approximated by quenched boxes in Riemannian geometry.

The above perturbative results imply that if there is some $r < \deg(V)$, for which $d(\mathfrak{A}_r, -\mathfrak{A}) = 0$, the flow $\tilde{\Phi}_t$ blows up the quenched boxes, and thus the Carnot-Caratheodory ϵ-ball around $a \in \mathfrak{C}_{gen}$, linearly in t, and along the direction of V_r. In fact, $B_\epsilon^{C-C}(a)$ is widened along V_r at a rate linear in t. For $t = O(\frac{1}{\epsilon})$, $\tilde{\Phi}_t$ maps the Carnot-Caratheodory ϵ-ball containing the initial condition to a Carnot-Caratheodory ball of radius $O(1)$. Thus, in the context of Carnot-Caratheodory geometry, these instabilities, which have no counterpart in systems with integrable constraints, are far more significant than in the Riemannian picture.

4. AUTONOMOUS NON-HOLONOMIC SYSTEMS IN CLASSICAL MECHANICS

In this main section, we focus on the analysis of non-holonomic mechanical systems, and their relationship to the constrained Hamiltonian systems considered previously. The discussion is restricted to linear non-holonomic, *Pfaffian* constraints.

Let (Q, g, U) be a Hamiltonian mechanical system, where Q is a smooth Riemannian n-manifold with a C^∞ metric tensor g, and where $U \in C^\infty(Q)$ denotes the potential energy. No gyroscopic forces are taken into consideration. Let g^* denote the induced Riemannian metric on the cotangent bundle T^*Q. For $X \in \Gamma(TM)$, let θ_X be the 1-form defined by $\theta_X(Y) = g(X,Y)$ for all $Y \in \Gamma(TQ)$. Clearly, $g(X,Y) = g^*(\theta_X, \theta_Y)$ for all $X, Y \in \Gamma(TQ)$.

The Kähler metric of the previous discussion, also denoted by g, will not appear in this section. From here on, g will denote the Riemannian metric on Q, which should not give rise to any confusion.

In a local trivialization of T^*Q, a point $x \in T^*Q$ is represented by a tuple (q^i, p_j), where q^i are coordinates on Q, and p_k are fibre coordinates in T_q^*Q, with $i, j = 1, \dots, n$. The natural symplectic 2-form associated to T^*Q, can be written in coordinates as

$$\omega_0 = \sum_i dq^i \wedge dp_i \ = \ -d\theta_0 \ .$$

$\theta_0 = p_i dq^i$ is referred to as the symplectic 1-form.

We will only consider Hamiltonians of the form

$$H(q,p) \ = \ \frac{1}{2}\, g_q^*(p,p) \ + \ U(q) \ . \tag{54}$$

In local bundle coordinates, the corresponding Hamiltonian vector field X_H is given by

$$X_H = \sum_i \big((\partial_{p_i} H)\partial_{q^i} - (\partial_{q^i} H)\partial_{p_i}\big) .$$

The orbits of the associated Hamiltonian flow Φ_t satisfy

$$\dot{q}^i = \partial_{p_i} H(q,p) \quad , \quad \dot{p}_j = -\partial_{q^j} H(q,p) . \tag{55}$$

The superscript dot abbreviates ∂_t, and will be used throughout the discussion.

Let $\mathcal{A}_I$ denote the space of smooth curves $\gamma : I \subset \mathbb{R} \to T^*Q$, with I compact and connected, and let t denote a coordinate on $\mathbb{R}$. The basis one form dt defines a measure on $\mathbb{R}$. The action functional is defined by $\mathcal{I} : \mathcal{A}_I \to \mathbb{R}$,

$$\begin{aligned} \mathcal{I}[\gamma] &= \int_I dt \ (\gamma^*\theta_0 - H \circ \gamma) \\ &= \int_I dt \ \Big(\sum p_i(t)\dot{q}^i(t) - H(q(t),p(t))\Big) , \end{aligned} \tag{56}$$

with $\dot{\gamma} = \sum(\dot{q}^i\partial_{q^i} + \dot{p}_i\partial_{p_i})$. Denoting the base point projection by $\pi : T^*Q \longrightarrow Q$, let $c := (\pi \circ \gamma) : I \longrightarrow Q$ denote the projection of γ to Q. We assume that $\|c(I)\|$ is sufficiently small so that solutions of (55) exist, which connect the end points $c(\partial I)$. Among all curves $\gamma : I \to T^*Q$ with fixed projected endpoints $c(\partial I)$, the ones that extremize $\mathcal{I}$ are physical orbits of the system.

4.1. **Linear Non-Holonomic Constraints.** Let us next impose linear, 'Pfaffian' constraints on the Hamiltonian mechanical system (Q, g, U), by adding a rank k distribution W over Q to the existing data, and invoke the Hölder variational principle, [3], that generates the correct physical flow on T^*Q. The orbits of the resulting constrained dynamical system possess W-horizontal projections to Q.

We introduce the g-symmetric projection tensor associated to W given by $\rho_W = \rho_W^2 : TQ \to TQ$, with

$$\mathrm{Ker}(\rho_W) = W^\perp \quad , \quad \rho_W(X) = X \quad \forall\, X \in \Gamma(TQ) ,$$

and its orthogonal complement $\bar{\rho}_W = \mathbf{1} - \rho_W$. We note that in local coordinates, ρ_W is represented by a $n \times n$ matrix of rank k. The dual of W, denoted by W^*, is defined as the image of W under the isomorphism $g : TQ \to T^*Q$, and likewise for $(W^*)^\perp := g \circ W^*$. The corresponding g^*-orthogonal projection tensors on T^*Q are denoted by $\rho_W^\dagger$ and $\bar{\rho}_W^\dagger$, respectively. Our inspiration to introduce ρ_W and $\bar{\rho}_W$ for this analysis stems from [11].

4.1.1. *Dynamics of the Constrained Mechanical System.* Next, we derive the equations of motion of the constrained mechanical system from the Hölder variational principle. For a closely related approach to the Lagrangian theory of constrained mechanical systems, cf. [13].

Definition 4.1. *A projective W-horizontal curve in T^*Q is an embedding $\gamma : I \subset \mathbb{R} \hookrightarrow T^*Q$ whose image $c = \pi \circ \gamma$ under base point projection $\pi : T^*Q \to Q$ is tangent to W.*

Let $\gamma_s : I \to T^*Q$, with $s \in [0,1]$, be a smooth one parameter family of curves for which the end points $c_s(\partial I)$ are independent of s (where $c_s := \pi \circ \gamma_s$).

Definition 4.2. *A W-horizontal variation of a projective W-horizontal curve γ is a smooth one parameter family $\gamma_s : \mathbb{R} \to T^*Q$, with $s \in [0,1]$, for which $\frac{\partial}{\partial s}(\pi \circ \gamma_s)$ is tangent to W, and $\gamma_0 = \gamma$.*

Let

$$\phi^i(t) := \partial_s|_{s=0}\, q^i(s,t) \quad , \quad \chi_k(t) := \partial_s|_{s=0}\, p_k(s,t) \; .$$

To any W-horizontal variation γ_s of a W-horizontal curve γ_0 with fixed projections of the boundaries

$$(\pi \circ \gamma_s)(\partial I) = (\pi \circ \gamma_0)(\partial I) \; , \tag{57}$$

so that $\phi^i|_{\partial I} = 0$, we associate the action functional

$$\mathcal{I}[\gamma_s] = \int_I \Big(\sum p_i(s,t) \dot{q}^i(s,t) - H(q(s,t), p(s,t)) \Big)\, dt \; .$$

Definition 4.3. *(Hölder principle) A physical orbit of the constrained mechanical system (Q, g, U, W) is a projective W-horizontal curve $\gamma_0 : I \to T^*Q$ that extremizes $\mathcal{I}[\gamma_s]$ among all W-horizontal variations γ_s which satisfy (57).*

Hence, if

$$\delta \mathcal{I}[\gamma_s] \;=\; \sum p_i \phi^i|_{\partial I} + \int_I \sum \big((\dot{p}_i - \partial_{q^i} H)\phi^i - (\dot{q}^i + \partial_{p_i} H)\chi_i \big) \;=\; 0 \tag{58}$$

for all W-horizontal variations of γ_0 that satisfy $\phi^i|_{\partial I} = 0$, then γ_0 is a physical orbit.

Theorem 4.1. *In the given local bundle chart, the Euler-Lagrange equations of the Hölder variational principle are the differential-algebraic relations*

$$\dot{q} = \rho_W(q)\partial_p H(q,p) \tag{59}$$

$$\rho_W^\dagger(q)\dot{p} = -\rho_W^\dagger(q)\partial_q H(q,p) \tag{60}$$

$$\bar{\rho}_W(q)\partial_p H(q,p) = 0\,. \tag{61}$$

Proof. The boundary term vanishes due to $\phi^i|_{\partial I} = 0$.

For any fixed value of t, one can write $\phi(t)$ as

$$\phi(t) = \sum_{\alpha=1}^{k} f_\alpha(q(t))\, Y_\alpha(q(t))\,,$$

where Y_α is a g-orthonormal family of vector fields over $c(I)$ that spans $W_{c(I)}$. Furthermore, $f_\alpha \in C^\infty(c(I))$ are test functions obeying the boundary condition $f_\alpha(c(\partial I)) = 0$.

Since f_α and χ are arbitrary, the terms in (58) that are contracted with ϕ, and the ones that are contracted with χ vanish independently. In case of ϕ, one finds

$$\int_I dt\; f_\alpha \;\; (\dot{p} + \partial_q H)_i \; Y_\alpha^i \;= 0$$

for all test functions f_α. Thus, $(\dot{p} + \partial_q H)_i\, Y_\alpha^i = 0$ for all $\alpha = 1, \ldots, k$, or equivalently, $\rho_W^\dagger(\dot{p} + \partial_q H) = 0$, which proves (60).

Since γ_0 is W-horizontal, $\bar{\rho}_W(q)\dot{q} = 0$, so the χ-dependent term in $\delta\mathcal{I}[\gamma_s]$ gives

$$\int_I dt\; (\dot{q} - \rho_W \partial_p H)^i \; (\rho_W^\dagger \chi)_i \;\; + \;\; \int_I dt\; (\bar{\rho}_W \partial_p H)^i \; (\bar{\rho}_W^\dagger \chi)_i \;=\; 0\,.$$

The components of χ in the images of $\rho_W^\dagger(q)$ and $\bar{\rho}_W^\dagger(q)$ can be varied independently. Thus, both terms on the second line must vanish separately, as a consequence of which one obtains (59) and (61). □

Definition 4.4. *The smooth submanifold*

$$\mathcal{S} := \Big\{ (q,p) \Big| \bar{\rho}_W(q)\partial_p H(q,p) = 0 \Big\} \subset T^*Q$$

locally characterized by (61) is called the physical leaf.

$\mathcal{S}$ contains all physical orbits of the system, that is, all smooth paths $\gamma : \mathbb{R} \to \mathcal{S} \subset T^*Q$ that satisfy the differential-algebraic relations of theorem 4.1.

Theorem 4.2. *Let H be of the form (54). Then, there exists a unique physical orbit $\gamma : \mathbb{R}^+ \to \mathcal{S}$ with $\gamma(0) = x$ for every $x \in \mathcal{S}$.*

Proof. We cover $\mathcal{S}$ with local bundle charts of T^*Q with coordinates (q,p). For the Hamiltonian (54), (61) reduces to

$$\bar{\rho}_W(q)\, g^{-1}(q)\, p \;=\; g^{-1}(q)\, \bar{\rho}_W^\dagger(q)\, p \;=\; 0\,,$$

where one uses the g-orthogonality of $\bar{\rho}_W$. Hence, (61) is equivalent to $\bar{\rho}_W^\dagger(q)p = 0$. Since $\mathcal{S}$ is the common zero level set of the n component functions $(\bar{\rho}_W^\dagger(q)p)_i$, every section

$$X \;=\; v^r(q,p)\,\partial_{q^r} \;+\; w_s(q,p)\,\partial_{p_s}$$

of $T\mathcal{S}$ is annihilated by the 1-forms

$$d(\bar{\rho}_W^\dagger p)_i \;=\; \partial_{q^r}(\bar{\rho}_W^\dagger p)_i\, dq^r \;+\; \partial_{p_s}(\bar{\rho}_W^\dagger p)_i\, dp_s$$

for $i = 1,\dots,n$ (of which only $n-k$ are linearly independent), on $\mathcal{S}$.

This is expressed by

$$\begin{aligned} 0 \;&=\; (v^r\partial_{q^r})\bar{\rho}_W^\dagger p \;+ (w_s\partial_{p_s})\bar{\rho}_W^\dagger p \\ &=\; (v^r\partial_{q^r})\bar{\rho}_W^\dagger p \;+\; \bar{\rho}_W^\dagger w\,, \end{aligned}$$

which shows that the components v of X determine the projection $\bar{\rho}_W^\dagger w$. Hence, the components v and $\rho_W^\dagger w$ suffice to uniquely reconstruct X. Consequently, the right hand sides of (59) and (60) determine a unique section X of $T\mathcal{S}$, so that every curve $\gamma : \mathbb{R}^+ \to \mathcal{S}$, with arbitrary $\gamma(0) \in \mathcal{S}$, that satisfies $\partial_t\gamma(t) = X(\gamma(t))$ automatically fulfills (59) $\sim$ (61). This proves the assertion. □

4.1.2. *Equilibria.* The constrained Hamiltonian mechanical system (Q, g, U, W) possesses

(62) $$\mathfrak{C}_Q := \left\{ q \in Q \,\middle|\, \rho_W^\dagger(q)\partial_q U(q) = 0 \right\}$$

as its critical set. An application of Sard's theorem fully analogous to the proof of theorem 2.1 shows that generically, this is a piecewise smooth, $n-k$-dimensional submanifold of Q, (recall that the rank of $\rho_W(q)$ is k).

4.1.3. *Symmetries.* Let G be a Lie group, and let $\psi : G \to \mathrm{Diff}(Q)$, $h \mapsto \psi_h$ with $\Psi_e = \mathrm{id}$, denote a group action. The constrained Hamiltonian mechanical system (Q, g, U, W) is said to exhibit a G-symmetry if the following hold. (1) Invariance of the Riemannian metrics: $g \circ \psi_h = g$ and $g^* \circ \psi_h = g^*$ for all $h \in G$. (2) Invariance of the potential energy: $U \circ \psi_h = U$ for all $h \in G$. (3) Invariance of the distributions: $\psi_{h\,*}W = W$ and $\psi_h^* W^* = W^*$ for all $h \in G$.

4.2. **Construction of the Auxiliary Extension.** We are now prepared to embed the nonholonomic mechanical system into a constrained Hamiltonian system of the type considered in the previous sections.

To this end, we will introduce a set of generalized Dirac constraints over the symplectic manifold (T^*Q, ω_0) in the way presented in section 1. They define a symplectic distribution V, in a manner that the constrained Hamiltonian system (T^*Q, ω_0, H, V), with H given by (54), contains the constrained mechanical system as a dynamical subsystem. Thus, the auxiliary constrained Hamiltonian system (T^*Q, ω_0, H, V) extends the mechanical system in the sense announced in the introduction. An early inspiration for this construction stems from [32]. We require the following properties to be satisfied by (T^*Q, ω_0, H, V).

i. $\mathcal{S}$ is an invariant manifold under the flow $\tilde{\Phi}_t$ generated by (6).
ii. All orbits $\tilde{\Phi}(x)$ with initial conditions $x \in \mathcal{S}$ satisfy the Euler-Lagrange equations of the Hölder principle.
iii. $\mathcal{S}$ is marginally stable under $\tilde{\Phi}_t$.
iv. The critical set $\mathfrak{C}$ of $\tilde{\Phi}_t$ is a vector bundle over $\mathfrak{C}_Q$, hence equilibria of the constrained mechanical system are obtained from equilibria of the extension by base point projection.
v. Symmetries of the constrained mechanical system extend to those of $\tilde{\Phi}_t$.

Let us briefly comment on (iii) $\sim$ (v). (iii) is of importance for numerical simulations of the mechanical system. (iv) makes it easy to extract information about the behaviour of the mechanical system from solutions of the auxiliary system. Condition (v) allows to apply reduction theory to the auxiliary system, in order to reduce the constrained mechanical system by a group action, if present. The choice for V is by no means unique, and depending on the specific problem at hand, other conditions than (iii) $\sim$ (v) might be more useful.

4.2.1. *Construction of V.* Guided by the above requirements, we shall now construct V.

To this end, we pick a smooth, g^*-orthonormal family of 1-forms $\{\zeta_I\}_{I=1}^{n-k}$ with

$$\zeta_I \ = \ \zeta_{Ik}(q)\, dq^k \ ,$$

so that locally,

$$\langle\{\zeta_1, \dots, \zeta_{n-k}\}\rangle = (W^*)^{\perp} \ .$$

The defining relationship $\bar{\rho}_W^{\dagger}(q)p = 0$ for $\mathcal{S}$ is equivalent to the condition

$$f_I(q,p) := g_q^*(p, \zeta_I(q)) = 0 \quad \forall I = 1, \dots, n-k \ . \tag{63}$$

It is clear that $f_I \in C^\infty(T^*Q)$.

(1) To satisfy conditions (i) and (iii), we require that the level surfaces

$$\mathcal{M}_{\underline{\mu}} := \{(q,p) | f_I(q,p) = \mu_I ; I = 1, \dots, n-k\} , \tag{64}$$

with $\underline{\mu} := (\mu_1, \dots, \mu_{n-k})$, are integral manifolds of $V_{\deg(V)}$. Here, $\deg(V)$ denotes the degree of non-holonomy of V, and evidently, $\mathcal{M}_{\underline{0}} = \mathcal{S}$.

Condition (iii) is satisfied because

$$L(q,p) \ := \ \sum_I \ |f_I(q,p)|^2$$

is an integral of motion for orbits of $\tilde{\Phi}_t$. Since L grows monotonically with increasing $|\underline{\mu}|$, and attains its (degenerate) minimum of value zero on $\mathcal{S}$, it is a Lyapunov function for $\mathcal{S}$. Anything better than marginal stability is prohibited by energy conservation.

(2) To satisfy condition (ii), we demand that $\bar{\rho}_W(q)\dot{q} = 0$, or equivalently, that

$$\zeta_I(\dot{q}) \ = \ 0 \quad , \quad \forall I = 1, \dots, n-k \ , \tag{65}$$

shall be satisfied along all orbits $(q(t), p(t))$ of (6), owing to (59).

(3) If the constrained mechanical system exhibits a G-symmetry, characterized by a group action $\psi : G \to \mathrm{Diff}(Q)$ so that $\psi_{h*}W = W \ \forall h \in G$, the local family of 1-forms $\{\zeta_I\}$ can be picked in a manner that $\psi_h^*\zeta_I = \zeta_I$ is satisfied for all $h \in G$ in a vicinity of the unit element e. Consequently, the functions $f_I(q,p) = h_q^*(\zeta_I, p)$ and their level sets $\mathcal{M}_{\underline{\mu}}$ are invariant under the group action.

The condition that (64) are integral manifolds of $V_{\deg(V)} \supset V$ implies that all sections of V are annihilated by the 1-forms df_I, for $I = 1, \dots, n-k$. Furthermore, the condition (65) requires V to be annihilated by the 1-forms

$$\xi_I \ := \ \zeta_{Ir}(q) \ dq^r \ + \ \sum_s \ 0 \ dp_s \tag{66}$$

that are obtained from lifting ζ_I to $T^*(T^*Q)$, with $I = 1, \dots, n-k$.

Proposition 4.1. *The distribution*

$$V \ := \ \Big(\bigcap_I \ker \, df_I\Big) \ \ \bigcap \ \ \Big(\bigcap_I \ker \, \xi_I\Big) \ \ \subset \ T(T^*Q)$$

is symplectic.

Proof. V is symplectic iff its symplectic complement $V^{\perp}$ is. With the given data, the latter condition is more convenient to check. $V^{\perp}$ is locally spanned by the vector fields $(Y_1, \ldots, Y_{2k})$ obtained from

$$\omega_0(Y_I, \cdot) \;=\; \xi_I(\cdot) \quad , \quad \omega(Y_{I+k}, \cdot) \;=\; df_I(\cdot), \tag{67}$$

where $I = 1, \ldots, k$, and $\omega_0 = -dp_i \wedge dq^i$.

$V^{\perp}$ is symplectic if and only if $D := \;[\omega(Y_I, Y_J)]$ has values in $GL_{\mathbb{R}}(2(n-k))$.

We remark that in the present notation, capital indices range from 1 to k if they label 1-forms, and from 1 to $2k$ if they label vector fields.

In local bundle coordinates,

$$df_I \;=\; (\partial_{q^i} f_I)(q,p)\, dq^i \;\;+\;\; \zeta_{Ii}(q)\, g^{ij}(q)\, dp_j\;,$$

where g_{ij} are the components of the metric tensor g on Q, as before. Let us introduce the functions $E(q) := [\zeta_{Ji}(q)]$ and $F(q,p) := [\partial_{q^j} f_K(q,p)]$, both with values in $\mathrm{Mat}_{\mathbb{R}}(n \times (n-k))$, which we use to assemble

$$K \;:=\; \begin{pmatrix} E^{\dagger} & 0 \\ F^{\dagger} & E^{\dagger} g^{-1} \end{pmatrix} \quad : \; T^*Q \;\longrightarrow\; \mathrm{Mat}_{\mathbb{R}}(2(n-k) \times 2n)\;.$$

Any component vector $v : T^*Q \to \mathbb{R}^{2n}$ that locally represents an element of $\Gamma(V)$ satisfies $Kv = 0$. The symplectic structure ω_0 is locally represented by J, defined in (7). One can easily verify that the I-th row vector of the matrix $K\mathcal{J}^{-1}$ is the component vector of Y_I. In conclusion, introducing the matrices

$$\begin{aligned} G(q) \;&:=\; E^{\dagger}(q)\, g^{-1}(q)\, E(q) \\ S(q,p) \;&:=\; F^{\dagger}(q,p)\, g^{-1}(q)\, E(q) \;-\; E^{\dagger}(q)\, g^{-1}(q)\, F(q,p)\;, \end{aligned}$$

one immediately arrives at

$$D \;=\; K\mathcal{J}K^{\dagger} \;=\; \begin{pmatrix} 0 & G \\ -G & S \end{pmatrix}. \tag{68}$$

Since ζ_I has been picked a g^*-orthonormal family of 1-forms on Q, it is clear that $G(q) = \mathbf{1}_{n-k}$. Thus, D is invertible. This proves that $V^{\perp}$ is symplectic. $\square$

4.2.2. *Construction of the projection tensors.* Next, we determine the matrix of the ω_0-orthogonal projection tensor π_V, which is associated to V, in the present bundle chart. Again, it is more convenient to carry out the construction for its complement first.

Proposition 4.2. *The matrix of the ω_0-orthogonal projection tensor $\bar{\pi}_V$ associated to $V^\perp$ (considered as a tensor field that maps $\Gamma(T(T^*Q))$ to itself, with kernel V) is given by*

$$\bar{\pi}_V = \begin{pmatrix} \bar{\rho}_W & 0 \\ T & \bar{\rho}_W^\dagger \end{pmatrix}$$

in the local bundle chart (q,p). The matrix $T = T(q,p)$ is defined in (70).

Proof. The proof of lemma 1.1 can be used for this proof. The inverse of (68) is

$$D^{-1} = \begin{pmatrix} S & -\mathbf{1}_{n-k} \\ \mathbf{1}_{n-k} & 0 \end{pmatrix} ,$$

where we recall that $G(q) = \mathbf{1}_{n-k}$. The I-th column vector of the matrix $K\mathcal{J}^{-1}$ is the component vector of Y_I (we have required that $\{Y_1, \ldots Y_{2(n-k)}\}$ spans $V^\perp$). This implies that $\bar{\pi}_V = \mathcal{J}K^\dagger D^{-1}K$.

Lemma 4.1. *The matrix of $\bar{\rho}_W$ in the given chart is given by*

$$\bar{\rho}_W(q) \;=\; g^{-1}(q)\; E(q)\; E^\dagger(q) \; . \tag{69}$$

Proof. The construction presently carried out for $\bar{\pi}_V$ can also be applied to $\bar{\rho}_W$. One simply replaces $V^\perp$ by $W^\perp$, and ω_0 by the Riemannian metric g on Q. An easy calculation immediately produces the asserted formula. The matrix of ρ_W is subsequently obtained from $\rho_W + \bar{\rho}_W = \mathbf{1}$. For more details, cf. [11]. □

Introducing

$$T(q,p) \;:=\; E(q)\; F^\dagger(q,p)\; \rho_W(q) \;\;-\;\; \rho_W^\dagger(q)\; F(q,p)\; E^\dagger(q) \; , \tag{70}$$

a straightforward calculation produces the asserted formula for $\bar{\pi}_V$. □

Corollary 4.1. *In the given bundle coordinates, the matrix of π_V is*

$$\pi_V = \begin{pmatrix} \rho_W & 0 \\ -T & \rho_W^\dagger \end{pmatrix} ,$$

where $T = T(q,p)$ is defined in (70).

Proof. This is obtained from $\pi_V + \bar{\pi}_V = \mathbf{1}_{2n}$. □

In this chart, $\pi_V(x)\mathcal{J} = \mathcal{J}\pi_V^\dagger(x)$, by ω_0-skew orthogonality of π_V.

Theorem 4.3. *Let H be as in (54). Then, the dynamical system locally represented by*

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & \rho_W \\ -\rho_W^\dagger & -T \end{pmatrix} \begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix} , \tag{71}$$

corresponding to the contrained Hamiltonian system (T^*Q, ω_0, H, V), is an extension of the constrained mechanical system (Q, g, U, W).

Proof. By construction, $\mathcal{S}$ is an invariant manifold of the associated flow $\tilde{\Phi}_t$, hence (61) is fulfilled for all orbits of (71) with initial conditions in $\mathcal{S}$.

The equation $\dot{q} = \rho_W \partial_p H$ in (71) obviously is (59).

Next, using the notation $\underline{f} := (f_1, \dots, f_{n-k})^\dagger$,

$$\underline{f} \;=\; E^\dagger \, g^{-1} \, p \;,$$

and substituting (70) for $T(q,p)$, the equation for $\dot{p}$ in (71) becomes

$$\dot{p} \;=\; -\rho_W^\dagger \, \partial_q H \;-\; E \, F^\dagger \dot{q} \;+\; \rho_W^\dagger F \underline{f} \;.$$

Since $M_{\underline{\mu}}$ are invariant manifolds of the flow $\tilde{\Phi}_t$ generated by (71), $\partial_t f_I(q(t), p(t))$ vanishes along all orbits of (71), so that $F^\dagger \dot{q} + E^\dagger g^{-1} \dot{p} = 0$. This implies that

$$\dot{p} \;=\; -\rho_W^\dagger \, \partial_q H \;+\; E \, E^\dagger \, g^{-1} \, \dot{p} \;+\; \rho_W^\dagger \, \partial_q \Big(\frac{1}{2} \underline{f}^\dagger \underline{f} \Big) \;. \tag{72}$$

Recalling that $\bar{\rho}_W = g^{-1} E E^\dagger$ from (69), and using the fact that $\underline{f} = \underline{0}$ on $\mathcal{S}$, one arrives at (60) by multiplication with $\rho_W^\dagger$ from the left. $\square$

4.2.3. *Equilibria of the extension.* The critical set of the extension constructed above is characterized by the following theorem.

Theorem 4.4. *The critical set of (71) is given by the vector bundle*

$$\mathfrak{C} \;=\; \bigcup_{q \in \mathfrak{C}_Q} \{q\} \times (W_q^*)^\perp$$

with base space $\mathfrak{C}_Q$, cf. (62).

Proof. Let us first consider (72). As has been stated above, the second term on its right hand equals $\bar{\rho}_W^\dagger(q)\dot{p}$, and moreover, from (63), one concludes that

$$\underline{f}^\dagger \underline{f} \;=\; \|\bar{\rho}_W^\dagger p\|_{g^*}^2 \;.$$

The Hamiltonian (54) can be decomposed into

$$H(q,p) \;=\; H(q, \rho_W^\dagger p) \;+\; \frac{1}{2} \|\bar{\rho}_W^\dagger p\|_{g^*}^2 \;,$$

due to the g^*-orthogonality of $\rho_W^\dagger$ and $\bar{\rho}_W^\dagger$, so that (72) can be written as

$$\dot{p} \;=\; -\, \rho_W^\dagger \partial_q H(q, \rho_W^\dagger p) \;+\; \bar{\rho}_W^\dagger \dot{p} \;.$$

The equilibria of (71) are therefore determined by the conditions

$$\rho_W^\dagger(q) p \;=\; 0 \quad , \quad \rho_W^\dagger(q) \, \partial_q H(q, \rho_W^\dagger p) \;=\; 0 \;.$$

Because H depends quadratically on $\rho_W^\dagger p$, the second condition can be reduced to

$$\rho_W^\dagger(q)\,\partial_q U(q) \;=\; 0$$

using the first condition. Comparing this with (62), the assertion follows. $\square$

In particular, this fact implies that every equilibrium (q_0, p_0) of the extension defines a unique equilibrium q_0 on $\mathfrak{C}_Q$ by base point projection.

To analyze the stability of a given equilibrium solution $q_0 \in \mathfrak{C}_Q$, it is necessary to determine the spectrum of the linearization of X_H^V at $a = (q_0, 0)$.

A straightforward calculation along the lines of the previous discussion shows that in the present bundle chart,

$$DX_H^V(a) \;=\; \begin{pmatrix} 0 & \rho_W g^{-1}\rho_W^\dagger \\ -\rho_W^\dagger D_{q_0}^2 U \rho_W - R & 0 \end{pmatrix}(a)\;, \tag{73}$$

where

$$[R_{jk}] \;:=\; \left[\,\partial_{q^i} U (\rho_W)_j^r\,(\rho_W)_k^s\,\partial_{q^s}(\rho_W)_r^i\,\right] \quad \in\; \mathrm{Mat}_{\mathbb{R}}(n\times n)\;. \tag{74}$$

Furthermore, $D_{q_0}^2 U$ is the matrix of second derivatives of U. The stability discussion in the previous section can now straightforwardly be applied to $DX_H^V(a)$.

4.2.4. *Extension of symmetries.* Let us assume that the constrained mechanical system (Q, g, U, W) exhibits a G-symmetry $\psi : G \to \mathrm{Diff}(Q)$. Then, we claim that it is extended by (T^*Q, ω_0, H, V). To this end, we recall that the 1-forms ζ_I satisfy $\psi_h^*\zeta_I$ for all $h \in G$ close to the unit.

Via its pullback, ψ induces the group action

$$\Psi := \psi^* \;\;:\;\; G\times T^*Q \;\longrightarrow\; T^*Q$$

on T^*Q. This group action is symplectic, that is, $\Psi_h^*\omega_0 = \omega_0$ for all $h \in G$. For a proof, consider for instance [1].

The 1-forms ξ_I, defined in (66), satisfy $\Psi_h^*\xi_I = \xi_I$, and likewise, $f_I \circ \Psi_h = f_I$ is satisfied for all $h \in G$ close to the unit. The definition of V in proposition 4.1 thus implies that

$$\Psi_{h\,*} V \;=\; V$$

is satisfied for all $h \in G$. Due to the fact that ω and V are both G-invariant, π_V and $\bar\pi_V$ are also invariant under the G-action Ψ.

The Hamiltonian H in (54) is G-invariant under Ψ, by assumption on the constrained Hamiltonian mechanical system. Thus, X_H fulfills $\Psi_{h*}X_H = X_H$ for all $h \in G$, which implies that $X_H^V = \pi_V(X_H)$ is G-invariant.

4.3. **The Topology of the Critical Manifold.** Since $\mathfrak{C}$ is not a compact submanifold of T^*Q, our previous results cannot be applied directly. However, owing to the vector bundle structure of $\mathfrak{C}$ and T^*Q, the result

$$\sum_{i,p} \lambda^{p+\mu_i} \mathrm{dim} H_c^p(\mathfrak{C}_i) = \sum_p \lambda^p \mathrm{dim} H_c^p(T^*Q) + (1+\lambda)\mathcal{Q}(\lambda) \tag{75}$$

still holds, where H_c^* denotes the de Rham cohomology based on differential forms with compact supports. The polynomial $\mathcal{Q}(t)$ has non-negative integer coefficients.

In a first step, the arguments of section 2 can be straightforwardly applied to $\mathfrak{C}_Q$. $\mathfrak{C}_Q$ is normal hyperbolic with respect to the gradient-like flow ψ_t generated by

$$\partial_t q(t) = -\rho_W(q(t))\nabla_g U(q(t)),$$

it contains all critical points of the Morse function U, but no other conditional extrema of $U|_{\mathfrak{C}_Q}$ apart from those (it is gradient-like because along all of its non-constant orbits, $\frac{d}{dt}U(t) = -g(\rho_W \nabla_g U, \rho_W \nabla_g U)|_{q(t)} < 0$ holds, since ρ_W is an orthogonal projection tensor with respect to the Riemannian metric g on Q). This can be proved by substituting $M \to Q$, $H \to U$, $\pi_V \to \rho_W$, $g_{(\mathrm{Kahler})} \to g$, and $\mathfrak{C} \to \mathfrak{C}_Q$ in section 2, and by applying the arguments used there. Hence, letting μ_i denote the index of the connectivity component $\mathfrak{C}_{Qi}$ of $\mathfrak{C}_Q$, (19) implies that for compact, closed Q,

$$\sum_{i,p} \lambda^{p+\mu_i} \mathrm{dim} H^p(\mathfrak{C}_{Qi}) = \sum_p \lambda^p \mathrm{dim} H^p(Q) + (1+\lambda)\mathcal{Q}(\lambda) \ , \tag{76}$$

where $\mathcal{Q}(t)$ is a polynomial with non-negative integer coefficients.

$\mathfrak{C}_Q$, being the zero section of $\mathfrak{C}$, is a deformation retract of $\mathfrak{C}$, and likewise, Q is a deformation retract of T^*Q. Thus, (75) follows trivially from the invariance of the de Rham cohomology groups under retraction, $H_c^p(\mathfrak{C}_i) \cong H^p(\mathfrak{C}_{Qi})$, $H_c^p(T^*Q) \cong H^p(Q)$. Hence, (75) is equivalent to

$$\sum_{i,p} \lambda^{p+\mu_i} b_p(\mathfrak{C}_{Qi}) = \sum_p \lambda^p b_p(Q) + (1+\lambda)\mathcal{Q}(\lambda), \tag{77}$$

where b_p is the p-th Betti number.

Consequently, one finds $\sum_i b_{p-\mu_i}(\mathfrak{C}_{Qi}) \geq b_p$, and in particular, for $\lambda = -1$, one obtains

$$\sum_{i,p} (-1)^{p+\mu_i} b_p(\mathfrak{C}_{Qi}) = \sum_i (-1)^{\mu_i} \chi(\mathfrak{C}_{Qi}) = \chi(Q),$$

where χ denotes the Euler characteristic.

5. Applications, Illustrations and Examples

Let us conclude our analysis with the discussion of some simple applications and examples.

5.1. **A Computational Application.**

Let us first formulate an application of our analysis for the computational problem of finding the equilibria in a large constrained multibody system. It is in this context also desirable to determine whether a given set of parameters and constraints implies the existence of non-generic critical points. This is due to the circumstance that in practice, manufacturing imprecisions can have a significant effect on the latter.

For large multibody systems, equilibria can realistically only be determined by numerical routines. The strategy presented in chapters 2 and 4 suggests the following method.

If U is a Morse function whose critical points are known, and if Q is compact and closed, it is possible to numerically construct all generic connectivity components of $\mathfrak{C}_Q$. This is because generic components of $\mathfrak{C}_Q$ are smooth, $n-k$-dimensional submanifolds of Q containing all critical points of U, and no other critical points of $U|_{\mathfrak{C}_Q}$. This information can be exploited to find sufficiently many points on $\mathfrak{C}_Q$, so that a suitable interpolation routine enables the approximate reconstruction of an entire connectivity component. To this end, one chooses a vicinity of a critical point a of U, and uses a fixed point solver to determine neighboring zeros of $|\rho_W(q)\nabla_g U(q)|^2$, which are elements of $\mathfrak{C}_Q$ close to a. Iterating this procedure with the critical points found in this manner, pieces of $\mathfrak{C}_Q$ of arbitrary size can be determined.

If all critical points of U are a priori known, one can proceed in this manner to construct all connectivity components of $\mathfrak{C}_Q$ that contain critical points of U. Then, one is guaranteed to have found all of the generic components of $\mathfrak{C}$ if the numerically determined connectivity components are closed, compact, and contain all critical points of U.

We remark that determining the critical points of a Morse function $U : Q \to \mathbb{R}$ is a difficult numerical task by itself. Attempting to find critical points by simulating the gradient flow generated by $-\nabla_g U$ is time costly, because the critical points define a thin set in M. Their existence, however, is of course ensured by the topology of Q.

Another remark is that all critical points a at which $D(\rho_W \nabla_g U)(a)$ has a reduced rank, are elements of the non-generic part of $\mathfrak{C}_Q$. Thus, the latter condition is an indicator for non-genericity. If there are such exceptional critical points in a technically relevant region of Q, they can be removed by a small local modification of the system parameters or constraints.

5.2. **A disc in a periodic potential, sliding on the plane.** Let us consider a mechanical example, consisting of a thin disk of radius r and mass m on the plane $\mathbb{R}^2$, which is attached to a massless skate. The connecting line between the center of the disc and the contact point at the center of the skate with the plane is normal to the plane, precisely if the disc is horizontal. We assume that the disk remains horizontal during its motion, and that the translational motion of the disc is only possible in the direction of the skate.

Let (x_1, x_2) denote the position of the center of mass of the disc with respect to some Euclidean coordinate system on $\mathbb{R}^2$, and let ϕ denote the angle enclosed by the skate and the x_1-axis.

The kinetic energy of this system is given by

$$T = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}\frac{mr^2}{2}\dot{\phi}^2 \ ,$$

which defines a Riemannian metric on TQ with metric tensor

$$[g_{ij}(\phi, \theta, \psi)] = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{mr^2}{2} \end{pmatrix} .$$

Furthermore, we assume that it moves against the background of a $(2\pi\mathbb{Z})^3$-periodic potential energy

$$U(x_1, x_2, \phi) = \sum_{i=1,2} c_i(1 - \cos x_i) + c_\phi(1 - \cos\phi) \ ,$$

where c_1, c_2, and c_ϕ are coupling constants.

Dividing out the translational symmetry with respect to $(2\pi\mathbb{Z})^3$, the configuration manifold of this mechanical system is given by $Q = [0, 2\pi]^3 \cong T^3$ (periodic boundary conditions). Clearly, $U : T^3 \to \mathbb{R}$ is a real analytic Morse function, with 8 critical points in the corners of $[0, \pi]^3$, while each of the remaining critical points in $[0, 2\pi]^3$ is identified with one of the former by periodicity. Correspondingly, we will from here on consider (x_1, x_2) as coordinates on T^2, that is, mod 2π.

The requirement that the disk shall slide in the direction of the skate is expressed by the non-holonomic constraint

$$\dot{x}_1 \sin\phi - \dot{x}_2 \cos\phi = 0.$$

The matrix $E^\dagger(x_1, x_2, \phi)$, introduced in the proof of theorem 4.1, thus corresponds to

$$E^\dagger(x_1, x_2, \phi) = (\sin\phi, -\cos\phi, 0) \ ,$$

so that $E^\dagger g^{-1} E = \frac{1}{m}$.

The orthoprojectors $\bar{\rho}_W$ and ρ_W are thus straightforwardly obtained as

$$\begin{aligned} \bar{\rho}_W(x_1, x_2, \phi) &= \begin{pmatrix} \sin^2\phi & -\sin\phi\cos\phi & 0 \\ -\sin\phi\cos\phi & \cos^2\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \rho_W(x_1, x_2, \phi) &= \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi & 0 \\ \sin\phi\cos\phi & \sin^2\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \end{aligned}$$

The critical set is given by

$$\mathfrak{C}_Q = \Big\{ (x_1, x_2, \phi) \Big| \big(\rho_W^\dagger \nabla U\big)(x_1, x_2, \phi) = 0 \Big\}$$

(where $\nabla := (\partial_{x_1}, \partial_{x_2}, \partial_\phi)$). Let

$$\mathfrak{C}_{a,b} := \Big\{ (x_1, x_2, \phi) \Big| x_1 = a \ , \ x_2 \in [0, 2\pi] \ , \ \phi = b \Big\} \ .$$

Then,

$$\mathfrak{C}_Q = \bigcup_{a,b \in \{0,\pi\}} \mathfrak{C}_{a,b} \ .$$

It is trivially clear that $\mathfrak{C}_Q$ contains all critical points of U. Let $q_c \in \mathfrak{C}_{a,b}$, where $a, b \in \{0, \pi\}$. Noting that $\rho_W = \mathrm{diag}(1, 0, 1)$ on $\mathfrak{C}_Q$, we have

$$\begin{aligned} \Big(\nabla \otimes \big(\rho_W^\dagger \nabla U\big)\Big)(q_c) &= \Big(\rho_W^\dagger \big(\nabla \otimes \nabla U\big) \rho_W\Big)(q_c) + \tilde{R}(q_c) \\ &= \begin{pmatrix} c_1 \cos a & 0 & c_2 \sin x_2 \\ 0 & 0 & 0 \\ 0 & 0 & c_\phi \cos b \end{pmatrix} . \end{aligned} \tag{78}$$

Clearly,

$$\mathrm{spec}\Big(\Big(\nabla \otimes \big(\rho_W^\dagger \nabla U\big)\Big)(q_c)\Big) = \Big\{ 0, c_1 \cos a, c_\phi \cos b \Big\} \ ,$$

which is, for each fixed a, b, independent of x_2. Thus, the indices of the connectivity components $\mathfrak{C}_{a,b}$ with respect to the gradient-like flow generated by $-\rho_W^\dagger \nabla U$ are given by

$$\mu(\mathfrak{C}_{0,0}) = 2 \ \ , \ \ \mu(\mathfrak{C}_{0,\pi}) = \mu(\mathfrak{C}_{\pi,0}) = 1 \ \ , \ \ \mu(\mathfrak{C}_{\pi,\pi}) = 0 \ , \tag{79}$$

and clearly, $\mathfrak{C}_{a,b} \cong S^1$ for all $a, b \in \{0, \pi\}$. Since the Betti numbers of T^3 are given by $b_0 = b_3 = 1$, $b_1 = b_2 = 3$, and those of $\mathfrak{C}_{a,b}$ by $b_0(\mathfrak{C}_{a,b}) = b_1(\mathfrak{C}_{a,b}) = 1$, $b_2(\mathfrak{C}_{a,b}) = b_3(\mathfrak{C}_{a,b}) = 0$, one finds that

$$\sum_{a,b} b_{p-\mu(\mathfrak{C}_{a,b})}(\mathfrak{C}_{a,b}) = b_p(Q)$$

for $p = 0, \ldots, 3$, or explicitly,

$$\begin{aligned}
b_{3-2}(\mathfrak{C}_{0,0}) &= 1 = b_3(T^3) \\
b_{2-2}(\mathfrak{C}_{0,0}) + b_{2-1}(\mathfrak{C}_{0,\pi}) + b_{2-1}(\mathfrak{C}_{\pi,0}) &= 3 = b_2(T^3) \\
b_{1-0}(\mathfrak{C}_{\pi,\pi}) + b_{1-1}(\mathfrak{C}_{0,\pi}) + b_{1-1}(\mathfrak{C}_{\pi,0}) &= 3 = b_1(T^3) \\
b_{0-0}(\mathfrak{C}_{\pi,\pi}) &= 1 = b_0(T^3) \ ,
\end{aligned}$$

in agreement with (77).

Next, we determine the spectrum of the linearization of X_H^V at $(q_c, 0) \in T^*Q$, cf. (73). To this end,

$$\Big(\rho_W g^{-1} \rho_W^\dagger\Big)(q_c) = \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{2}{mr^2} \end{pmatrix} ,$$

and multiplying this matrix from the right with (78) yields

$$\Omega(q_c, \theta) := \begin{pmatrix} \frac{c_1 \cos a}{m} & 0 & \frac{c_2 \sin x_2}{m} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{2c_\phi \cos b}{mr^2} \end{pmatrix} .$$

Clearly,

$$\mathrm{spec}\Big(\Omega(q_c, \theta)\Big) = \Big\{0, \frac{c_1 \cos a}{m}, \frac{2c_\phi \cos b}{mr^2}\Big\} .$$

From (73), it is easy to see that

$$\mathrm{spec}\Big(DX_H^V(q_c, 0)\Big) = \Big\{0, \pm\sqrt{\frac{c_1 \cos a}{m}}, \pm\sqrt{\frac{2c_\phi \cos b}{mr^2}}\Big\} ,$$

hence critical stability occurs for the case $a = b = \pi$, while in all other cases, there is an asymptotically unstable direction.

We conclude that all components $\mathfrak{C}_{a,b}$, where $a + b \leq \pi$, are unstable. In the critically stable case $a = b = \pi$, the linear problem is oscillatory, and the eigenfrequencies are given by $\sqrt{\frac{c_1}{m}}$ and $\sqrt{\frac{2c_\phi}{mr^2}}$, independently of x_2. Since $\mu(\mathfrak{C}_{\pi,\pi}) = 0$, our discussion in section 3 suggests that the connectivity component $\mathfrak{C}_{\pi,\pi}$ of $\mathfrak{C}_Q$ is stable in the sense of Nekhoroshev if the ratio $\sqrt{\frac{c_1 mr^2}{2c_\phi}}$ is irrational.

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