The Beth's experiment is under review

Radi I. Khrapko*
Moscow Aviation Institute, 125993, Moscow, Russia

Abstract

It is shown that the Maxwell electrodynamics cannot explain the result of the Beth's experiment. So, the electrodynamics is not complete. A spin tensor is introduced into the electrodynamics. A theory of the Beth's experiment is presented.

PACS number: 03.05.De
Keywords: classical spin; Belinfante's procedure; torque

1 The Beth's experiment

The Beth's classical experiment [1] was made almost 70 years ago.

> A beam of circularly polarized light should exert a torque on a doubly refracting plate which changes the state of polarization of the light beam.
>
> The apparatus used involves a torsional pendulum with about a ten-minute period consisting of a round quartz half-wave plate one inch in diameter suspended with its plane horizontal from a quartz fiber about 25 centimeters long. About 4 millimeters above this is mounted a fixed quartz quarter-wave plate. The *top* side of the upper plate was coated by evaporation with a reflecting layer of aluminum. The rotation of the pendulum is observed by a telescope.

A circularly polarized light beam (power, $\mathcal{P} = 80$ mW; $\lambda = 1.2$ mu; $\omega = 1.6 \cdot 10^{15}$ 1/s) passes through the half-wave plate, then it passes twice through the quarter-wave plate, and then returns through the half-wave plate. The torque, τ, exerting on the half-wave plate is 20 dyne-cm. This result agrees with the formula

$$\tau = 4\mathcal{P}/\omega. \tag{1}$$

2 The standard explanation of the Beth's result

According to Maxwell's theory (see, for example, [2]), the Poynting vector $\mathbf{E}\times\mathbf{H}$ is interpreted as the density of momentum of the field (we put speed of light, $c = 1$). We can then also define an angular momentum relative to a given point O or to a given axis,

$$\mathbf{J} = \int \mathbf{r}\times(\mathbf{E}\times\mathbf{H})dV. \tag{2}$$

*Email addresses: khrapko_ri@mai.ru, khrapko_ri@hotmail.com

A circularly polarized plane wave without an azimuthal phase structure travelling in the z-direction and with infinite extension in the xy-directions can have no angular momentum about the z-axis, because $\mathbf{E} \times \mathbf{H}$ is in the z-direction and $(\mathbf{r} \times (\mathbf{E} \times \mathbf{H}))_z = 0$. However, this is no longer the case for a beam. Consider a cylindrical beam. At the wall of the cylinder we let the field drop to zero. It can then be shown that the wall of a beam gives a finite contribution to J_z [3, 4, 5, 6, 7, 8].

For example, Ohanian wrote [6]

> In a wave of finite transverse extent, the **E** and **B** fields have a component parallel to the wave vector (the field lines are closed loops) and the energy flow has components perpendicular to the wave vector. ... The circulating energy flow in the wave implies the existence of angular momentum, whose direction is along the direction of propagation. This angular momentum is the spin of the wave.

The angular momentum (2) of a beam was repeatedly calculated,

$$J = \int E_0^2 dV/\omega.$$

The energy of the beam is

$$\mathcal{E} = \int E_0^2 dV,$$

where E_0 is the amplitude of the **E** field in the inner region of the beam. So, the ratio $\mathcal{E}/J = \omega$ appears as the ratio $\mathcal{E}/S$, i.e. energy/spin, for a photon.

A circularly polarization of the Beth's beam is being reversed when the beam passes through the half-wave plate. So, according to the paradigm, the plate gets

$$J = 2\mathcal{E}/\omega$$

when the beam passes through the plate. The plate gets the same angular momentum from the reflected beam. So, having divided by time we arrive to Eq. (1).

3 The Beth's experiment is a puzzle

At the same time, it is evident that the Poynting vector in the Beth's experiment equals zero because the passed beam interferes with the reflected one. Indeed, let us start from the Jackson's expression for a circularly polarized beam [3]. ($\omega = k = 1$).

$$\mathbf{E} = [\mathbf{x}\cos(z-t) - \mathbf{y}\sin(z-t) + \mathbf{z}(-\sin(z-t)\partial_x - \cos(z-t)\partial_y)]E_0(x,y),$$

$$\mathbf{H} = -\int \mathrm{rot}\mathbf{E}dt = [\mathbf{x}\sin(z-t) + \mathbf{y}\cos(z-t) + \mathbf{z}(\cos(z-t)\partial_x - \sin(z-t)\partial_y)]E_0(x,y),$$

Here $E_0(x,y) = \text{Const}$ inside the beam, and $E_0(x,y) = 0$ outside the beam.

The reflected beam may be got by changing the sign of z and y. Adding up the passed and reflected beams, we get interesting expressions,

$$\mathbf{E}_{\mathrm{tot}} = 2[(\mathbf{x}\cos z - \mathbf{y}\sin z)E_0 - \mathbf{z}(\sin z\,\partial_x E_0 + \cos z\,\partial_y E_0)]\cos t, \tag{3}$$

$$\mathbf{H}_{\mathrm{tot}} = -2[(\mathbf{x}\cos z - \mathbf{y}\sin z)E_0 - \mathbf{z}(\sin z\,\partial_x E_0 + \cos z\,\partial_y E_0)]\sin t. \tag{4}$$

The $\mathbf{E}_{\mathrm{tot}}$ and $\mathbf{H}_{\mathrm{tot}}$ fields are parallel to each other everywhere. So, the Poynting vector is zero, and, according to (2), angular momentum of the light in the Beth's experiment is zero! So, the light should not exert a torque on a plate. Why did the torque exert the plate then?

As a matter of fact, the all-recognized equation (2) seems to be incorrect. As a matter of fact, the Maxwell electrodynamics is not complete [9, 10, 11].

4 Electrodynamics spin

The Poynting vector, $\mathbf{E} \times \mathbf{H}$, is a component of the Maxwell-Minkowski energy-momentum tensor,

$$T^{\alpha}_{\mu} = -F_{\mu\nu}F^{\alpha\nu} + \delta^{\alpha}_{\mu}F_{\sigma\nu}F^{\sigma\nu}/4, \tag{5}$$

and the Eq. (2) looks as

$$J^{\mu\nu} = \int 2r^{[\mu}T^{\nu]\alpha}dV_{\alpha} \tag{6}$$

in the Minkowski space.

There are no simple theoretical method to obtain the Maxwell-Minkowski tensor. The canonical Lagrangian,

$$\mathcal{L} = -F_{\mu\nu}F^{\mu\nu}/4, \quad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]},$$

according to Noether's theorem, gives the canonical energy-momentum and spin tensors [12, Sec. 7g]:

$$\underset{c}{T}^{\mu\alpha} = \partial^{\mu}A_{\sigma}\frac{\partial\mathcal{L}}{\partial(\partial_{\alpha}A_{\sigma})} - g^{\mu\alpha}\mathcal{L} = -\partial^{\mu}A_{\sigma}F^{\alpha\sigma} + g^{\mu\alpha}F_{\sigma\rho}F^{\sigma\rho}/4, \tag{7}$$

$$\underset{c}{\Upsilon}^{\mu\nu\alpha} = -2A^{[\mu}\delta^{\nu]}_{\sigma}\frac{\partial\mathcal{L}}{\partial(\partial_{\alpha}A_{\sigma})} = -2A^{[\mu}F^{\nu]\alpha}. \tag{8}$$

But the canonical energy-momentum tensor is asymmetric and has an incorrect divergence,

$$\partial_{\alpha}\underset{c}{T}^{\mu\alpha} = -\partial^{\mu}A_{\nu}j^{\nu}.$$

The divergence of a true energy-momentum tensor must be equal to $-F_{\mu\nu}j^{\nu}$.

To symmetrize the canonical energy-momentum tensor and to turn it to the Maxwell-Minkowski tensor, theorists has to add a term

$$\partial_{\beta}A_{\mu}F^{\alpha\beta} \tag{9}$$

to the canonical energy-momentum tensor (7). This term divides in two parts:

$$T^{\alpha}_{\mu} = \underset{c}{T}^{\alpha}_{\ \mu} + \partial_{\beta}A_{\mu}F^{\alpha\beta} = \underset{c}{T}^{\alpha}_{\ \mu} + \partial_{\beta}(A_{\mu}F^{\alpha\beta}) + A_{\mu}j^{\alpha}.$$

The second part,

$$A_{\mu}j^{\alpha}, \tag{10}$$

repairs the divergence of the canonical tensor, and the first part,

$$\partial_{\beta}(A_{\mu}F^{\alpha\beta}), \tag{11}$$

symmetrizes the contravariant form of the tensor.

The only reason for adding the term (9) to the canonical tensor (7) is to obtain the Maxwell-Minkowski tensor (5) which was known beforehand. The term (9) is not even a divergence.

Theorists ignore the second part (10) of the term. They do not see it. But Belinfante and Rosenfeld [13, 14] made a considerable study of the first part (11). They pointed out that an antisymmertization of the first part yields the divergence of the canonical spin tensor (8) with the minus sign,

$$2\partial_{\beta}(A^{[\mu}F^{\nu]\beta}) = -\partial_{\beta}\underset{c}{\Upsilon}^{\mu\nu\beta}. \tag{12}$$

We shall use the fact below, but now we have to emphasize that the Belinfante's part (11) itself turns the canonical energy-momentum tensor (7) not to the Maxwell-Minkowski tensor (5), but to a tensor that may be named the Belinfante's tensor,

$$\underset{B}{T}^{\mu\alpha} = \underset{c}{T}^{\mu\alpha} + \partial_\beta (A^\mu F^{\alpha\beta}) = T^{\mu\alpha} - A^\mu j^\alpha.$$

Nevertheless, theorists believe that they obtain the Maxwell-Minkowski tensor (5) by adding the Belinfante's divergence (11) to the canonical tensor (7). Moreover, on the grounds of (12), theorists add $(-\underset{c}{\Upsilon}^{\mu\nu\beta})$ to the canonical spin tensor (8) and arrive to a zero as the electrodynamics' spin tensor. So, the Belinfante's divergence (11) does not lead to the Maxwell-Minkowski tensor (5), but it eliminates spin tensor. That is why classical spin is absent in the Maxwell electrodynamics. The classical spin tensor is considered zero. That is why they consider a circularly polarized plane wave has no angular momentum [15, 3, 4, 5, 6, 7, 8, 16, 17].

Here a problem arises: what is an electrodynamics' true spin tensor. What must we add to the canonical spin tensor to get the true spin tensor?

Our answer is as follows: a spin addition, $\Delta\underset{c}{\Upsilon}^{\mu\nu\alpha}$, and the energy-momentum addition (9) must satisfy an equation of type (12) which uses (9) instead of (11),

$$2\partial_\beta A^{[\mu} F^{\nu]\beta} = \partial_\beta \Delta \underset{c}{\Upsilon}^{\mu\nu\beta}. \tag{13}$$

A simple expression satisfies the Eq. (13),

$$\Delta \underset{c}{\Upsilon}^{\mu\nu\alpha} = 2A^{[\mu}\partial^{\nu]}A^\alpha.$$

So, we obtain [18, 19, 20]

$$\Upsilon^{\mu\nu\alpha} = \underset{c}{\Upsilon}^{\mu\nu\alpha} + \Delta\underset{c}{\Upsilon}^{\mu\nu\alpha} = 2A^{[\mu}\partial^{|\alpha|}A^{\nu]}. \tag{14}$$

This result was submitted to "JETP Letters" on May 14, 1998.

The spin tensor (14) is a function of the vector potential A_μ and is not gauge invariant. We greet this fact [20]. As is shown, A^μ must satisfy the Lorentz condition, $\partial_\mu A^\mu = 0$.

The expression (14) is not final. As a matter of fact, the electrodynamics is asymmetric. Magnetic induction is closed, but magnetic field strength has electric current as a source:

$$\partial_{[\alpha} F_{\beta\gamma]} = 0, \quad \partial_\nu F^{\mu\nu} = j^\mu.$$

So, a magnetic vector potential exists, but, generally speaking, an electric vector potential does not exist. However, when currents are absent the symmetry is restored, and a possibility to introduce an electric multivector potential $\Pi^{\mu\nu\sigma}$ appears. The electric multivector potential satisfies the equation

$$\partial_\sigma \Pi^{\mu\nu\sigma} = F^{\mu\nu}.$$

A covariant vector, dual relative to the multivector potential,

$$\Pi_\alpha = \epsilon_{\alpha\mu\nu\sigma}\Pi^{\mu\nu\sigma},$$

is an analog of the magnetic vector potential A_α. We name it the electric vector potential. Using vector notations, it can be introduced as follows.

If $\mathrm{div}\mathbf{D} = 0$, then $\mathbf{D} = \mathrm{curl}\Pi$. If also $\mathrm{curl}\mathbf{H} = \partial\mathbf{D}/\partial t$, then $\mathbf{H} = \partial\mathbf{\Pi}/\partial\mathbf{t}$.

This procedure is similar to an introduction of the magnetic vector potential:

If div$\mathbf{B} = 0$, then $\mathbf{B} = $ curl$\mathbf{A}$. If also curl$\mathbf{E} = - \partial\mathbf{B}/\partial t$, then $\mathbf{E} = -\partial\mathbf{A}/\partial t$.

Scalar potentials may participate in both cases still, but we may consider they are zero.

The symmetry of the electrodynamics forces us to offer a symmetric expression for the spin tensor consisting of two parts, electric and magnetic [18, 19].

$$\Upsilon^{\mu\nu\alpha} = \underset{e}{\Upsilon}^{\mu\nu\alpha} + \underset{m}{\Upsilon}^{\mu\nu\alpha} = A^{[\mu}\partial^{|\alpha|}A^{\nu]} + \Pi^{[\mu}\partial^{|\alpha|}\Pi^{\nu]}. \tag{15}$$

5 The theory of the Beth's experiment

Now we have to replace Eqs. (2) and (6) by

$$J^{\mu\nu} = \int 2r^{[\mu}T^{\nu]\alpha}dV_\alpha + \int \Upsilon^{\mu\nu\alpha}dV_\alpha. \tag{16}$$

We have to name the first term *orbital* angular momentum of electromagnetic waves. It is zero in the Beth's experiment. The second term is spin angular momentum of the waves. Ohanian et al. were not right. The spin part of the total angular momentum of the light (16) exerts a torque on the Beth's plate. The part is absent in the Maxwell electrodynamics. So, the electrodynamics is incomplete.

We consider the Beth's light beams as plane waves because the Poynting vector is zero, and the wall effects are of no importance. In front of the half-wave plate we have (3) and (4) with $E_0 = 1$,

$$\mathbf{E} = 2(\mathbf{x}\cos z - \mathbf{y}\sin z)\cos t,$$

$$\mathbf{A} = -\int \mathbf{E}dt = 2(\mathbf{x}\cos z - \mathbf{y}\sin z)(-\sin t).$$

So, the electrical part of spin current density is uniform, but pulses,

$$\underset{e}{\Upsilon}^{xyz} = 2\sin^2 t.$$

The magnetic part is

$$\mathbf{H} = 2(-\mathbf{x}\cos z + \mathbf{y}\sin z)\sin t,$$

$$\Pi = \int \mathbf{H}dt = 2(\mathbf{x}\cos z - \mathbf{y}\sin z)\cos t, \quad \underset{m}{\Upsilon}^{xyz} = 2\cos^2 t.$$

So, the total spin current density is constant, $\Upsilon^{xyz} = \underset{e}{\Upsilon}^{xyz} + \underset{m}{\Upsilon}^{xyz} = 2$.

The same calculation for the domain behind the plate gives $\Upsilon^{xyz} = -2$. This means that the S^{xy}-component of the spin moves opposite the z-direction, i.e. towards the plate also. In result, the plate receives the spin current density of 4 with the absence of an energy current!

Conclusion

The Maxwell electrodynamics is not complete. We have to put a spin tensor in the modern electrodynamics. Theorists did not catch sight of the classical spin because of ignoring a localization of energy-momentum and due to the concept of the ambiguity of an energy-momentum tensor. A significance of the Lagrange formalism was overestimated.

I am deeply grateful to Professor Robert H. Romer for publishing my question [15].

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