The nature of the essential spectrum in curved quantum waveguides

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Abstract

We study the nature of the essential spectrum of the Dirichlet Laplacian in tubes of constant radius about infinite curves embedded in Euclidean spaces. Under suitable assumptions about the decay of curvatures at infinity, we prove the absence of singular continuous spectrum and state properties of possible embedded eigenvalues. The argument is based on Mourre conjugate operator method developed for acoustic multistratified domains by Benbernou in [3] and Dermenjian *et al.* in [7]. As a technical preliminary, we carry out a spectral analysis for Schrödinger-type operators in straight Dirichlet tubes. We also apply the result to the strips embedded in abstract surfaces.

1 Introduction

A strong physical motivation to study the Dirichlet Laplacian in infinitely stretched tubular regions comes from the fact it constitutes a reasonable model for the Hamiltonian of a non-relativistic quantum particle in mesoscopic systems called *quantum waveguides* [8, 21, 15]. Since there exists a close relation between spectral and scattering properties of Hamiltonians, one is naturally interested in carrying out the spectral analysis of the Laplacian in order to understand the quantum dynamics in waveguides. For instance, the crucial step in most proofs of asymptotic completeness is to show that the Hamiltonian has no singular continuous spectrum [23]. The Laplacian in a tube has attracted considerable attention since it was shown in [12] that there may be discrete eigenvalues in curved waveguides. However, a detailed analysis of the essential part of the spectrum has been left aside up to know. The purpose of the present paper is to fill in this gap.

The usual model for a curved quantum waveguide, which we adopt in this paper, is as follows. Let $s \mapsto p(s)$ be an infinite unit-speed smooth curve in $\mathbb{R}^d$, $d \geq 2$ (the physical cases corresponding to $d = 2, 3$), and let κ_i, $i \in \{1, \dots, d-1\}$, denote its i^{th} curvature (w.r.t. to an appropriate Frenet frame) as a function of the arc-length s. Given $a > 0$, we identify the configuration space Γ of the waveguide with the a-tubular neighbourhood of p, *i.e.*

$$\Gamma := \left\{ x \in \mathbb{R}^d \,:\, \text{distance}\,(x,p) < a \right\}. \tag{1.1}$$

We always assume that the tube does not overlap and that its radius fulfils the condition $a\|\kappa_1\|_\infty < 1$ so that Γ is a Riemannian submanifold of $\mathbb{R}^d$. Our object of interest is the Dirichlet Laplacian associated with the tube, *i.e.*

$$-\Delta_{\mathrm{D}}^{\Gamma} \qquad \text{on} \qquad L^2(\Gamma). \tag{1.2}$$

If p is a straight line, *i.e.* all $\kappa_i = 0$, then Γ may be identified with the straight tube $\Omega := \mathbb{R} \times \mathcal{B}_a$, where $\mathcal{B}_a$ is the open ball of radius a in $\mathbb{R}^{d-1}$. Then, it is easy to see that the spectrum of (1.2) is purely *absolutely continuous* and equal to the interval $[\nu_1, \infty)$, where ν_1 denotes the first eigenvalue of the Dirichlet Laplacian in the ball $\mathcal{B}_a$.

On the other hand, if p is non-trivially curved and straight asymptotically, in the sense that the curvature κ_1 vanishes at infinity, then the essential spectrum of (1.2) remains equal to $[\nu_1, \infty)$. However, there are always *discrete eigenvalues* below ν_1. When $d = 2$, the latter was proved for the first time in [12] for a rapidly decaying curvature and sufficiently small a. Numerous subsequent studies improved and generalised this initial result [13, 24, 8, 19, 11]. The generalisation to tubes in $\mathbb{R}^3$ was done in [13] (see also [8]) and the case of arbitrary dimension $d \geq 2$ can be found in [11].

The main goal of the present work is a thorough analysis of the *essential spectrum* of (1.2). In particular, we find sufficient conditions which guarantee that the essential spectrum of a curved tube "does not differ too much" from the straight case (for simplicity, we present here our results only for $d = 2$, see Theorem 3.4 with Assumption 3.4 for the d-dimensional case):

Theorem 1.1 (d=2). *Let Γ be as above ($\kappa := \kappa_1$) and $\mathcal{T} := \{n^2\nu_1\}_{n=1}^{\infty}$ with $\nu_1 := \pi^2/(2a)^2$ (the set of eigenvalues of the Dirichlet Laplacian in the 1-dimensional ball $\mathcal{B}_a$). Suppose*

1. $\kappa(s)$, $\ddot{\kappa}(s) \longrightarrow 0$ *as* $|s| \to \infty$,
2. $\exists \vartheta \in (0,1]$ *s.t.* $\dot{\kappa}(s)$, $\dddot{\kappa}(s) = \mathcal{O}\left(|s|^{-(1+\vartheta)}\right)$.

Then

(i) $\sigma_{\mathrm{ess}}\left(-\Delta_{\mathrm{D}}^{\Gamma}\right) = [\nu_1, \infty)$,

(ii) $\sigma_{\mathrm{sc}}\left(-\Delta_{\mathrm{D}}^{\Gamma}\right) = \varnothing$,

(iii) $\sigma_{\mathrm{p}}\left(-\Delta_{\mathrm{D}}^{\Gamma}\right) \cup \mathcal{T}$ *is closed and countable*,

(iv) $\sigma_{\mathrm{p}}\left(-\Delta_{\mathrm{D}}^{\Gamma}\right) \setminus \mathcal{T}$ *is composed of finitely degenerated eigenvalues which can accumulate at points of* $\mathcal{T}$ *only*.

To prove this theorem (and the general Theorem 3.4), we use the conjugate operator method introduced by [22] E. Mourre and lastly developed by [1] W. Amrein *et al.*

Actually, the property (i) holds true whenever the first curvature vanishes at infinity, without assuming any decay of the derivatives (they may not even exist), see [19] for $d = 2$ and [11] for the general case. Our second result (ii) can be compared only with [10] (see also [9]), where the problem of resonances is investigated for $d = 2$. Assuming that there exists $\vartheta \in (0,1]$ such that $\kappa(s), \dot{\kappa}(s)^2, \ddot{\kappa}(s) = \mathcal{O}\left(|s|^{-(1+\vartheta)}\right)$, the authors proved the absence of singular continuous spectrum as a consequence of the completeness of wave operators obtained by standard smooth perturbation methods of scattering theory. Notice that our and their results are independent. Indeed, while we need to require a faster decay of $\dot{\kappa}$ and also impose a condition on $\dddot{\kappa}$, our decay assumptions on κ and $\ddot{\kappa}$ are on the contrary much weaker. Our other spectral results (iii) and (iv) (and (ii) for $d \geq 3$) are new.

The organisation of this paper is as follows. Given a bounded open connected set ω in $\mathbb{R}^{d-1}$ with $d \geq 2$, we define Ω as the straight tube $\mathbb{R} \times \omega$. In Section 2, we consider the Schrödinger-type operator

$$H := -\partial_i G^{ij} \partial_j + V \qquad \text{on} \qquad \mathcal{H}(\Omega) := L^2(\Omega)\,, \tag{1.3}$$

subject to Dirichlet boundary conditions, $G \equiv (G^{ij})$ being a real symmetric matrix-valued measurable function on Ω and V the multiplication operator by a real-valued measurable function on Ω (repeated indices convention is adopted, with the range of Latin indices being $1, \ldots, d$). We make Assumption 2.1 and Assumption 2.2 stated below. Adapting the approach of [3, 7] to non-zero V and G different from a multiple of the identity, we study the nature of the essential spectrum of the operator H. In particular, we prove the absence of singular continuous spectrum and state properties of possible embedded eigenvalues. The result is contained in Theorem 2.14 and is of independent interest. In Section 3, we apply it to the case of curved tubes (1.1). Using geodesic coordinates and a unitary transformation (ideas which go back to [12]), we cast the Laplacian (1.2) into a unitarily equivalent operator of the form (1.3) for which Theorem 2.14 can be used. The obtained spectral results can be found in Theorem 3.4 (the general version of Theorem 1.1 above). Finally, in Section 4, we similarly investigate the essential spectrum of the Dirichlet Laplacian in an infinite strip in an abstract two-dimensional Riemannian manifold of curvature K. The general result is contained in Theorem 4.1, while the case of flat strips, *i.e.* with $K = 0$, is summarised in Theorem 4.2 (the latter involves the curved strips in $\mathbb{R}^2$ as a special case).

For the conjugate operator method and notation used in Section 2, the reader is referred to [1] and particularly to short well-arranged reviews of the abstract theory in [3, Sec. 2] or [7, Sec. 1]. A more detailed geometric background for Section 3 and Section 4 can be found in [17, 11] and [14, 18], respectively.

We use the standard component notation of tensor analysis, the range of indices being $1, \dots, d$ for Latin and $2, \dots, d$ for Greek. The indices are associated in a natural way with the components of $x \in \mathbb{R} \times \omega$; the corresponding partial derivatives are often denoted by commas. The brackets $(\cdot)$ are used in order to distinguish a matrix from its coefficients. The symbols δ_{ij} are reserved for the components of the Euclidean metric matrix 1.

2 Schrödinger-type operators in straight tubes

2.1 Preliminaries

Let ω be a bounded open connected set in $\mathbb{R}^{d-1}$, $d \geq 2$, with locally Lipschitz boundary and consider the straight tube $\Omega := \mathbb{R} \times \omega$. Our object of interest in this section is the operator given formally by (1.3), subject to Dirichlet boundary conditions. In addition to the basic properties required for the matrix G and function V, we make the following assumptions.

Assumption 2.1.

1. $\exists C_\pm \in (0, \infty)$ *s.t.* $C_- 1 \leq G(x) \leq C_+ 1$ *for a.e.* $x \in \Omega$,
2. $\forall i, j \in \{1, \dots, d\}$, $\lim_{R\to\infty} \operatorname{ess\,sup}_{x\in(\mathbb{R}\setminus[-R,R])\times\omega} \left|G^{ij}(x) - \delta^{ij}\right| = 0$,
3. $\exists \vartheta_1 \in (0, 1]$, $C \in (0, \infty)$ *s.t.* $\left(\left|G^{ij}_{\ ,1}(x)\right|\right) \leq C \, \langle x^1\rangle^{-(1+\vartheta_1)} \, 1$ *for a.e.* $x \in \Omega$.

Here $\langle x\rangle := \left(1 + |x|^2\right)^{1/2}$ for $x \in \Omega$ and the inequalities must be understood in the sense of matrices.

Assumption 2.2.

1. $V \in L^\infty(\Omega)$,
2. $\lim_{R\to\infty} \operatorname{ess\,sup}_{x\in(\mathbb{R}\setminus[-R,R])\times\omega} |V(x)| = 0$,

3. $\exists \vartheta_2 \in (0,1]$, $C \in (0,\infty)$ *s.t.* $|V_{,1}(x)| \leq C \langle x^1 \rangle^{-(1+\vartheta_2)}$ *for a.e.* $x \in \Omega$.

We now give a meaning to the formal expression (1.3). Notice that the Sobolev spaces $\mathcal{H}^\nu(\Omega)$ and $\mathcal{H}_0^\nu(\Omega)$, $\nu \in \mathbb{R}$, present all the usual characteristics, since Ω has a locally Lipschitz boundary [2, Sec. 4.7]. Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $\mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$, respectively $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$, the set of bounded, respectively compact, operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. We also define $\mathscr{B}(\mathcal{H}_1) := \mathscr{B}(\mathcal{H}_1, \mathcal{H}_1)$ and $\mathcal{K}(\mathcal{H}_1) := \mathcal{K}(\mathcal{H}_1, \mathcal{H}_1)$. We write $(\cdot,\cdot)$ and $\|\cdot\|$ for the inner product and the norm in $\mathcal{H}(\Omega)$. We start by introducing the sesquilinear form Q_0 on $\mathcal{H}(\Omega)$ defined by

$$Q_0(\varphi,\psi) := \big(\varphi_{,i}, \delta^{ij}\psi_{,j}\big), \qquad \varphi,\psi \in \mathcal{D}(Q_0) := \mathcal{H}_0^1(\Omega), \tag{2.1}$$

which is densely defined, symmetric, non-negative and closed. Consequently, there exists a unique self-adjoint operator H_0 associated with it, which is just the Dirichlet Laplacian $-\Delta_{\mathrm{D}}^\Omega$ on $L^2(\Omega)$. We have

$$H_0\psi = -\Delta\psi, \qquad \psi \in \mathcal{D}(H_0) = \big\{\psi \in \mathcal{H}_0^1(\Omega) \,:\, \Delta\psi \in \mathcal{H}(\Omega)\big\}.$$

We consider H as an operator obtained by perturbing the free Hamiltonian H_0. Since the matrix G is uniformly positive and bounded by Assumption 2.1.1, the sesquilinear form $(\varphi,\psi) \mapsto \big(\varphi_{,i}, G^{ij}\psi_{,j}\big)$ defined on $\mathcal{D}(Q_0) \times \mathcal{D}(Q_0)$ is also densely defined, symmetric, non-negative and closed. At the same time, the potential V is supposed to be bounded by Assumption 2.2.1, which means that the sesquilinear form Q defined by

$$Q(\varphi,\psi) := \big(\varphi_{,i}, G^{ij}\psi_{,j}\big) + (\varphi, V\psi), \qquad \varphi,\psi \in \mathcal{D}(Q) := \mathcal{H}_0^1(\Omega), \tag{2.2}$$

gives rise to a self-adjoint operator H. We verify that on

$$\mathcal{D}(H) = \big\{\psi \in \mathcal{H}_0^1(\Omega) \,:\, \partial_i G^{ij}\partial_j\psi \in \mathcal{H}(\Omega)\big\},$$

where the derivatives must be interpreted in the distributional sense, H acts as in (1.3).

For any $z \in \mathbb{C} \setminus \sigma(H_0)$, respectively $z \in \mathbb{C} \setminus \sigma(H)$, let $R_0(z) := (H_0 - z)^{-1}$, respectively $R(z) := (H - z)^{-1}$.

2.2 Localisation of the essential spectrum

The Dirichlet Laplacian $-\Delta_{\mathrm{D}}^\omega$ on $L^2(\omega)$, *i.e.* the operator associated with

$$q(\varphi,\psi) := (\varphi_{,\mu}, \delta^{\mu\nu}\psi_{,\nu}), \qquad \varphi,\psi \in \mathcal{D}(q) := \mathcal{H}_0^1(\omega),$$

has a purely discrete spectrum consisting of eigenvalues $\nu_1 < \nu_2 \leq \nu_3 \leq \ldots$ with $\nu_1 > 0$. We set $\mathcal{T} := \{\nu_n\}_{n=1}^\infty$. Since H_0 is naturally decoupled in the following way:

$$H_0 = -\Delta^{\mathbb{R}} \otimes 1 + 1 \otimes (-\Delta_{\mathrm{D}}^\omega) \qquad \text{on} \qquad L^2(\mathbb{R}) \otimes L^2(\omega),$$

where "$\otimes$" denote the closed tensor product, 1 the identity operators on appropriate spaces and $-\Delta^{\mathbb{R}}$ the Laplacian on $L^2(\mathbb{R})$, one has

$$\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = [\nu_1, \infty). \tag{2.3}$$

Under our assumptions, H possesses the same essential spectrum:

Proposition 2.1. *One has*

(i) $\forall z \in \mathbb{C} \setminus (\sigma(H) \cup \sigma(H_0)),\ R(z) - R_0(z) \in \mathcal{K}\big(\mathcal{H}(\Omega)\big)$,

(ii) $\sigma_{\mathrm{ess}}(H) = [\nu_1, \infty)$.

Proof. We use ideas of [3, Thm. 1.1]. Since the second claim is a direct consequence of the first one and (2.3) (see [1, Lemma 7.2.8]), it is enough to prove (i) for some (and hence for all) value of $z \in \mathbb{C} \setminus (\sigma(H) \cup \sigma(H_0))$. Let $z \in (-\infty, \nu_1) \setminus \sigma(H)$. Then, $R_0(z)$ is an isomorphism of $\mathcal{H}(\Omega)$ onto $\mathcal{D}(H_0)$, $R(z)$ extends to an isomorphism of $\mathcal{D}(H)^*$ onto $\mathcal{H}(\Omega)$ and

$$R(z) - R_0(z) = -R(z)(H - H_0)R_0(z)$$

if $H - H_0$ is viewed as an operator in $\mathscr{B}\left(\mathcal{D}(H_0), \mathcal{D}(H)^*\right)$. Thus, it is sufficient to show that $H - H_0 \in \mathcal{K}\left(\mathcal{D}(H_0), \mathcal{D}(H)^*\right)$. Let $\varphi_0 \in C^\infty(\mathbb{R})$ be such that

$$\varphi_0(x^1) = \begin{cases} 1 & \text{if} \quad |x^1| \le 1 \\ 0 & \text{if} \quad |x^1| \ge 2\,. \end{cases}$$

Let $n \in \mathbb{N} \setminus \{0\}$. Set $\varphi_n(x^1) := \varphi_0(x^1/n)$ for $x^1 \in \mathbb{R}$, $\phi_n := \varphi_n \otimes 1$ on Ω and

$$K_n\psi := (-\partial_i F^{ij}\,\partial_j + V)\,\phi_n\,\psi\,, \qquad \psi \in \mathcal{D}(H_0)\,,$$

where $(F^{ij}) := (G^{ij} - \delta^{ij})$. Clearly, $K_n \in \mathscr{B}\left(\mathcal{D}(H_0), \mathcal{D}(H)^*\right)$ and

$$\begin{aligned}
&\|K_n - (H - H_0)\|_{\mathscr{B}(\mathcal{D}(H_0),\mathcal{D}(H)^*)} \\
&\equiv \sup_{\|\psi\|_{\mathcal{D}(H_0)}=1} \left\| \left(I + H^2\right)^{-1/2} (-\partial_i F^{ij}\,\partial_j + V)(\phi_n - 1)\,\psi \right\| \\
&= \sup_{\|\psi\|_{\mathcal{D}(H_0)}=1} \left\| \left(I + H^2\right)^{-1/2} \left\{\partial_i \left[F^{i1}\phi_{n,1} + F^{ij}(\phi_n - 1)\,\partial_j\right] + V(\phi_n - 1)\right\} \psi \right\| \\
&\le \sup_{\|\psi\|_{\mathcal{D}(H_0)}=1} \Big[\left\| \left(I + H^2\right)^{-1/2} \partial_i \right\| \left\| F^{i1}\phi_{n,1} + F^{ij}(\phi_n - 1) \right\|_\infty \|\partial_j \psi\| \\
&\qquad + \left\| \left(I + H^2\right)^{-1/2} \right\| \|V(\phi_n - 1)\|_\infty \|\psi\| \Big] \xrightarrow[n\to\infty]{} 0\,,
\end{aligned}$$

where we have used Assumptions 2.1.2 and 2.2.2 in the final step. So, it only remains to show that $R(z)K_nR_0(z) \in \mathcal{K}\left(\mathcal{H}(\Omega)\right)$, which is equivalent to $K_n \in \mathcal{K}\left(\mathcal{D}(H_0), \mathcal{D}(H)^*\right)$. Let $z_1 \in (-\infty, 0)$ and $z_2 \in (-\infty, \nu_1)$ be such that $z = z_1 + z_2$. Then, using some standard results on tensor products of operators [16, Chap. 11], one can write

$$(\varphi_n \otimes 1)\,R_0(z) = \left[\varphi_n\,R_{\shortparallel}^{1/2}(z_1) \otimes R_\perp^{1/2}(z_2)\right]\left[R_{\shortparallel}^{-1/2}(z_1) \otimes R_\perp^{-1/2}(z_2)\right]R_0(z)$$

where φ_n is seen as a multiplication operator in $L^2(\mathbb{R})$, $R_{\shortparallel}(z_1) := \left(-\Delta^{\mathbb{R}} - z_1\right)^{-1}$ and $R_\perp(z_2) := (-\Delta_{\mathrm{D}}^{\omega} - z_2)^{-1}$. The first factor on the r.h.s. is a compact operator because $\varphi_n\,R_{\shortparallel}^{1/2}(z_1) \in \mathcal{K}\left(L^2(\mathbb{R})\right)$ by [1, Thm. 4.1.3] and $-\Delta_{\mathrm{D}}^{\omega}$ has a compact resolvent. The remaining factor can be rewritten as

$$\Psi(X_1, X_2) := X_1^{1/2} X_2^{1/2} (X_1 + X_2)^{-1}$$

with $X_1 := \left(-\Delta^{\mathbb{R}} - z_1\right) \otimes 1$ and $X_2 := 1 \otimes (-\Delta_{\mathrm{D}}^{\omega} - z_2)$ (both self-adjoint and mutually commuting). Then, one can estimate

$$\|\Psi(X_1, X_2)\| \leq \sup_{x_1, x_2 \in (0,\infty)} (x_1 x_2)^{1/2} (x_1 + x_2)^{-1} < \infty \, .$$

Consequently, $(\varphi_n \otimes 1)\, R_0(z)$ is in $\mathcal{K}\left(\mathcal{H}(\Omega)\right)$ with range in $\mathcal{D}(H_0)$. This implies that $R(z) K_n R_0(z) \in \mathcal{K}\left(\mathcal{H}(\Omega)\right)$. $\square$

Remark 2.2. *Notice that* Assumptions 2.1.3 *and* 2.2.3 *are not used in the proof of* Proposition 2.1.

2.3 Nature of the essential spectrum

This part is devoted to a more detailed analysis of the essential spectrum of H. In particular, we show that the singular continuous spectrum is empty. The strategy adapted from [3] is the following. Firstly, we construct a dilation operator A such that $H_0 \in C^\infty(A)$ and $H \in C^{1+\vartheta}(A)$ with $\vartheta := \min\{\vartheta_1, \vartheta_2\} \in (0, 1]$ (see [1], [3, Sec. 2] or [7, Sec. 1] for definitions of the spaces involved here and in the sequel). Secondly, we prove that A is *strictly conjugate* (in Mourre's sense) to H_0 on $\mathbb{R} \setminus \mathcal{T}$. Finally, since $R(i) - R_0(i)$ is compact by the first claim of Proposition 2.1 and both H and H_0 are of class $C_u^1(A) \supseteq C^{1+\vartheta}(A) \supseteq C^\infty(A)$, it follows that A is conjugate to H on $\mathbb{R} \setminus \mathcal{T}$ as well.

2.3.1 The dilation operator

Let q^1 be the multiplication operator by the coordinate x^1 in $\mathcal{H}(\Omega)$. Let

$$A := \tfrac{1}{2}\left(q^1 p_1 + p_1 q^1\right) \qquad \text{with} \qquad p_1 := -i\,\partial_1 \tag{2.4}$$

be the dilation operator in $\mathcal{H}(\Omega)$ w.r.t. x^1, *i.e.* the self-adjoint extension of the operator defined by the expression (2.4) with $C_0^\infty(\Omega)$ as initial domain.

Remark 2.3. *The group* $\left\{\mathrm{e}^{iAt}\right\}_{t\in\mathbb{R}}$ *leaves invariant* $\mathcal{H}_0^1(\Omega)$. *Indeed, using the natural isomorphism* $\mathcal{H}_0^1(\Omega) \simeq \mathcal{H}^1(\mathbb{R}) \otimes \mathcal{H}_0^1(\omega)$, *one can write*

$$\left\{\mathrm{e}^{iAt}\right\}_{t\in\mathbb{R}} = \left\{\mathrm{e}^{iAt} \otimes 1\right\}_{t\in\mathbb{R}} \tag{2.5}$$

where A on the r.h.s. must be viewed as an operator acting in $L^2(\mathbb{R})$. Then, the affirmation follows from the fact [1, Prop. 4.2.4] *that $\mathcal{H}^1(\mathbb{R})$ is stable under* $\left\{\mathrm{e}^{iAt}\right\}_{t\in\mathbb{R}}$.

In order to deal with the commutator $i\,[H, A]$, we need the following family of operators

$$\left\{p_1(\varepsilon) := p_1(1 + i\varepsilon p_1)^{-1}\right\}_{\varepsilon>0} \, , \tag{2.6}$$

which regularises the momentum operator p_1:

Lemma 2.4. *One has*

(i) $\{p_1(\varepsilon)\}_{\varepsilon>0} \subset \mathscr{B}\big(\mathcal{H}(\Omega)\big)$,

(ii) $\text{s-lim}_{\varepsilon\to 0}\, p_1(\varepsilon) = p_1$ *in* $\mathscr{B}\big(\mathcal{H}^1(\Omega), \mathcal{H}(\Omega)\big)$,

(iii) $\forall \varepsilon > 0\,,\ p_1(\varepsilon)\,\mathcal{D}(\langle q^1\rangle) \subseteq \mathcal{D}(\langle q^1\rangle)$,

(iv) $\forall \varepsilon > 0\,,\ p_1(\varepsilon)\,\mathcal{H}_0^1(\Omega) \subseteq \mathcal{H}_0^1(\Omega)$.

Proof. The first three assertions are established in [3, Lemma 4.1]. Consequently, it only remains to prove the last statement. Let $\psi \in \mathcal{H}_0^1(\Omega)$. Using the isomorphism mentioned in Remark 2.3, one can write

$$\forall \varepsilon > 0\,, \qquad p_1(\varepsilon)\psi = -i\varepsilon^{-1}\big\{\big[1 + i\varepsilon^{-1}(p_1 - i\varepsilon^{-1})^{-1}\big] \otimes 1\big\}\,\psi\,,$$

where p_1 must be viewed as an operator acting in $L^2(\mathbb{R})$. With this last relation, it is clear that $\mathcal{H}_0^1(\Omega)$ is left invariant by the family $\{p_1(\varepsilon)\}_{\varepsilon>0}$. ☐

Now, we can compute the commutator $i\,[H, A]$.

Proposition 2.5. *Let* $\mathcal{D}(H)_c := \{\psi \in \mathcal{D}(H)\ :\ \mathrm{supp}(\psi)$ *is compact*$\}$. *Then*

$$B := i\,[H, A] = -\partial_j\, G^{1j}\partial_1 - \partial_1 G^{1j}\partial_j + \partial_i\, q^1 G^{ij}_{,1}\partial_j - q^1 V_{,1} \tag{2.7}$$

on $\mathcal{D}(H)_c$, *and* B *admits a continuous extension in* $\mathscr{B}\big(\mathcal{H}_0^1(\Omega), \mathcal{H}^{-1}(\Omega)\big)$.

Proof. We remark first that $\mathcal{D}(H)_c$ is dense in the form domain $\mathcal{H}_0^1(\Omega)$ of H and second that $\mathcal{D}(H)_c$ is a subset of $\mathcal{D}(A)$. So, let $\psi \in \mathcal{D}(H)_c$. Using the identity $A = q^1 p_1 - \frac{i}{2}$ valid on $\mathcal{D}(H)_c$, we have

$$\begin{aligned}
(\psi, B\psi) &\equiv i\,[(H\psi, A\psi) - (A\psi, H\psi)] \\
&= (\psi, H\psi) + i\,\big[\big(-\partial_i\, G^{ij}\partial_j\psi, q^1 p_1\psi\big) - \big(q^1 p_1\psi, -\partial_i\, G^{ij}\partial_j\psi\big)\big] \\
&\quad + \big(V\psi, q^1\psi_{,1}\big) + \big(q^1\psi_{,1}, V\psi\big)\,.
\end{aligned}$$

In order to justify the subsequent integration by parts, we employ the family (2.6). Since ψ has a compact support and belongs to $\mathcal{H}_0^1(\Omega)$, it follows by using the property (iv) of Lemma 2.4 that $q^1 p_1(\varepsilon)\psi \in \mathcal{H}_0^1(\Omega)$ for all $\varepsilon > 0$. So, we can write

$$\begin{aligned}
\big(-\partial_i\, G^{ij}\partial_j\psi, q^1 p_1\psi\big) &= \lim_{\varepsilon\to 0}\big(-\partial_i\, G^{ij}\partial_j\psi, q^1 p_1(\varepsilon)\psi\big) \\
&= \lim_{\varepsilon\to 0}\big(\psi_{,j}, G^{ij}\partial_i\, q^1 p_1(\varepsilon)\psi\big) \\
&= -\,i\,\big(\psi_{,j}, G^{1j}\psi_{,1}\big) + \lim_{\varepsilon\to 0}\big(\psi_{,i}, G^{ij} q^1 p_1(\varepsilon)\psi_{,j}\big)
\end{aligned}$$

and similarly for the conjugate integral

$$\begin{aligned}
\big(q^1 p_1\psi, -\partial_i\, G^{ij}\partial_j\psi\big) &= \lim_{\varepsilon\to 0}\big(q^1 p_1(\varepsilon)^*\psi, -\partial_i\, G^{ij}\partial_j\psi\big) \\
&= i\,\big(\psi_{,1}, G^{1j}\psi_{,j}\big) + \lim_{\varepsilon\to 0}\big(p_1(\varepsilon)^*\psi_{,i}, q^1 G^{ij}\psi_{,j}\big)\,.
\end{aligned}$$

Since

$$\begin{aligned}\lim_{\varepsilon\to 0}\big(p_1(\varepsilon)^*\psi_{,i},q^1G^{ij}\psi_{,j}\big) &= \lim_{\varepsilon\to 0}\big(\psi_{,i},p_1(\varepsilon)\,q^1G^{ij}\psi_{,j}\big)\\ &= -\,i\Big[\big(\psi_{,i},G^{ij}\psi_{,j}\big)+\big(\psi_{,i},q^1G^{ij}_{,1}\psi_{,j}\big)\Big]\\ &\quad+\lim_{\varepsilon\to 0}\big(\psi_{,i},q^1G^{ij}p_1(\varepsilon)\psi_{,j}\big)\,,\end{aligned}$$

and

$$\big(q^1\psi_{,1},V\psi\big)=-\big(\psi,\partial_1\,q^1V\psi\big)=-\,(\psi,V\psi)-\big(\psi,q^1V_{,1}\psi\big)-\big(\psi,q^1V\psi_{,1}\big)\,,$$

we finally obtain that

$$(\psi,B\psi)=\big(\psi_{,j},G^{1j}\psi_{,1}\big)+\big(\psi_{,1},G^{1j}\psi_{,j}\big)-\big(\psi_{,i},G^{ij}_{,1}q^1\psi_{,j}\big)-\big(\psi,q^1V_{,1}\psi\big)\,.$$

This shows the first affirmation. To prove the second one, it is sufficient to remark that B defined on $\mathcal{D}(H)_c$ is continuous for the topology induced by $\mathcal{H}^1(\Omega)$. Thus, it defines (by continuity) an operator in $\mathscr{B}(\mathcal{H}^1_0(\Omega),\mathcal{H}^{-1}(\Omega))$ which we denote by the same symbol. □

2.3.2 Strict Mourre estimate for the free Hamiltonian

Now we prove that H_0 is of class $C^\infty(A)$ and A strictly conjugate to it on $\mathbb{R}\setminus\mathcal{T}$. So, let us first recall the following definition [1, Sec. 7.2.1 & 7.2.2]:

Definition 2.6. *Let A, H be self-adjoint operators in a Hilbert space $\mathcal{H}$ with H of class $C^1(A)$. Furthermore, if S, $T\in\mathscr{B}(\mathcal{H})$, we write $S\gtrsim T$ if there exists $K\in\mathcal{K}(\mathcal{H})$ so that $S\geq T+K$. Then, $\forall\lambda\in\mathbb{R}$,*

$$\begin{aligned}\varrho^A_H(\lambda)&:=\sup\big\{a\in\mathbb{R}\,:\,\exists\varepsilon>0\ \textit{s.t.}\ \ E^H(\lambda;\varepsilon)\,i\,[H,A]E^H(\lambda;\varepsilon)\geq a\,E^H(\lambda;\varepsilon)\big\}\,,\\ \widetilde{\varrho}^A_H(\lambda)&:=\sup\big\{a\in\mathbb{R}\,:\,\exists\varepsilon>0\ \textit{s.t.}\ \ E^H(\lambda;\varepsilon)\,i\,[H,A]E^H(\lambda;\varepsilon)\gtrsim a\,E^H(\lambda;\varepsilon)\big\}\end{aligned}$$

where $E^H(\lambda;\varepsilon):=E^H((\lambda-\varepsilon,\lambda+\varepsilon))$ designates the spectral projection of H for the interval $(\lambda-\varepsilon,\lambda+\varepsilon)$.

We also need the following natural generalisation of [4, Thm. 3.4].

Theorem 2.7. *Let H_1, H_2 be two self-adjoint, bounded from below operators in the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$. Assume that $A_\jmath$, $\jmath=1,2$, is a self-adjoint operator in $\mathcal{H}_\jmath$ such that $H_\jmath$ is of class $C^k(A_\jmath)$, $k\in\mathbb{N}\setminus\{0\}$. Let $H:=H_1\otimes 1+1\otimes H_2$ and $A:=A_1\otimes 1+1\otimes A_2$, which are self-adjoint operators in $\mathcal{H}_1\otimes\mathcal{H}_2$. Then H is of class $C^k(A)$ and $\forall\lambda\in\mathbb{R}$:*

$$\varrho^A_H(\lambda)=\inf_{\lambda=\lambda_1+\lambda_2}\Big[\varrho^{A_1}_{H_1}(\lambda_1)+\varrho^{A_2}_{H_2}(\lambda_2)\Big]\,.$$

Corollary 2.8. *$H_0\in C^\infty(A)$ and*

$$\forall\lambda\in\mathbb{R}\,,\qquad \varrho^A_{H_0}(\lambda)=\begin{cases}2\rho(\lambda) & \textit{if}\quad \lambda\geq\nu_1\\ +\infty & \textit{if}\quad \lambda<\nu_1\,,\end{cases}\tag{2.8}$$

where $\rho(\lambda):=\lambda-\sup\{\zeta\in\mathcal{T}\,:\,\zeta\leq\lambda\}$ is strictly positive on $\mathbb{R}\setminus\mathcal{T}$.

Proof. $A_1 := A$, $A_2 := 0$ are self-adjoint in $L^2(\mathbb{R})$, respectively $L^2(\omega)$. $H_1 := p_1^2$, $H_2 := -\Delta_{\mathrm{D}}^{\omega}$ are self-adjoint, bounded from below in $L^2(\mathbb{R})$, respectively $L^2(\omega)$. Clearly, [1, Ex. 6.2.8] $p_1^2 \in C^{\infty}(A)$ and $-\Delta_{\mathrm{D}}^{\omega} \in C^{\infty}(0)$. The first part of the claim and (2.8) then follows from Theorem 2.7. The expression for $\rho(\lambda)$ is a direct consequence of the respective behaviours of [1, Sec. 7.2.1] $\varrho_{p_1^2}^{A}$ and $\varrho_{-\Delta_{\mathrm{D}}^{\omega}}^{0}$:

$$\begin{bmatrix} \varrho_{p_1^2}^{A}(\lambda_1) \\ \varrho_{-\Delta_{\mathrm{D}}^{\omega}}^{0}(\lambda_2) \end{bmatrix} = \begin{cases} \begin{bmatrix} 2\lambda_1 \\ +\infty \end{bmatrix} & \text{if} \quad \begin{bmatrix} \lambda_1 \geq 0 \\ \lambda_1 < 0 \end{bmatrix} \\[2ex] \begin{bmatrix} 0 \\ +\infty \end{bmatrix} & \text{if} \quad \begin{bmatrix} \lambda_2 \in \mathcal{T} \\ \lambda_2 \in \mathbb{R} \setminus \mathcal{T} \end{bmatrix}. \end{cases}$$

□

2.3.3 Regularity of the Hamiltonian

In order to prove the regularity of H, we need two technical lemmas.

Lemma 2.9. $\forall z \in \mathbb{R} \setminus \sigma(H)$, $\forall \vartheta \leq 1$, $\left[R(z), \langle q^1 \rangle^{\vartheta}\right] \in \mathscr{B}\left(\mathcal{H}^{-1}(\Omega), \mathcal{H}_0^1(\Omega)\right)$.

This is easily established by mimicking the proof of [3, Lemma 4.3] while next Lemma follows from the use of [3, Proof of Prop. 4.2] or [6, Sec. 6].

Lemma 2.10. *Let $S \in \mathscr{B}(\mathcal{H}(\Omega))$ be self-adjoint and $\vartheta \in (0,1]$, then*

$$\langle q^1 \rangle^{\vartheta} S \in \mathscr{B}\left(\mathcal{H}(\Omega), \mathcal{H}^{\vartheta}(\mathbb{R}) \otimes L^2(\omega)\right) \implies S \in C^{\vartheta}(A).$$

Remark 2.11. *The facts that $B \in \mathscr{B}\left(\mathcal{H}_0^1(\Omega), \mathcal{H}^{-1}(\Omega)\right)$ and $\mathcal{H}_0^1(\Omega)$ is stable under $\left\{\mathrm{e}^{iAt}\right\}_{t\in\mathbb{R}}$ imply* [1, Sec. 6.3] *that $H \in C^1(A)$.*

Proposition 2.12. $\exists \vartheta \in (0,1]$ *such that* $H \in C^{1+\vartheta}(A)$.

Proof. Let $z \in \mathbb{R} \setminus \sigma(H)$. Using Proposition 2.5 and Remark 2.11, one can write, as an operator identity in $\mathscr{B}(\mathcal{H}^{-1}(\Omega), \mathcal{H}_0^1(\Omega))$,

$$\begin{aligned} [iA, R(z)] = R(z)BR(z) = & -R(z)\left(\partial_1 G^{1j}\partial_j + \partial_j G^{j1}\partial_1\right)R(z) \\ & + R(z)\,\partial_i\, q^1 G^{ij}_{,1}\partial_j R(z) \\ & - R(z)\, q^1 V_{,1}\, R(z) \end{aligned} \tag{2.9}$$

with the last equality holding on $(H-z)\,\mathcal{D}(H)_{\mathrm{c}}$. Now, $\mathcal{D}(H)_{\mathrm{c}}$ is dense in $\mathcal{H}_0^1(\Omega)$ and $H-z$, seen as an operator from $\mathcal{H}_0^1(\Omega)$ onto $\mathcal{H}^{-1}(\Omega)$, is an isomorphism. Consequently, to prove the affirmation, it is enough to show that the closures of the terms on the r.h.s. of (2.9) are of class $C^{\vartheta}(A)$ for some $\vartheta \in (0,1]$. From now on, it will be implicit that each operator identity shown must be interpreted as the closure of calculation initially conducted on $(H-z)\,\mathcal{D}(H)_{\mathrm{c}}$.

It is obvious by a direct argument that the first term on the r.h.s. of (2.9) is of class $C^1(A)$. In detail, writing

$$\begin{aligned}\left[iA, R(z)\left(\partial_1 G^{1j}\partial_j + \partial_j G^{j1}\partial_1\right)R(z)\right] &= [iA, R(z)]\left(\partial_1 G^{1j}\partial_j + \partial_j G^{j1}\partial_1\right)R(z)\\ &\quad + R(z)\left(\partial_1 G^{1j}\partial_j + \partial_j\, G^{j1}\partial_1\right)[iA, R(z)]\\ &\quad + R(z)\left[iA, \partial_1 G^{1j}\partial_j + \partial_j G^{j1}\partial_1\right]R(z)\,,\end{aligned}$$

we see that the first two terms on the r.h.s. of this equality belong to $\mathscr{B}(\mathcal{H}(\Omega))$ because of Proposition 2.5 and Remark 2.11. It remains to examine the third one. However, an explicit calculation (analogous to that of the proof of Proposition 2.5) gives

$$\begin{aligned}&\left[-\partial_1 G^{1j}\partial_j - \partial_j\, G^{j1}\partial_1, iA\right]\\ &\qquad = -2\,\partial_1 G^{11}\partial_1 - \partial_1 G^{1j}\partial_j - \partial_j\, G^{j1}\partial_1 + \partial_j\, q^1 G^{j1}_{\ ,1}\partial_1 + \partial_1 q^1 G^{1j}_{\ ,1}\partial_j\end{aligned}$$

in $\mathscr{B}(\mathcal{H}^1_0(\Omega), \mathcal{H}^{-1}(\Omega))$.

In order to deal with the remaining terms on the r.h.s. of (2.9), we shall employ Lemma 2.10. Let $\vartheta := \min\{\vartheta_1, \vartheta_2\}$. Using some commutation relations, we have

$$\begin{aligned}\langle q^1\rangle^{\vartheta} R(z)\,\partial_i\, q^1 G^{ij}_{\ ,1}\partial_j\, R(z) &= R(z)\,\partial_i \langle q^1\rangle^{\vartheta} q^1 G^{ij}_{\ ,1}\partial_j\, R(z)\\ &\quad - \left[R(z), \langle q^1\rangle^{\vartheta}\right]\partial_i\, q^1 G^{ij}_{\ ,1}\partial_j\, R(z)\\ &\quad - R(z)\left[\partial_i, \langle q^1\rangle^{\vartheta}\right] q^1 G^{ij}_{\ ,1}\partial_j\, R(z)\,.\end{aligned}$$

Under Assumption 2.1.3, the first term on the r.h.s. is in $\mathscr{B}(\mathcal{H}(\Omega), \mathcal{H}^1_0(\Omega))$. The second and the last ones are in $\mathscr{B}(\mathcal{H}(\Omega), \mathcal{H}^1_0(\Omega))$ by Lemma 2.9 and the boundedness of $\langle q^1\rangle^{\vartheta}_{,1}$, respectively. Finally,

$$\langle q^1\rangle^{\vartheta} R(z)\, q^1 V_{,1}\, R(z) = R(z)\langle q^1\rangle^{\vartheta} q^1 V_{,1}\, R(z) + \left[\langle q^1\rangle^{\vartheta}, R(z)\right] q^1 V_{,1}\, R(z)$$

is in $\mathscr{B}\left(\mathcal{H}(\Omega), \mathcal{H}^1_0(\Omega)\right)$ by Assumption 2.2.3 and Lemma 2.9. □

2.3.4 The main result

Proposition 2.13. $\forall \lambda \in \mathbb{R}\setminus\mathcal{T},\ \ \widetilde{\varrho}^{A}_{H}(\lambda) > 0\,.$

Proof. Corollary 2.8 and Proposition 2.12 imply that both H_0 and H are of class $C^1_u(A)$. Furthermore, $R(i) - R_0(i)$ is compact by Proposition 2.1, with the result that $\widetilde{\varrho}^{A}_{H} = \widetilde{\varrho}^{A}_{H_0}$ due to [1, Thm. 7.2.9]. Finally, since [1, Prop. 7.2.6] $\widetilde{\varrho}^{A}_{H_0} \geq \varrho^{A}_{H_0}$, we can conclude using Corollary 2.8. □

Summing up, we result in the following spectral properties of H.

Theorem 2.14. *Let ω be a bounded open connected set in $\mathbb{R}^{d-1}$, $d \geq 2$, with locally Lipschitz boundary and denote by $\mathcal{T}$ the set of eigenvalues of $-\Delta^{\omega}_{\mathrm{D}}$. Let H be the operator* (1.3) *with $\Omega := \mathbb{R}\times\omega$, subject to Dirichlet boundary conditions, and satisfying* Assumptions 2.1 *and* 2.2. *Then*

(i) $\sigma_{\mathrm{ess}}(H) = [\kappa, \infty)$, *where* $\kappa := \inf \mathcal{T}$,

(ii) $\sigma_{\mathrm{sc}}(H) = \varnothing$,

(iii) $\sigma_{\mathrm{p}}(H) \cup \mathcal{T}$ *is closed and countable*,

(iv) $\sigma_{\mathrm{p}}(H) \setminus \mathcal{T}$ *is composed of finitely degenerated eigenvalues, which can accumulate at the points of* $\mathcal{T}$ *only*,

Proof. The claim (i) is included in Proposition 2.1. Since A is conjugate to H on $\mathbb{R} \setminus \mathcal{T}$ by Proposition 2.13, the assertions (ii)–(iv) follow by the abstract conjugate operator method [1, Thm. 7.4.2]. ☐

To conclude this section, let us remark that Assumptions 2.1.3 and 2.2.3 could be weakened. Firstly, we recall that the situation with $V = 0$ and $G = \rho\, 1$, ρ being a real-valued function greater than a strictly positive constant, is investigated in [3, 7] where the authors admit local singularities of ρ. More specifically, one assumes that $\rho = \rho_{\mathrm{s}} + \rho_\ell$, where ρ_ℓ is the part satisfying a condition analogous to Assumption 2.1.3, while ρ_{s} need not be differentiable. (In [3], $\mathrm{supp}(\rho_{\mathrm{s}})$ is assumed to be compact. The result of [7] is better in the sense that ρ_s is only supposed to be a short-range perturbation there. However, this requires strengthening of the condition analogous to Assumption 2.1.2 about the decay of ρ at infinity.) Secondly, the optimal conditions on the potential of a Schrödinger operator are [5, 1] known.

3 Curved tubes

In this part, we use Theorem 2.14 in order to find geometric sufficient conditions which guarantee that the spectral results of the theorem hold true for curved tubes.

3.1 Geometric preliminaries

3.1.1 The reference curve

Given $d \geq 2$, let $p : \mathbb{R} \to \mathbb{R}^d$ be a regular unit-speed smooth (*i.e.* C^∞-smooth) curve satisfying the following hypothesis.

Assumption 3.1. *There exists a collection of* d *smooth mappings* $e_i : \mathbb{R} \to \mathbb{R}^d$ *with the following properties:*

1. $\forall i, j \in \{1, \ldots, d\}$, $\forall s \in \mathbb{R}$, $e_i(s) \cdot e_j(s) = \delta_{ij}$,
2. $\forall i \in \{1, \ldots, d-1\}$, $\forall s \in \mathbb{R}$, *the* i^{th} *derivative* $p^{(i)}(s)$ *of* $p(s)$ *lies in the span of* $e_1(s), \ldots, e_i(s)$,
3. $e_1 = \dot{p}$,
4. $\forall s \in \mathbb{R}$, $\{e_1(s), \ldots, e_d(s)\}$ *has the positive orientation*,

5. $\forall i \in \{1, \dots, d-1\}$, $\forall s \in \mathbb{R}$, $\dot{e}_i(s)$ lies in the span of $e_1(s), \dots, e_{i+1}(s)$.

Here and in the sequel, "$\cdot$" denotes the inner product in $\mathbb{R}^d$.

Remark 3.1. *A vector field with the property 1 is called a* moving frame *along p and it is a* Frenet frame *if it satisfies 2 in addition,* cf. [17, Sec. 1.2]. *A sufficient condition to ensure the existence of the frame of* Assumption 3.1 *is to require that* [17, Prop. 1.2.2]*, for all $s \in \mathbb{R}$, the vectors $\dot{p}(s), p^{(2)}(s), \dots, p^{(d-1)}(s)$ are linearly independent. Furthermore, such a frame is called a* distinguished Frenet frame *if it satisfies, in addition to 1–4,*

5'. $\forall i \in \{1, \dots, d-1\}$, $\forall s \in \mathbb{R}$, $\left\{\dot{p}(s), \dots, p^{(i)}(s)\right\}$ and $\{e_1(s), \dots, e_i(s)\}$ have the same orientation.

However, we prefer not to assume a priori *the above non-degeneracy condition because it excludes curves with subsets lying in lower-dimensional subspaces of $\mathbb{R}^d$ as soon as $d \geq 3$.*

The properties of $\{e_1, \dots, e_d\}$ summarised in Assumption 3.1 yield [17, Sec. 1.3] the Serret-Frenet formulae,

$$\dot{e}_i = \mathcal{K}_i^{\ j}\, e_j \tag{3.1}$$

with $\mathcal{K} \equiv \left(\mathcal{K}_i^{\ j}\right)$ being a skew-symmetric $d \times d$ matrix defined by

$$\mathcal{K} := \begin{pmatrix} 0 & \kappa_1 & & \mathbf{0} \\ -\kappa_1 & \ddots & \ddots & \\ & \ddots & \ddots & \kappa_{d-1} \\ \mathbf{0} & & -\kappa_{d-1} & 0 \end{pmatrix}. \tag{3.2}$$

Here κ_i is called the i^{th} *curvature* of p. Under our Assumption 3.1, the curvatures are smooth functions of the arc-length parameter $s \in \mathbb{R}$.

3.1.2 The appropriate moving frame

We introduce another moving frame along p which better reflects the geometry of the curve and is convenient for our further purposes. Let the $(d-1) \times (d-1)$ matrix $(\mathcal{R}_\mu^{\ \nu})$ be defined by the system of differential equations

$$\dot{\mathcal{R}}_\mu^{\ \nu} + \mathcal{R}_\mu^{\ \alpha}\, \mathcal{K}_\alpha^{\ \nu} = 0 \tag{3.3}$$

with $(\mathcal{R}_\mu^{\ \nu}(s_0))$ being a rotation matrix in $\mathbb{R}^{d-1}$ for some $s_0 \in \mathbb{R}$ as initial condition, *i.e.*

$$\det\left(\mathcal{R}_\mu^{\ \nu}(s_0)\right) = 1 \qquad \text{and} \qquad \delta_{\alpha\beta}\, \mathcal{R}_\mu^{\ \alpha}(s_0)\, \mathcal{R}_\nu^{\ \beta}(s_0) = \delta_{\mu\nu}\,. \tag{3.4}$$

The solution of (3.3) exists and is unique and smooth by standard arguments in the theory of differential equations (*cf.* [20, Sec. 4]). Furthermore, the conditions (3.4) are satisfied for *all* $s_0 \in \mathbb{R}$. Indeed, by means of Liouville's formula [20, Thm. 4.7.1] and $\mathrm{tr}(\mathcal{K}) = 0$, one checks that $\det\left(\mathcal{R}_\mu^{\ \nu}\right) = 1$ identically, while the validity of the second condition for all $s_0 \in \mathbb{R}$ is obtained via

$$\left(\delta_{\alpha\beta}\mathcal{R}_\mu^{\ \alpha}\mathcal{R}_\nu^{\ \beta}\right)^{\cdot} = -\mathcal{R}_\mu^{\ \alpha}\left(\delta_{\gamma\beta}\mathcal{K}_\alpha^{\ \gamma} + \delta_{\alpha\gamma}\mathcal{K}_\beta^{\ \gamma}\right)\mathcal{R}_\nu^{\ \beta} = 0\,.$$

We set

$$\mathcal{R} \equiv \left(\mathcal{R}_i^{\ j}\right) := \begin{pmatrix} 1 & 0 \\ 0 & \left(\mathcal{R}_\mu^{\ \nu}\right) \end{pmatrix}$$

and define a moving frame $\{\widetilde{e}_1, \ldots, \widetilde{e}_d\}$ along p by

$$\widetilde{e}_i := \mathcal{R}_i^{\ j}\, e_j\,. \tag{3.5}$$

Combining (3.1) with (3.3), one easily finds

$$\dot{\widetilde{e}}_1 = \kappa_1\, e_2 \qquad \text{and} \qquad \dot{\widetilde{e}}_\mu = \mathcal{R}_\mu^{\ \alpha}\mathcal{K}_\alpha^{\ 1} e_1 = -\kappa_1\, \mathcal{R}_\mu^{\ 2}\, e_1\,. \tag{3.6}$$

3.1.3 The tube

Given $a > 0$, let $\mathcal{B}_a$ denote the open ball of radius a in $\mathbb{R}^{d-1}$ centered at the origin and consider the straight tube $\Omega := \mathbb{R} \times \mathcal{B}_a$. We define the curved tube Γ of radius a about p as the image of the mapping

$$\mathscr{L} : \Omega \to \mathbb{R}^d\,, \qquad \left(s, u^2, \ldots, u^d\right) \mapsto p(s) + \widetilde{e}_\mu(s)\, u^\mu\,, \tag{3.7}$$

i.e. $\Gamma := \mathscr{L}(\Omega)$. We always assume

Assumption 3.2.

1. $\kappa_1 \in L^\infty(\mathbb{R})$ *and* $a\|\kappa_1\|_\infty < 1$,
2. Γ *does not overlap*.

Then the mapping $\mathscr{L} : \Omega \to \Gamma$ is a diffeomorphism. Indeed, by virtue of the implicit function theorem, the first condition guarantees that it is a local diffeomorphism which is global through the injectivity induced by the second condition. Consequently, $\mathscr{L}^{-1}$ determines a system of global (*geodesic* or *Fermi*) "coordinates" (s, u), $u \equiv (u^2, \ldots, u^d)$. At the same time, the tube Γ can be identified with the Riemannian manifold (Ω, g), where $g \equiv (g_{ij})$ is the metric tensor induced by the immersion (3.7). That is, $g_{ij} := \mathscr{L}_{,i} \cdot \mathscr{L}_{,j}$. The formulae (3.6) yield

$$g = \mathrm{diag}\left(h^2, 1, \ldots, 1\right) \qquad \text{with} \qquad h(s,u) := 1 + u^\mu\, \mathcal{R}_\mu^{\ \alpha}\mathcal{K}_\alpha^{\ 1}\,. \tag{3.8}$$

Note that the metric tensor is diagonal due to our special choice of the "transverse" frame $\{\widetilde{e}_2, \ldots, \widetilde{e}_d\}$. In particular, $|g| := \det g = h^2$ defines through $\mathrm{d}v := h(s,u)\,\mathrm{d}s\,\mathrm{d}u$ the volume element of Γ; here $\mathrm{d}u$ denotes the $(d-1)$-dimensional Lebesgue measure in $\mathcal{B}_a$.

Remark 3.2 (Low-dimensional examples). *When $d = 2$, the ball $\mathcal{B}_a$ is just the interval $(-a, a)$, the curve p has only one curvature $\kappa := \kappa_1$, the rotation matrix $(\mathcal{R}_\mu{}^\nu)$ equals (the scalar) 1 and*

$$h(s, u) = 1 - \kappa(s)\, u\,.$$

If $d = 3$, it is convenient to make the Ansatz

$$(\mathcal{R}_\mu{}^\nu) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

where α is a real-valued differentiable function. Then it is easy to see that (3.3) *reduces to the differential equation $\dot\alpha = \tau$, where τ is the torsion of p,* i.e. *one puts $\kappa := \kappa_1$ and $\tau := \kappa_2$. Choosing α as an integral of τ, we can write*

$$h(s, u) = 1 - \kappa(s)\left[u^2 \cos\alpha(s) + u^3 \sin\alpha(s)\right].$$

3.2 The Laplacian

Our object of interest is the Dirichlet Laplacian (1.2), with Γ defined by (3.7). We construct it as follows. Using the diffeomorphism (3.7), we identify the Hilbert space $L^2(\Gamma)$ with $L^2(\Omega, \mathrm{d}v)$ and consider on the latter the Dirichlet form

$$\widetilde{Q}(\varphi, \psi) := \int_\Omega \overline{\varphi_{,i}}\, g^{ij} \psi_{,j}\, \mathrm{d}v\,, \qquad \varphi, \psi \in \mathcal{D}(\widetilde{Q}) := \mathcal{H}_0^1(\Omega, \mathrm{d}v)\,, \tag{3.9}$$

where $(g^{ij}) := g^{-1}$. The form $\widetilde{Q}$ is clearly densely defined, non-negative, symmetric and closed on its domain. Consequently, there exists a unique non-negative self-adjoint operator $\widetilde{H}$ satisfying $\mathcal{D}(\widetilde{H}) \subset \mathcal{D}(\widetilde{Q})$ associated with $\widetilde{Q}$. We have

$$\widetilde{H}\psi = -|g|^{-1/2}\partial_i\, |g|^{1/2} g^{ij} \partial_j\, \psi\,, \tag{3.10}$$

$$\psi \in \mathcal{D}(\widetilde{H}) = \left\{\psi \in \mathcal{H}_0^1(\Omega, \mathrm{d}v) \,:\, \partial_i\, |g|^{1/2} g^{ij} \partial_j\, \psi \in L^2(\Omega, \mathrm{d}v)\right\}. \tag{3.11}$$

That is, $\widetilde{H}$ is the Laplacian (1.2) expressed in our coordinates (s, u).

In order to apply Theorem 2.14, we transform $\widetilde{H}$ into a unitarily equivalent operator H of the form (1.3) acting on the Hilbert space $\mathcal{H}(\Omega) := L^2(\Omega)$, without the additional weight $|g|^{1/2}$ in the volume element. This is achieved by means of the unitary mapping $\mathcal{U} : L^2(\Omega, \mathrm{d}v) \to \mathcal{H}(\Omega), \psi \mapsto |g|^{1/4}\psi$. Defining $H := \mathcal{U}\widetilde{H}\mathcal{U}^{-1}$, one has

$$H\psi = -|g|^{-1/4}\partial_i\, |g|^{1/2} g^{ij} \partial_j |g|^{-1/4}\, \psi\,, \tag{3.12}$$

$$\psi \in \mathcal{D}(H) = \left\{\psi \in \mathcal{H}_0^1(\Omega) \,:\, \partial_i |g|^{1/2} g^{ij} \partial_j |g|^{-1/4} \psi \in L^2(\Omega)\right\}. \tag{3.13}$$

Commuting $|g|^{-1/4}$ with the gradient components in the expression for H, we obtain on $\mathcal{D}(H)$

$$H = -\partial_i\, g^{ij} \partial_j + V\,, \tag{3.14}$$

where

$$V := -\frac{5}{4}\,\frac{(h_{,1})^2}{h^4} + \frac{1}{2}\,\frac{h_{,11}}{h^3} - \frac{1}{4}\,\frac{\delta^{\mu\nu}\,h_{,\mu}\,h_{,\nu}}{h^2} + \frac{1}{2}\,\frac{\delta^{\mu\nu}\,h_{,\mu\nu}}{h}\,. \tag{3.15}$$

Actually, (3.14) with (3.15) is a general formula valid for any smooth metric of the form $g = \mathrm{diag}(h^2, 1, \ldots, 1)$. In our special case with h given by (3.8), we find that $h_{,\mu\nu} = 0$, $\delta^{\mu\nu}h_{,\mu}h_{,\nu} = \delta^{\alpha\beta}\mathcal{K}_\alpha^{\;1}\mathcal{K}_\beta^{\;1}$ by (3.4), while (3.3) gives

$$\begin{aligned} h_{,1}(\cdot,u) &= u^\mu\,\mathcal{R}_\mu^{\;\alpha}\big(\dot{\mathcal{K}}_\alpha^{\;1} - \mathcal{K}_\alpha^{\;\beta}\mathcal{K}_\beta^{\;1}\big)\,, \\ h_{,11}(\cdot,u) &= u^\mu\,\mathcal{R}_\mu^{\;\alpha}\big(\ddot{\mathcal{K}}_\alpha^{\;1} - \dot{\mathcal{K}}_\alpha^{\;\beta}\mathcal{K}_\beta^{\;1} - 2\mathcal{K}_\alpha^{\;\beta}\dot{\mathcal{K}}_\beta^{\;1} + \mathcal{K}_\alpha^{\;\beta}\mathcal{K}_\beta^{\;\gamma}\mathcal{K}_\gamma^{\;1}\big)\,. \end{aligned} \tag{3.16}$$

3.3 Results

It remains to impose decay conditions on the curvatures of p (and their derivatives) in order that the operator (3.14) satisfies Assumption 2.1 and Assumption 2.2.

Let us first consider the more general situation where the matrix (g^{ij}) is equal to $\mathrm{diag}(h^{-2}, 1, \ldots, 1)$ with the explicit dependence of h on s and u not specified. One shows that it is sufficient to impose the following hypotheses.

Assumption 3.3. *Uniformly for* $u \in \mathcal{B}_a$,

1. $h(s,u) \longrightarrow 1$ *as* $|s| \to \infty$,
2. $h_{,11}(s,u)$, $(\delta^{\mu\nu}\,h_{,\mu}h_{,\nu})(s,u)$, $\delta^{\mu\nu}\,h_{,\mu\nu}(s,u) \longrightarrow 0$ *as* $|s| \to \infty$,
3. $\exists\vartheta \in (0,1]$ *s.t.*

$$h_{,1}(s,u),\ h_{,111}(s,u),\ (\delta^{\mu\nu}\,h_{,\mu}h_{,\nu})_{,1}(s,u),\ \delta^{\mu\nu}\,h_{,1\mu\nu}(s,u) = \mathcal{O}\big(|s|^{-(1+\vartheta)}\big)\,.$$

Indeed, the first hypothesis supplies Assumption 2.1.2, while Assumption 2.1.1 is fulfilled due to basic Assumption 3.2. Next, since h is a smooth function, Assumption 3.3.2 together with the behaviour of $h_{,1}$ in Assumption 3.3.3 are sufficient to ensure both Assumption 2.2.1 and Assumption 2.2.2. It is also clear that the asymptotic behaviour of $h_{,1}$ in Assumption 3.3.3 supplies Assumption 2.1.3. It remains to check Assumption 2.2.3. This is easily done by calculating the derivative of the potential (3.15):

$$\begin{aligned} V_{,1} = {} & 5\,\frac{(h_{,1})^3}{h^5} - 4\,\frac{h_{,1}\,h_{,11}}{h^4} + \frac{h_{,111}}{2\,h^3} \\ & + \frac{\delta^{\mu\nu}}{2}\left[\frac{h_{,1}\,h_{,\mu}\,h_{,\nu}}{h^3} - \frac{h_{,1}\,h_{,\mu\nu} + h_{,1\mu}\,h_{,\nu}}{h^2} + \frac{h_{,1\mu\nu}}{h}\right]. \end{aligned}$$

With h given by (3.8), we find in addition to (3.16) that $h_{,1\mu\nu} = 0$ and

$$\begin{aligned} (\delta^{\mu\nu}h_{,\mu}h_{,\nu})_{,1} &= 2\,\delta^{\alpha\beta}\dot{\mathcal{K}}_\alpha^{\;1}\mathcal{K}_\beta^{\;1} \\ h_{,111}(\cdot,u) &= u^\mu\mathcal{R}_\mu^{\;\alpha}\big(\dddot{\mathcal{K}}_\alpha^{\;1} - \ddot{\mathcal{K}}_\alpha^{\;\beta}\mathcal{K}_\beta^{\;1} - 3\,\mathcal{K}_\alpha^{\;\beta}\ddot{\mathcal{K}}_\beta^{\;1} - 3\,\dot{\mathcal{K}}_\alpha^{\;\beta}\dot{\mathcal{K}}_\beta^{\;1} + \dot{\mathcal{K}}_\alpha^{\;\beta}\mathcal{K}_\beta^{\;\gamma}\mathcal{K}_\gamma^{\;1} \\ &\qquad + 2\,\mathcal{K}_\alpha^{\;\beta}\dot{\mathcal{K}}_\beta^{\;\gamma}\mathcal{K}_\gamma^{\;1} + 3\,\mathcal{K}_\alpha^{\;\beta}\mathcal{K}_\beta^{\;\gamma}\dot{\mathcal{K}}_\gamma^{\;1} - \mathcal{K}_\alpha^{\;\beta}\mathcal{K}_\beta^{\;\gamma}\mathcal{K}_\gamma^{\;\delta}\mathcal{K}_\delta^{\;1}\big)\,. \end{aligned}$$

Since $|u^\mu \mathcal{R}_\mu^{\ \alpha}| < a$, Assumption 3.3 holds true provided we impose the following conditions on the curvatures

Assumption 3.4.

1. $\forall \alpha \in \{2, \ldots, d\},\ \mathcal{K}_\alpha^{\ 1}(s),\ \ddot{\mathcal{K}}_\alpha^{\ 1}(s) \longrightarrow 0$ *as* $|s| \to \infty$,
2. $\forall \alpha, \beta \in \{2, \ldots, d\},\ \mathcal{K}_\alpha^{\ \beta},\ \dot{\mathcal{K}}_\alpha^{\ 2} \in L^\infty(\mathbb{R})$,
3. $\exists \vartheta \in (0, 1]$ *s.t.* $\forall \alpha \in \{2, \ldots, d\}$,

$$\dot{\mathcal{K}}_\alpha^{\ 1}(s),\ \dddot{\mathcal{K}}_\alpha^{\ 1}(s),\ \mathcal{K}_\alpha^{\ 2}(s),\ \ddot{\mathcal{K}}_\alpha^{\ 2}(s),\ \big(\dot{\mathcal{K}}_\alpha^{\ \beta}\mathcal{K}_\beta^{\ 2}\big)(s),\ \big(\mathcal{K}_\alpha^{\ \beta}\dot{\mathcal{K}}_\beta^{\ 2}\big)(s) = \mathcal{O}\big(|s|^{-(1+\vartheta)}\big)\,.$$

Remark 3.3. *These conditions reduce to those of* Theorem 1.1 *provided* $d = 2$. *When* $d = 3$, *it is sufficient to assume the conditions of* Theorem 1.1 *for the first curvature, and* $\dot{\kappa}_2 \in L^\infty(\mathbb{R})$ *and* $\kappa_2(s), \ddot{\kappa}_2(s) = \mathcal{O}\left(|s|^{-(1+\vartheta)}\right)$ *for some* $\vartheta \in (0, 1]$.

We conclude this section by applying Theorem 2.14 (with $\mathcal{T} = \{\nu_n\}_{n=1}^\infty$, the set of the eigenvalues of the Dirichlet Laplacian in the $(d-1)$-dimensional ball $\mathcal{B}_a$).

Theorem 3.4. *Let* Γ *be a tubular neighbourhood of radius* $a > 0$ *about a smooth infinite curve embedded in* $\mathbb{R}^d$ *with curvatures* κ_i. *Suppose* Assumptions 3.1, 3.2 *and* 3.4. *Then all the spectral results* (i)–(iv) *of* Theorem 2.14 *hold true for the Dirichlet Laplacian on* $L^2(\Gamma)$.

4 Curved strips on surfaces

In this final section, we investigate the situation where the ambient space is a general Riemannian manifold instead of the Euclidean space $\mathbb{R}^d$. We restrict ourselves to $d = 2$, *i.e.* Γ is a strip around an infinite curve in an (abstract) two-dimensional surface. We refer to [18] for basic spectral properties of $-\Delta_D^\Gamma$ and geometric details.

4.1 Preliminaries

Let $\mathcal{A}$ be a smooth connected complete non-compact two-dimensional Riemannian manifold of bounded Gauss curvature K. Let $p : \mathbb{R} \to \mathcal{A}$ be a smooth unit-speed curve embedded in $\mathcal{A}$ with (geodesic) curvature κ and denote by $n : \mathbb{R} \to T_{p(\cdot)}\mathcal{A}$ a smooth unit normal vector field along p. Given $a > 0$, we consider the straight strip $\Omega := \mathbb{R} \times (-a, a)$ and define a curved strip Γ of same width over p as a a-tubular neighbourhood of p in $\mathcal{A}$ by

$$\Gamma := \mathscr{L}(\Omega)\,, \qquad \text{where} \qquad \mathscr{L} : (s, u) \mapsto \exp_{p(s)}(u\, n(s))\,. \tag{4.1}$$

Note that $s \mapsto \mathscr{L}(s, u)$ traces the curves parallel to p at a fixed distance $|u|$, while the curve $u \mapsto \mathscr{L}(s, u)$ is a unit-speed geodesic orthogonal to p for any fixed s. We always assume

Assumption 4.1. $\mathscr{L} : \Omega \to \Gamma$ *is a diffeomorphism.*

Then $\mathscr{L}^{-1}$ determines a system of Fermi "coordinates" (s, u), *i.e.* the geodesic coordinates based on p. The metric tensor of Γ in these coordinates acquires [14, Sec. 2.4] the diagonal form

$$g(s,u) = \operatorname{diag}\left(h^2(s,u), 1\right), \tag{4.2}$$

where h is a smooth function satisfying the Jacobi equation

$$h_{,22} + K\,h = 0 \qquad \text{with} \qquad \begin{cases} h(\cdot, 0) = 1 \\ h_{,2}(\cdot, 0) = -\kappa\,. \end{cases} \tag{4.3}$$

Here K and κ are considered as functions of the Fermi coordinates (the sign of κ being uniquely determined up to the re-parameterisation $s \mapsto -s$ or the choice of n). The determinant of the metric tensor, $|g| := \det g = h^2$, defines through $\mathrm{d}v := h(s,u)\,\mathrm{d}s\,\mathrm{d}u$ the area element of the strip.

Assuming that the metric g is uniformly elliptic in the sense that

Assumption 4.2. $\exists c_\pm \in (0, \infty)$ *s.t.* $\forall (s,u) \in \Omega,\ c_- \leq h(s,u) \leq c_+$

holds true, the Dirichlet Laplacian corresponding to Γ can be defined in the same way as in Section 3.2, *i.e.* as the operator $\widetilde{H}$ associated with the form (3.9), satisfying (3.10). At the same time, we may introduce the unitarily equivalent operator H on $L^2(\Omega)$ given by (3.12) and satisfying (3.14) with (3.15).

4.2 Results

In view of the more general approach in the beginning of Section 3.3, we see that Assumption 3.3 (with $d = 2$) guarantees Assumptions 2.1 and 2.2 also in the present case. Applying Theorem 2.14, we obtain, with $\mathcal{T} = \{n^2\nu_1\}_{n=1}^\infty$ where $\nu_1 := \pi^2/(2a)^2$, the following result

Theorem 4.1. *Let* Γ *be a tubular neighbourhood of radius* $a > 0$ *about a smooth infinite curve, which is embedded in a smooth connected complete non-compact surface. Suppose* Assumptions 4.1*,* 4.2 *and* 3.3*. Then all the spectral results* (i)–(iv) *of* Theorem 2.14 *hold true for the Dirichlet Laplacian on* $L^2(\Gamma)$*.*

Assume now that the strip is *flat* in the sense of [18], *i.e.* the curvature K is equal to zero everywhere on Γ. Then the Jacobi equation (4.3) has the explicit solution (*cf.* (3.8) for $d = 2$)

$$h(s,u) = 1 - \kappa(s)\,u \tag{4.4}$$

and Assumption 3.3 can be replaced by some conditions on the decay of the curvature κ at infinity, namely, we adopt Assumption 3.4 with $\kappa_1 \equiv \kappa$ and $\mathcal{K}_\mu^{\ \nu} = 0$ (*cf.* the assumptions of Theorem 1.1). At the same time, it easy to see that Assumption 4.1 and 4.2 are satisfied if Assumption 3.2 holds true.

Theorem 4.2 (Flat strips). *Let Γ be a tubular neighbourhood of radius $a > 0$ about a smooth infinite curve of curvature κ, which is embedded in a smooth connected complete non-compact surface of curvature K such that $K \restriction \Gamma = 0$. Suppose* Assumption 3.2 *and*

1. $\kappa(s)$, $\ddot{\kappa}(s) \longrightarrow 0$ *as* $|s| \to \infty$,
2. $\exists \vartheta \in (0,1]$ *s.t.* $\dot{\kappa}(s)$, $\dddot{\kappa}(s) = \mathcal{O}\left(|s|^{-(1+\vartheta)}\right)$.

Then all the spectral results (i)–(iv) *of* Theorem 2.14 *hold true for the Dirichlet Laplacian on* $L^2(\Gamma)$.

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