Variational Principle for Non-Equilibrium Steady States of the XX Model

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We show that non-equilibrium steady states of the one dimensional exactly solved XY model can be characterized by the variational principle of free energy of a long range interaction and that they cannot be a KMS state for any C*-dynamical system.

KEY WORDS: ; variational principle; non-equilibrium steady state; the XX model

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1 Introduction

In the research of statistical mechanics on equilibrium states, one requirement of the theory is to reproduce the thermodynamics. As a macroscopic phenomenological theory, the thermodynamics has macroscopic functions which are transformed among them by Legendre transformation. The first task for statistical mechanics was to introduce various thermodynamical functions from microscopic description and to reproduce their relation of corresponding to thermodynamics . For finite systems, the Gibbs formalism with the distribution $\mathrm{e}^{-\beta H}$ embodies this task. For in infinite quantum systems, the Gibbs formalism was generalized to the KMS state. The main tool of the investigation in this direction is the C^*-algebraic method. (c.f. [6] and [7].) In the C^*-algebraic formalism, various justification of the KMS state as the equilibrium state has been done. For quantum spin systems, the equivalence of variational principle, Gibbs condition, and the KMS condition was proved under some additional conditions on the decay of the potential(interaction). [16],[13],[2]. Recently, H.Araki and H.Moriya established the equivalence of these conditions for a conceivably widest class of potentials and they extended their results for Fermion lattice systems in [4]. The Gibbs condition requires that the product property of the state in question is obtained by bounded perturbation. This gets along with the spilt of the extensivity of normal thermodynamics. On the other hand, the variational principle is the thermodynamical statement. Other justification are the equivalence with the passivity [19], which reveals the second low, structural stability , and dynamical stability of KMS state under bounded perturbation, [9]. Hence the KMS condition is a reasonable way to characterize the extensive equilibrium state.

An attractive attempt is to characterize non-equilibrium steady states by a new type of thermodynamics. So far , the characterization of non-equilibrium steady states was not established for general infinite systems. In the early stage of research of non-equilibrium steady states, D.N.Zubarev proposed his interpretation of the non-equilibrium steady state as an equilibrium state of some non-equilibrium effective Hamiltonian. See [21]. Let us carry out a very heuristic argument for a while. Suppose that an initial state is given by the density matrix $\rho = \mathrm{e}^{-\beta H_0}$ and that the state evolves under a Hamiltonian H. Then the state at a time t is described by the density matrix $\rho(t) = \mathrm{e}^{-\beta H(t)}$, with $H(t) = \mathrm{e}^{itH} H_0 \mathrm{e}^{-itH}$. We may regard this state as a state in equilibrium for the Hamiltonian $H(t)$. Thus the non-equilibrium steady state is an equilibrium state for the Hamiltonian $H(\infty)$. That this effective Hamiltonian $H(\infty)$ make sense is a subtle question, which D.N.Zubarev left untouched. In particular, for an infinite system, whether a state can be interpreted as an equilibrium state (i.e., KMS state) for some dynamics is a highly non-trivial matter. In this paper,we investigate this question of characterization of non-equilibrium steady states for a concrete model, the exactly solved XX-model.

After publication of papers by D.Ruelle [17], [18] on non-equilibrium steady states, results on concrete models are obtained on entropy production by V.Jaksic and A.C.Pillet in [11] and [12] and on the three or higher dimensional free Bose

gas by J.Froehlich ,M.Merkli, and D.Ueltschi in [8] . In a jointwork with S.Tasaki ,one of the author of this article (T.M.) considered relationship between the fluctuation of entropy production and the time reversal symmetry as well as some basic property of non-equilibrium steady states of L_1 asymptotic abelian systems [20]

T.G.Ho and H.Araki derived explicitly the non-equilibrium steady state of XX-model [10]. The non-equilibrium steady states T.G.Ho and H.Araki considered are in the same class of invariant states proposed by D.Ruelle in [17]. (See also [18].) W.H. Aschbacher C.-A. Pillet proved the strict positivity of the entropy production and independence of choice of the initial states [5]. Note that ,for the XX-model , the dynamics does not satisfy L_1 asymptotic abelian property and a part of the results of. [17] cannot be applied.

Formally the Jordan-Wigner transformation transforms the XX model to the free Fermion. We consider the XX-model and the corresponding Fermion system bu use of C^*-algebraic methods. We will see that the non-equilibrium steady state of XX-model can be not a KMS state of any one-parameter group of automorphisms of the C^*-algebra of local observables. Our main result is that the non-equilibrium steady state of XX-model is the unique solution of variational principle for a translation invariant long range potential. On the other hand, for the corresponding Fermion system is a KMS state for the quasi free motion and satisfies the Gibbs condition and variational equality for the corresponding potential [15]. This shows that the heuristic equivalence of Pauli spin and Fermion systems breaks down in our context. In Section 2, we recall the Jordan Wigner transformation, and the non-equilibrium steady state is introduced in Section 3. In Section 4, we show that the non-equilibrium steady state of XX-model can not be a KMS state of any C^*-dynamical system. In Section 5, we consider corresponding the Fermi system, and, we show the variational equality of non equilibrium steady state of XX-system.

2 The model

Following H.Araki ([3]), we introduce the Jordan-Wigner transformation between the one-dimensional lattice spin system $\mathfrak{A}_S$ and the one-dimensional lattice Fermion system $\mathfrak{A}_F$. $\mathfrak{A}_S$ is generated by Pauli matrices $\sigma_x^n, \sigma_y^n, \sigma_z^n$ located on the lattice site $n \in \mathbb{Z}$ and $\mathfrak{A}_F$ is generated by Fermion creation annihilation operators a_n^*, a_n $n \in \mathbb{Z}$ satisfying the canonical anti-commutation relations. $\mathfrak{A}_F$ is referred to as the CAR algebra . Both algebras $\mathfrak{A}_S$ and $\mathfrak{A}_F$ are realized as the C^*-subalgebras of a lager algebra $\hat{\mathfrak{A}}$ defined as follows. $\hat{\mathfrak{A}}$ is the crossed product of $\mathfrak{A}_F$ by a $\mathbb{Z}_2$-action Θ_- where the $*-$automorphism Θ_- on $\mathfrak{A}_F$ is determined by the following equations:

$$\Theta_-(a_n) = \begin{cases} a_n & n \geq 1 \\ -a_n & n \leq 0. \end{cases}$$

It is the C^*-algebra generated by $\mathfrak{A}_F$ and T which satisfies

$$\begin{aligned} &T = T^*, \\ &T^2 = 1, \\ &TAT = \Theta_-(A), \quad A \in \mathfrak{A}_F. \end{aligned}$$

The spin algebra $\mathfrak{A}_S$ is the C^*-subalgebra defined by

$$\begin{aligned} \sigma_z^n &= 2a_n^* a_n - 1 \\ \sigma_x^n &= TS^n(a_n + a_n^*) \\ \sigma_x^n &= iTS^n(a_n - a_n^*). \end{aligned}$$

where

$$S^n = \begin{cases} \sigma_z^1 \cdots \sigma_z^{n-1} & n > 1 \\ 1 & n = 1 \\ \sigma_z^0 \cdots \sigma_z^n & n < 1 \end{cases}.$$

Next let us define an automorphism Θ on $\hat{\mathfrak{A}}$ as

$$\begin{aligned} &\Theta(a_n) = -a_n \quad \forall n \in \mathbb{Z} \\ &\Theta(T) = T. \end{aligned}$$

Let $\mathfrak{A}_+$ (resp.$\mathfrak{A}_-$) be an even (odd) part of $\hat{\mathfrak{A}}_S$ i.e.,

$$\mathfrak{A}_+ \equiv \{A \in \mathfrak{A}_S; \Theta(A) = A\}, \quad \mathfrak{A}_- \equiv \{A \in \mathfrak{A}_S; \Theta(A) = -A\}.$$

$\hat{\mathfrak{A}}$ is decomposed into four subspaces as

$$\begin{aligned} \hat{\mathfrak{A}} &= \mathfrak{A}_+ + \mathfrak{A}_- + \mathfrak{A}_{+T} + \mathfrak{A}_{-T} \\ \mathfrak{A}_{+T} &= \mathfrak{A}_+ T, \quad \mathfrak{A}_{-T} = \mathfrak{A}_- T, \end{aligned}$$

and

$$\begin{aligned} \mathfrak{A}_S &= \mathfrak{A}_+ + \mathfrak{A}_- \\ \mathfrak{A}_F &= \mathfrak{A}_+ + \mathfrak{A}_{-T}. \end{aligned}$$

In this paper, we investigate the non-equilibrium steady state of the XX-model and the corresponding Fermion system. The Hamiltonian of the XX-model has the following form:

$$\mathcal{H} \quad = \quad \frac{1}{4} \sum_{n=-\infty}^{\infty} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y \right) + \frac{\gamma}{2} \sum_{n=-\infty}^{\infty} \sigma_n^z$$

where the real parameter γ is an external magnetic field. The above equation means the generator of the dynamics τ_t^S is determined by the potential $\Phi_S^0(I)$ via the following equation:

$$\frac{d}{dt} \tau_t^S|_{t=0}(A) = \sum_{J \cap I \neq \varnothing} i[\Phi_S^0(J), A] \quad A \in \mathfrak{A}_S(I),$$

for each interval I. Here $\Phi_S^0(I)$ is given by

$$\Phi_S^0(\{n\}) = \frac{\gamma}{2}\sum_{n=-\infty}^{\infty}\sigma_n^z$$
$$\Phi_S^0(\{n,n+1\}) = \frac{1}{4}\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y$$
$$\Phi_S^0(I) = 0 \quad \text{others.}$$

As each $\Phi_S^0(I) \in \mathfrak{A}_S \cap \mathfrak{A}_F$, it can be expressed by the Fermion operators:

$$\Phi(\{n\}) = \gamma\left(a_n^* a_n - \frac{1}{2}\right)$$
$$\Phi(\{n,n+1\}) = -\frac{1}{2}\left[a_{n+1}^* a_n + a_n^* a_{n+1}\right]$$
$$\Phi(I) = 0 \quad \text{others.}$$

The corresponding Hamiltonian is

$$\mathcal{H} = -\frac{1}{2}\sum_{n=-\infty}^{\infty}\left[a_{n+1}^* a_n + a_n^* a_{n+1}\right] + \frac{\gamma}{2}\sum_{n=-\infty}^{\infty}(2a_n^* a_n - 1)$$

where a_n and a_n^* are the fermionic annihilation and creation operators of the nth site. The automorphism τ_t^S can be extended to the whole $\hat{\mathfrak{A}}$ [3] and the extension τ_t gives rise to the quasi free motion of Fermions:

$$\tau_t\left(a^*(f)\right) = a^*\left(\mathrm{e}^{itb}f\right), \tag{1}$$

where

$$(bf)(n) = -\frac{1}{2}(f(n-1) + f(n+1)) + \gamma f(n).$$

3 Non-Equilibrium Steady State

The non-equilibrium steady state we consider here is obtained as the limit state starting from inhomogeneous initial state. In our system, the spin chain is initially separated from the left and the right infinite systems(reservoirs) with different temperature and the evolution of states is governed by the XX Hamiltonian as the time goes to infinite. The existence of the limit state itself is not an obvious matter . However, as the set of states of $\mathcal{O}$ of $\mathfrak{A}_S$ is weak $*$ -compact by Alaoglu's Theorem, the sets $\omega \circ \alpha_t$ $t \in \mathbb{R}$ has the accumulation points which are not necessary to be invariant under the dynamics α_t. To make the invariant state, we take the long time average $\omega \circ \alpha_t$ as follows;

$$\omega^T = \frac{1}{T}\int_0^T \omega \circ \alpha_t dt$$

Again by Alaoglu's Theorem, ω^T has the accumulation points which, this time, are invariant under α_t, i.e., there exists sequence T_n which converges as

$$\omega(A) = \lim_n \omega^{T_n}(A) \quad \forall A \in \mathcal{O}.$$

For the XX model, (1), we do not have to carry out such time average in order to construct the steady state. T.G.Ho and H.Araki derived the non-equilibrium steady state of the XX-chain explicitly in [10];

$$\omega|_{\mathfrak{A}_+} = \omega_\rho|_{\mathfrak{A}_+}, \quad \omega|_{\mathfrak{A}_-} = 0. \tag{2}$$

Here ω_ρ is the quasifree state on Fermion algebra $\mathfrak{A}_F$ defined by

$$\omega_\rho(a(f_n)^* \cdots a(f_1)^* a(g_1) \cdots a(g_m)) = \delta_{nm} \det(\langle f_i, \rho g_j \rangle),$$

with ρ is multiplication operator in the Fourier representation;

$$\rho_r(k) = \begin{cases} \left(1 + e^{-\beta_+ \epsilon(k)}\right)^{-1} & k \in [0, \pi) \\ \left(1 + e^{-\beta_- \epsilon(k)}\right)^{-1} & k \in [-\pi, 0). \end{cases}$$
$$\epsilon(k) = \cos(k) - \gamma$$

Let α_t^F be a one parameter group of automorphism of $\mathfrak{A}_F$ such that

$$\alpha_t^F(a(f)) = a(\mathrm{e}^{ith} f) \tag{3}$$

with

$$(\widehat{hf})(k) = \begin{cases} -\dfrac{2\beta_+}{\beta_+ + \beta_-}(\cos(k) - \gamma)\hat{f}(k) \\ -\dfrac{2\beta_-}{\beta_+ + \beta_-}(\cos(k) - \gamma)\hat{f}(k) \end{cases}. \tag{4}$$

It is easily verified that ω_ρ is the (α, β)-KMS state of $\mathfrak{A}_F$, with $\beta = (\beta_+ + \beta_-)/2$. As α_F is a quasifree motion, ω_ρ is the unique (α_F, β)-KMS state. (c.f. [1])

In spite of this fact, the state ω on $\mathfrak{A}_S$, cannot be a KMS state of any one-parameter group of automorphisms on $\mathfrak{A}_S$. In the next section, we will show this impossibility.

4 NESS and the KMS condition

As we claimed in the previous section, we show that the state ω of (2) cannot be a KMS state of any one-parameter group of automorphisms on $\mathfrak{A}_S$ even though it is a faithful factor state. We denote the GNS triple associated with the state φ on $\mathfrak{A}$ by $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$. The natural extension $\hat{\varphi}$ of φ to $\pi_\varphi(\mathfrak{A})''$ is defined by

$$\hat{\varphi}(x) = \langle \Omega_\varphi, x\Omega_\varphi \rangle, \quad x \in \pi_\varphi(\mathfrak{A})''.$$

Lemma 4.1 *Let $\hat{\omega}$ be the natural extension of the non-equilibrium steady state ω of the XX model to the von Neumann algebra $\pi_\omega(\mathfrak{A}_S)''$. Then $\hat{\omega}$ is faithful for $\pi_\omega(\mathfrak{A}_S)''$.*

proof
We can extend ω to the state ω' on $\hat{\mathfrak{A}}$, by $\omega'|_{\mathfrak{A}_+T} = \omega'|_{\mathfrak{A}_-T} = 0$. It is sufficient to show that $\hat{\omega}'$ is faithful for $\pi_{\hat{\omega}'}(\hat{\mathfrak{A}})''$. The Hilbert space $\mathcal{H}_{\omega'}$ is orthogonally decomposed as

$$\mathcal{H}_{\omega'} = \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \mathcal{H}_{+T} \oplus \mathcal{H}_{-T},$$

where

$$\mathcal{H}_+ = \overline{\pi_{\omega'}(\mathfrak{A}_+)\Omega}, \quad \mathcal{H}_- = \overline{\pi_{\omega'}(\mathfrak{A}_-)\Omega}, \quad \mathcal{H}_{+T} = \overline{\pi_{\omega'}(\mathfrak{A}_{+T})\Omega}, \quad \mathcal{H}_{-T} = \overline{\pi_{\omega'}(\mathfrak{A}_{-T})\Omega}.$$

First note that due to strict positivity of ρ, $\omega_F = \omega'|_{\mathfrak{A}_F}$ is faithful for $\pi_{\omega_F}(\mathfrak{A}_F)''$. That is, $\Omega_{\omega'}$ is separating for $\pi_{\omega'}(\mathfrak{A}_F)''$. Next, let $x \in \pi_{\omega'}(\mathfrak{A}_F T)''$. If $x\Omega_{\omega'} = 0$, then

$$\pi_{\omega'}(T)(x\pi_{\omega'}(T))\pi_{\omega'}(T)\Omega_{\omega'} = 0.$$

As $\pi_{\omega'}(T)(x\pi_{\omega'}(T))\pi_{\omega'}(T) \in \pi_{\omega'}(\mathfrak{A}_F)''$, we get $\pi_{\omega'}(T)(x\pi_{\omega'}(T))\pi_{\omega'}(T) = 0$. Then by invertibility of T, $x = 0$. That is, $\Omega_{\omega'}$ is separating for $\pi_{\omega'}(\mathfrak{A}_F T)''$. All $x \in \pi_{\omega'}(\hat{\mathfrak{A}})''$ is decomposed as

$$x = x_1 + x_2, \quad x_1 \in \pi_{\omega'}(\mathfrak{A}_F)'', \quad x_2 \in \pi_{\omega'}(\mathfrak{A}_F T)''.$$

If $x\Omega_{\omega'} = 0$, then $x_1\Omega_{\omega'} + x_2\Omega_{\omega'} = 0$. As $x_1\Omega_{\omega'} \in \mathcal{H}_+ \oplus \mathcal{H}_{-T}$ and $x_2\Omega_{\omega'} \in \mathcal{H}_- \oplus \mathcal{H}_{+T}$ are orthogonal, we get $x_1\Omega_{\omega'} = x_2\Omega_{\omega'} = 0$. So by the above arguments, we obtain $x_1 = x_2 = 0$, i.e., $x = 0$. That is, $\hat{\omega}'$ is faithful for $\pi_{\hat{\omega}'}(\hat{\mathfrak{A}})''$. □
Now suppose that there exists a one-parameter group of automorphisms α^S of $\mathfrak{A}_S$ such that ω is (α^S, β)-KMS state.

Lemma 4.2 *Suppose that there exists a one-parameter group of automorphisms of $\mathfrak{A}_S$ α^S such that ω is (α^S, β)-KMS state. Then, $\alpha_t^S \circ \Theta = \Theta \circ \alpha_t^S$, and $\alpha_t^S|_{\mathfrak{A}_+}$ is given by*

$$\alpha_t^S(a^*(f)a(g)) = a^*(u_t f)a(u_t g).$$

proof
By the definition of ω, $\omega \circ \Theta = \omega$. As ω is α^S-KMS state, $\omega \circ \alpha^S = \omega$. Hence Θ and α^S are both extendible to the von Neumann algebra $\pi_{\hat{\omega}}(\mathfrak{A}^S)''$, which we denote by the same symbol and ω is $(\hat{\alpha}^S, \beta)$-KMS state on $\pi_{\hat{\omega}}(\mathfrak{A}^S)''$. By Lemma 4.1, ω is faithful for $\pi_{\hat{\omega}}(\mathfrak{A}^S)''$. By the uniqueness of automorphism which makes a faithful state on von Neumann algebra be a KMS state, [7] we obtain $\alpha_t \circ \Theta = \Theta \circ \alpha_t$. Hence $\alpha|_{\mathfrak{A}_+}$ gives an automorphism of $\mathfrak{A}_+$, and by restriction, $\omega|_{\mathfrak{A}_+}$ is $\alpha - KMS$ state of $\mathfrak{A}$. On the other hands, ω satisfies the KMS condition with the automorphism given by $\alpha'(a^*(f)a(g)) = a^*(u_t f)a(u_t g)$. As $\omega|_{\mathfrak{A}_+}$ is faithful, we again obtain the uniqueness $\alpha^S|_{\mathfrak{A}_+} = \alpha'$. □

Lemma 4.3 *Let V_t be*

$$V_t \equiv \alpha_t^S(B_2 T) \cdot \alpha_t^F(B_2) T$$

with $B_2 = a_2^* + a_2$. Then we have

$$V_t Q V_t^* = \alpha_t^S \circ \Gamma \circ \Theta_- \circ \Gamma \circ \alpha_t^{S-1} \circ \Theta_-(Q), \quad \forall Q \in \mathfrak{A}_+.$$

where $\Gamma(Q) = B_2 Q B_2^{-1}$.

proof

$$V_t \Theta_-(\alpha_t^S(Q)) = \alpha_t^S(B_2 T)\alpha_t^F(B_2)\alpha_t^F(Q)T = \alpha_t^S(B_2 T)\alpha_t^F(\Gamma(Q))\alpha_t^F(B_2)T$$
$$= \alpha_t^S(B_2 T)\alpha_t^S(\Gamma(Q))\alpha_t^S(T B_2^{-1})V_t = \alpha_t^S(B_2 T \Gamma(Q) T B_2^{-1})V_t = \alpha_t^S \circ \Gamma \circ \Theta_- \circ \Gamma(Q) V_t . \square$$

Note that $V_t \in \mathfrak{A}_+$ Let us consider an arbitrary gauge invariant quasi-free state φ_q which is given by two point function

$$\varphi_q(a^*(f)a(g)) = \langle f, qg \rangle.$$

For any A in $\mathfrak{A}_- T \subset \mathfrak{A}_F$, $V_t A V_t^*$ is an element in $\mathfrak{A}_- T$, and we obtain $\varphi_q(V_t A V_t^*) = 0$. On the other hand, by Lemma 4.3, we obtain for $A \in \mathfrak{A}_+$

$$\varphi_q(V_t A V_t^*) = \varphi_q(\alpha_t^S \circ \Gamma \circ \Theta_- \circ \Gamma \circ \alpha_t^{S-1} \circ \Theta_-(A)).$$

Then we conclude that $\varphi_q(V_t \cdot V_t^*)$ is quasifree, which is given by the two point function

$$\varphi_q(V_t a^*(f) a(g) V_t^*) = \langle w_t f, q w_t g \rangle.$$

where $S = P_+ - P_-$, $w_t = u_t S u_{-t} S$. It is generally known that an automorphism which maps any quasifree state to a quasifree state is a Bogoliubov automorphism. It turns out that there exists a one parameter family of unitary operators v_t such that

$$V_t B(f) V_t^* = B(v_t f).$$

This equation suggests that v_t is continuous in t as the left hand side is continuous. By the equality

$$\langle v_t f, q v_t g \rangle = \varphi_q(a^*(v_t f) a(v_t g)) = \varphi_q(V_t a^*(f) a(g) V_t^*) = \langle w_t f, q w_t g \rangle,$$

we obtain

$$v_t w_t^* q = q v_t w_t^*,$$

for all $q \in B(l^2)$. As $v_t w_t^*$ is unitary, we obtain a continuous complex function $f(t)$ such that

$$v_t w_t^* = f(t) \cdot 1.$$

By the equation

$$a(w_t f)a(w_t g) = V_t a(f)a(g)V_t^* = a(v_t f)a(v_t g) = f(t)^2 a(w_t f)a(w_t g),$$

we obtain $f(t) = \pm 1$. But due to continuity of $f(t)$, we have $f(t) = 1$. Hence we obtain

$$V_t B(f) V_t^* = B(w_t f), \quad w_t = u_t S u_{-t} S.$$

The gauge invariant Bogolibov automorphism associated with w_t is inner if and only if $1 - w_t$ is of traceclass. (c.f. [1]) In the following, we show this is not the case. If $1 - w_t$ is trace class, it is Hilbert Schmidt class i.e.,

$$Tr((1 - u_t S u_{-t} S)^*(1 - u_t S u_{-t} S)) = \sum_n \langle n|(1 - u_t S u_{-t} S)^*(1 - u_t S u_{-t} S)|n\rangle < \infty.$$

As

$$(1 - u_t S u_{-t} S)^*(1 - u_t S u_{-t} S) = 2P_- u_t P_+ u_{-t} + 2P_+ u_t P_- u_{-t} + 2u_t P_+ u_{-t} P_- + 2u_t P_- u_{-t} P_+,$$

for all $N > 0$, we have

$$0 \leq \sum_{1 \leq n \leq N, -N \leq m \leq 0} 8|\langle m|u_t|n\rangle|^2 \leq Tr((1 - u_t S u_{-t} S)^*(1 - u_t S u_{-t} S)) < \infty.$$

That is, $\sum_{1 \leq n \leq N, -N \leq m \leq 0} |\langle m|u_t|n\rangle|^2$ is a upper bounded monotonically increasing sequence. So it has a limit

$$\sum_{m \leq 0, 1 \leq n} |\langle m|u_t|n\rangle|^2 < \infty.$$

Let us consider $|\langle m|u_t|n\rangle|$. By partial integral, we obtain

$$\begin{aligned}\langle m|u_t|n\rangle = &\frac{1}{2\pi i(m-n)}(\mathrm{e}^{i\beta_+(1-\gamma)t} - \mathrm{e}^{i\beta_+(-1-\gamma)t} \cdot (-1)^{m-n}) - f_{mn}^+ \\ &+ \frac{1}{2\pi i(m-n)}(-\mathrm{e}^{i\beta_-(1-\gamma)t} + \mathrm{e}^{i\beta_-(-1-\gamma)t} \cdot (-1)^{m-n}) - f_{mn}^-,\end{aligned}$$

where

$$\begin{aligned}f_{mn}^+ &= \frac{\beta_+ t}{2\pi(m-n)} \int_0^\pi dk \sin k \mathrm{e}^{i\beta_+ \epsilon(k)t} \cdot \mathrm{e}^{-ik(m-n)} \\ f_{mn}^- &= \frac{\beta_- t}{2\pi(m-n)} \int_{-\pi}^0 dk \sin k \mathrm{e}^{i\beta_- \epsilon(k)t} \cdot \mathrm{e}^{-ik(m-n)}.\end{aligned}$$

Again using partial integral, we obtain a bound

$$|f_{mn}^+| \leq \frac{c}{(n-m)^2}, \quad |f_{mn}^-| \leq \frac{c}{(n-m)^2}$$

for some constant c. Hence we obtain

$$\left| |\langle m|u_t|n\rangle|^2 - B_{mn} \right| \leq \frac{8c}{\pi(n-m)^3} + \frac{4c^2}{(n-m)^4},$$

where

$$B_{mn} \equiv \frac{1}{4\pi^2(n-m)^2} \left| \mathrm{e}^{i\beta_+(1-\gamma)t} - \mathrm{e}^{i\beta_+(-1-\gamma)t} \cdot (-1)^{m-n} - \mathrm{e}^{i\beta_-(1-\gamma)t} + \mathrm{e}^{i\beta_-(-1-\gamma)t} \cdot (-1)^{m-n} \right|^2.$$

It is summable

$$\sum_{m\leq 0, 1\leq n} \left| |\langle m|u_t|n\rangle|^2 - B_{mn} \right| \leq \sum_{m\leq 0, 1\leq n} \frac{8c}{\pi(n-m)^3} + \frac{4c^2}{(n-m)^4} < \infty.$$

As B_{mn} is

$$B_{mn} \equiv \frac{1}{\pi^2(n-m)^2} \left[1 - \cos(\gamma t(\beta_+ - \beta_-)) \cdot \cos(t(\beta_+ - \beta_-)) \right.$$
$$\left. +2(-1)^{m-n} \cos(t(\beta_+ + \beta_-)) \cdot \sin\left(\frac{t}{2}(\beta_+ - \beta_-)(1-\gamma)\right) \cdot \sin\left(\frac{t}{2}(\beta_+ - \beta_-)(1+\gamma)\right) \right],$$

in case $\beta_+ \neq \beta_-$, we can take t so that there exists $d > 0$ such that

$$B_{mn} > \frac{d}{(n-m)^2} > 0,$$

for all $m \leq 0$ and $n \leq 1$. Hence we have

$$\sum_{m\leq 0, n\geq 1} B_{mn} > \sum_{m\leq 0, n\geq 1} \frac{d}{(n-m)^2} = +\infty.$$

Then we obtain

$$\sum_{m\leq 0, 1\leq n} B_{mn} - |\langle m|u_t|n\rangle|^2 + \sum_{m\leq 0, 1\leq n} |\langle m|u_t|n\rangle|^2 = \sum_{m\leq 0, 1\leq n} B_{mn} = +\infty.$$

The left handside is finite by the previous arguments, while the right handside is infinite. This is a contradiction and we conclude that $1 - u_t S u_{-t} S$ is not trace class.

As a consequence, we obtain the following Theorem,

Theorem 4.1 *The non-equilibrium steady state ω of the XX model ω cannot be a KMS state of any one-parameter group automorphism of the C^*-algebra $\mathfrak{A}_S$.*

5 The Gibbs Condition and The variational principle

In this section, we prove that the NESS ω_ρ of the Fermion system satisfies the Gibbs condition and the variational principle for a long range interaction. As a result the NESS ω of XX-model satisfies variational equality. We apply the equivalence of the KMS-condition, the Gibbs condition, and the variational principle, for long range interactions which is recently established by H.Araki and H.Moriya in [4]. We collect here the essence of their results which we will use in our proof later.

Before stating the details of the conditions, let us recall the notion of standard potentials of [4]. A potential Φ is a map defined on the set of finite subsets I of $\mathbb{Z}$ with values in the set of selfadjoint elements of $\mathcal{A}$. We define the translation invariant standard potential as follows.

Definition 5.1 (Translation Invariant Standard Potential) *The translation invariant standard potential $\mathcal{P}_\tau$ is defined as the set of potentials which satisfies the following conditions;*

1. $\Phi(I) \in A(I), \Phi(\varnothing) = 0$,
2. $\Phi(I)^* = \Phi(I)$,
3. $\Theta(\Phi(I)) = \Phi(I)$,
4. $E_J(\Phi(I)) = 0$, *if* $J \subset I$ *and* $J \neq I$,
5. *For each finite* $I \subset \mathbb{Z}$, *the limit*

$$\lim_{J \nearrow \mathbb{Z}} \sum_K \{\Phi(K); K \cap I \neq \varnothing, K \subset J\}$$

   exists in $\mathfrak{A}_F$,

6. $\tau_k \Phi(I) = \Phi(I + k)$.

E_J is the conditional expectation from $\mathfrak{A}_F$ to $\mathfrak{A}_F(I)$ which is uniquely determined by the condition $E_J(a) \in A(J)$ and

$$\tau(ab) = \tau(E_J(a)b)$$

for all $a \in \mathfrak{A}_F$ and $b \in A(J)$. Here, τ is the unique trace state of $\mathfrak{A}_F$. τ_k is k-translation automorphism. We define the internal energy $U(I)$ and the surface energy $W(I)$ as

$$\begin{aligned} H(I) &= \sum_K \{\Phi(K); K \cap I \neq \varnothing\} \\ U(I) &= \sum_{K \subset I} \Phi(K) \\ W(I) &= \sum_K \{\Phi(K); K \cap I \neq \varnothing, K \cap I^c \neq \varnothing\}. \end{aligned}$$

The convergence of the summation in the definition of $W(I)$ and $H(I)$ is guaranteed by the fifth condition .

Now let us turn to the Gibbs condition. To define the Gibbs condition we need some preparations. Let $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ be the GNS triple of φ. Suppose that φ is a faithful state for $\pi_\varphi(\mathfrak{A}_F)''$. Then, Ω_φ is a cyclic and separating vector for $\pi_\varphi(\mathfrak{A}_F)''$ and modular operator Δ_φ corresponding to Ω_φ is well defined. We denote the modular automorphism as σ_t^φ;

$$\sigma_t^\varphi(x) = \Delta_\varphi^{it} x \Delta_\varphi^{-it}, \quad x \in \pi_\varphi(\mathfrak{A}_F)''.$$

If φ is a α-KMS state of $\mathfrak{A}_F$, we have the relation

$$\sigma_t^\varphi(\pi_\varphi(A)) = \pi_\varphi(\alpha_t(A)), \quad A \in \mathfrak{A}_F.$$

Now we define the bounded perturbation of the dynamics α_t, σ_t^φ and the state φ. Let $(\tau_t, \mathcal{O})$ be a C^*-dynamical system or W^*-dynamical system, and $v \in \mathcal{O}$ be a bounded operator. There exists the perturbed dynamics τ_t^v of τ_t with generator

$$\delta^v(A) \equiv \delta(A) + i[v, A], \quad A \in D(\delta),$$

where δ is the generator of τ_t. The perturbed dynamics α_t^v and $(\sigma_t^\varphi)^{\pi_\varphi(v)}$ are in relation that

$$(\sigma_t^\varphi)^{\pi_\varphi(v)}(\pi_\varphi(A)) = \pi_\varphi(\alpha_t^v(A)), \quad A \in \mathfrak{A}_F.$$

The bounded perturbation of Ω_φ by v is defined as

$$\Omega_\varphi^v \equiv \sum_{m=0}^{\infty} \int_0^{\frac{1}{2}} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_m \Delta_\varphi^{t_m} \pi_\varphi(v) \Delta_\varphi^{t_{m-1}-t_m} \pi_\varphi(v) \cdots \Delta_\varphi^{t_1-t_2} \pi_\varphi(v) \Omega.$$

The perturbed state φ^v is defined by

$$\varphi^v(A) = \frac{\langle \Omega_\varphi^v, \pi_\varphi(A)\Omega_\varphi^v \rangle}{\langle \Omega_\varphi^v, \Omega_\varphi^v \rangle}.$$

The sum converges absolutely. It is known that Ω_φ^v is cyclic and separating for $\pi_\varphi(\mathfrak{A}_F)''$. Its modular automorphism group $\sigma_t^{\varphi^v}$ coincides with $(\sigma_t^\varphi)^{\pi_\varphi(v)}$. Let $\Phi : I \subset \mathbb{Z} \to \Phi(I) \in \mathfrak{A}_F$ be an interaction which satisfies

$$\delta_\alpha(A) = \sum_{J \cap I \neq \varnothing} i[\Phi(J), A], \quad A \in \mathfrak{A}_F(I)$$

Suppose that $W(I)$ converges in the norm topology of $\mathfrak{A}_F$ for any finite I. In [4] H.Araki and H.Moriya introduced the Gibbs condition of states of quantum lattice models in a different manner from the standard definition [7].The definition of H.Araki and H.Moriya is convenient for study of Fermion lattice models as follows;

Definition 5.2 *Suppose $\mathfrak{A}_{F0} \subset D(\delta)$, where $\mathfrak{A}_{F0}$ is the union of finite local subalgebras. Let $W(\beta, I) \equiv \pi_\varphi(\beta W(I))$. A state φ satisfies the (δ, β)-Gibbs condition if φ is faithful and $\sigma_t^{\varphi^{W(\beta,I)}}$ satisfies*

$$\sigma_t^{\varphi^{W(\beta,I)}}(\pi_\varphi(A)) = \pi_\varphi\left(\mathrm{e}^{-i\beta U(I)t} A \mathrm{e}^{i\beta U(I)t}\right), \quad A \in \mathfrak{A}_F(I),$$

for any finite $I \subset \mathbb{Z}$.

The above definition is equivalent to the following product property, which is chosen as the definition of the Gibbs state in quantum spin systems.

Proposition 5.1 *Assume that α_t is even, i.e., $\alpha_t \circ \Theta = \Theta \circ \alpha_t$ for any $t \in \mathbb{R}$, and $\mathfrak{A}_{F0} \subset D(\delta)$. Set $W(\beta, I) \equiv \pi_\varphi(\beta W(I)) \in \mathfrak{A}_F$. Suppose that $\varphi \circ \Theta = \varphi$. φ satisfies the Gibbs condition if and only if φ^w has the following product property ;*

$$\varphi^{W(\beta,I)}(AB) = \varphi^{W(\beta,I)}(A)\varphi^{W(\beta,I)}(B), \quad A \in \mathcal{A}(I), \quad B \in \mathcal{A}(I^c),$$

for any finite subset $I \subset \mathbb{Z}$.

Next let us turn to the variational principle. Before treating the thermodynamical limit, let us recall the definition of entropy of a state restricted to finite system. Let ω_I be a restriction of ω to $\mathfrak{A}_F(I)$. Then there exists a density matrix ρ_I which satisfies

$$\omega_I(A) = \mathrm{Tr}(\rho_I A), \quad A \in \mathfrak{A}_F(I).$$

The entropy $S(\omega_I)$ of ω_I is defined as

$$S(\omega_I) \equiv -\omega_I(\log \rho_I).$$

For the translation invariant potential $\Phi \in \mathcal{P}_\tau$, and translation invariant state ω, there exist the thermodynamical limit of pressure, entropy and mean energy;

$$\begin{aligned} P(\Phi) &\equiv v.H. \lim_{I\to\infty} \frac{1}{|I|} \log \mathrm{Tr}_I(\mathrm{e}^{-H(I)}) \\ e_\Phi(\omega) &\equiv v.H. \lim_{I\to\infty} \frac{1}{|I|} \omega(H(I)) \\ s(\omega) &\equiv v.H. \lim_{I\to\infty} \frac{1}{|I|} S(\omega_I). \end{aligned}$$

Here the limit $v.H.$ is taken in the sense of van Hove . For any translation invariant state φ, the following variational inequality is valid;

$$P(\beta\Phi) \geq s(\varphi) - \beta e_\Phi(\varphi).$$

It can be shown that there exists a state which achieves the equality;

Theorem 5.1 *For any $\Phi \in \mathcal{P}_\tau$, there exists a state φ which satisfies*

$$P(\beta\Phi) = s(\varphi) - \beta e_\Phi(\varphi).$$

The KMS-condition is equivalent to Gibbs condition and variational equality under the following assumptions. c.f. [4].

Theorem 5.2 *Let α_t be a strongly continuous one parameter group of automorphisms with the generator δ_α. Suppose that α_t satisfies $\alpha_t\Theta = \Theta\alpha_t$, and the domain $D(\delta_\alpha)$ of δ_α contains the strict local algebra $\mathfrak{A}_{F0}$. Let δ be the restriction of δ_α to $\mathfrak{A}_{F0}$.*

There exists a translation invariant standard potential $\Phi \in \mathcal{P}_\tau$ such that

$$\delta(A) = \sum_{I \cap J \neq \varnothing} i[\Phi(J), A], \quad A \in \mathfrak{A}_F(I).$$

Suppose further that the strict local algebra $\mathfrak{A}_{F0}$ is a core of the generator δ_α. Then, for any translation invariant state φ, the following conditions are equivalent.

1. *φ is the (α_t, β)-KMS state,*
2. *φ satisfies the (δ, β)-Gibbs condition,*
3. *$P(\beta\Phi) = s(\varphi) - \beta e_\Phi(\varphi)$.*

Let us return to our system.

Lemma 5.1 *Let α_t^F be a one-parameter group of Bogoliubov automorphisms on $\mathfrak{A}_F$ such that the generator δ of α_t on the one-particle space $l_2(\mathbf{Z})$ is bounded. The strict local algebra $\mathfrak{A}_{F0}$ is a core for the generator δ of α_t^F.*

proof
This follows from the following fact. (c.f. Corollary 3.1.20. [6])
Let $\mathcal{D}$ be a dense α_t invariant subalgebra of $\mathfrak{A}_F$ such that any element of $\mathcal{D}$ is in the domain of the generator δ of α_t^F. Then $\mathcal{D}$ is a core of δ .

For our purpose $\mathcal{D}$ is the set of all polynomials in creation and annihilation operators with all smearing functions in $l_2(\mathbf{Z})$. As the generator of α_t on the one-particle space $l_2(\mathbf{Z})$ is bounded , the graph closure of $(\mathfrak{A}_{F0}, \delta(\mathfrak{A}_{F0})$ contains $(\mathcal{D}, \delta(\mathcal{D}))$. This shows the claim of the lemma. □

Obviously $\alpha_t^F\Theta = \Theta\alpha_t^F$, $\mathfrak{A}_{F0} \subset D(\delta_{\alpha^F})$, and ω_ρ is translation invariant. Now let us check that the potential of our Fermion system is a translation invariant standard potential. Requiring

$$\frac{d}{dt}\alpha_t^F(a_m)\Big|_{t=0} = \sum_{I \subset \mathbb{Z}, m \in I} i[\Phi_F(I), a_m],$$

we obtain the potential $\Phi_F(I)$ as follows;

$$\Phi_F(\{m\}) = \gamma(a_m^* a_m - \frac{1}{2})$$

$$\Phi_F(\{n, n+1\}) = -\frac{1}{2}(a_n^* a_{n+1} + a_{n+1}^* a_n)$$

$$\Phi_F(\{n, m : n \neq m\}) = \begin{cases} \frac{2i}{\pi}\frac{\beta_+ - \beta_-}{\beta_+ + \beta_-}\frac{n-m}{(n-m)^2-1}(a_n^* a_m - a_m^* a_n), \text{ for } n-m \text{ even} \\ -\frac{2i}{\pi}\gamma\frac{\beta_+ - \beta_-}{\beta_+ + \beta_-}\frac{1}{n-m}(a_n^* a_m - a_m^* a_n), \text{ for } n-m > 1 \text{ odd} \end{cases}$$

$$\Phi_F(I) = 0 \; for \; |I| \geq 3 \tag{5}$$

Obviously, the conditions 1,2,3, and 6 of the standard Potential are satisfied. For $b \in A(J)$, and $n \in J^c$ the conditional expectation satisfies

$$\tau(E_J(a_n)b) = \tau(a_n b) = 0.$$

So we have $E_J(a_n) = 0$, which implies the condition 4 by

$$E_J(a_m^* a_n) = a_m^* E_J(a_n) = 0, \quad m \in J, \; n \in J^c.$$

The condition 5 can be proved by noting that the internal energy $U_F(I)$ and the surface energy $W_F(I)$ are elements of $\mathfrak{A}_F$ and given by

$$U_F(I) = \frac{1}{2}\sum_{m\in I} a^*\left(P(I)f_m\right)a_m + a_m^* a\left(P(I)f_m\right) + \gamma\sum_{m\in I}(a_m^* a_m - \frac{1}{2})$$

$$W_F(I) = \sum_{m\in I} a^*\left(P(I^c)f_m\right)a_m + a_m^* a\left(P(I^c)f_m\right). \tag{6}$$

Here f_m is given by

$$f_m = \frac{i}{\pi}\frac{\beta_- - \beta_+}{\beta_- + \beta_+}\sum_{l=-\infty}^{\infty}\frac{4l}{4l^2-1}|2l+m\rangle - \frac{1}{2}(|m+1\rangle + |m-1\rangle)$$
$$- \frac{2i}{\pi}\gamma\frac{\beta_- - \beta_+}{\beta_- + \beta_+}\sum_{l=-\infty}^{\infty}\frac{1}{2l+1}|2l+1+m\rangle,$$

where the sum converges in the norm of $l_2(\mathbf{Z})$. $P(I)$ is a projection onto I. Hence the conditions required in Theorem 5.2 are satisfied. Recall that ω_ρ is the (α_F, β)-KMS state, with $\beta = (\beta_+ + \beta-)/2$. In general, the KMS state of a quasi free motion is unique, and ω_ρ is the only (α^F, β)-KMS state. Due to Theorem 5.2 ω_ρ is the only translation invariant state which satisfies (δ, β)-Gibbs condition and (Φ_F, β)-variational equality as well.

Theorem 5.3 *Let α_t^F be the one-parameter group of automorphisms of $\mathfrak{A}_F$ determined by (3)(4). Let $\Phi_F(I)$ be the standard potential of (5), then ,*

$$\delta(A) = \sum_{I\cap J\neq\varnothing} i[\Phi_F(J), A], \quad A \in \mathfrak{A}_F(I),$$

where δ is the restriction of the generator of α_t^F. The non-equilibrium steady state ω_ρ is the unique (α_t^F, β)-KMS state, satisfies the (δ, β)-Gibbs condition and (Φ_F, β)-variational equality. No other translation invariant state satisfies neither (δ, β)-Gibbs condition nor (Φ_F, β)-variational equality.

The above result in our Fermion system implies the variational equality of ω_ρ, because all the potentials $\Phi(I)$ belong to $\mathfrak{A}_+$. However for the uniqueness of the state which satisfies variational equality for the spin system of the XX model, we need some more preparations. Suppose that ω and ω' are translation invariant states of $\mathfrak{A}_S$ that satisfy the (Φ_S, β)-variational principle. Then their restriction to $\mathfrak{A}_+$, $\omega|_{\mathfrak{A}_+}$ and $\omega'|_{\mathfrak{A}_+}$ satisfies the (Φ_S, β)-variational principle as a state on $\mathfrak{A}_+$. Note that we are using the fact that the mean entropy is affine and and that $s(\omega) = s(1/2(\omega + \omega \circ \Theta))$ On the other hand, the statement of the Theorem 5.2 is satisfied even if $\mathfrak{A}_F$ is replaced by $\mathfrak{A}_+$. (The proof is the same as the $\mathfrak{A}_F$ case.) So any translation invariant state which satisfies the $(\Phi_S|_{\mathfrak{A}_+}, \beta)$-variational equality is the $(\alpha_t|_{\mathfrak{A}_+}, \beta)$-KMS state. Let ω_+ be the restriction of our NESS ω_ρ to $\mathfrak{A}_+$

Lemma 5.2 *ω_+ is the unique $(\beta, \alpha_t|_{\mathfrak{A}_+})$-KMS state on $\mathfrak{A}_+$.*

proof
We apply Theorem 5.4.3 of [7]. $(\mathfrak{A}_F, \mathfrak{A}_+, \mathbb{Z}_2, \alpha_F, \gamma)$ is the field system, where γ is the group action of $\mathbb{Z}_2$ with $\gamma_1 = \Theta$. Let φ_+ be the extremal (α_F, β)-KMS state over $\mathfrak{A}_+$, Then Theorem 5.4.3 of [7]. implies that any extremal α^F invariant extension φ of φ_+ to $\mathfrak{A}_F$ is a $(\alpha_t^F \gamma_{\xi_t}, \beta)$-KMS state, where $t \to \xi_t$ is a continuous one-parameter group of $\mathbb{Z}_2$. However, as $\mathbb{Z}_2$ is discrete, $\xi_t = 1$. So the extension is a (α_t^F, β)-KMS state. But as the KMS state of our quasifree motion is unique, we conclude that ω_+ is the unique $(\beta, \alpha_t|_{\mathfrak{A}_+})$-KMS state on $\mathfrak{A}_+$.
□
By this Lemma, we obtain $\omega|_{\mathfrak{A}_+} = \omega'|_{\mathfrak{A}_+}$. Furthermore we have $\omega = \frac{\omega + \omega \circ \Theta}{2} = \frac{\omega' + \omega' \circ \Theta}{2}$ because they are even states. We need another lemma to remove the restriction to $\mathfrak{A}_+$.

Lemma 5.3 *The NESS ω on $\mathfrak{A}_S$ is a factor state.*

proof
We use the notation of Lemma 4.1. The GNS representation of ω is unitary equivalent to $(\mathcal{H}_S, \pi_S, \Omega_S)$. Here, Hilbert space $\mathcal{H}_S$ is

$$\mathcal{H}_S = \mathcal{H}_+ \oplus \mathcal{H}_-$$

where

$$\mathcal{H}_+ = \overline{\pi_\omega(\mathfrak{A}_+)\Omega}, \quad \mathcal{H}_- = \overline{\pi_\omega(\mathfrak{A}_-)\Omega},$$

and

$$\Omega_S \equiv \Omega \oplus 0,$$

where $B \equiv B_2 T$. Let us define σ on $\mathfrak{A}_+$ by the following equation;

$$\sigma(A) \equiv BAB^{-1} \quad A \in \mathfrak{A}_+.$$

Set $U\mathcal{H}_S U^* \equiv \mathcal{H}_+ \oplus \mathcal{H}_+$. By U we denote the unitary operator from $\mathcal{H}_S$ to $U\mathcal{H}_S U^*$ determined by

$$UA\Omega = A\Omega, \quad U\sigma(A)B\Omega = A\Omega \quad A \in \mathfrak{A}_+.$$

We have

$$UAU^* = \begin{pmatrix} A & 0 \\ 0 & \sigma(A) \end{pmatrix} \quad A \in \mathfrak{A}_+, \quad UBU^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let us consider an element A of $U\pi_S(\mathfrak{A})^{''} \cap \pi_S(\mathfrak{A})^{'} U^*$. As $A \in U\pi_S(\mathfrak{A})^{'} U^*$, we obtain

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, d \in \mathbb{C}1, \quad xb = b\sigma(x), \quad cA = \sigma(A)c, \quad x \in \mathfrak{A}_+,$$

because $\pi_+(\mathfrak{A}_+)^{''}$ on $\mathcal{H}_+$ is a factor. If $b \neq 0$ or $c \neq 0$ the relation implies that the representation π_+ of $\mathfrak{A}_+$ is equivalent to $\pi_+ \circ \sigma$ We will see that this leads to a contradiction. As a consequence, we conclude $b = c = 0$. Similarly, as A commutes with UBU^*, we obtain $a = d$. Hence we obtain $A \in \mathbb{C}1$. Thus, ω is a factor state. □

Lemma 5.4 *The quasifree states ω and $\omega \circ \sigma$ restricted to $\mathfrak{A}_+$ are disjoint.*

We can apply the results of [14]. It suffices to show that $\rho^{1/2} - \theta_- \rho^{1/2} \theta_-$ and $(1-\rho)^{1/2} - \theta_-(1-\rho)^{1/2}\theta_-$ are not of Hilbert Schmidt class. This claim can be proved in the same way as the proof of Lemma 4.3. We omit the proof.

The previous argument shows that $\omega = \frac{\omega' + \omega' \circ \Theta}{2}$ if ω' is a translation invariant state satisfying the variational equality. As ω is a translation invariant factor state, ω is ergodic (extremal in the set of translation invariant states) so $\omega = \omega' = \omega' \circ \Theta$. As a consequence we arrive at our main result of this article.

Theorem 5.4 *The non-equilibrium stay state ω of XX-model is the unique translation invariant state which satisfies the (Φ_S, β)-variational equality;*

$$P(\beta\Phi_S) = s(\omega) - \beta e_{\Phi_S}(\omega),$$

where

$$\Phi_S(\{m\}) = \frac{1}{2}\gamma\sigma_z^m, \quad , \quad \Phi_S(\{n, n+1\}) = -\frac{1}{2}\left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y\right)$$

$$\Phi_S([m,n]) = \begin{cases} -\frac{1}{\pi}\frac{\beta_+ - \beta_-}{\beta_+ + \beta_-}\frac{n-m}{(n-m)^2 - 1}\prod_{k=m+1}^{n-1}\sigma_z^k \cdot (\sigma_y^m \sigma_y^n - \sigma_x^m \sigma_x^n) \text{ for } n-m \text{ even} \\ \frac{1}{\pi}\gamma\frac{\beta_+ - \beta_-}{\beta_+ + \beta_-}\frac{1}{n-m}\prod_{k=m+1}^{n-1}\sigma_z^k \cdot (\sigma_y^m \sigma_y^n - \sigma_x^m \sigma_x^n) \text{ for } n-m > 1 \text{ andodd} \end{cases}.$$

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